Math 215 - Introduction to Advanced Mathematics

Sets and Functions Problem Set Solutions

Fall 2017

1. *Determine the following sets:*

   - $\{m \in \mathbb{Z}^+ : \exists n \in \mathbb{Z}^+, m \leq n\}$
   - $\{m \in \mathbb{Z}^+ : \forall n \in \mathbb{Z}^+, m \leq n\}$
   - $\{m \in \mathbb{Z}^+ : \exists n \in \mathbb{Z}^+, n \leq m\}$
   - $\{m \in \mathbb{Z}^+ : \forall n \in \mathbb{Z}^+, n \leq m\}$

   The sets are $\mathbb{Z}^+$, $\{1\}$, $\mathbb{Z}^+$, and $\emptyset$.

2. *For each* $n \in \mathbb{N}$ *let*
$$A_n = \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{n}\right).$$
   Find $\bigcup_{n\in\mathbb{N}} A_n$ *and* $\bigcap_{n\in\mathbb{N}} A_n$.

   First, $\bigcup_{n\in\mathbb{N}} A_n = \left(\frac{1}{2}, \frac{3}{2}\right)$.

   Proof. The fact that
$$\left(\frac{1}{2}, \frac{3}{2}\right) \subseteq \bigcup_{n\in\mathbb{N}} A_n$$
   is trivial. So we need only show that the opposite is true. Let
$$x \in \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{n}\right)$$
   for some $n \in \mathbb{N}$. Then since $1/n \leq 1$ for any such n, it follows that
$$x \in \left(\frac{1}{2}, \frac{3}{2}\right)$$
   and we are done. $\square$

Next, $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$.

Proof. Assume, towards a contradiction, that there exists some number $x \in \bigcap_{n\in\mathbb{N}} A_n$. Certainly, if $x \leq 1/2$, then this is an immediate contradiction so assume that $1/2 < x$. Therefore, $x = 1/2 + \epsilon$ for some positive ϵ. Since we know that $1/n \to 0$ as $n \to \infty$, then there must exist some $N \in \mathbb{N}$ for which

$$\frac{1}{N} < \epsilon.$$

Therefore,

$$x = \frac{1}{2} + \epsilon > \frac{1}{2} + \frac{1}{N}.$$

Hence,

$$x \notin \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{N}\right).$$

Therefore, x cannot be in the intersection, a contradiction. □

3. *Prove de Morgan's laws, that for any sets A and B (in some universe U), the following hold:*

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c.$$

First, we'll show that $(A \cup B)^c = A^c \cap B^c$.

Proof. Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$. If either $x \in A$ or $x \in B$, then $x \in A \cup B$. Therefore, $x \notin A$ and $x \notin B$. So $x \in A^c$ and $x \in B^c$. Thus, $x \in A^c \cap B^c$. So $(A \cup B)^c \subseteq A^c \cap B^c$.

Conversely, if $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$. Therefore, $x \notin A$ and $x \notin B$. So $x \notin A \cup B$. Hence, $x \in (A \cup B)^c$. □

Next, we'll show that $(A \cap B)^c = A^c \cup B^c$.

Proof. Let $x \in (A \cap B)^c$, then $x \notin A \cap B$. So either $x \notin A$ or $x \notin B$. In the first case, $x \in A^c$ and therefore, $x \in A^c \cup B^c$. In the other case, $x \in B^c$ and so $x \in A^c \cup B^c$.

Conversely, if $x \in A^c \cup B^c$, then either $x \in A^c$ or $x \in B^c$. So either $x \notin A$ or $x \notin B$. In either case, $x \notin A \cap B$. Hence, $x \in (A \cap B)^c$. □

4. *Prove that for any two sets, A and B, the following statements hold:*

   - *$A \subseteq B$ if and only if $A \cup B = B$.*

Proof. (⇒) Assume that $A \subseteq B$. If $x \in A \cup B$, then either $x \in B$ and we're done or $x \in A$ and by the assumption that $A \subseteq B$, then $x \in B$. Conversely, if $x \in B$, then $x \in A \cup B$ by definition of union. So $A \cup B = B$.
(⇐) Assume that $A \cup B = B$. Let $x \in A$. Then $x \in A \cup B$. Hence, $x \in B$. So $A \subseteq B$. □

- *$A \subseteq B$ if and only if $A \cap B = A$.*

  Proof. (⇒) Assume that $A \subseteq B$. Let $x \in A \cap B$, then $x \in A$ by definition of intersection. So $A \cap B \subseteq A$. Conversely, let $x \in A$, then by assumption, $x \in B$. Therefore, $x \in A \cap B$. So $A \subseteq A \cap B$ and we are done.
  (⇐) Now assume that $A \cap B = A$. Let $x \in A$. Then $x \in A \cap B$ by assumption. So $x \in B$ by definition of intersection. Thus, $A \subseteq B$. □

