Geometry and dynamics of surface homeomorphisms.

Chris Leininger (UIUC)

October 1, 2011

Surfaces

Surfaces are everywhere

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- Surface of the earth – a *sphere*.

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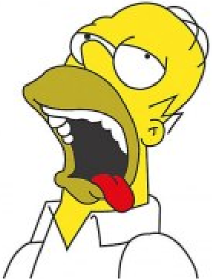
- Surface of the earth – a *sphere*.
- Surface of a doughnut – a *torus*.

Surfaces



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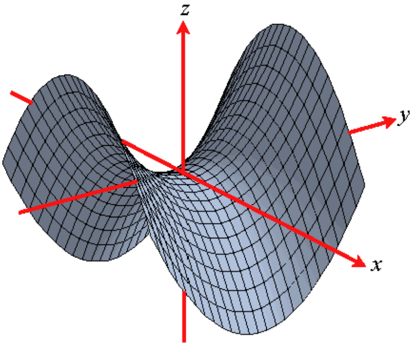

Surfaces are everywhere

- Surface of the earth – a *sphere*.
- Surface of a doughnut – a *torus*.
- Solutions to $z = y^2 - x^2$ in $\mathbb{R}^3$.

First definition

"Definition": A *surface* is a subset $S \subset \mathbb{R}^3$ that "looks like" the plane near any point.

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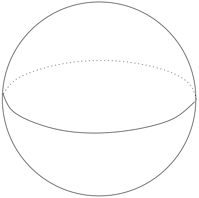

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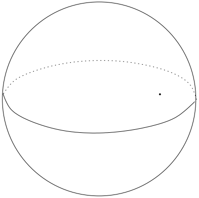

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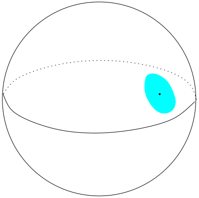

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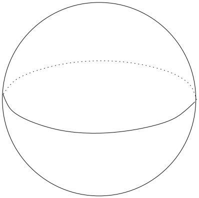

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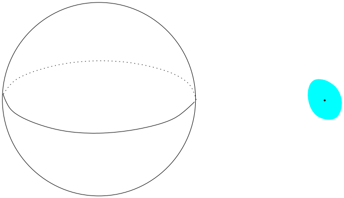

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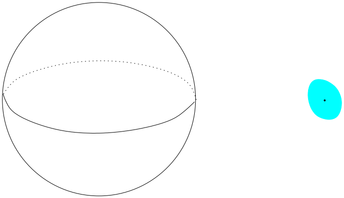

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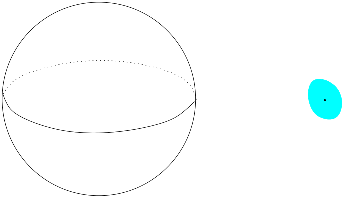

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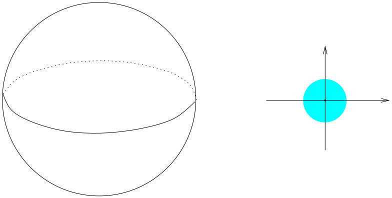

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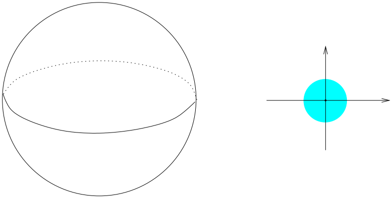

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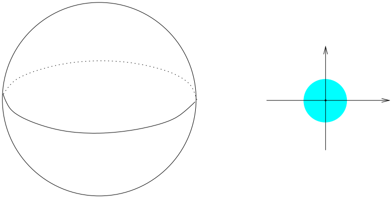

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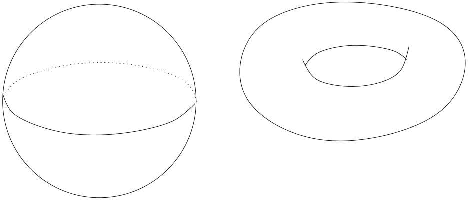

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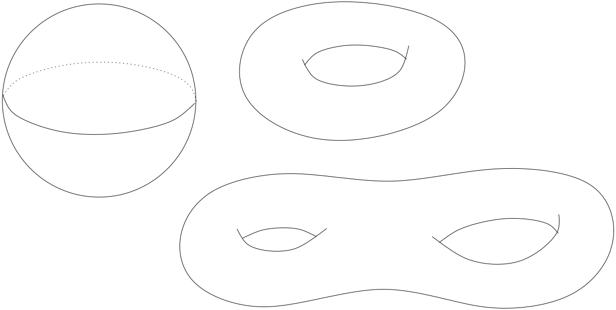

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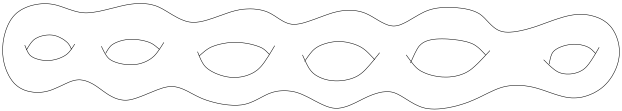

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More examples... a *genus* 6 surface.

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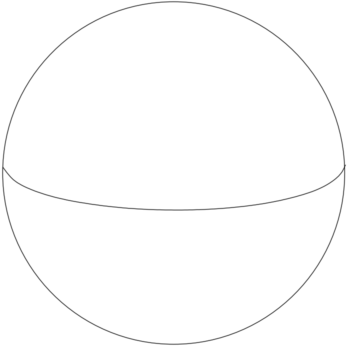

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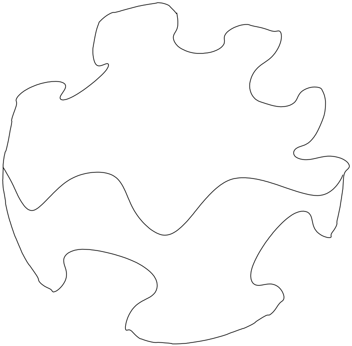

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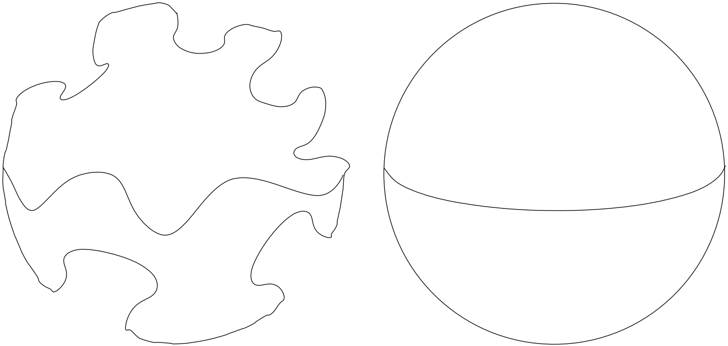

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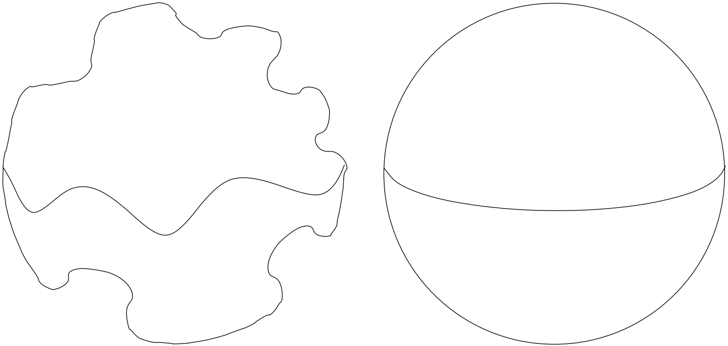

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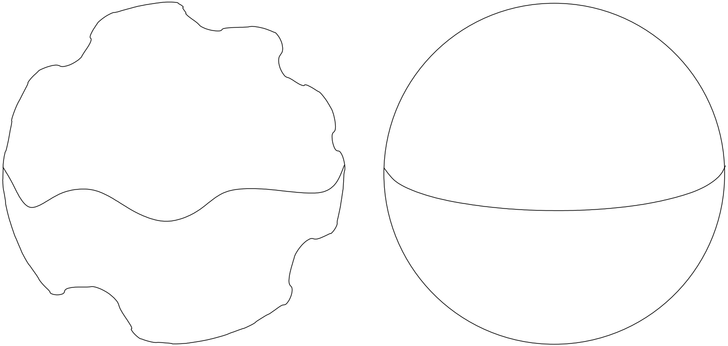

