Math 215 - Introduction to Advanced Mathematics

Sets and Functions Quiz-Exam, Take 2

Fall 2017

1. We use standard interval notation in what follows. That is, for real numbers $a < b$,

$$\begin{aligned}(a,b) &= \{r \in \mathbb{R} : a < r < b\}\\ [a,b) &= \{r \in \mathbb{R} : a \leq r < b\}\\ (a,b] &= \{r \in \mathbb{R} : a < r \leq b\}\\ [a,b] &= \{r \in \mathbb{R} : a \leq r \leq b\}\end{aligned}$$

For any $n \in \mathbb{N}$, define the set A_n as

$$A_n = \left(0, \frac{1}{n}\right).$$

You may assume the following fact is true without proof: for any real number $\epsilon \in (0,1)$, there exists a natural number k such that $\frac{1}{k+1} \leq \epsilon < \frac{1}{k}$.

(a) What is

$$\bigcup_{n\in\mathbb{N}} A_n?$$

Prove that your answer is correct.

Solution:

$$\bigcup_{n\in\mathbb{N}} A_n = (0,1)$$

Proof. Let $x \in \bigcup_{n\in\mathbb{N}} A_n$, then $x \in (0, 1/n)$ for some n. Since $(0,1/n) \subseteq (0,1)$ for any $n \in \mathbb{N}$, then $x \in (0,1)$. Conversely, if $x \in (0,1)$, then x belongs to the first set of this union. Hence, by definition of union, $x \in \bigcup_{n\in\mathbb{N}} A_n$. □

(b) What is

$$\bigcap_{n=1}^{5} A_n?$$

Solution:

$$\bigcap_{n=1}^{5} A_n = \left(1, \frac{1}{5}\right).$$

(c) What is

$$\bigcap_{n\in\mathbb{N}} A_n?$$

Prove that your answer is correct.

Solution:

$$\bigcap_{n\in\mathbb{N}} A_n = \emptyset$$

Proof. Suppose $\bigcap_{n\in\mathbb{N}} A_n$ is not empty. Then there exists some $\epsilon \in \bigcap_{n\in\mathbb{N}} A_n$. So $\epsilon \in A_1 = (0,1)$ by definition of intersection. Therefore, by the fact above, we know that there exists a natural number k such that

$$\frac{1}{k+1} \leq \epsilon < \frac{1}{k}.$$

Hence,

$$\epsilon \notin \left(0, \frac{1}{k+1}\right),$$

a contradiction. □

(d) For any n, what is $A_n \setminus A_{n+1}$?

Solution:

$$A_n \setminus A_{n+1} = \left[\frac{1}{n+1}, \frac{1}{n}\right).$$

(e) What is

$$\bigcup_{n\in\mathbb{N}} (A_n \setminus A_{n+1})?$$

Prove that your answer is correct.

Solution:

$$\bigcup_{n\in\mathbb{N}} (A_n \setminus A_{n+1}) = (0,1)$$

Proof. Let $x \in \bigcup_{n \in \mathbb{N}} (A_n \setminus A_{n+1})$, then

$$x \in A_n \setminus A_{n+1} = \left[\frac{1}{n+1}, \frac{1}{n}\right)$$

for some $n \in \mathbb{N}$. So $x \in (0,1)$. Conversely, if $x \in (0,1)$, then by the fact we assume above, there exists some natural number k such that

$$\frac{1}{k+1} \leq x < \frac{1}{k}.$$

Hence,

$$x \in \bigcup_{n \in \mathbb{N}} (A_n \setminus A_{n+1}).$$

□

2. Let $A = \{a, b, c\}$ and $B = \{b, d\}$.

(a) What is $A \cap B$?

Solution:

$$\{b\}$$

(b) What is $\mathcal{P}(A)$?

Solution:

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{c, b\}, \{a, b, c\}\}$$

(c) What is $\mathcal{P}(B)$?

Solution:

$$\{\emptyset, \{b\}, \{d\}, \{b, d\}\}$$

(d) What is $\mathcal{P}(A \cap B)$?

Solution:

$$\{\emptyset, \{b\}\}$$

(e) What is $\mathcal{P}(A) \cap \mathcal{P}(B)$?

