Math 215 - Introduction to Advanced Mathematics

Quiz-Exam 4

Fall 2017
Due by 4pm on Wednesday, December 6

1. Let $f : X \to Y$ be a function. Prove that f is an injection if and only if for any subsets $A, B \subseteq X$,
$$f(A \cap B) = f(A) \cap f(B).$$

2. Give the negation of each of the following statements.

   - $\forall x \exists y \, [(A(x, y) \wedge B(x, y)) \implies (C(x) \vee \neg D(y))]$.
   - $\forall x \neg \exists y \, [(A(x, y) \wedge B(x, y)) \implies (C(x) \vee \neg D(y))]$.
   - For all $x, y \in V$, there exists an xy-path in G of length at least k.
   - There exists a $\mathcal{B}$-regular integer $x \in S$ such that for any integer $y \in S$, either $y = x$, x eliminates y, or y is not $\mathcal{B}$-regular.

3. For each of the following implications, give the converse and the contrapositive of the statement. Then list the starting assumptions you would make and the conclusions you would want to draw were you to prove each statement directly, through contraposition, and through contradiction.

   - $A \implies B$.
   - If $a|b$ and $a|c$, then $a|(b + c)$.
   - If p is a prime number and $p|ab$, then $p|a$ or $p|b$.
   - If S is $\mathcal{L}$-categorical and $T \leq S$, then for any $\mathcal{L}$-categorical X, $T \oplus X \leq S \oplus X$.

4. Let $P(x, y) =$ "y is adjacent to x". For each of the following, give an example of a graph that satisfies the statement and one that does not.

   - $\exists x \in V \exists y \in V \setminus \{x\} P(x, y)$.
   - $\forall x \in V \forall y \in V \setminus \{x\} P(x, y)$.
   - $\exists x \in V \forall y \in V \setminus \{x\} P(x, y)$.

- $\forall x \in V \exists y \in V \setminus \{x\} P(x, y)$.

Now prove or disprove the following statements.

- Let $G = (V, E)$ be a graph. If $\forall x \in V \exists y \in V \setminus \{x\} P(x, y)$, then $\exists x \in V \forall y \in V \setminus \{x\} P(x, y)$.
- Let $G = (V, E)$ be a graph. If $\exists x \in V \forall y \in V \setminus \{x\} P(x, y)$, then $\forall x \in V \exists y \in V \setminus \{x\} P(x, y)$.

5. Look back through the previous worksheets, problem sets, and quiz-exams given in this class. Give an example (with full proof) of each of the following.

   - A direct proof
   - A proof by contraposition
   - A proof by contradiction
   - An existence proof
   - A proof using weak induction
   - A proof using strong induction

   Many examples we have seen will satisfy more than one of these, but please use different examples for each.