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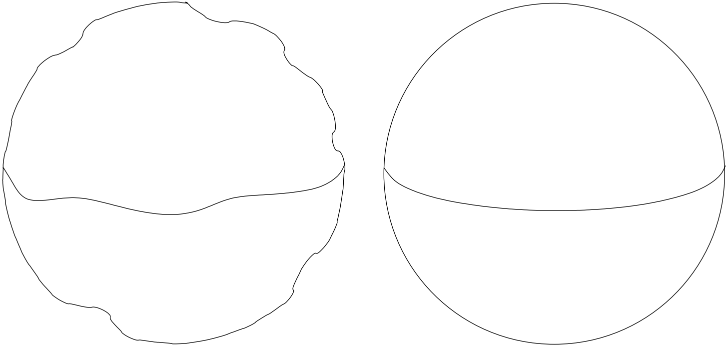

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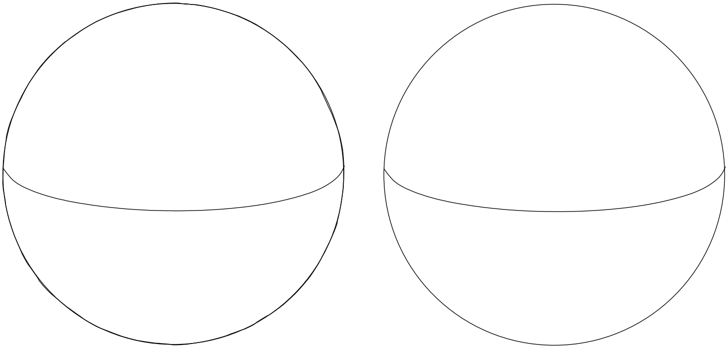

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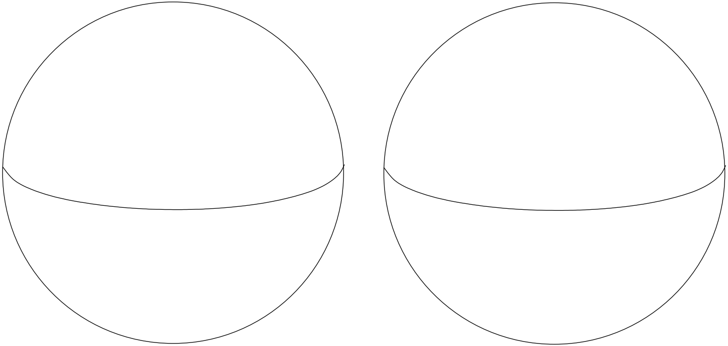

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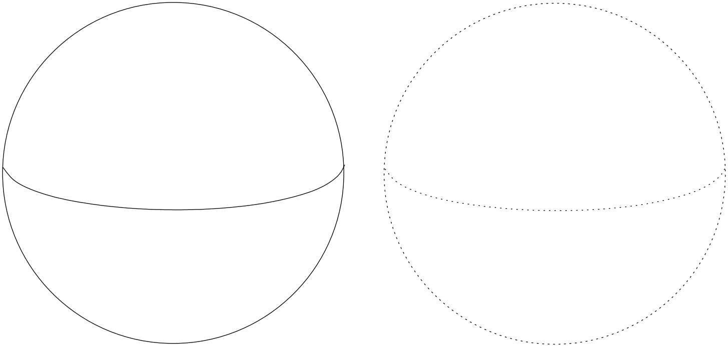

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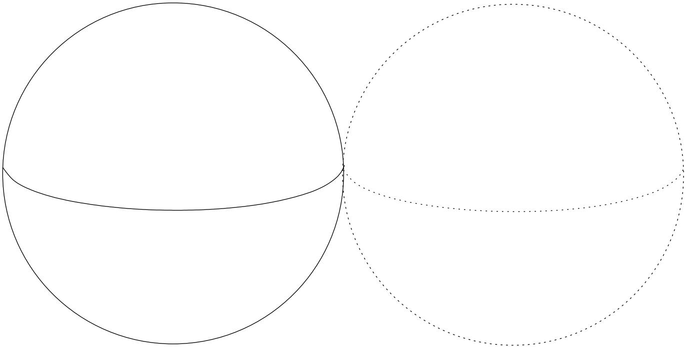

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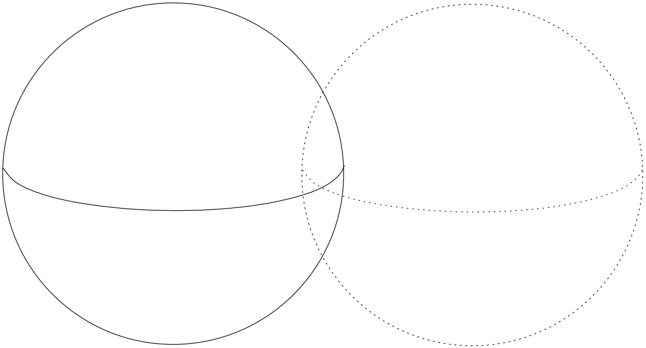

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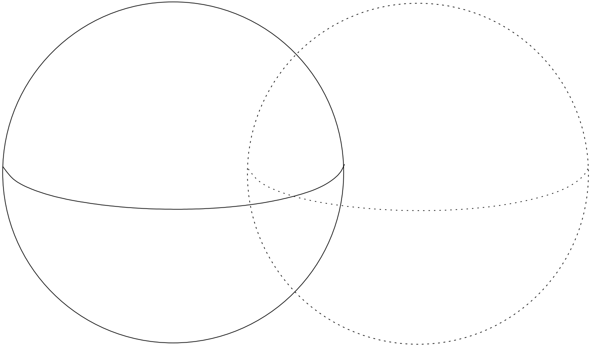

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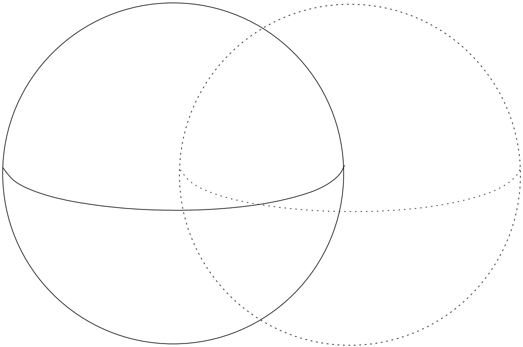

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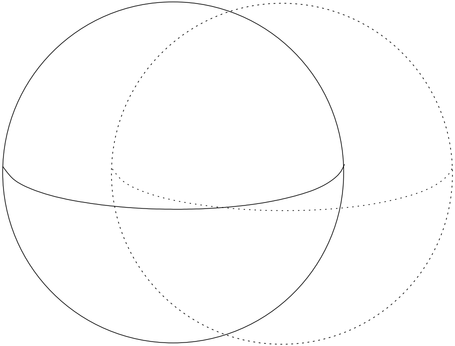

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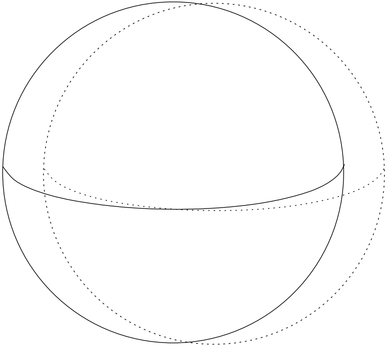

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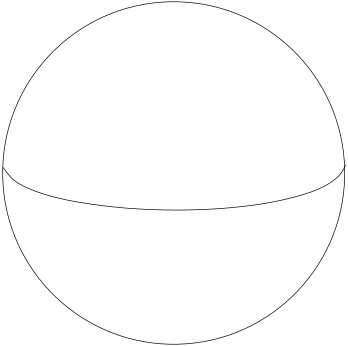

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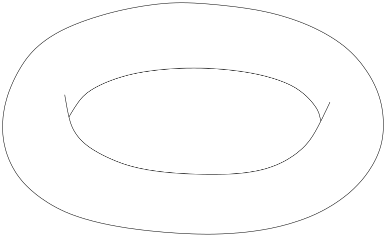
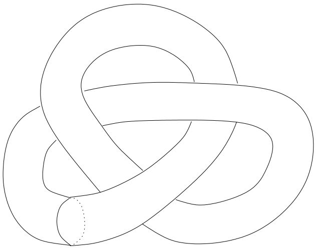
Not general enough...

