Math 215 - Introduction to Advanced Mathematics

Graph Theory

Fall 2017

Divide yourselves into teams of 1-4 people for this part of the course. As we go through this sheet, I will present all definitions to you. Each time we reach a problem, proposition, question, conjecture, theorem, or lemma, I will give any team the opportunity to send someone up and present a solution to the class to earn bonus points for their team on the next Quiz-Exam. If no team wants to present within a reasonable amount of time, then I will present my solution to the class and we will move on.
Here are the point values:

- Questions are worth 1 bonus point
- Problems, Propositions, and Conjectures are worth 5 bonus points
- Lemmas are worth 10 bonus points
- Theorems are worth 20 bonus points

By my count there are 143 points up for grabs on this sheet so you should fight for them! To earn the points for a solution you will have to convince me (and hopefully the rest of the class) that you're correct. If multiple teams volunteer to give a solution, I will pick the one who hasn't been to the board in a while. Teams *must* rotate through the members that they send to the board though helping will be allowed. Please work on these problems ahead of time or run the risk of one or two teams gobbling up all of the points and getting a free ride on the next test.

Notation 1 *Let V be a set. Then $\binom{V}{2}$ denotes the set of all subsets of V with exactly two elements. For any natural number n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. For any set S and natural number k, let S^k be the set of ordered k-tuples of elements of S,*

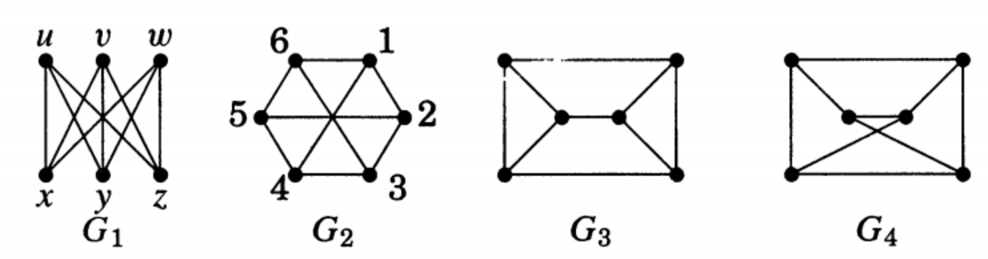
$$S^k = \{(s_1, \ldots, s_k) : s_1, \ldots, s_k \in S\}.$$

Definition 2 *A **graph** G is a pair of sets (V, E) where V is a finite set of **vertices** and $E \subseteq \binom{V}{2}$ is a set of **edges**.*

Definition 3 *Let $G = (V, E)$ be a graph and let $x, y \in V$ be vertices of G. We say that x and y are **adjacent** if and only if there is an edge between them, $xy \in E$.*

Definition 4 *Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there exists a bijection $\varphi : V_1 \to V_2$ such that for every pair of distinct vertices $x, y \in V_1$, x and y are adjacent in G_1 if and only if $\varphi(x)$ and $\varphi(y)$ are adjacent in G_2.*

Problem 5 *Which of the following graphs are isomorphic?*



Problem 6 *Draw all possible graphs on 3 or 4 vertices up to isomorphism.*

Definition 7 *The* ***Petersen graph*** *is the graph with vertex set* $V = \binom{[5]}{2}$ *and edge set*

$$E = \{AB : A \cap B = \emptyset\}.$$

Problem 8 *Draw the Petersen graph.*

Proposition 9 *If two vertices are not adjacent in the Petersen graph, then there is exactly one vertex that is adjacent to both of them.*

Definition 10 *Let* $G = (V, E)$ *and* $G' = (V', E')$ *be graphs. We say that* G' *is a* ***subgraph*** *of* G *if* $V' \subseteq V$ *and* $E' \subseteq E$. *We call a subgraph* H *of* G ***induced*** *if for every* $x, y \in V'$, $xy \in E$ *implies that* $xy \in E'$.

Definition 11 *Let* G *be a graph and let* x *be a vertex of* G. *The* ***degree*** *of* x, $d(x)$, *is the number of other vertices of* G *that are adjacent to* x. *If* $d(x) = 0$, *then we say that* x *is an* ***isolated*** *vertex.*

Problem 12 *Make a conjecture about what the sum of the degrees of a graph equals in general for all graphs,*

$$\sum_{x \in V} d(x) = ???,$$

and prove your conjecture is true.
A hint to get you started: Calculate this sum for the Petersen graph, for each of the graphs pictured in Problem 5, and for each that you drew in Problem 6 until you see the answer.

Definition 13 *Let* $G = (V, E)$ *be a graph, and let* $S \subseteq V$. *We say that* S *is an* ***independent set*** *of vertices if for every* $x, y \in S$, $xy \notin E$. *We say that* S *is a* ***clique*** *if for every distinct pair of vertices* $x, y \in S$, $xy \in E$.

