Chapter 1

Static Games of Complete Information

In this chapter we consider games of the following simple form: first the players simultaneously choose actions; then the players receive payoffs that depend on the combination of actions just chosen. Within the class of such static (or simultaneous-move) games, we restrict attention to games of *complete information*. That is, each player's payoff function (the function that determines the player's payoff from the combination of actions chosen by the players) is common knowledge among all the players. We consider dynamic (or sequential-move) games in Chapters 2 and 4, and games of incomplete information (games in which some player is uncertain about another player's payoff function—as in an auction where each bidder's willingness to pay for the good being sold is unknown to the other bidders) in Chapters 3 and 4.

In Section 1.1 we take a first pass at the two basic issues in game theory: how to describe a game and how to solve the resulting game-theoretic problem. We develop the tools we will use in analyzing static games of complete information, and also the foundations of the theory we will use to analyze richer games in later chapters. We define the *normal-form representation* of a game and the notion of a *strictly dominated strategy*. We show that some games can be solved by applying the idea that rational players do not play strictly dominated strategies, but also that in other games this approach produces a very imprecise prediction about the play of the game (sometimes as imprecise as "anything could ıappen"). We then motivate and define *Nash equilibrium*—a so-ution concept that produces much tighter predictions in a very ›road class of games.

In Section 1.2 we analyze four applications, using the tools leveloped in the previous section: Cournot's (1838) model of im-›erfect competition, Bertrand's (1883) model of imperfect com-›etition, Farber's (1980) model of final-offer arbitration, and the ›roblem of the commons (discussed by Hume [1739] and others). n each application we first translate an informal statement of the ›roblem into a normal-form representation of the game and then olve for the game's Nash equilibrium. (Each of these applications ıas a unique Nash equilibrium, but we discuss examples in which his is not true.)

In Section 1.3 we return to theory. We first define the no-ion of a *mixed strategy*, which we will interpret in terms of one ›layer's uncertainty about what another player will do. We then tate and discuss Nash's (1950) Theorem, which guarantees that a Vash equilibrium (possibly involving mixed strategies) exists in a ›road class of games. Since we present first basic theory in Sec-ion 1.1, then applications in Section 1.2, and finally more theory n Section 1.3, it should be apparent that mastering the additional heory in Section 1.3 is not a prerequisite for understanding the ıpplications in Section 1.2. On the other hand, the ideas of a mixed trategy and the existence of equilibrium do appear (occasionally) n later chapters.

This and each subsequent chapter concludes with problems, uggestions for further reading, and references.

l.1 Basic Theory: Normal-Form Games and Nash Equilibrium

.1.A Normal-Form Representation of Games

n the normal-form representation of a game, each player simul-aneously chooses a strategy, and the combination of strategies hosen by the players determines a payoff for each player. We llustrate the normal-form representation with a classic example – *The Prisoners' Dilemma*. Two suspects are arrested and charged vith a crime. The police lack sufficient evidence to convict the sus-›ects, unless at least one confesses. The police hold the suspects in

separate cells and explain the consequences that will follow from the actions they could take. If neither confesses then both will be convicted of a minor offense and sentenced to one month in jail. If both confess then both will be sentenced to jail for six months. Finally, if one confesses but the other does not, then the confessor will be released immediately but the other will be sentenced to nine months in jail—six for the crime and a further three for obstructing justice.

The prisoners' problem can be represented in the accompanying bi-matrix. (Like a matrix, a bi-matrix can have an arbitrary number or rows and columns; "bi" refers to the fact that, in a two-player game, there are two numbers in each cell—the payoffs to the two players.)

		Prisoner 2	
		Mum	Fink
Prisoner 1	Mum	−1, −1	−9, 0
	Fink	0, −9	−6, −6

The Prisoners' Dilemma

In this game, each player has two strategies available: confess (or fink) and not confess (or be mum). The payoffs to the two players when a particular pair of strategies is chosen are given in the appropriate cell of the bi-matrix. By convention, the payoff to the so-called row player (here, Prisoner 1) is the first payoff given, followed by the payoff to the column player (here, Prisoner 2). Thus, if Prisoner 1 chooses Mum and Prisoner 2 chooses Fink, for example, then Prisoner 1 receives the payoff −9 (representing nine months in jail) and Prisoner 2 receives the payoff 0 (representing immediate release).