Another example



We also want this surface

Another example



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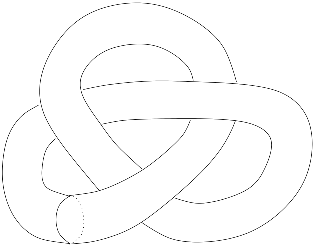

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To see this, deform this surface to

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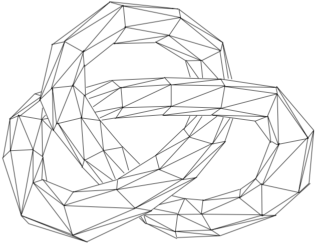

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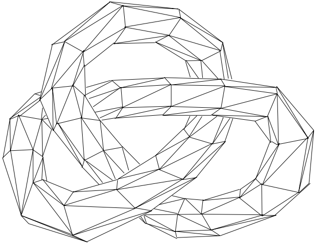

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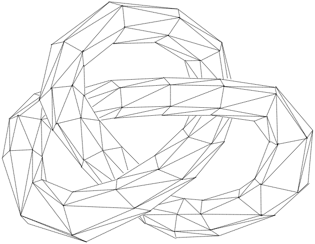

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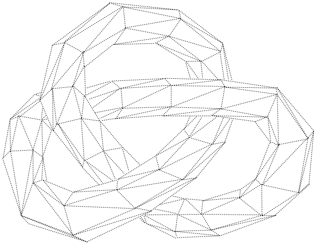

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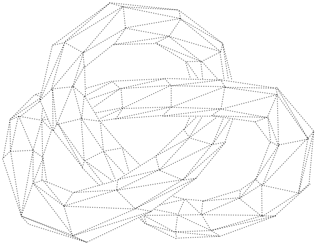

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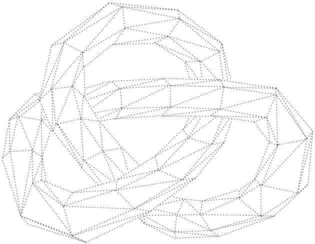

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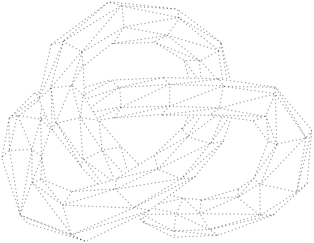

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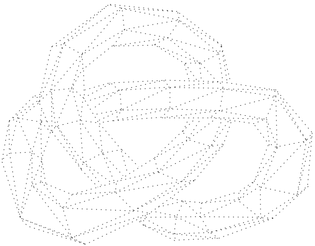

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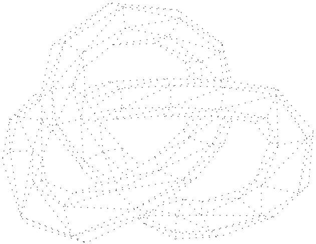

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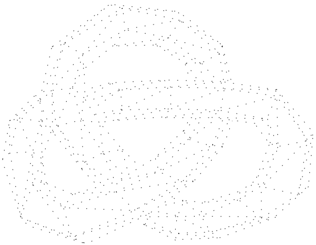

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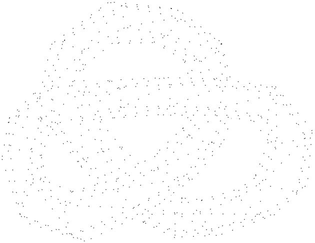

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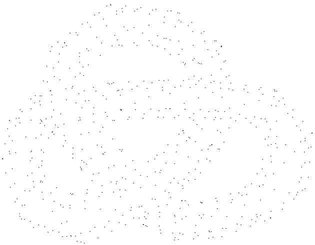

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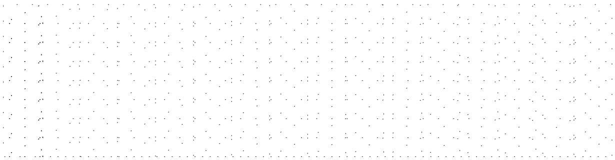

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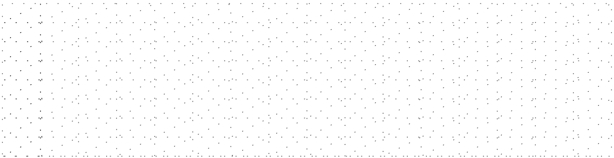

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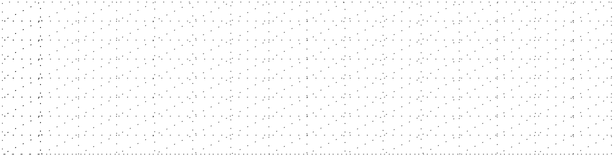

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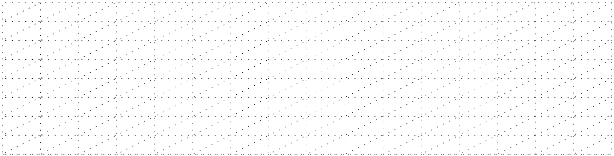

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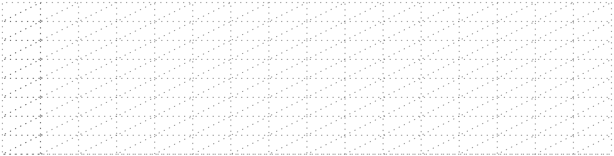

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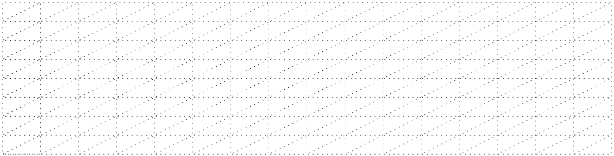

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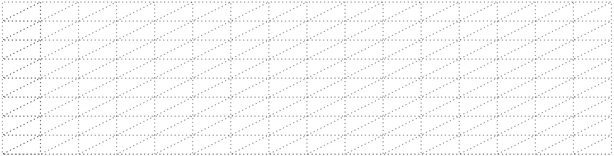

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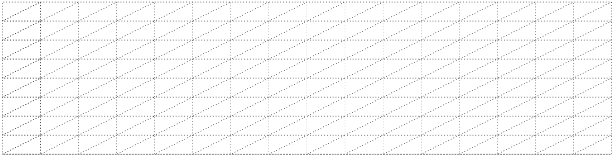

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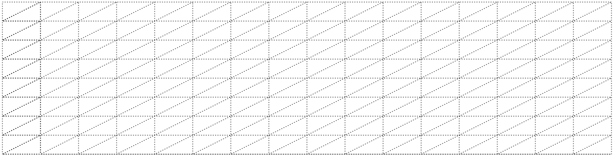

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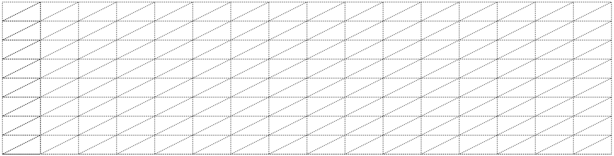

