VII

THE PROJECTIVE PLANE AND PROJECTIVE SPACE

We shall now investigate a geometry which is more fundamental than Euclidean geometry, in the sense that there will be no mention of metrical properties, distances or angle-measure, and the only concept we shall make use of at first is that of *incidence.* This geometry is called *projective geometry*, since its properties are invariant under *projection*, which is a geometrical transformation we shall investigate later. We shall enquire into the properties of projective space of n dimensions, after a special study of the cases $n = 2$ and $n = 3$, and since our treatment will be algebraic (non-algebraic treatments are also possible), we must say what kind of algebraic construct our numbers are to be taken from.

We shall work in a *commutative field*, although some of our theorems are also valid in a *skew* (non-commutative) field. For most of our purposes it will suffice to assume that our field is that of *the complex numbers*.

The outstanding property of this field which we may want to use is sometimes called *the fundamental theorem of algebra*: any polynomial with coefficients in the field of complex numbers splits into linear factors with coefficients which are complex numbers.

60.1 The complex projective plane

We must define the points of our geometry. A *point* is defined as an ordered triad of complex numbers (y_0, y_1, y_2) *not all zero*, with the understanding that the ordered triad (y_0k, y_1k, y_2k), where k is *any non-zero complex number*, represents the same point.

We must show that this relation is *an equivalence relation*, which is easily done, but this becomes evident if we remark that the ordered triad (y_0, y_1, y_2) represents a point distinct from the origin in three-dimensional complex affine space, and the set of points (y_0k, y_1k, y_2k) as k varies is the *ray*, or line through the origin joining the origin to the point. We are therefore defining a *line* or *ray* through the origin to be a *point* in our geometry, which is that of the complex projective plane.

The coordinates (y_0, y_1, y_2) we are using are usually called *homogeneous coordinates*, since only the ratios $y_0:y_1:y_2$ of the coordinates are significant, these being the same for the point (y_0k, y_1k, y_2k).

A *line* in our complex projective plane is defined to be the subset of points in the plane whose coordinates satisfy a linear homogeneous equation

$$u^0X_0 + u^1X_1 + u^2X_2 = 0,$$

where not all the u^i $(i = 0, 1, 2)$ are zero. Since, multiplying on the right by $k \neq 0$ we also have

$$u^0(X_0k) + u^1(X_1k) + u^2(X_2k) = 0,$$

it follows that if one member of the equivalence class representing a point of our projective plane satisfies the equation, so does every member.

We note that in three-dimensional affine space the equation above is that of *a plane through the origin.* Hence rays through the origin are to be points of the projective plane, and *planes* through the origin are to be *lines* of the projective plane. A *point* of the projective plane is *incident* with, or *lies on* a *line* of the projective plane if and only if the *ray* corresponding to the point lies in the *plane* corresponding to the line.

We now prove two incidence theorems which explain why our algebraic formulation is chosen in what may appear, at first sight, to be a curious manner.

Theorem I *In the projective plane, two distinct points determine a unique line with which they are incident.*

Theorem II *In the projective plane two distinct lines intersect in a unique point.*

Remark. It is the second theorem which distinguishes this geometry from ordinary Euclidean geometry, where lines which are distinct may not intersect. In this geometry distinct lines *always* intersect.

Proof. We prove the second theorem first. Let

$$u^0X_0 + u^1X_1 + u^2X_2 = 0 = v^0X_0 + v^1X_1 + v^2X_2$$

be the equations of the two lines. Since they are distinct, we do *not* have

$$u^0:u^1:u^2 = v^0:v^1:v^2.$$

Solving the two homogeneous equations for the point of intersection of the two lines gives us as point of intersection:

$$(u^1v^2 - u^2v^1,\ u^2v^0 - u^0v^2,\ u^0v^1 - u^1v^0).$$

This triad could only fail to be a point if *each* of the elements of the triad displayed above were zero. This could only be the case if

$$[u^0, u^1, u^2] = k[v^0, v^1, v^2] \qquad (k \neq 0),$$

which would mean that the two lines are identical, and this possibility is contrary to hypothesis.