We now turn to the general case. The *normal-form representation* of a game specifies: (1) the players in the game, (2) the strategies available to each player, and (3) the payoff received by each player for each combination of strategies that could be chosen by the players. We will often discuss an n-player game in which the players are numbered from 1 to n and an arbitrary player is called player i. Let S_i denote the set of strategies available to player i (called i's *strategy space*), and let s_i denote an arbitrary member of this set. (We will occasionally write $s_i \in S_i$ to indicate that the trategy s_i is a member of the set of strategies S_i.) Let $(s_1, \ldots, s_n)$ enote a combination of strategies, one for each player, and let $_i$ denote player i's *payoff function*: $u_i(s_1, \ldots, s_n)$ is the payoff to layer i if the players choose the strategies $(s_1, \ldots, s_n)$. Collecting ll of this information together, we have:

efinition *The* ***normal-form representation*** *of an n-player game spec-ies the players' strategy spaces $S_1, \ldots, S_n$ and their payoff functions $_1, \ldots, u_n$. We denote this game by $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$.*

Although we stated that in a normal-form game the players ıoose their strategies simultaneously, this does not imply that the arties necessarily *act* simultaneously: it suffices that each choose is or her action without knowledge of the others' choices, as 'ould be the case here if the prisoners reached decisions at ar-itrary times while in their separate cells. Furthermore, although ı this chapter we use normal-form games to represent only static ames in which the players all move without knowing the other layers' choices, we will see in Chapter 2 that normal-form repre-ntations can be given for sequential-move games, but also that ı alternative—the *extensive-form* representation of the game—is ten a more convenient framework for analyzing dynamic issues.

1.B Iterated Elimination of Strictly Dominated Strategies

aving described one way to represent a game, we now take a rst pass at describing how to solve a game-theoretic problem. 'e start with the Prisoners' Dilemma because it is easy to solve, sing only the idea that a rational player will not play a strictly ominated strategy.

In the Prisoners' Dilemma, if one suspect is going to play Fink, en the other would prefer to play Fink and so be in jail for six onths rather than play Mum and so be in jail for nine months. milarly, if one suspect is going to play Mum, then the other ould prefer to play Fink and so be released immediately rather an play Mum and so be in jail for one month. Thus, for prisoner playing Mum is dominated by playing Fink—for each strategy at prisoner j could choose, the payoff to prisoner i from playing um is less than the payoff to i from playing Fink. (The same ould be true in any bi-matrix in which the payoffs 0, −1, −6,

and −9 above were replaced with payoffs T, R, P, and S, respectively, provided that $T > R > P > S$ so as to capture the ideas of temptation, reward, punishment, and sucker payoffs.) More generally:

Definition *In the normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, let s_i' and s_i'' be feasible strategies for player i (i.e., s_i' and s_i'' are members of S_i). Strategy s_i' is* ***strictly dominated*** *by strategy s_i'' if for each feasible combination of the other players' strategies, i's payoff from playing s_i' is strictly less than i's payoff from playing s_i'':*

$$u_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n) < u_i(s_1, \ldots, s_{i-1}, s_i'', s_{i+1}, \ldots, s_n) \quad \text{(DS)}$$

for each $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ that can be constructed from the other players' strategy spaces $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n$.

Rational players do not play strictly dominated strategies, because there is no belief that a player could hold (about the strategies the other players will choose) such that it would be optimal to play such a strategy.[1] Thus, in the Prisoners' Dilemma, a rational player will choose Fink, so (Fink, Fink) will be the outcome reached by two rational players, even though (Fink, Fink) results in worse payoffs for both players than would (Mum, Mum). Because the Prisoners' Dilemma has many applications (including the arms race and the free-rider problem in the provision of public goods), we will return to variants of the game in Chapters 2 and 4. For now, we focus instead on whether the idea that rational players do not play strictly dominated strategies can lead to the solution of other games.