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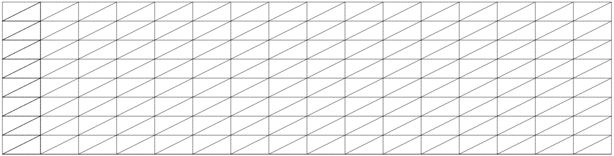

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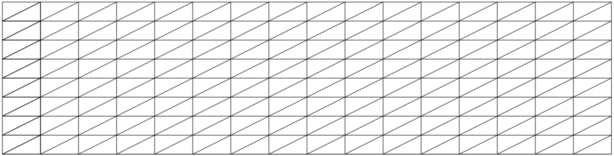

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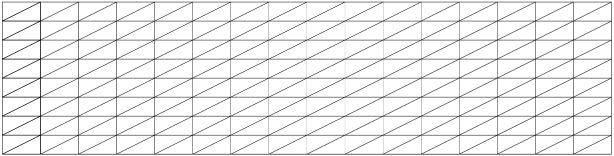

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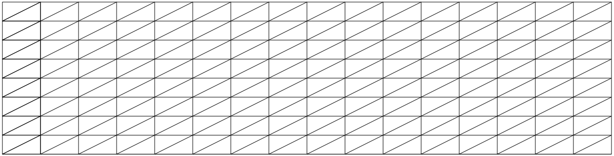

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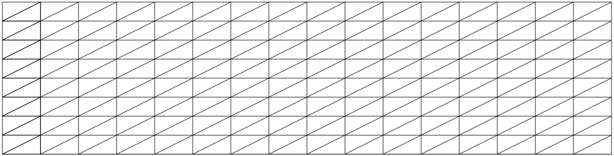

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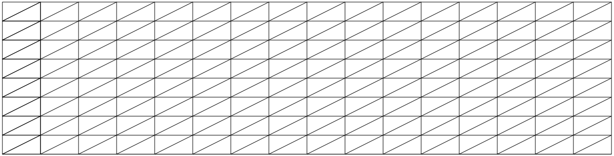

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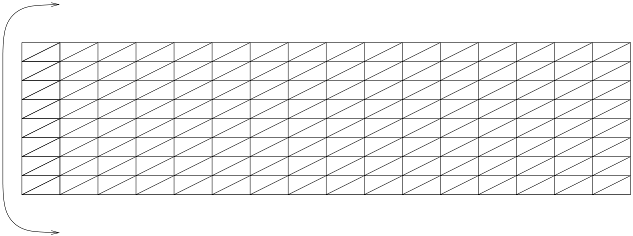

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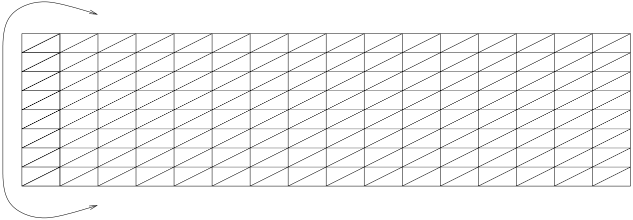

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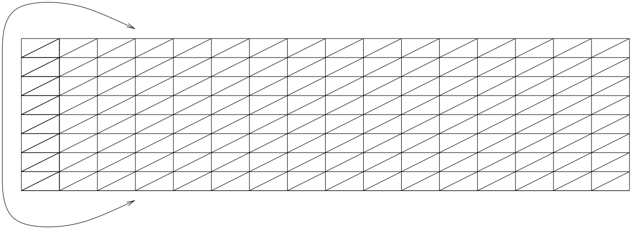

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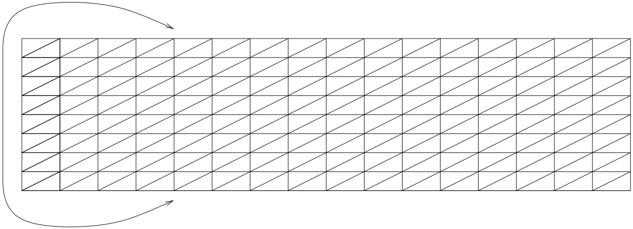

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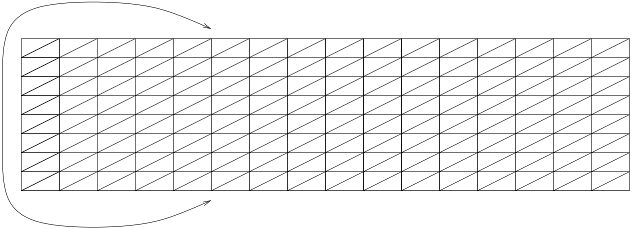

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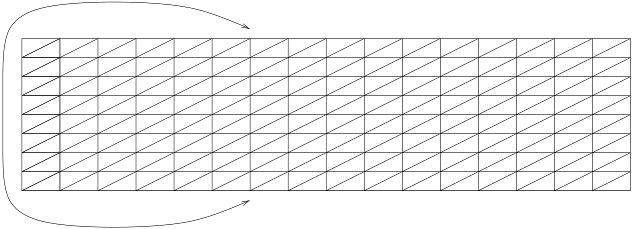

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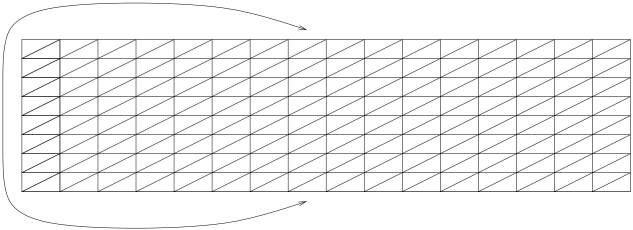

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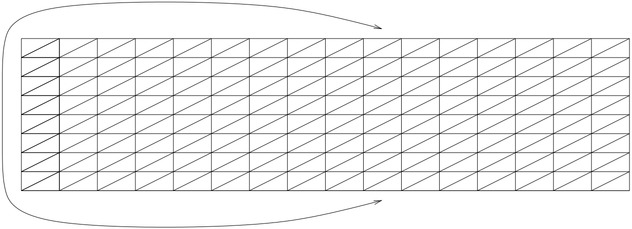

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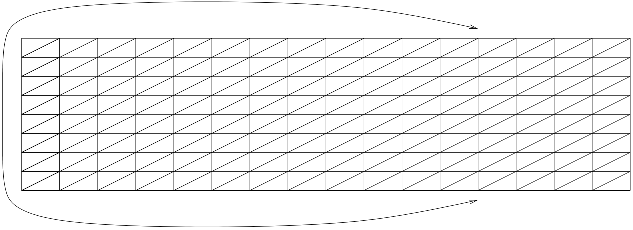

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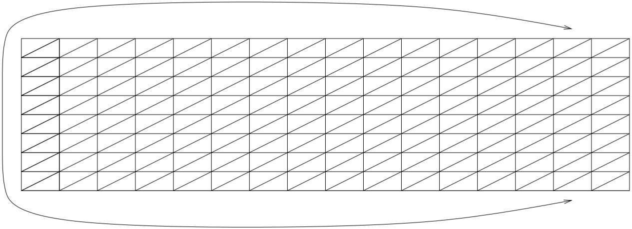

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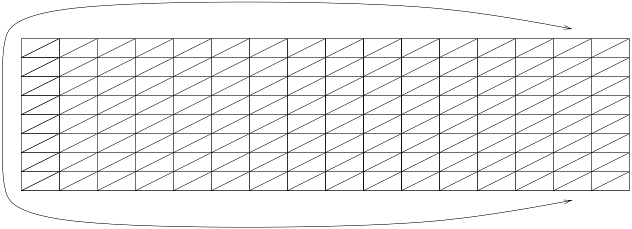

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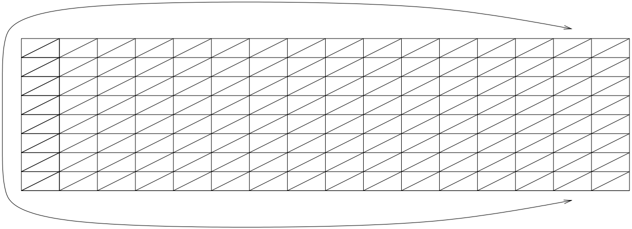