To prove Thm I we remark that if the two distinct points are

$$(x_0, x_1, x_2) \quad \text{and} \quad (y_0, y_1, y_2),$$

then we do *not* have

$$(x_0, x_1, x_2) = (y_0 k, y_1 k, y_2 k) \qquad (k \neq 0),$$

and the line

$$u^0 X_0 + u^1 X_1 + u^2 X_2 = 0$$

contains both points, where

$$u^0 = x_1 y_2 - x_2 y_1, \quad u^1 = x_2 y_0 - x_0 y_2, \quad \text{and} \quad u^2 = x_0 y_1 - x_1 y_0,$$

as can be immediately verified. This line exists, since we do not have $u^0 = u^1 = u^2 = 0$, the two points (x_0, x_1, x_2) and (y_0, y_1, y_2) being distinct points. Hence there is *a* line through the two points. There cannot be more than one, since we should then have two distinct lines intersecting in more than one point, contrary to Thm II.

To Theorems I and II we add a third, designed to ensure that the geometry we are considering is not trivial.

Theorem III The projective plane contains at least four distinct points, no three of which are collinear.

Proof. The points (1, 0, 0), (0, 1, 0), (0, 0, 1) and (1, 1, 1) are easily seen to satisfy this condition over the field of the complex numbers, since the first three points lie, in pairs, on the lines $X_2 = 0$, $X_0 = 0$ and $X_1 = 0$ respectively, and the fourth point does not lie on any of these lines.

It will be noticed that we use capital letters for the variables in an equation, lower-case letters for actual coordinates of points, and also for what we may evidently call *the coordinates of a line*: $[u^0, u^1, u^2]$, the line being

$$u^0 X_0 + u^1 X_1 + u^2 X_2 = 0.$$

We note, that just as with points, the ordered triad $[ku^0, ku^1, ku^2]$ $(k \neq 0)$ represents the same line as does the ordered triad $[u^0, u^1, u^2]$.

60.2 The principle of duality in the projective plane

The symmetrical nature of the incidence relation

$$u^0 y_0 + u^1 y_1 + u^2 y_2 = 0,$$

which expresses the fact that the line $[u^0, u^1, u^2]$ *contains* the point (y_0, y_1, y_2), or equivalently, that the point (y_0, y_1, y_2) *lies on* the line $[u^0, u^1, u^2]$ gives rise to the *Principle of Duality* in the projective plane. This asserts that by an automatic interchange of the terms *point* and *line*, *lying on* and *passing through*, *join* and *intersection*, *collinear* and *concurrent*, and so on, any theorem in the projective plane which involves only incidence properties of points and lines becomes, on transliteration with the help of the dictionary of interchanges we have just listed, a theorem involving lines and points.

To give an immediate example, the theorem of Desargues, which we have already encountered (§5.2) and which is essentially a theorem of the projective plane, says that if ABC, $A'B'C'$ are two triangles in the projective plane which are such that the lines AA', BB' and CC' are concurrent (that is, pass through the same point), then the three points of intersection of corresponding sides, $BC \cap B'C'$, $CA \cap C'A'$ and $AB \cap A'B'$ are collinear (lie on a line).

The theorem obtained by applying the Principle of Duality says that if abc, $a'b'c'$ are triangles (we should change the term *triangle*, which refers to a configuration formed by three *points*, to *triline*, to be precise, since we now consider two configurations formed by triads of lines, a, b and c and a', b' and c', these not being concurrent: but we shall not do this) in the projective plane which are such that the points aa', bb' and cc' are collinear (lie on a line) then the lines joining corresponding vertices, $bc \cup b'c'$, $ca \cup c'a'$ and $ab \cup a'b'$ are concurrent (pass through a point).

We shall see, in fact, that exactly the same algebra which we shall use to establish the Desargues Theorem will establish the dual theorem, which is the converse of the Desargues Theorem (§62.2).

It will be noted that we are using the symbol AB to denote the *line* joining two distinct points A and B, and dually the symbol ab to denote the *point* of intersection of the two distinct lines a and b. Since we use the symbol $\cap$ for the intersection of two point-sets, the point of intersection of two lines AB and $A'B'$ is written as $AB \cap A'B'$, and therefore dually we write $ab \cup a'b'$ for the line joining the points ab and $a'b'$, although the $\cup$ symbol here does *not* stand for the join of two point-sets. Another symbol, if used, would also create difficulties. Some authors use the symbol $A + B$ for the set of points on the line joining A to B. We shall not do this, and we do not think that there will be any misunderstanding in interpreting our symbolism.

