Math 215 - Introduction to Advanced Mathematics

Graph Theory Quiz-Exam Solutions

Fall 2017

1. Let $G_k = (V, E)$ be a graph with vertex set $V = \{0,1\}^k$ and edge set

$$E = \{xy : x \text{ and } y \text{ differ in exactly one position}\}.$$

Is G_k bipartite? Prove or disprove your answer.

Solution: Yes, G is bipartite. One way to show this is by induction on k:

Proof. We need to prove that there is a proper two-coloring of G. Proceed by induction on k.

Base Case: If $k = 1$, then there are only two vertices, 0 and 1, connected by an edge. Color one of the vertices with color 1 and the other with color 2, and we are done.

Induction Step: Now assume that $k > 1$ and that we have a proper two-coloring ϕ of the vertices of G_{k-1}. We define a new vertex coloring, $\phi' : \{0,1\}^k \to [2]$, like this

$$\phi'((x_1, x_2, \ldots, x_k)) = \begin{cases} \phi((x_1, \ldots, x_{k-1})) & \text{if } x_k = 0 \\ \delta(\phi((x_1, \ldots, x_{k-1}))) & \text{if } x_k = 1 \end{cases}$$

where $\delta : [2] \to [2]$ is a function that just switches 1 and 2. That is, $\delta(1) = 2$ and $\delta(2) = 1$.

So this is our vertex coloring that uses only two colors and we claim that it is proper. To prove this we can take two adjacent vertices of G_k and show that they received different colors. So take two vertices, $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$. Since they are adjacent, then they differ in exactly one position, say the ith position. That is, $x_i \neq y_i$ but $x_j = y_j$ for all $j \neq i$.

If $i < k$, then $x_k = y_k$. Therefore, the $(k-1)$-tuples $(x_1, \ldots, x_{k-1})$ and $(y_1, \ldots, y_{k-1})$ differ in exactly one position so

$$\phi((x_1, \ldots, x_{k-1})) \neq \phi((y_1, \ldots, y_{k-1}))$$

by induction. Therefore, if $x_k = y_k = 0$,

$$\phi'((x_1, \ldots, x_k)) = \phi((x_1, \ldots, x_{k-1})) \neq \phi((y_1, \ldots, y_{k-1})) = \phi'((y_1, \ldots, y_k)),$$

and if $x_k = y_k = 1$, then

$$\phi'((x_1, \ldots, x_k)) = \delta(\phi((x_1, \ldots, x_{k-1}))) \neq \delta(\phi((y_1, \ldots, y_{k-1}))) = \phi'((y_1, \ldots, y_k)).$$

If $i = k$, then without loss of generality we can say that $x_k = 0$ and $y_k = 1$. Then

$$\begin{aligned}\phi'((x_1, \ldots, x_k)) &= \phi((x_1, \ldots, x_{k-1})) \\ &\neq \delta(\phi((x_1, \ldots, x_{k-1}))) \\ &= \delta(\phi((y_1, \ldots, y_{k-1}))) \\ &= \phi'((y_1, \ldots, y_k)).\end{aligned}$$

This exhausts all of the cases. So the coloring is proper. □

Another solution is the following:

Proof. Define the following coloring of the vertices of G_k:

$$\phi(x) = \begin{cases} 1 & \text{if } x \text{ has an odd number of ones} \\ 2 & \text{if } x \text{ has an even number of ones} \end{cases}$$

We must show that this coloring is proper. Let x and y be two adjacent vertices of G_k. We need to show that x and y get different colors. Since they are adjacent, then they differ in only one position. If x is one in this position, then y is zero. Since y and x are the same everywhere else, then x has exactly one more one than y. Hence, if y has an odd number of ones, then x has an even number and if y has an even number of ones, then x has an odd number. So they receive two different colors. The same argument applies if x has a zero in the position where the two are different. □

2. Prove König's Theorem: "A graph is bipartite if and only if it has no odd cycle." You may use either the way we proved it in class or the way you were asked to prove it in the problem set (induction on the number of edges).

Solution: I will prove this one by induction on the number of edges for the $\Leftarrow$ direction.

Proof. ($\implies$ by contraposition) Assume that G is a graph that contains an odd cycle. Since any proper vertex coloring of an odd cycles requires at least three colors, then G is not 2-colorable. Hence, G is not bipartite by definition.

($\Leftarrow$) Assume that $G = (V, E)$ is a graph containing no odd cycles. Let G have m edges.

(*Base Case:*) If $m = 0$, then $\chi(G) = 1$ and so it is bipartite.

(*Induction Step:*) So let $m > 0$ and assume that any graph H with fewer than m edges and no odd cycles is bipartite. Remove any edge xy from G to get a subgraph $G' = (V, E \setminus \{xy\})$. Since G' has fewer than m edges, then by induction we know that G' is bipartite. So there is a proper coloring $\phi : V \to [2]$ of G'.

