Math 215 - Introduction to Advanced Mathematics

Graph Theory Problem Set

Fall 2017

1. Let $G_k = (V, E)$ be a graph with vertex set $V = \{0, 1\}^k$ and edge set

$$E = \{xy : x \text{ and } y \text{ differ in exactly one position}\}.$$

   Is G_k bipartite? Prove or disprove your answer.

2. Let $G = (V, E)$ be a graph with no cycles smaller than C_5. Fix some positive integer k. If $d(x) \geq k$ for all $x \in V$, then prove that G must have at least $k^2 + 1$ vertices.

3. Remove opposite corner squares from an 8×8 checkerboard. Prove that you cannot cover every square of the board using dominos (1×2 pieces).

4. For a natural number n let S_n be the set of all **permutations** of $[n]$. That is, S_n is the set of all possible ordered n-tuples of the numbers $1, 2, \ldots, n$ such that no number repeats. For example,

$$S_3 = \{123, 132, 213, 231, 312, 321\}.$$

   Let $G_n = (V, E)$ be the graph with vertex set $V = S_n$ such that $ab \in E$ if and only if we can get the permutation b from the permutation a by swapping two numbers that are next to each other. For instance, 312 and 321 are adjacent as vertices of G_3, but 123 and 321 are not.

   What is the degree of any vertex in G_n? Prove that G_n is connected.

5. Reprove König's Theorem by using induction on the number of edges to show that if a graph G contains no odd cycles, then it is bipartite.

6. Let $G = (V, E)$ be a graph. Prove that G is bipartite if and only if every subgraph $H = (V', E')$ of G has an independent set consisting of at least half of the vertices from V'.

7. Prove that every graph on n vertices with at least n edges contains a cycle.