Definition 14 *Let* n *be some positive integer. The* ***complete graph*** *on* n *vertices, denoted* K_n, *is the graph with vertex set* $V = [n]$ *and edge set* $E = \binom{V}{2}$.

Problem 15 *What is the size of the largest independent set of the Petersen graph?*

Definition 16 *Let* $G = (V, E)$ *be a graph. The* ***complement*** *of* G *is a graph* $\bar{G} = (V, E')$ *on the same set of vertices but with the opposite edge set,* $E' = \binom{V}{2} \setminus E$.

Question 17 *What is the complement of K_n?*

Proposition 18 *The graphs G and H are isomorphic if and only if their complements are isomorphic.*

Definition 19 *Let G be a graph. A* ***walk*** *in G is a alternating sequence of vertices and edges,*

$$x_1 e_1 x_2 e_2 \cdots x_{k-1} e_{k-1} x_k,$$

such that each edge $e_i = x_i x_{i+1}$. We say that the walk is ***even*** *if k is even or* ***odd*** *if k is odd. We say that the walk is* ***closed*** *if $x_1 = x_k$. If x and y are vertices of G and there exists a walk that begins at vertex x and ends at vertex y, then we call this an x, y-walk. A walk that never repeats an edge is called a* ***trail****. A walk that never repeats an edge and never repeats a vertex is called a* ***path****. A closed walk that never repeats an edge and never repeats a vertex other than the first and last vertices is called a* ***cycle****. A paths and cycles are denoted by P_k and C_k respectively where k gives the number of vertices.*

Conjecture 20 *If every vertex of a graph G has degree 2, then G is a cycle.*

Problem 21 *What is the length of the Petersen graph's shortest cycle?*

Definition 22 *A graph $G = (V, E)$ is* ***connected*** *if for any two distinct vertices $x, y \in V$, there exists an x, y-path. Otherwise, we say that G is* ***disconnected****. In any graph, a* ***connected component*** *is a set of vertices $S \subseteq V$ such that there exists an x, y-path for any $x, y \in S$, and there exists no x, z-path for any $x \in S$ and $z \in V \setminus S$.*

Conjecture 23 *The complement of a disconnected graph is connected.*

Definition 24 *Let $G = (V, E)$ be a graph. Let $\phi : V \to [k]$ be a function. Then ϕ is called a* ***coloring*** *of the vertices of G. If ϕ is such that $xy \in E$ implies that $\phi(x) \neq \phi(y)$, then ϕ is called a* ***proper coloring*** *of the vertices of G. The smallest possible natural number k for which such a proper coloring of the vertices of G exists is called the* ***chromatic number*** *of G, $\chi(G)$.*

Problem 25 *Determine the following and prove your answer is correct:*

- $\chi(K_n)$
- $\chi(P_n)$
- $\chi(C_n)$

Definition 26 *Let G be a graph with chromatic number $\chi(G) \leq 2$, then we call G* ***bipartite****.*

Question 27 *Which bipartite graphs are also complete graphs?*

Conjecture 28 *The Petersen graph is bipartite.*

Proposition 29 *Let G be a graph and let u and v be vertices of G. Then every u, v-walk contains a u, v-path.*

Proposition 30 *Every graph with n vertices and k edges has at least $n-k$ connected components.*

Problem 31 *Give an example of a graph G such that $\chi(G) = 4$ but G is not the complete graph K_4.*

Problem 32 *Prove or disprove the following*

- *Every disconnected graph has an isolated vertex.*
- *A graph is connected if and only if some vertex it connected to all other vertices.*

Problem 33 *Prove or disprove the following*

- *The edge set of every closed trail can be partitioned into edge sets of cycles.*
- *If a maximal trail in a graph is not closed, then its endpoints have odd degree.*

Problem 34 *Prove or disprove the following*

- *K_4 contains a trail that is not closed and is not a path.*
- *K_4 contains a closed trail that is not a cycle.*

Problem 35 *Let G be a graph with vertex set $V = [15]$ and edge set*

$$E = \{ij : gcd(i,j) > 1\}.$$

How many connected components does G have and what is the length of its longest path?

Proposition 36 *Let G be a bipartite graph. There are exactly two proper 2-colorings of the vertices of G if and only if G is connected.*

Proposition 37 *Prove that a graph is connected if and only if for every partition of the vertices into two nonempty parts there exists an edge that goes between the parts.*

Proposition 38 *Let G be a graph with no isolated vertices and no induced subgraph with exactly two edges. Then G is a complete graph.*

Lemma 39 *Every closed odd walk contains an odd cycle.*

Question 40 *Does a closed even walk necessarily contain a cycle?*

Theorem 41 (König, 1936) *A graph is bipartite if and only if it has no odd cycle.*