Local Structure: Forbidden Subgraphs

Graph Theory (Fall 2011)
Rutgers University
Swastik Kopparty

We now start systematically investigating the local structure of graphs. Local structure refers to the intrinsic relations that hold between the answers to the questions "which small subgraphs appear in G?" and "how many of them are there?"; relations that hold between these answers in every graph G.

A trivial example is: if G has n vertices and $\binom{n}{2}$ edges, then G contains a K_4 as a subgraph.

The first serious result of this kind is Mantel's theorem from the 1907, which studies the maximum number of edges that a graph with n vertices can have without having a triangle as a subgraph.

Theorem 1. *Let G be a graph with n vertices and m edges. If $m > \lfloor n/2 \rfloor \lceil n/2 \rceil$, then G contains a triangle as a subgraph.*

Remark: The bound on m is tight; if G is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, then G has n vertices and $\lfloor n/2 \rfloor \lceil n/2 \rceil$ edges, and G has no triangles (or any odd cycles) because it is bipartite.

The idea of the following proof of Mantel's theorem is to compare every triangle free graph with this complete bipartite graph.

Proof. Given a triangle-free $G = (V, E)$, we will see how to modify G so that it is bipartite and has at least as many edges. This will imply the result.

Let $v \in G$ be a vertex of maximum degree. Let S be the set of neighbors of v. Since G is triangle free, we must have that there are no edges between vertices of S (i.e., S is an independent set).

Consider the graph $G' = (V, E')$ which is the complete bipartite graph on S and $V \setminus S$.

We claim that for every vertex $w \in V$, the degree of w in G' is at least as large as the degree of w in G. Indeed, if $w \notin S$, then the degree of w in G' equals $|S|$, which is at least the degree of w in G (by choice of v). If $w \in S$, then the degree of w in G' equals $|V| - |S|$, which is at least as large as the degree of w in G (because in G, w is not adjacent to any vertex in S).

Thus the number of edges in G' is at least as large as the number of edges in G. Therefore, the number of edges in G is at most $\max_{0<i<n} \lfloor n/2 \rfloor \lceil n/2 \rceil$, as required.

From this proof, one can verify that $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is the unique triangle free graph with these many edges. □

Turan's theorem generalizes Mantel's theorem to K_r for arbitrary r.

Theorem 2. *Let G be a graph with n vertices and m edges. If $m > \frac{n^2}{2} \cdot \left(1 - \frac{1}{r}\right)$, then G contains a K_{r+1} as a subgraph.*

Remark: Notice that this theorem is also tight. Consider the complete r-partite graph, with each part having n/r vertices. This graph is K_r-free, and the total number of edges in this graph is $\left(\frac{n}{r}\right)^2 \cdot \binom{r}{2} = \frac{n^2}{2} \cdot \left(1 - \frac{1}{r}\right)$.

The proof below compares an arbitrary K_{r+1}-free graph with a suitable complete r-partite graph.

Proof. We will prove by induction on r that all K_{r+1}-free graphs with the largest number of edges are complete r-partite graphs. This will imply the result.

Let G be a K_{r+1}-free graph. Let $v \in G$ be a vertex of maximum degree. Let S be the set of neighbors of v. Since G is K_{r+1}-free, we must have $G|_S$ is K_r-free.

We now consider a new graph $G' = (V, E')$ on the same vertex set as follows. We let $G'|_S$ be the $(r-1)$-partite graph with the maximum number of edges which is K_r-free. Note that $G'|_S$ has at least as many edges as $G|_S$. Next, we make $G'|_{V \setminus S}$ an independent set. Finally, we introduce an edge in G' between every vertex in S and every vertex in $V \setminus S$.

The main point is that G' has at least as many edges as G does. This comes from the following observations. First, $G'|_S$ has at least as many edges as $G|_S$. Next, every vertex in $V \setminus S$ has its G' degree equal to $|S|$, which is at least as large as its G-degree (by choice of v). Finally, every vertex of S is joined to as many vertices of $V \setminus S$ as possible, namely $|V \setminus S|$.

Now notice that G' is r-partite. Thus the maximum number of edges in a K_{r+1}-free graph is achieved by an r-partite graph.

Inspecting the proof, we see that unless G itself is r-partite, G' has strictly more edges that G does. This shows that the only K_{r+1}-free graphs with the maximum number of edges are r-partite. $\square$

Turan's theorem has many many proofs, based on many different principles. You should try to come up with another.

Turan's theorem in the 1940s marked the birth of extremal graph theory.

One basic question that this directly raises is: for a fixed graph F, what is the largest number of edges that an n vertex graph can have without having a subgraph isomorphic to F? This quantity is called the Turan number of F, and is denoted by $\mathsf{ex}(n, F)$.

Another direct question that arises: for a collection of graphs $\mathcal{F}$, what is the maximum number of edges in an n-vertex graph which does not contain a copy of F for every $F \in \mathcal{F}$? The answer to this question is denoted $\mathsf{ex}(n, \mathcal{F})$ and is called the Turan number of the collection $\mathcal{F}$.

Many deep and profound theorems about Turan numbers are known today (and many are not too). We will see (and not see) some of them in the next few lectures.

More generally, one can ask about the relationship between the number of copies of one graph F_1 and the number of copies of another graph F_2 in every graph G. We will also explore such questions in our study of the local structure of graphs.