Consider the abstract game in Figure 1.1.1.[2] Player 1 has two strategies and player 2 has three: $S_1 = \{\text{Up}, \text{Down}\}$ and $S_2 = \{\text{Left}, \text{Middle}, \text{Right}\}$. For player 1, neither Up nor Down is strictly

[1] A complementary question is also of interest: if there is no belief that player i could hold (about the strategies the other players will choose) such that it would be optimal to play the strategy s_i, can we conclude that there must be another strategy that strictly dominates s_i? The answer is "yes," *provided* that we adopt appropriate definitions of "belief" and "another strategy," both of which involve the idea of mixed strategies to be introduced in Section 1.3.A.

[2] Most of this book considers economic applications rather than abstract examples, both because the applications are of interest in their own right and because, for many readers, the applications are often a useful way to explain the underlying theory. When introducing some of the basic theoretical ideas, however, we will sometimes resort to abstract examples that have no natural economic interpretation.

		Player 2		
		Left	Middle	Right
Player 1	Up	1,0	1,2	0,1
	Down	0,3	0,1	2,0

Figure 1.1.1.

ominated: Up is better than Down if 2 plays Left (because $1 > 0$), ut Down is better than Up if 2 plays Right (because $2 > 0$). For layer 2, however, Right is strictly dominated by Middle (because > 1 and $1 > 0$), so a rational player 2 will not play Right. hus, if player 1 knows that player 2 is rational then player 1 can liminate Right from player 2's strategy space. That is, if player knows that player 2 is rational then player 1 can play the game ı Figure 1.1.1 *as if* it were the game in Figure 1.1.2.

		Player 2	
		Left	Middle
Player 1	Up	1,0	1,2
	Down	0,3	0,1

Figure 1.1.2.

In Figure 1.1.2, Down is now strictly dominated by Up for layer 1, so if player 1 is rational (and player 1 knows that player 2 rational, so that the game in Figure 1.1.2 applies) then player 1 ·ill not play Down. Thus, if player 2 knows that player 1 is ra- onal, *and* player 2 knows that player 1 knows that player 2 is ıtional (so that player 2 knows that Figure 1.1.2 applies), then layer 2 can eliminate Down from player 1's strategy space, leav- ıg the game in Figure 1.1.3. But now Left is strictly dominated y Middle for player 2, leaving (Up, Middle) as the outcome of ıe game.

This process is called *iterated elimination of strictly dominated rategies*. Although it is based on the appealing idea that ratio- al players do not play strictly dominated strategies, the process as two drawbacks. First, each step requires a further assumption

		Player 2	
		Left	Middle
Player 1	Up	1,0	1,2

Figure 1.1.3.

about what the players know about each other's rationality. If we want to be able to apply the process for an arbitrary number of steps, we need to assume that it is *common knowledge* that the players are rational. That is, we need to assume not only that all the players are rational, but also that all the players know that all the players are rational, and that all the players know that all the players know that all the players are rational, and so on, *ad infinitum*. (See Aumann [1976] for the formal definition of common knowledge.)

The second drawback of iterated elimination of strictly dominated strategies is that the process often produces a very imprecise prediction about the play of the game. Consider the game in Figure 1.1.4, for example. In this game there are no strictly dominated strategies to be eliminated. (Since we have not motivated this game in the slightest, it may appear arbitrary, or even pathological. See the case of three or more firms in the Cournot model in Section 1.2.A for an economic application in the same spirit.) Since all the strategies in the game survive iterated elimination of strictly dominated strategies, the process produces no prediction whatsoever about the play of the game.

	L	*C*	*R*
T	0,4	4,0	5,3
M	4,0	0,4	5,3
B	3,5	3,5	6,6

Figure 1.1.4.