The reader may wonder why we talk of the Principle of Duality, when our statement describing it amounts to a proof. Historically the statement was put in the form of a Principle. For us it is almost a self-evident theorem.

Exercises

60.1 Show that the equation of the line joining the two points (x_0, x_1, x_2) and (y_0, y_1, y_2) is the result of eliminating U^0, U^1, U^2 from the three equations $U^0 X_0 + U^1 X_1 + U^2 X_2 = 0$, $U^0 x_0 + U^1 x_1 + U^2 x_2 = 0$, $U^0 y_0 + U^1 y_1 + U^2 y_2 = 0$, and is therefore the determinantal equation

$$\begin{vmatrix} X_0 & X_1 & X_2 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix} = 0.$$

60.2 Write out the dual of the Exercise above, changing the phrase 'equation of the line' to 'equation of the point'. (Note that the equation of a *line* is the condition that a point (X_0, X_1, X_2) lie on a line $[u^0, u^1, u^2]$. The equation of a *point* is the condition that a line $[U^0, U^1, U^2]$ pass through a point (x_0, x_1, x_2).)

60.3 Four distinct points A, B, C and D, no three of which are collinear, are said to form a complete quadrangle (or four-point), with *sides* formed by joining the points in pairs, and with *diagonal triangle* formed by the *diagonal points*, which are the intersections of the sides, other than A, B, C and D. Write down the dual of this definition, drawing a diagram in each case.

60.4 The Pappus Theorem says that if A, B, C are three distinct points on a line l, and A', B', C' three distinct points on a distinct line m, then the three points $BC' \cap B'C$, $CA' \cap C'A$ and $AB' \cap A'B$ are collinear. Write down the dual of this theorem, drawing a diagram in each case.

60.5 A set of points $(P, Q, R, \ldots)$ on a line l is said to be *in perspective* with a set of points $(P', Q', R', \ldots)$ on a line l', if the joins PP', QQ', RR', ... all pass through a point V. Formulate a dual concept for lines $(p, q, r, \ldots)$ on a point L, and lines $(p', q', r', \ldots)$ on a point L'.

60.6 The points on the surface of a given sphere which are diametrically opposed are identified as Points, and great circles cut on the surface of the sphere by planes through the center are identified as Lines. Show that these Points and Lines satisfy Theorems I, II and III of §60.1, the ground field being that of the real numbers.

60.7 Show that the points $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$, $D = (0, 1, 1)$, $E = (1, 0, 1)$, $F = (1, 1, 0)$ and $G = (1, 1, 1)$, where the field is that of the integers, mod 2 (that is $1 + 1 = 0$) form a finite projective geometry, with three points on every line and three lines through every point (See Pedoe, **12**, p. 85).

60.8 If all the elements involved, with the exception of the symbol ∞, are real numbers, show that the set of points and lines described satisfy Thms. I, II and III of §60.1. The points are: (*i*) ordinary points with coordinates (a, b); (*ii*) infinite points (m), together with the point ∞. The lines are given by the equations $X = c$, and $Y = mX + b$, and there is also the line at infinity ω. The line at infinity contains the point ∞ and all points (m), the line $X = c$ contains the point ∞ and all points (c, d), and the line $Y = mX + b$ contains the point (m) and all the points $(a, ma + b)$.

61.1 A model for the projective plane

We have already associated a *ray* through the origin of a three-dimensional space with a *point* of our projective plane. We pursue this further. Let V_3 be the complex vector space of vectors $y = (y_0, y_1, y_2)$, where the y_i are complex numbers. If $y \neq (0, 0, 0)$, the zero vector, then we may define a *ray* of V_3 as the set of vectors $yk = (y_0k, y_1k, y_2k)$ $(k \neq 0)$. We say that a *point* of our complex projective plane is a *ray* of V_3.

Two vectors $y = (y_0, y_1, y_2)$ and $z = (z_0, z_1, z_2)$ of V_3 are said to be *linearly dependent* if there exist complex numbers λ and μ, not both zero, so that

$$y\lambda + z\mu = 0,$$

where 0 is the zero vector. If such a relation necessarily implies that $\lambda = \mu = 0$, we say that y and z are *linearly independent*. Since a point of the complex projective plane is defined to be a ray of V_3, we see that if two non-zero vectors y and z are linearly dependent, so that they define the same ray of V_3, then they also define the same point of our complex projective plane.

