Math 215 - Introduction to Advanced Mathematics

Review for Midterm Exam

Spring 2018

The following are representative of the kinds of problems that you will see on the midterm exam. You should know the basic definitions needed to work these: set operations like intersection, union, power set, Cartesian product, etc; function things: surjective, injective, bijective, etc; and proof/logic things: contradiction, contrapositive, converse, negation, etc.

1. Let $A = \{a, b, c\}$ and $B = \{b, d\}$.

   (a) What is $A \cap B$?

   (b) What is $\mathcal{P}(A)$?

   (c) What is $\mathcal{P}(B)$?

   (d) What is $\mathcal{P}(A \cap B)$?

   (e) What is $\mathcal{P}(A) \cap \mathcal{P}(B)$?

   (f) What is $A \times B$?

   (g) What is $B \times A$?

   (h) What is $A - B$?

   (i) What is $B - A$?

   (j) What is $(B \cup A) \cap (A \cap \emptyset)$?

2. Prove that in general, for any two sets S and T,

$$\mathcal{P}(S \cap T) = \mathcal{P}(S) \cap \mathcal{P}(T).$$

3. In the following questions determine whether the function f is injective, surjective, both (bijective), or neither. Prove your answer is correct.

   (a) $f : \mathbb{R} \to \mathbb{Z}$ defined by $f(x) = \lceil x \rceil$ (the ceiling function returns the integer n for which $n - 1 < x \leq n$

(b) Let S be the set of *finite* subsets of $\mathbb{Z}$. Then $f : S \to \mathbb{Z}$ is defined by $f(A) = |A|$ (the number of elements in A).

(c) Let U be a set. Define $f : \mathcal{P}(U) \to \mathcal{P}(U)$ by $f(A) = U \setminus A$.

4. Prove that if $f : X \to Y$ and $g : Y \to Z$ are surjections, then the function

$$g \circ f : X \to Z$$

is also a surjection.

5. Prove that if $f : X \to Y$ and $g : Y \to Z$ are injections, then the function

$$g \circ f : X \to Z$$

is also an injection.

6. Show by induction that the sum of the first k odd integers is equal to k^2.

7. Let a_n be a sequence of numbers defined by $a_1 = 1$, $a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Prove that

$$a_n = 3 \cdot 2^{n-1} + 2(-1)^n$$

for all $n \in \mathbb{N}$. **Hint:** You actually need two base cases for the induction proof.

8. Let a, b, c be integers. Prove that if $a|b$ or $a|c$, then $a|bc$. What is the converse of this statement? Is the converse true? Why or why not?

9. Determine the following sets (just give a simple description for each one, no proof necessary):

- $\{m \in \mathbb{Z}^+ : \exists n \in \mathbb{Z}^+, m \leq n\}$
- $\{m \in \mathbb{Z}^+ : \forall n \in \mathbb{Z}^+, m \leq n\}$
- $\{m \in \mathbb{Z}^+ : \exists n \in \mathbb{Z}^+, n \leq m\}$
- $\{m \in \mathbb{Z}^+ : \forall n \in \mathbb{Z}^+, n \leq m\}$

10. Prove that if $A \cap B \subseteq C$ and $x \in B$, then $x \notin A - C$.

11. Prove that intersection distributes over union. That is, prove that for any sets A, B, and C:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

12. For each $r \in \mathbb{Q}$, let K_r be the set containing all real numbers except for r. Find $\bigcup_{r \in \mathbb{Q}} K_r$ and $\bigcap_{r \in \mathbb{Q}} K_r$.

13. Prove de Morgan's laws, that for any sets A and B (in some universe U), the following hold:

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c.$$

14. Prove that for any two sets, A and B, the following statements hold:

- $A \subseteq B$ if and only if $A \cup B = B$.
- $A \subseteq B$ if and only if $A \cap B = A$.
- $A \cup B = B$ if and only if $A \cap B = A$.

15. Find functions $f_i : \mathbb{R} \to \mathbb{R}$ with the following images (the range):

- $\mathrm{Im}(f_1) = \mathbb{R}$
- $\mathrm{Im}(f_2) = \mathbb{R}^+$
- $\mathrm{Im}(f_3) = \mathbb{R} \setminus \mathbb{Z}$
- $\mathrm{Im}(f_4) = \mathbb{Z}$

16. Determine whether each of the following functions $f_i : \mathbb{R} \to \mathbb{R}$ is injective, surjective, or bijective:

- $f_1(x) = 2x + 5$
- $f_2(x) = x^2 + 2x + 1$
- $f_3(x) = x^3$
- $f_4(x) = e^x$

17. Give the negation of each of the following statements.

- $\forall x \exists y \left[(A(x,y) \wedge B(x,y)) \implies (C(x) \vee \neg D(y)) \right]$.
- $\forall x \neg \exists y \left[(A(x,y) \wedge B(x,y)) \implies (C(x) \vee \neg D(y)) \right]$.
- For all $x, y \in V$, there exists an xy-path in G of length at least k.
- There exists a $\mathcal{B}$-regular integer $x \in S$ such that for any integer $y \in S$, either $y = x$, x eliminates y, or y is not $\mathcal{B}$-regular.

18. For each of the following implications list the starting assumptions you would make and the conclusions you would want to draw were you to prove each statement directly, through contraposition, and through contradiction.

- $A \implies B$.

- If $a|b$ and $a|c$, then $a|(b+c)$.
- If p is a prime number and $p|ab$, then $p|a$ or $p|b$.
- If S is $\mathcal{L}$-categorical and $T \leq S$, then for any $\mathcal{L}$-categorical X, $T \oplus X \leq S \oplus X$.