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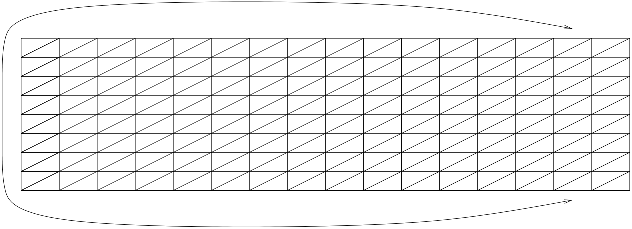

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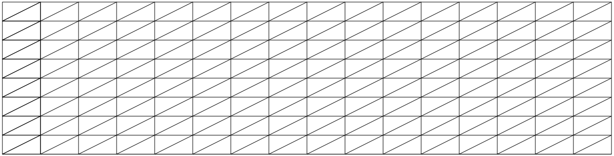

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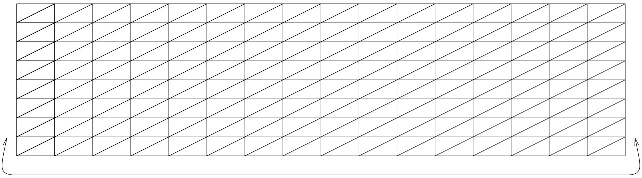

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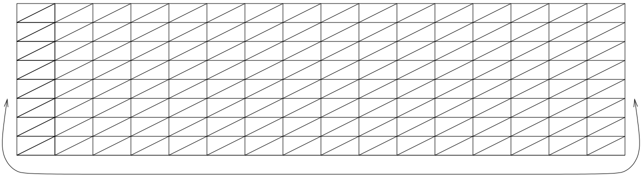

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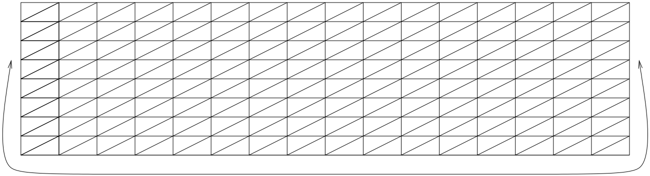

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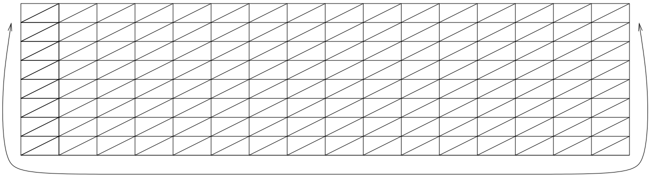

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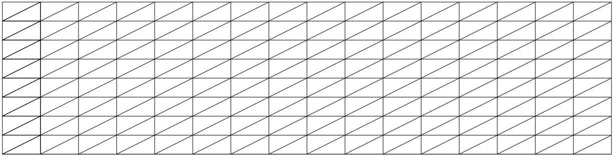

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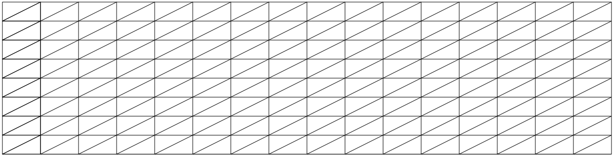

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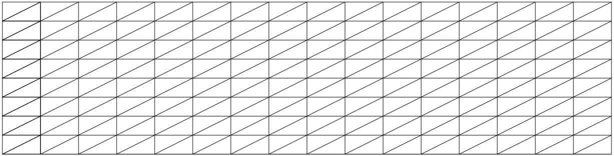

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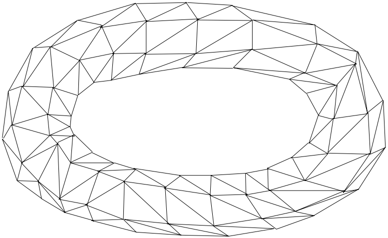

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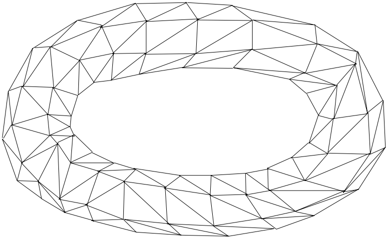

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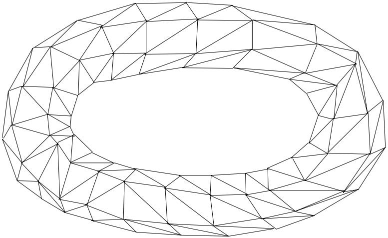

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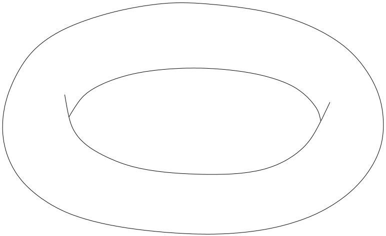

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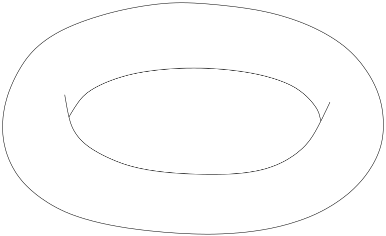

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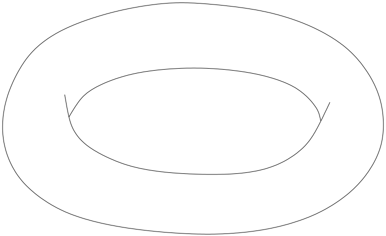

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Given two subsets $S_1, S_2 \subset \mathbb{R}^3$, a *homeomorphism* $f : S_1 \to S_2$ is a continuous bijection with continuous inverse. If such f exists, we say S_1 and S_2 are *homeomorphic*, written $S_1 \cong S_2$.

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$S \subset \mathbb{R}^3$ is a *surface* if for all $x \in S$ there is an $\epsilon > 0$ so that the set

$$B_\epsilon(x) = \{z \in \mathbb{R}^3 \mid |z - x| < \epsilon\}$$

is homeomorphic to the unit disk in $\mathbb{R}^2$.

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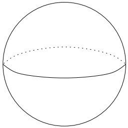

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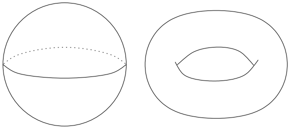

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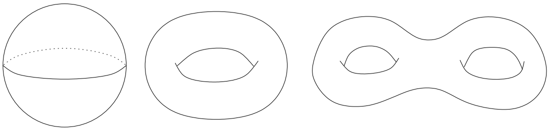

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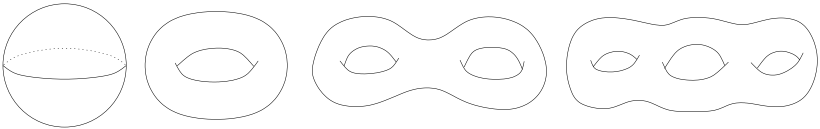

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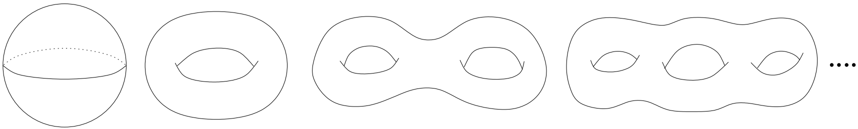

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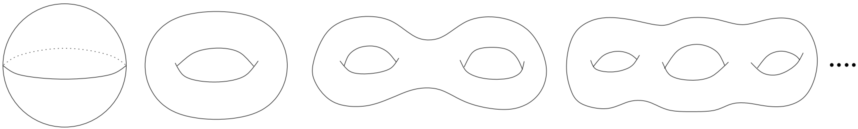