- *$A \cup B = B$ if and only if $A \cap B = A$.*

  Proof. By the first and second parts of this problem $A \cup B = B$ if and only if $A \subseteq B$ if and only if $A \cap B = A$. □

5. *Prove that if $A \cap B \subseteq C$ and $x \in B$, then $x \notin A \setminus C$.*

Proof. Assume that $A \cap B \subseteq C$ and that $x \in B$. Assume towards a contradiction that $x \in A \setminus C$. Then $x \in A$ and $x \notin C$ by definition of set difference. Since $x \in A$ and $x \in B$, then $x \in A \cap B$. So by assumption that $A \cap B \subseteq C$, we see that $x \in C$, a contradiction. □

6. *Prove that for any two sets, A and B,*

$$A \subseteq B \iff \bar{B} \subseteq \bar{A},$$

where the complement is taken with respect to some universal set U.

Proof. (⇒) Let $A \subseteq B$ and assume towards a contradiction that $\bar{B} \not\subseteq \bar{A}$. Therefore, there exists some element $x \in \bar{B}$ such that $x \notin \bar{A}$. Hence, $x \notin B$ and $x \in A$. Thus, $A \not\subseteq B$, a contradiction.

(⇐) We've just shown that for any two sets A and B that

$$A \subseteq B \Rightarrow \bar{B} \subseteq \bar{A}.$$

Therefore,

$$\bar{B} \subseteq \bar{A} \Rightarrow \bar{\bar{A}} \subseteq \bar{\bar{B}} \Rightarrow A \subseteq B.$$

□

7. *Prove for any sets A, B, C, and D, that*

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Show that these two sets are not necessarily equal.

Proof. Let $(x, y) \in (A \times B) \cup (C \times D)$. So either $x \in A$ and $y \in B$ or $x \in C$ and $x \in D$. In either case $x \in A \cup C$ and $y \in B \cup D$. Hence, $(x, y) \in (A \cup C) \times (B \cup D)$. □

The two sets are not always equal as the following example shows:

$$\begin{aligned} A = B &= \{1\} \\ C = D &= \{2\} \\ (A \times B) \cup (C \times D) &= \{(1,1),(2,2)\} \\ (A \cup C) \times (B \cup D) &= \{(1,1),(1,2),(2,1),(2,2)\} \end{aligned}$$

8. *Let A be a finite set with exactly n elements. How many elements are in the power set $\mathcal{P}(A)$?*

We can construct any subset of A by looking at each element of A in turn and deciding whether or not it belongs to the subset. So for each element we have two choices, in or out. Therefore, at the end of the n elements there were 2^n possible combinations of decisions we could have made. Therefore, there are 2^n different subsets of A.

We can prove this more formally by using a proof technique called **induction** that we will talk a lot more about in class.

Proposition 0.1. *Let A be a finite set with $|A| = n$ (this is notation for A has n elements), then $|\mathcal{P}(A)| = 2^n$.*

Proof. First, when $n = 1$, then $A = \{a_1\}$ for some element a_1. So $\mathcal{P}(A) = \{\emptyset, \{a_1\}\}$. Therefore, $|\mathcal{P}(A)| = 2^1$ and the proposition is true for $n = 1$.

Next, assume that $n > 1$ and that the proposition is true for any set with $n - 1$ elements. Let

$$A = \{a_1, \ldots, a_n\}.$$

We will count the number of subsets of A by counting the number of subsets that do not contain the element a_n and then counting the ones that do.

Consider the set $A \setminus \{a_n\}$. It has $n - 1$ elements. Therefore, there are exactly 2^{n-1} different subsets of $A \setminus \{a_n\}$ according to our assumption. Therefore, there are 2^{n-1} subsets of A that do not contain the element a_n.

So now we need to count the number of subsets of A that contain the element a_n. Each of these subsets can be thought of as $\{a_n\} \cup B$ for some subset $B \subseteq A \setminus \{a_n\}$. Moreover, if $B_1, B_2 \subseteq A \setminus \{a_n\}$ are two distinct subsets ($B_1 \neq B_2$), then

$$B_1 \cup \{a_n\} \neq B_2 \cup \{a_n\}.$$

So there must be 2^{n-1} subsets of A that contain a_n. Therefore,

$$|\mathcal{P}(A)| = 2^{n-1} + 2^{n-1} = 2^n.$$

□

Do you see why this kind of argument works?