If $\phi(x) \neq \phi(y)$, then ϕ is also a proper coloring of G and so G must also be bipartite. So let's assume that $\phi(x) = \phi(y)$. If x is connected to y in G', then there is an x, y-path in G'. This path must have an even number of edges since x and y belong to the same part of G'. But this means that adding the edge xy creates an odd cycle in G, a contradiction. Therefore, x and y cannot be connected in G', but then we can simply reverse the colors given to the vertices of the connected component containing y and still have a proper coloring of G'. This coloring will also be proper for G since now x and y have opposite colors. $\square$

3. Prove that every graph on n vertices with at least n edges contains a cycle.

Solution: We can prove this by induction on the number of vertices, n.

Proof. (*Base Case:*) If $n = 1$, then the statement is vacuously true since a graph with one vertex cannot have an edge.

(*Induction Step:*) Let $n > 1$ and assume that any graph with m or vertices and at least m edges for $1 \leq m < n$ contains a cycle. Let $G = (V, E)$ be a graph with n vertices and n edges.

First, if G is disconnected, then it has $k \geq 2$ connected components with $1 \leq c_1, c_2, \ldots, c_k < n$ vertices in each. If the ith component contains at least c_i edges, then by induction it contains a cycle and we are finished. Otherwise, each component contains at most $c_i - 1$ edges. Since no edges exist between components, the total number of edges in G is

$$(c_1 - 1) + \cdots + (c_k - 1) = (c_1 + \cdots + c_k) - k = n - k \leq n - 2 < n,$$

a contradiction. Therefore, G must contain a cycle.

Now, assume that G is connected and fix some vertex x. Consider the graph G' formed by removing x and all of the edges incident to it from G. If G' contains a cycle we are finished. Otherwise, by the induction assumption, G' has at most $n - 2$ edges. Therefore, x was adjacent to at least 2 vertices in G. Call these y and z.

If y and z are connected in G', then there exists a y, z-path in G'. Combining this path with the edges xy and zx shows us that there is a cycle in G. So we must assume that y and z are disconnected in G'. Moreover, any two vertices adjacent to x must be disconnected in G' for the same reason.

Suppose $d(x) = k$. We know that $k \geq 2$ and each edge goes to a different connected component of G'. Again, let each of these components have $1 \leq c_1, \ldots, c_k < n$ vertices. Since we assume that there are no cycles in G', then the ith component has at most $c_i - 1$ edges. This gives a total of

$$k + (c_1 - 1) + \cdots + (c_k - 1) = k + (c_1 + \cdots + c_k) - k = n - 1$$

edges in G, a contradiction. Therefore, G must contain a cycle. $\square$

4. Let $G = (V, E)$ be a graph such that $d(x) = m$ for every $x \in V$. Prove that if G contains no cycles smaller than C_4, then G has at least $2m$ vertices. For 5 bonus points list all such graphs up to isomorphism which have exactly $2m$ vertices.

Solution: We will prove both the main statement and the fact that the only possible graph that G could be up to isomorphism is the complete bipartite graph $K_{m,m}$ (two equal parts both of size m with every possible edge between them).

Proof. Let x be a vertex of G and let $y_1, \ldots, y_m$ be the vertices we know are adjacent to it since $d(x) = m$. If $y_i y_j \in E$ for any $i \neq j$, then G would contain a 3-cycle, a contradiction of our assumption that it has no cycles smaller than a C_4. The vertex y_1 must therefore be adjacent to $m-1$ additional vertices that are not x or any of the y_i, call them $z_1, \ldots, z_{m-1}$. So there must be at least $1 + m + (m - 1) = 2m$ vertices in G.

If there are exactly this many vertices in G, then it follows by the same argument that each additional y_i must be adjacent to every z_j. This gives us a bipartite graph with parts $\{y_1, \ldots, y_m\}$ and $\{x, z_1, \ldots, z_m\}$. $\square$

5. Prove that every graph has an even number of vertices with odd degree.

Solution: Proof by contradiction.

Proof. Suppose that $G = (V, E)$ is a graph that contains an odd number of vertices with odd degree. Let A equal the sum of the odd degrees and let B equal the sum of the even degrees. Then

$$\sum_{x \in V} d(x) = A + B.$$

Since the sum of a bunch of even numbers is even, then B is even. The sum of any two odd numbers is also even, but the sum of an even number and an odd number is odd. Therefore, the sum of an odd number of odd numbers is also odd. Hence, A is odd. So $\sum_{x \in V} d(x) = A + B$ is odd. However, we have shown that

$$2|E| = \sum_{x \in V} d(x).$$

This means that $\sum_{x \in V} d(x)$ is actually even, a contradiction. $\square$