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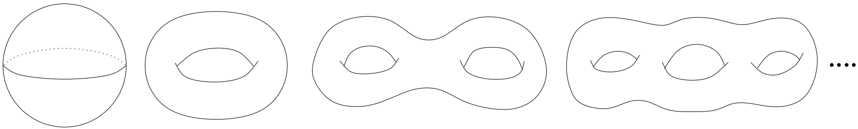

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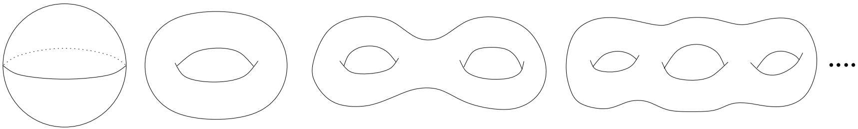

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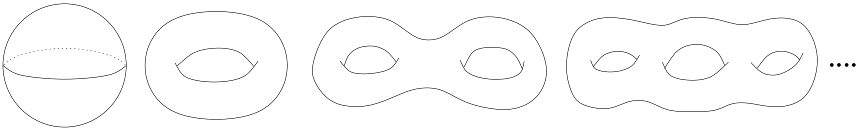

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- Connected $\Rightarrow$ any 2 points connected by a path.

3-dimensions

Can also define n–dimensional analogues of surfaces:

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Can use surfaces to study 3–manifolds.

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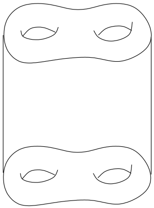

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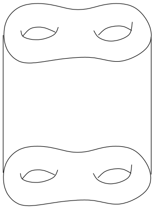

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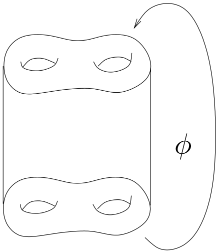

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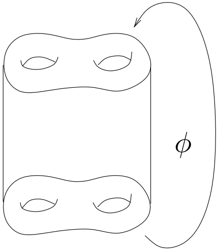

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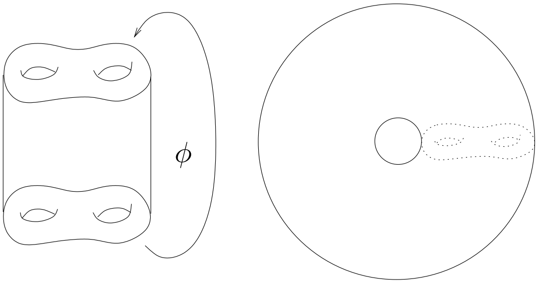

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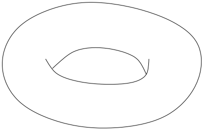

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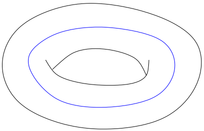

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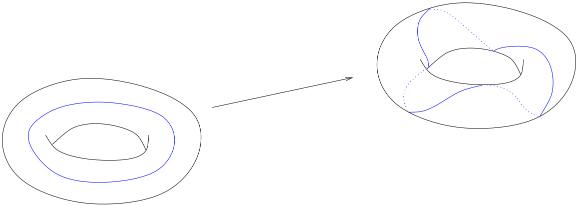

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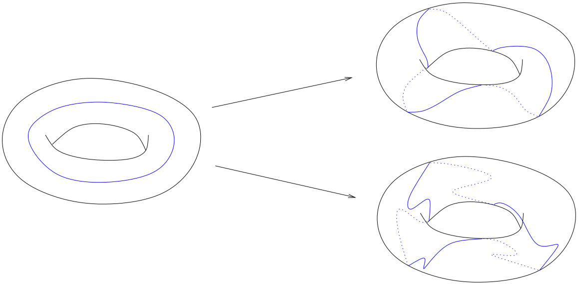

Homeomorphisms and isotopy

Definition

Homeomorphisms $\phi_0, \phi_1 : S \to S$ are *isotopic* if there is a continuous family of homeomorphisms

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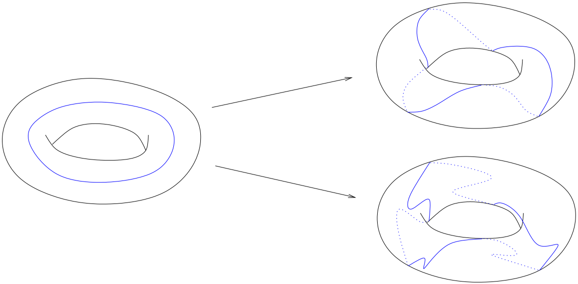
interpolating between ϕ_0 and ϕ_1.



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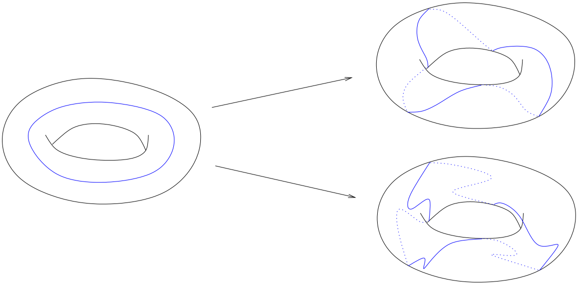

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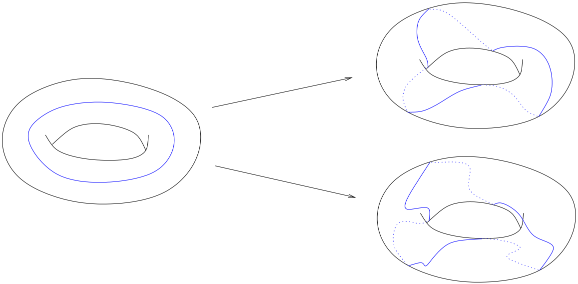

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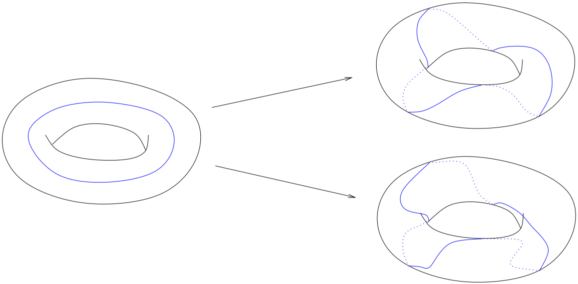

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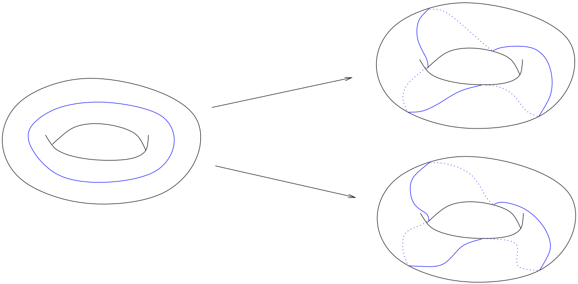

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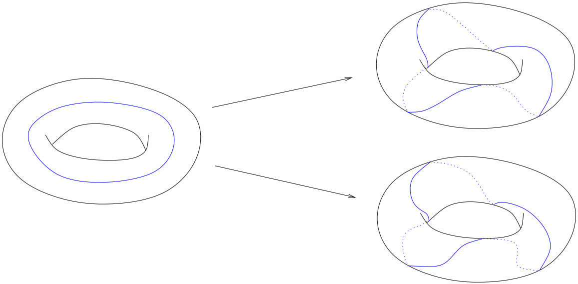

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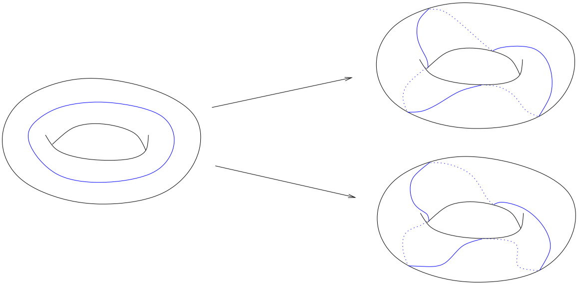