Solution:

$$\{\emptyset, \{b\}\}$$

(f) Prove that in general, for any two sets S and T,

$$\mathcal{P}(S \cap T) = \mathcal{P}(S) \cap \mathcal{P}(T).$$

Solution:

Proof. Let $A \in \mathcal{P}(S \cap T)$. Then $A \subseteq S \cap T$. So $A \subseteq S$ and $A \subseteq T$. So $A \in \mathcal{P}(S)$ and $A \in \mathcal{P}(T)$. Hence, $A \in \mathcal{P}(S) \cap \mathcal{P}(T)$.
Conversely, let $A \in \mathcal{P}(S) \cap \mathcal{P}(T)$. Then $A \in \mathcal{P}(S)$ and $A \in \mathcal{P}(T)$. So $A \subseteq S$ and $A \subseteq T$. Therefore, $A \subseteq S \cap T$. So $A \in \mathcal{P}(S \cap T)$. $\square$

3. In the following questions determine whether the function f is injective, surjective, both (bijective), or neither. Prove your answer is correct.

(a) $f : \mathbb{R} \to \mathbb{Z}$ defined by $f(x) = \lceil x \rceil$ (the ceiling function returns the integer n for which $n - 1 < x \leq n$

Solution: f is not injective since $f(0.9) = 1 = f(1)$.
f is surjective

Proof. Let $y \in \mathbb{Z}$. Then $y \in \mathbb{R}$ as well and $f(y) = y$. $\square$

(b) Let S be the set of finite subsets of $\mathbb{Z}$. Then $f : S \to \mathbb{Z}$ is defined by $f(A) = |A|$ (the number of elements in A).

Solution: f is neither injective or surjective. It is not surjective since $-1 \in \mathbb{Z}$, but no set can contain -1 elements. It is not injective since $f(\{1\}) = 1 = f(\{2\})$.

(c) Let $S = \{1, x, x^2, \ldots, x^k, \ldots\}$, the set of all monic monomials in a single variable. Define $f : S \to \mathbb{Z}$ by $f(p(x)) = \deg(p(x))$, the degree of the polynomial.
Solution: f is not surjective since $-1 \in \mathbb{Z}$, but no polynomial has degree -1.
f is injective.

Proof. Let $p(x), q(x) \in S$ such that $f(p(x)) = f(q(x))$. Since $p(x) = x^k$ and $q(x) = x^{k'}$ for some nonnegative integers k, k', then $f(x^k) = f(x^{k'})$. So k=k'. Hence, $p(x) = q(x)$. So f is injective. $\square$

(d) Let U be a set. Define $f : \mathcal{P}(U) \to \mathcal{P}(U)$ by $f(A) = U \setminus A$.

Solution: f is a bijection.

Proof. Let $A, B \in \mathcal{P}(U)$ such that $f(A) = f(B)$. Then $U \setminus A = U \setminus B$. We want to show that $A = B$ to prove this function is injective. Let $x \in A$. Then $x \in U$. Hence, $x \notin U \setminus A$. Therefore, $x \notin U \setminus B$. Since $x \in U$ this means that $x \in B$ by definition of the set difference. Therefore, $A \subseteq B$. The same argument shows that $B \subseteq A$. Hence, $A = B$ and so f is injective.
Now, let $A \in \mathcal{P}(U)$. Then

$$\begin{aligned}(U \setminus A) &= U \setminus (U \setminus A) \\ &= \{x \in U : x \notin U \setminus A\} \\ &= \{x \in U : x \notin U \text{ or } x \in U \cap A = A\} \\ &= \{x \in U : x \in A\} \\ &= A.\end{aligned}$$

So f is surjective. □

4. Prove that if $f : X \to Y$ and $g : Y \to Z$ are surjections, then the function

$$g \circ f : X \to Z$$

is also a surjection.

Solution:

Proof. Let $z \in Z$. Since g is a surjection, then there exists a $y \in Y$ such that $g(y) = z$. Since f is a surjection, then there exists an $x \in X$ such that $f(x) = y$. Therefore,

$$(g \circ f)(x) = z.$$

Hence, $g \circ f$ is surjective. □

5. Show by induction that the sum of the first k odd integers is equal to k^2.

Solution:

Proof. **Base case:** If $k = 1$, then this is clearly true.

Induction Step: Assume that $k \geq 2$, and that the sum of the first $k-1$ odd numbers is $(k-1)^2$. Then

$$\begin{aligned}1 + \cdots + (2k-3) &= (k-1)^2 \\ 1 + \cdots + (2k-3) + (2k-1) &= (k-1)^2 + (2k-1) \\ 1 + \cdots + (2k-3) + (2k-1) &= k^2 - 2k + 1 + 2k - 1 \\ 1 + \cdots + (2k-3) + (2k-1) &= k^2\end{aligned}$$

□