Math 215, Fall 2026, Problem Set 6

0) **[50 pts]** Discuss some topics relevant to this homework (e.g. rational and irrational numbers, infinite decimals, countable sets, or Cantor's diagonal argument) with an LLM of your choice. Ask some interesting questions, seek clarification, and/or probe for more depth. Make the conversation interactive and show that you have given some thought to the LLM's responses. Turn in that chat log as your answer to this question. If you use an LLM for any of the remaining questions on this problem set, also turn in the corresponding chat logs.

1) **[10 pts]** Recall that fractions a/b and c/d represent the same rational number precisely when

$$ad = bc.$$

Suppose

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} \qquad \text{and} \qquad \frac{c_1}{d_1} = \frac{c_2}{d_2},$$

where all denominators are nonzero. Prove that

$$\frac{a_1 c_1}{b_1 d_1} = \frac{a_2 c_2}{b_2 d_2}.$$

Explain briefly why this shows that multiplication of rational numbers is well-defined.

2) **[10 pts]** Let r be a rational number and let x be an irrational real number.

   (a) Prove that $r + x$ is irrational.

   (b) If $r \neq 0$, prove that rx is irrational.

3) **[10 pts]** Let

$$x = 0.27272727\ldots$$

where the block 27 repeats forever. Find a fraction representing x, and justify your answer.

4) **[10 pts]** Let A be a finite set and let B be a denumerable set. Prove that $A \cup B$ is denumerable.

Hint: First consider the elements of A which are not already in B.

5) **[10 pts]** Let S be the set of all infinite sequences of 0's and 1's:

$$S = \{(a_1, a_2, a_3, \ldots) : a_i \in \{0, 1\}\}.$$

Prove that S is uncountable.

Hint: Suppose the elements of S could be listed $s_1, s_2, s_3, \ldots$. Construct a new sequence whose nth entry differs from the nth entry of s_n.