If y and z are *independent*, and therefore neither is the zero vector, each defines a ray of V_3, and the set of vectors linearly dependent on y and z form a subspace of V_3.

In fact, if $x = y\lambda + z\mu$, and we write down the three equations for the coordinates of the vectors on each side of this equation, we have

$$y_0\lambda + z_0\mu - x_0 = 0,$$
$$y_1\lambda + z_1\mu - x_1 = 0,$$
$$y_2\lambda + z_2\mu - x_2 = 0,$$

and on eliminating λ and μ from these equations we have the determinantal equation

$$\begin{vmatrix} X_0 & X_1 & X_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix} = 0.$$

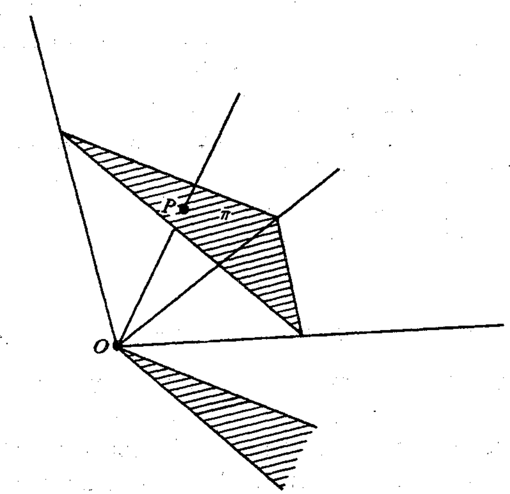

Fig. 61.1

This is a linear homogeneous equation in (X_0, X_1, X_2), and represents both the *plane* in V_3 defined by two distinct rays $y\lambda$ and $z\mu$ and the *line* in our projective plane which joins the two distinct points y and z.

It is clear that we may use a single symbol such as y or z for a point in our projective plane, with the understanding that $y\lambda$ represents the same point as y. We have seen that the set of points linearly dependent on the points y and z is the set $y\lambda + z\mu$. This gives the set of points on the *line* joining y and z.

If we assume that we are perfectly familiar with vector spaces V_3 over the complex number field, we have set up a *model* for the complex projective plane, since all the mathematical properties of the projective plane can be described or verified for V_3, and we assume that the mathematical properties of V_3 are free from contradiction. If we wish to *see* something, and this is what a model is for, let us consider how to set up homogeneous coordinates in a plane in real affine space. Call the plane π, and choose a system of three-dimensional coordinates for which the origin $O = (0, 0, 0)$ does not lie in π (Fig. 61.1). Then a ray through O which is not parallel to π intersects

π in a unique point P, and if the ray is the set of points (y_0k, y_1k, y_2k), where k varies over the real number field, the point P can be assigned the coordinates (y_0, y_1, y_2), where (y_0k, y_1k, y_2k) are also the coordinates of the same point P. This is precisely the development already given. Here there is a difference, since rays through O may be parallel to π. If such a ray is $(z_0, z_1, z_2)k$, we say that the point (z_0, z_1, z_2) is *an ideal point* in π, or *a point at infinity*. If the equation of the plane π is $u^0X_0 + u^1X_1 + u^2X_2 =$ constant, in our system of three-dimensional coordinates, the equation of the plane through O which is *parallel* to π is simply the equation $u^0X_0 + u^1X_1 + u^2X_2 = 0$, and we may take this same equation as *the equation of the line at infinity in the plane* π.

By introducing ideal points into the plane π, we have made it into a projective plane, in which all points have equal status, and two distinct lines always intersect.

Returning to our complex projective plane, which we shall also label π, we know that the vector space V_3 contains sets of 3 linearly independent vectors. One such set is $E_0 = (1, 0, 0)$, $E_1 = (0, 1, 0)$, $E_2 = (0, 0, 1)$. Vectors are, of course, linearly independent if there is no non-trivial linear homogeneous relation between them; that is, if the vectors be x, y and z, they are linearly independent when any linear relation $x\lambda + y\mu + z\nu = 0$ necessarily implies $\lambda = \mu = \nu = 0$. The vectors E_0, E_1 and E_2 are clearly independent. Since we may write any vector $y = (y_0, y_1, y_2)$ in V_3 as

$$y = E_0y_0 + E_1y_1 + E_2y_2,$$

and this same expression holds for points in π, we have shown that:

The system of homogeneous coordinates in π arises from the expression of a point P being linearly dependent on three fixed points which form a proper triangle in π.

