# Math 215 - Introduction to Advanced Mathematics

Sets and Functions Quiz-Exam Solutions and Rubric

Fall 2017

1. *Determine the following sets (just give a simple description for each one, no proof necessary):*

   - $\{m \in \mathbb{Z}^+ : \exists n \in \mathbb{Z}^+, m \leq n\}$
   - $\{m \in \mathbb{Z}^+ : \forall n \in \mathbb{Z}^+, m \leq n\}$
   - $\{m \in \mathbb{Z}^+ : \exists n \in \mathbb{Z}^+, n \leq m\}$
   - $\{m \in \mathbb{Z}^+ : \forall n \in \mathbb{Z}^+, n \leq m\}$

   The sets are $\mathbb{Z}^+$, $\{1\}$, $\mathbb{Z}^+$, and $\emptyset$.

2. *Prove that if $A \cap B \subseteq C$ and $x \in B$, then $x \notin A \setminus C$.*

   *Proof.* Assume that $A \cap B \subseteq C$ and that $x \in B$. Assume towards a contradiction that $x \in A \setminus C$. Then $x \in A$ and $x \notin C$ by definition of set difference. Since $x \in A$ and $x \in B$, then $x \in A \cap B$. So by assumption that $A \cap B \subseteq C$, we see that $x \in C$, a contradiction. □

3. *Prove that if $f : X \to Y$ and $g : Y \to Z$ are injections, then the function $g \circ f : X \to Z$ is also an injection.*

   *Proof.* Let $x_1, x_2 \in X$ such that

   $$(g \circ f)(x_1) = (g \circ f)(x_2).$$

   Then $g(f(x_1)) = g(f(x_2))$. Since $g$ is injective, then it follows that $f(x_1) = f(x_2)$. Since $f$ is injective, then if follows that $x_1 = x_2$. Thus, $g \circ f$ is injective. □

4. *Prove that intersection distributes over union. That is, prove that for any sets $A$, $B$, and $C$:*

   $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

*Proof.* Let $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C$. So either $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cap B$ and therefore, $x \in (A \cap B) \cup (A \cap C)$. If $x \in C$, then $x \in A \cap C$ and so again $x \in (A \cap B) \cup (A \cap C)$. Hence,

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$

In the other direction, let $x \in (A \cap B) \cup (A \cap C)$. Then either $x \in A \cap B$ or $x \in A \cap C$. In the first case, $x \in A$ and $x \in B$. Thus, $x \in A$ and $x \in B \cup C$. So $x \in A \cap (B \cup C)$. In the second case, $x \in A$ and $x \in C$. Thus, $x \in A$ and $x \in B \cup C$. So $x \in A \cap (B \cup C)$. Hence,

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C),$$

and so the two sets are equal. □

5. *Prove that if $f : X \to Y$ and $g : Y \to Z$ are surjections, then the function*

$$g \circ f : X \to Z$$

*is also a surjection.*

*Proof.* Let $z \in Z$. Since $g$ is a surjection, then there exists a $y \in Y$ such that $g(y) = z$. Since $f$ is a surjection, then there exists an $x \in X$ such that $f(x) = y$. Therefore,

$$(g \circ f)(x) = z.$$

Hence, $g \circ f$ is surjective. □