We turn next to Nash equilibrium—a solution concept that produces much tighter predictions in a very broad class of games. We show that Nash equilibrium is a stronger solution concept han iterated elimination of strictly dominated strategies, in the ense that the players' strategies in a Nash equilibrium always urvive iterated elimination of strictly dominated strategies, but he converse is not true. In subsequent chapters we will argue that n richer games even Nash equilibrium produces too imprecise a rediction about the play of the game, so we will define still-tronger notions of equilibrium that are better suited for these icher games.

.1.C Motivation and Definition of Nash Equilibrium

)ne way to motivate the definition of Nash equilibrium is to argue ıat if game theory is to provide a unique solution to a game-ıeoretic problem then the solution must be a Nash equilibrium, ı the following sense. Suppose that game theory makes a unique rediction about the strategy each player will choose. In order or this prediction to be correct, it is necessary that each player be rilling to choose the strategy predicted by the theory. Thus, each layer's predicted strategy must be that player's best response) the predicted strategies of the other players. Such a prediction ould be called *strategically stable* or *self-enforcing*, because no single layer wants to deviate from his or her predicted strategy. We will ıll such a prediction a Nash equilibrium:

efinition *In the n-player normal-form game* $G = \{S_1, \ldots, S_n; u_1, \ldots,$ $_n\}$*, the strategies* $(s_1^*, \ldots, s_n^*)$ *are a* ***Nash equilibrium*** *if, for each player* s_i^* *is (at least tied for) player i's best response to the strategies specified r the* $n-1$ *other players,* $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$:

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \qquad (NE)$$

r every feasible strategy s_i *in* S_i*; that is,* s_i^* *solves*

$$\max_{s_i \in S_i} u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*).$$

To relate this definition to its motivation, suppose game theory fers the strategies $(s_1', \ldots, s_n')$ as the solution to the normal-form ıme $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$. Saying that $(s_1', \ldots, s_n')$ is *not*

a Nash equilibrium of G is equivalent to saying that there exists some player i such that s_i' is *not* a best response to $(s_1', \ldots, s_{i-1}', s_{i+1}', \ldots, s_n')$. That is, there exists some s_i'' in S_i such that

$$u_i(s_1', \ldots, s_{i-1}', s_i', s_{i+1}', \ldots, s_n') < u_i(s_1', \ldots, s_{i-1}', s_i'', s_{i+1}', \ldots, s_n').$$

Thus, if the theory offers the strategies $(s_1', \ldots, s_n')$ as the solution but these strategies are not a Nash equilibrium, then at least one player will have an incentive to deviate from the theory's prediction, so the theory will be falsified by the actual play of the game. A closely related motivation for Nash equilibrium involves the idea of convention: if a convention is to develop about how to play a given game then the strategies prescribed by the convention must be a Nash equilibrium, else at least one player will not abide by the convention.

To be more concrete, we now solve a few examples. Consider the three normal-form games already described—the Prisoners' Dilemma and Figures 1.1.1 and 1.1.4. A brute-force approach to finding a game's Nash equilibria is simply to check whether each possible combination of strategies satisfies condition (NE) in the definition.[3] In a two-player game, this approach begins as follows: for each player, and for each feasible strategy for that player, determine the other player's best response to that strategy. Figure 1.1.5 does this for the game in Figure 1.1.4 by underlining the payoff to player j's best response to each of player i's feasible strategies. If the column player were to play L, for instance, then the row player's best response would be M, since 4 exceeds 3 and 0, so the row player's payoff of 4 in the (M, L) cell of the bi-matrix is underlined.

A pair of strategies satisfies condition (NE) if each player's strategy is a best response to the other's—that is, if both payoffs are underlined in the corresponding cell of the bi-matrix. Thus, (B, R) is the only strategy pair that satisfies (NE); likewise for (Fink, Fink) in the Prisoners' Dilemma and (Up, Middle) in

[3]In Section 1.3.A we will distinguish between pure and mixed strategies. We will then see that the definition given here describes *pure-strategy* Nash equilibria, but that there can also be *mixed-strategy* Nash equilibria. Unless explicitly noted otherwise, all references to Nash equilibria in this section are to pure-strategy Nash equilibria.

