Math 215 - Introduction to Advanced Mathematics

Sets and Functions Problem Set

Fall 2017

1. Determine (find an easier description for) the following sets:
   - $\{m \in \mathbb{Z}^+ : \exists n \in \mathbb{Z}^+, m \leq n\}$
   - $\{m \in \mathbb{Z}^+ : \forall n \in \mathbb{Z}^+, m \leq n\}$
   - $\{m \in \mathbb{Z}^+ : \exists n \in \mathbb{Z}^+, n \leq m\}$
   - $\{m \in \mathbb{Z}^+ : \forall n \in \mathbb{Z}^+, n \leq m\}$

2. For each $n \in \mathbb{N}$ let
   $$A_n = \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{n}\right).$$
   Find $\bigcup_{n\in\mathbb{N}} A_n$ and $\bigcap_{n\in\mathbb{N}} A_n$. In this problem, the notation (a, b) stands for the open interval of real numbers from a to b. So
   $$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$
   It does not mean a point or a set of two elements.

3. Prove de Morgan's laws, that for any sets A and B (in some universe U), the following hold:
   $$(A \cup B)^c = A^c \cap B^c$$
   $$(A \cap B)^c = A^c \cup B^c.$$

4. Prove that for any two sets, A and B, the following statements hold:
   - $A \subseteq B$ if and only if $A \cup B = B$.
   - $A \subseteq B$ if and only if $A \cap B = A$.
   - $A \cup B = B$ if and only if $A \cap B = A$.

5. Prove that if $A \cap B \subseteq C$ and $x \in B$, then $x \notin A \setminus C$.

6. Prove that for any two sets, A and B,

$$A \subseteq B \iff \bar{B} \subseteq \bar{A},$$

where the complement is taken with respect to some universal set U.

7. Prove for any sets A, B, C, and D, that

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Show that these two sets are not necessarily equal. Here $A \times B$ stands for the Cartesian Product (ordered pairs) of the sets A and B,

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

8. Let A be a finite set with exactly n elements. How many elements are in the power set $\mathcal{P}(A)$?

9. Find functions $f_i : \mathbb{R} \to \mathbb{R}$ with the following images (the range):

   - $\mathrm{Im}(f_1) = \mathbb{R}$
   - $\mathrm{Im}(f_2) = \mathbb{R}^+$
   - $\mathrm{Im}(f_3) = \mathbb{R} \setminus \mathbb{Z}$
   - $\mathrm{Im}(f_4) = \mathbb{Z}$

10. Determine whether each of the following functions $f_i : \mathbb{R} \to \mathbb{R}$ is injective, surjective, or bijective:

   - $f_1(x) = 2x + 5$
   - $f_2(x) = x^2 + 2x + 1$
   - $f_3(x) = x^3$
   - $f_4(x) = e^x$

11. Prove that if $f : X \to Y$ and $g : Y \to Z$ are injections, then the function $g \circ f : X \to Z$ is also an injection.

12. Prove that if $f : X \to Y$ and $g : Y \to Z$ are bijections, then the function $g \circ f : X \to Z$ is also a bijection, and that the two functions $(g \circ f)^{-1} : Z \to X$ and $f^{-1} \circ g^{-1} : Z \to X$ are equal.