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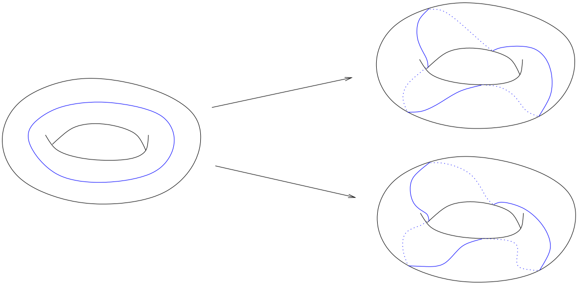

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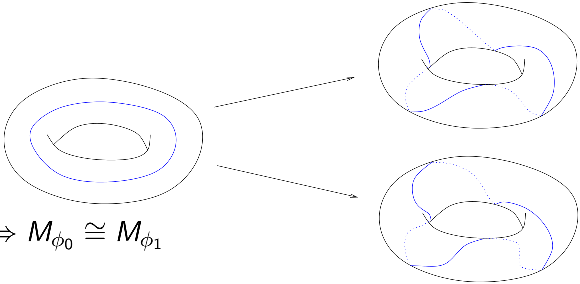

Fact: $\phi_0 \simeq \phi_1 \Rightarrow M_{\phi_0} \cong M_{\phi_1}$

Mapping class group

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Given a surface S, the *mapping class group* of S is defined by

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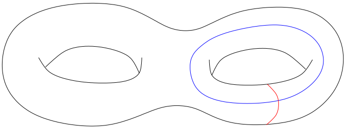

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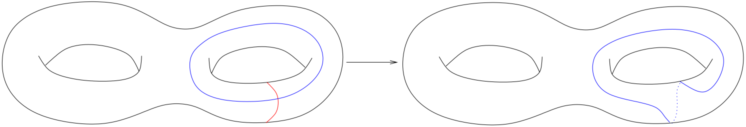

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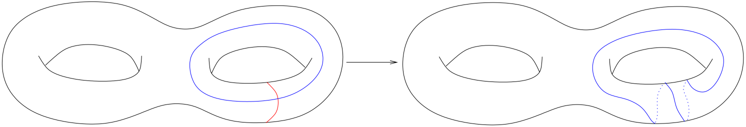

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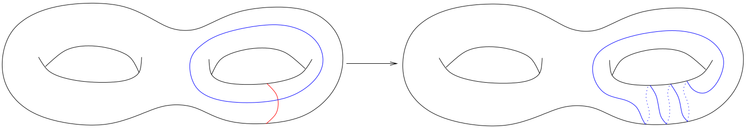
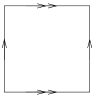


Torus

The torus T^2 is a square with sides glued.



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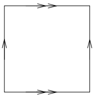

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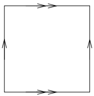

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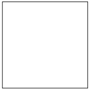

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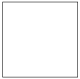

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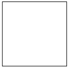

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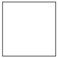

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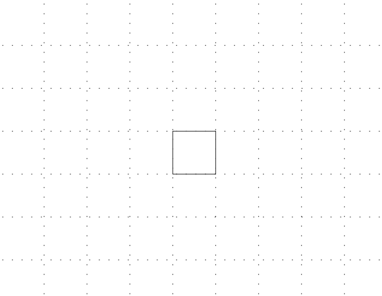

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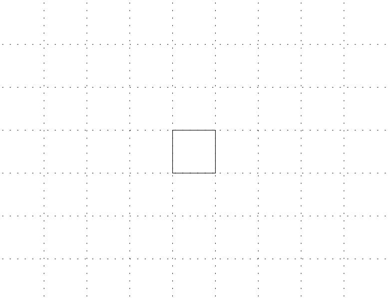

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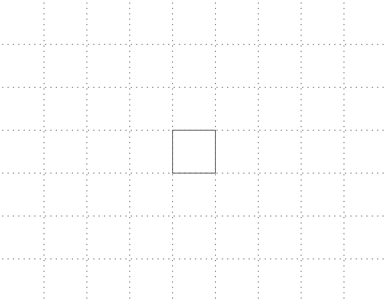

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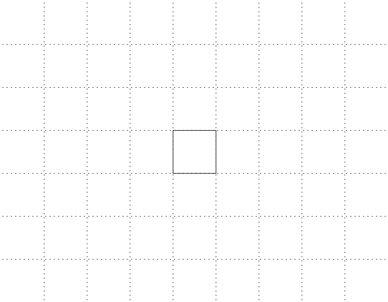

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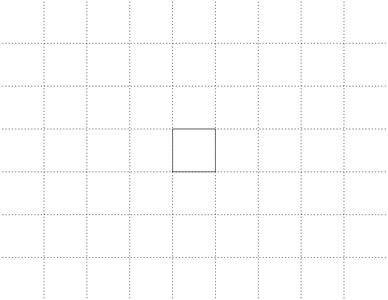

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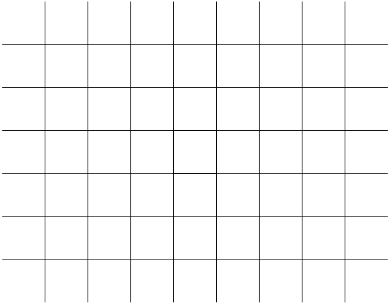

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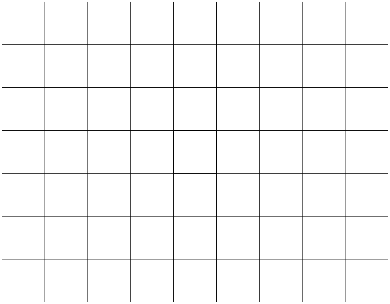

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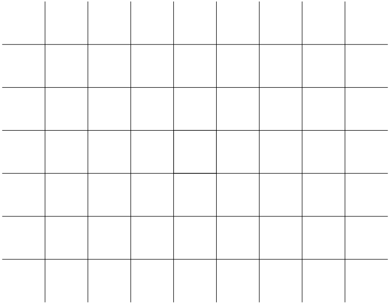

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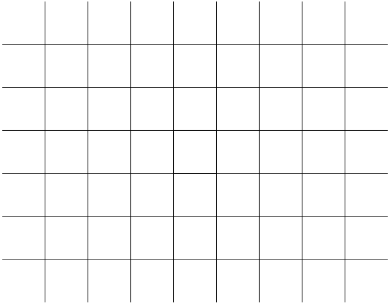

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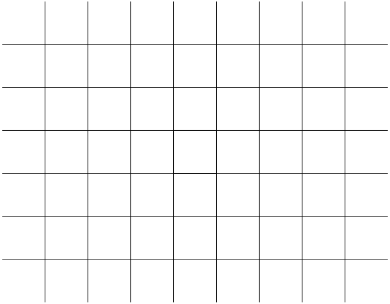

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Theorem

$Mod(T^2) = GL(2, \mathbb{Z})$

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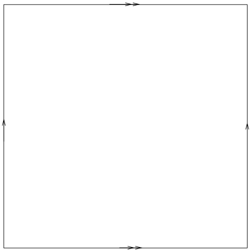

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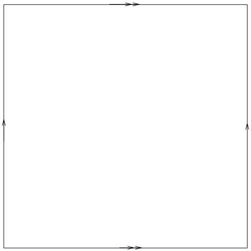

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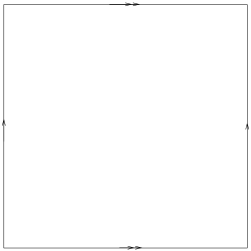

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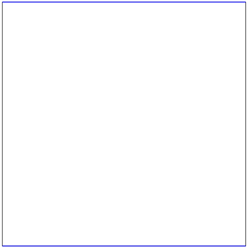

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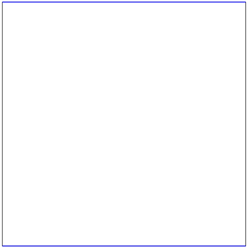