	L	C	R
T	0, $\underline{4}$	$\underline{4}$, 0	5, 3
M	$\underline{4}$, 0	0, $\underline{4}$	5, 3
B	3, 5	3, 5	$\underline{6}$, $\underline{6}$

Figure 1.1.5.

Figure 1.1.1. These strategy pairs are the unique Nash equilibria of these games.[4]

We next address the relation between Nash equilibrium and iterated elimination of strictly dominated strategies. Recall that the Nash equilibrium strategies in the Prisoners' Dilemma and Figure 1.1.1—(Fink, Fink) and (Up, Middle), respectively—are the only strategies that survive iterated elimination of strictly dominated strategies. This result can be generalized: if iterated elimination of strictly dominated strategies eliminates all but the strategies $(s_1^*, \ldots, s_n^*)$, then these strategies are the unique Nash equilibrium of the game. (See Appendix 1.1.C for a proof of this claim.) Since iterated elimination of strictly dominated strategies frequently does *not* eliminate all but a single combination of strategies, however, it is of more interest that Nash equilibrium is a stronger solution concept than iterated elimination of strictly dominated strategies, in the following sense. If the strategies $(s_1^*, \ldots, s_n^*)$ are a Nash equilibrium then they survive iterated elimination of strictly dominated strategies (again, see the Appendix for a proof), but there can be strategies that survive iterated elimination of strictly dominated strategies but are not part of any Nash equilibrium. To see the latter, recall that in Figure 1.1.4 Nash equilibrium gives the unique prediction (B, R), whereas iterated elimination of strictly dominated strategies gives the maximally imprecise prediction: no strategies are eliminated; anything could happen.

Having shown that Nash equilibrium is a stronger solution concept than iterated elimination of strictly dominated strategies, we must now ask whether Nash equilibrium is too strong a solution concept. That is, can we be sure that a Nash equilibrium exists? Nash (1950) showed that in any finite game (i.e., a game in which the number of players n and the strategy sets $S_1, \ldots, S_n$ are all finite) there exists at least one Nash equilibrium. (This equilibrium may involve mixed strategies, which we will discuss in Section 1.3.A; see Section 1.3.B for a precise statement of Nash's Theorem.) Cournot (1838) proposed the same notion of equilibrium in the context of a particular model of duopoly and demonstrated (by construction) that an equilibrium exists in that model; see Section 1.2.A. In every application analyzed in this book, we will follow Cournot's lead: we will demonstrate that a Nash (or stronger) equilibrium exists by constructing one. In some of the theoretical sections, however, we will rely on Nash's Theorem (or its analog for stronger equilibrium concepts) and simply assert that an equilibrium exists.

[4]This statement is correct even if we do not restrict attention to pure-strategy Nash equilibrium, because no mixed-strategy Nash equilibria exist in these three games. See Problem 1.10.

We conclude this section with another classic example—*The Battle of the Sexes*. This example shows that a game can have multiple Nash equilibria, and also will be useful in the discussions of mixed strategies in Sections 1.3.B and 3.2.A. In the traditional exposition of the game (which, it will be clear, dates from the 1950s), a man and a woman are trying to decide on an evening's entertainment; we analyze a gender-neutral version of the game. While at separate workplaces, Pat and Chris must choose to attend either the opera or a prize fight. Both players would rather spend the evening together than apart, but Pat would rather they be together at the prize fight while Chris would rather they be together at the opera, as represented in the accompanying bi-matrix.

		Pat: Opera	Pat: Fight
Chris	Opera	2, 1	0, 0
	Fight	0, 0	1, 2

The Battle of the Sexes

Both (Opera, Opera) and (Fight, Fight) are Nash equilibria.

We argued above that if game theory is to provide a unique solution to a game then the solution must be a Nash equilibrium. This argument ignores the possibility of games in which game theory does not provide a unique solution. We also argued that if a convention is to develop about how to play a given game, then the strategies prescribed by the convention must be a Nash equilibrium, but this argument similarly ignores the possibility of games for which a convention will not develop. In some games with multiple Nash equilibria one equilibrium stands out as the compelling solution to the game. (Much of the theory in later chapters is an effort to identify such a compelling equilibrium in different classes of games.) Thus, the existence of multiple Nash equilibria is not a problem in and of itself. In the Battle of the Sexes, however, (Opera, Opera) and (Fight, Fight) seem equally compelling, which suggests that there may be games for which game theory does not provide a unique solution and no convention will develop.[5] In such games, Nash equilibrium loses much of its appeal as a prediction of play.