We know that any three independent vectors in V_3 form *a basis* for the vectors of V_3: that is, any vector of V_3 can be expressed linearly in terms of three independent vectors. If the three independent vectors be named E_0^*, E_1^*, E_2^* then any vector y of V_3 has a unique expression in the form

$$y = E_0^*\lambda + E_1^*\mu + E_2^*\nu.$$

Independent vectors in V_3 correspond, if there are two vectors, to two distinct points of π, and if there are three vectors, to three non-collinear points of π, forming a proper triangle. The theory of the basis for vectors in V_3 shows us that if any three non-collinear points be chosen in π, with coordinates E_0^*, E_1^* and E_2^*, then any point in π may be expressed in the form

$$P = E_0^*\lambda + E_1^*\mu + E_2^*\nu,$$

where the ordered triad (λ, μ, ν) is any one of an equivalence class $(\lambda k, \mu k, \nu k)$ $(k \neq 0)$ (Fig. 61.2.)

If we call the triangle we have chosen *the triangle of reference*, we may call (λ, μ, ν) *the coordinates of the point P with respect to the given triangle of reference.*

We shall investigate, in §64.1 the effect of a change of triangle of reference on the coordinates of a given point. But for a given triangle of reference we now have enough

apparatus to prove theorems on points, lines, their joins and intersections in the projective plane π.

Since we shall be investigating the projective geometry of n dimensions in §66.1, we have introduced a suffix notation (y_0, y_1, y_2) for our coordinates, but we may now revert to a simpler notation, calling the triangle of reference ABC, and the coordinates of a point in π: (x, y, z).

We note that the coordinates of A, B and C are respectively $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ referred to ABC as triangle of reference. Any point on BC is $(0, 1, 0)y + (0, 0, 1)z = (0, y, z)$, and in fact the equation of the line BC is $X = 0$.

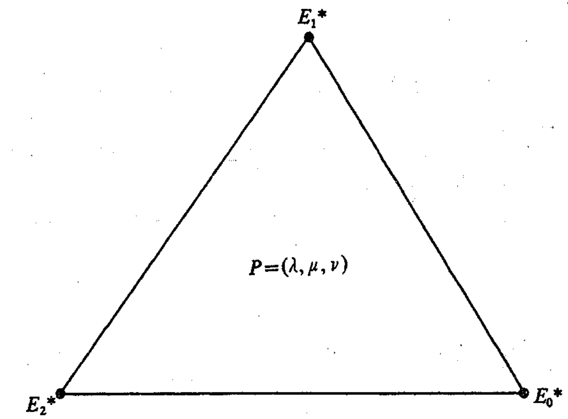

Fig. 61.2

If A' is the point $(0, q, r)$ on BC, any point on the line AA' is given by $(1, 0, 0)p + (0, q, r) = (p, q, r)$. Conversely, if the point (p, q, r) be joined to the vertex A, the intersection of this line with the side BC is the point $(0, q, r)$, and the equation of the line joining A to the point (p, q, r) is simply $rY - qZ = 0$.

The reader will find it interesting to compare the treatment of points and lines given here with that of points and lines in the real Euclidean plane given in Chapter I. Some of the results appear to be similar, but we do not deal with metrical results in the projective plane. We have a greater freedom, since our coordinates are not normalized (for instance, if A, B and C are three non-collinear points, any point P may be expressed as $P = Ax + By + Cz$, without having $x + y + z = 1$, as in Thm 4.2), but we do not have a metrical interpretation.

On the other hand, some theorems, such as the Theorem of Ceva, do have a corresponding theorem in the projective plane, since the metrical interpretation is not fundamental to the theorem.

Example. The lines AP, BP and CP cut the opposite sides of triangle ABC in the points L, M and N respectively. If $L = xB + x'C$, $M = yC + y'A$, $N = zA + z'B$, then

$$xyz = x'y'z'.$$

If $P = (p, q, r)$ with respect to ABC as triangle of reference, then $L = (0, q, r)$, $M = (p, 0, r)$ and $N = (p, q, 0)$, Since we are given that $L = (0, x, x')$, $M = (y', 0, y)$ and $N = (z, z', 0)$, we have

$$x:x' = q:r,\ y:y' = r:p,\ z:z' = p:q,$$

and therefore

$$(x:x')(y:y')(z:z') = (q:r)(r:p)(p:q) = 1.$$

Example. Theorem of Menelaus (compare Thm 4.1). A line cuts the sides BC, CA and AB of a triangle ABC in the points L, M and N respectively. If $L = Bx + Cx'$, $M = Cy + Ay'$ and $N = Az + Bz'$, then $xyz = -x'y'z'$.