9. *Find functions $f_i : \mathbb{R} \to \mathbb{R}$ with the following images:*

   - $\mathrm{Im}(f_1) = \mathbb{R}$
   - $\mathrm{Im}(f_2) = \mathbb{R}^+$
   - $\mathrm{Im}(f_3) = \mathbb{R} \setminus \mathbb{Z}$
   - $\mathrm{Im}(f_4) = \mathbb{Z}$

   Many different answers for this one. For example, the floor function, $\lfloor x \rfloor$, defined by taking the biggest integer less than or equal to x is a good f_4.

10. *Determine whether each of the following functions $f_i : \mathbb{R} \to \mathbb{R}$ is injective, surjective, or bijective:*

    - $f_1(x) = 2x + 5$

      This one is bijective.

      Proof. Let $f_1(x_1) = f_1(x_2)$, then

      $$\begin{aligned} 2x_1 + 5 &= 2x_2 + 5 \\ 2x_1 &= 2x_2 \\ x_1 &= x_2 \end{aligned}$$

      So it is injective.
      Let $y \in \mathbb{R}$. Then $\frac{y-5}{2} \in \mathbb{R}$, and

      $$f_1\left(\frac{y-5}{2}\right) = y.$$

      So f_1 is surjective. Therefore, f_1 is a bijection. □

- $f_2(x) = x^2 + 2x + 1$

  This one is neither. Since $x = 0$ and $x = -2$ both give $f_2(x) = 1$, then it is not injective. And setting $f_2(x) = -1$ gives complex values for x so it is not surjective.

- $f_3(x) = x^3$

  This is a bijection.

  Proof. Let $f_3(x_1) = f_3(x_2)$ for some $x_1, x_2 \in \mathbb{R}$, then

  $$x_1^3 = x_2^3$$
  $$x_1 = x_2$$

  So f_3 is an injection.
  Let $y \in \mathbb{R}$, then it is known that the cube root of y is also a real number. Hence, f_3 is surjective. Therefore, it is a bijection. □

- $f_4(x) = e^x$

  This function is not surjective since there exists no real number x for which $e^x = -1$. However, it is injective.

  Proof. Let $x_1, x_2 \in \mathbb{R}$ such that $f_4(x_1) = f_4(x_2)$. Then

  $$e^{x_1} = e^{x_2}$$
  $$x_1 = x_2$$

  by taking the natural log of both sides. So the function is injective. □

11. *Prove that if $f : X \to Y$ and $g : Y \to Z$ are injections, then the function $g \circ f : X \to Z$ is also an injection.*

Proof. Let $x_1, x_2 \in X$ such that

$$(g \circ f)(x_1) = (g \circ f)(x_2).$$

Then $g(f(x_1)) = g(f(x_2))$. Since g is injective, then it follows that $f(x_1) = f(x_2)$. Since f is injective, then if follows that $x_1 = x_2$. Thus, $g \circ f$ is injective. □

12. *Prove that if* $f : X \to Y$ *and* $g : Y \to Z$ *are bijections, then the function* $g \circ f : X \to Z$ *is also a bijection, and that the two functions* $(g \circ f)^{-1} : Z \to X$ *and* $f^{-1} \circ g^{-1} : Z \to X$ *are equal.*

Proof. In the previous problem we showed that $g \circ f$ must be injective, and on the Worksheet we showed that $g \circ f$ must be surjective. Therefore, it is a bijection. So it must have a unique inverse, $(g \circ f)^{-1}$ (again, from the worksheet). Therefore, to show that

$$f^{-1} \circ g^{-1} = (g \circ f)^{-1}$$

we need only show that composing the function $f^{-1} \circ g^{-1}$ with the function $g \circ f$ gives the identity functions (id_X or id_Z depending on the order of composition).

So for all $x \in X$,

$$\begin{aligned} \left((f^{-1} \circ g^{-1}) \circ (g \circ f)\right)(x) &= (f^{-1} \circ g^{-1})(g(f(x))) \\ &= f^{-1}(g^{-1}(g(f(x)))) \\ &= f^{-1}(f(x)) \\ &= x \end{aligned}$$

So

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = \mathrm{id}_X.$$

Similarly, for any $z \in Z$,

$$\begin{aligned} \left((g \circ f) \circ (f^{-1} \circ g^{-1})\right)(z) &= (g \circ f)(f^{-1}(g^{-1}(z))) \\ &= g(f(f^{-1}(g^{-1}(z)))) \\ &= g(g^{-1}(z)) \\ &= z \end{aligned}$$

So

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = \mathrm{id}_Z.$$

□