Appendix 1.1.C

This appendix contains proofs of the following two Propositions, which were stated informally in Section 1.1.C. Skipping these proofs will not substantially hamper one's understanding of later material. For readers not accustomed to manipulating formal definitions and constructing proofs, however, mastering these proofs will be a valuable exercise.

Proposition A *In the n-player normal-form game* $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, *if iterated elimination of strictly dominated strategies eliminates all but the strategies* $(s_1^*, \ldots, s_n^*)$, *then these strategies are the unique Nash equilibrium of the game.*

Proposition B *In the n-player normal-form game* $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, *if the strategies* $(s_1^*, \ldots, s_n^*)$ *are a Nash equilibrium, then they survive iterated elimination of strictly dominated strategies.*

[5]In Section 1.3.B we describe a third Nash equilibrium of the Battle of the Sexes (involving mixed strategies). Unlike (Opera, Opera) and (Fight, Fight), this third equilibrium has symmetric payoffs, as one might expect from the unique solution to a symmetric game; on the other hand, the third equilibrium is also inefficient, which may work against its development as a convention. Whatever one's judgment about the Nash equilibria in the Battle of the Sexes, however, the broader point remains: there may be games in which game theory does not provide a unique solution and no convention will develop.

Since Proposition B is simpler to prove, we begin with it, to warm up. The argument is by contradiction. That is, we will assume that one of the strategies in a Nash equilibrium is eliminated by iterated elimination of strictly dominated strategies, and then we will show that a contradiction would result if this assumption were true, thereby proving that the assumption must be false.

Suppose that the strategies $(s_1^*, \ldots, s_n^*)$ are a Nash equilibrium of the normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, but suppose also that (perhaps after some strategies other than $(s_1^*, \ldots, s_n^*)$ have been eliminated) s_i^* is the first of the strategies $(s_1^*, \ldots, s_n^*)$ to be eliminated for being strictly dominated. Then there must exist a strategy s_i'' that has not yet been eliminated from S_i that strictly dominates s_i^*. Adapting (DS), we have

$$u_i(s_1, \ldots, s_{i-1}, s_i^*, s_{i+1}, \ldots, s_n) < u_i(s_1, \ldots, s_{i-1}, s_i'', s_{i+1}, \ldots, s_n) \qquad (1.1.1)$$

for each $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ that can be constructed from the strategies that have not yet been eliminated from the other players' strategy spaces. Since s_i^* is the first of the equilibrium strategies to be eliminated, the other players' equilibrium strategies have not yet been eliminated, so one of the implications of (1.1.1) is

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) < u_i(s_1^*, \ldots, s_{i-1}^*, s_i'', s_{i+1}^*, \ldots, s_n^*). \qquad (1.1.2)$$

But (1.1.2) is contradicted by (NE): s_i^* must be a best response to $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$, so there cannot exist a strategy s_i'' that strictly dominates s_i^*. This contradiction completes the proof.

Having proved Proposition B, we have already proved part of Proposition A: all we need to show is that if iterated elimination of dominated strategies eliminates all but the strategies $(s_1^*, \ldots, s_n^*)$ then these strategies are a Nash equilibrium; by Proposition B, any other Nash equilibria would also have survived, so this equilibrium must be unique. We assume that G is finite.

The argument is again by contradiction. Suppose that iterated elimination of dominated strategies eliminates all but the strategies $(s_1^*, \ldots, s_n^*)$ but these strategies are not a Nash equilibrium. Then there must exist some player i and some feasible strategy s_i in S_i such that (NE) fails, but s_i must have been strictly dominated by some other strategy s_i' at some stage of the process. The formal