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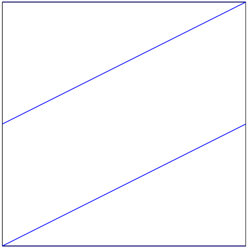

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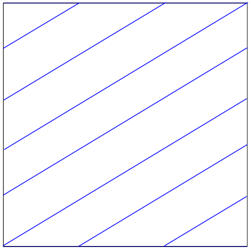

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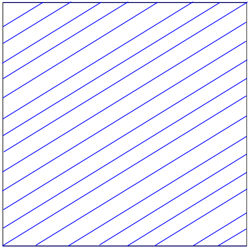

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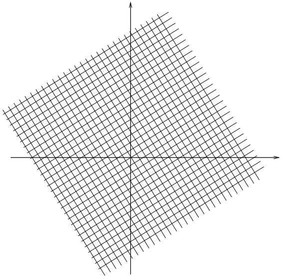

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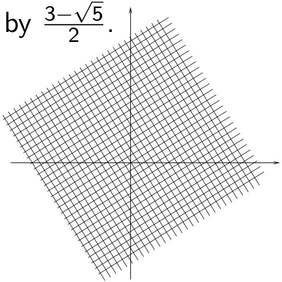

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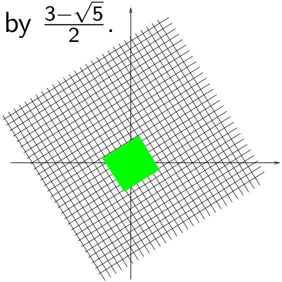

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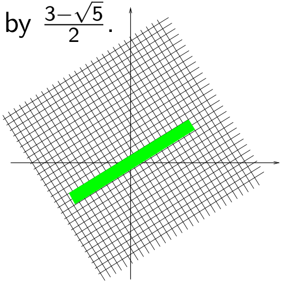

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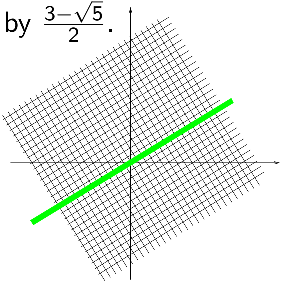
$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has two eigenvectors with eigenvalues $\lambda^{\pm 1} = \frac{3 \pm \sqrt{5}}{2}$.

Stretch by $\frac{3+\sqrt{5}}{2}$, contract by $\frac{3-\sqrt{5}}{2}$.



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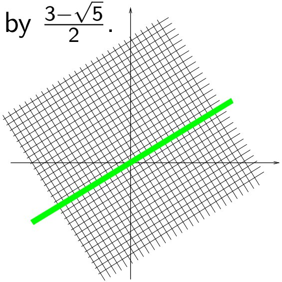

On T^2, $\lim_{n \to \infty} \dfrac{Length(\phi_A^n(\gamma))}{\lambda^n} = 1$ for all nontrivial curves γ.

Classification of mapping classes

Theorem (Thurston)

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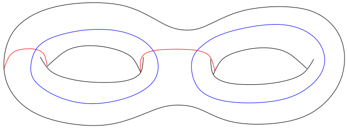
In third (generic) case, can actually assume ϕ looks like $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ locally. Third type is called *pseudo-Anosov*.

Example in genus 2



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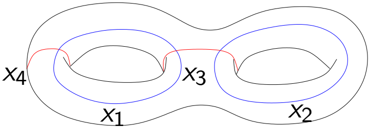

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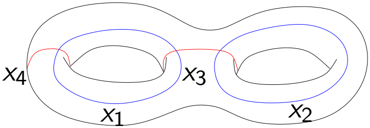

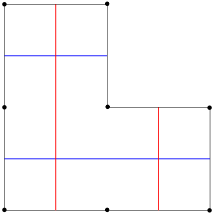

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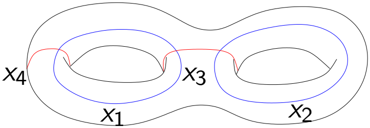

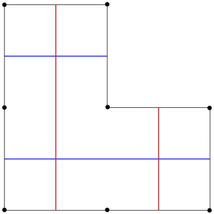

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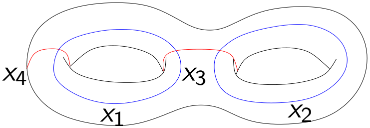

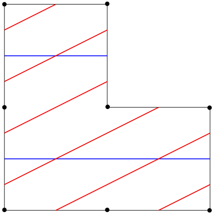

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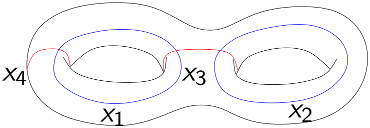

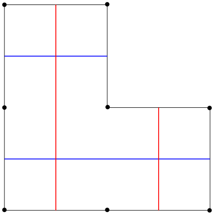

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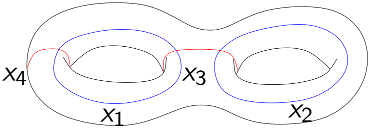

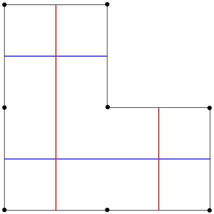

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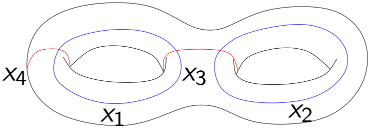

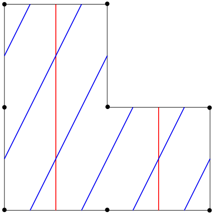

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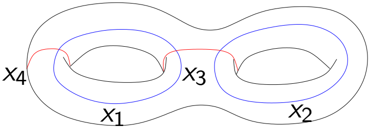

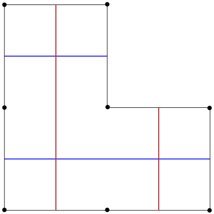

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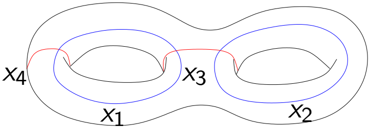

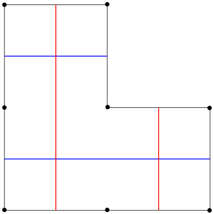

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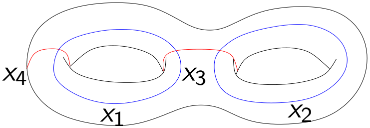

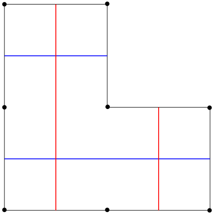

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The dilatation.

The number $\lambda(\phi) > 1$ is called the *dilatation* of ϕ, and it's a basic measure of the complexity of ϕ. Facts:

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- For a fixed surface S and $N > 0$, there are only finitely many* pseudo-Anosov homeomorphisms ϕ with $\lambda(\phi) < N$.
- For any $g \geq 1$ there are pseudo-Anosov homeomorphisms on S_g, a genus g surface, $\phi : S_g \to S_g$ for which

$$\lambda(\phi) \leq \sqrt[g]{4}$$

Mapping tori

How does $\lambda(\phi)$ affect M_ϕ?

Theorem

For any $L > 0$, the set of mapping tori

$$\bigcup_{g \geq 1} \{M_\phi \mid \phi : S_g \to S_g, \lambda(\phi) < \sqrt[g]{L}\}$$

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Fixed $g \Rightarrow$ finitely many $\phi : S_g \to S_g$ with $\lambda(\phi) \leq \sqrt[g]{L}$.

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$\forall\ L \geq 4, g \geq 1$ there is $\phi : S_g \to S_g$ with $\lambda(\phi) \leq \sqrt[g]{L}$.

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*technical point: must puncture surfaces.

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- Several results relating the geometry of M_ϕ to properties of ϕ. Still many interesting open questions about this relationship.

The End

Thanks!!