Let the equation of the line be $eX + fY + gZ = 0$. Then L is given by the equation $fY + gZ = 0$, so that $L = (0, -g, f) = B(-g) + C(f)$. Hence $x:x' = -g:f$. Similarly $y:y' = -e:g$, and $z:z' = -f:e$, and the theorem follows immediately.

Exercises

61.1 Prove that if $A' = B + Ck$ is a point on BC, and if M be taken on AA' distinct from A or A', then if BM intersects CA in Q and CM intersects AB in R, the line QR will intersect BC in the point $B - Ck$.

61.2 The sides BC, CA and AB of the triangle of reference have coordinates [1, 0, 0], [0, 1, 0] and [0, 0, 1] respectively. If we write

$$a = [1, 0, 0],\quad b = [0, 1, 0],\quad c = [0, 0, 1],$$

show that $a' = b + kc$ is a line through the point bc. Write down the theorem dual to that given in Exercise 61.1, and prove it, using essentially the same algebra.

61.3 If ABC be the triangle of reference, show that the equation $lX + nZ = 0$ represents a line through the point B, and the equation $pX + qY = 0$ represents a line through the point C. If the first line intersects the side AC in the point E, and the second line intersects the side AB in the point F, show that the equation of the line EF is $lpX + lqY + npZ = 0$.

61.4 If the lines $uX + vY + wZ = 0$, $u'X + v'Y + w'Z = 0$ intersect at the point P, show that the equation

$$(uX + vY + wZ)\lambda + (u'X + v'Y + w'Z)\mu = 0$$

represents a line which always passes through P, for all values of λ and μ except $\lambda = \mu = 0$. Find the equation of the line which joins P to the vertex A of the triangle of reference.

62.1 The normalization theorem for points on a line

Theorem *If P, Q and R are three distinct collinear points in π, then a coordinate system may be set up in which the homogeneous coordinates of P, Q and R, written as vectors y, z and t, satisfy the equation*

$$t = y + z.$$

Remark. We use a single symbol to represent a vector. When we use a single symbol to represent a point in π, this means that we have chosen a specific representative from the equivalence class of ordered triads which represent the point.

Proof. In any system of homogeneous coordinates where P, Q and R are represented by the vector symbols y', z' and t' respectively, we have a relation

$$y'\lambda + z'\mu + t'\nu = 0,$$

since the three points are linearly dependent. The λ, μ, ν are not all zero. In fact, none of the multipliers is zero, since this would involve two of the points under consideration being identical, and this is excluded. Hence $\nu \neq 0$, and we may write

$$t' = y'\rho + z'\sigma,$$

where neither ρ nor σ is zero. Now write

$$y'\rho = y,\quad z'\sigma = z \quad \text{and} \quad t' = t,$$

so that y is a specific representative of the point P, z a specific representative of the point Q and t a specific representative of the point R, and we have

$$t = y + z.$$

Remark. If we set up a coordinate system on the line for which P and Q are base points, any point on the line being $X = Px_0 + Qx_1$, our procedure assigns the unit point, with coordinates (1, 1), to R.

This tidying-up process enables us to prove the Desargues Theorem very simply.

62.2 The Desargues theorem

Theorem *If two triangles ABC, $A'B'C'$ in π are such that the lines AA', BB' and CC' pass through a point V, then the three points $BC \cap B'C'$, $CA \cap C'A'$ and $AB \cap A'B'$ all lie on a line.*

Proof. We assume that the seven points V, A, B, C, A', B', C' are distinct (Fig. 62.1). In a system of homogeneous coordinates let ε denote the point V, and let $A = \alpha$, $B = \beta$, $C = \gamma$, $A' = \alpha'$, $B' = \beta'$, and $C' = \gamma'$.

By the Normalization Theorem of §62.1, we may write

$$\varepsilon = \alpha + \alpha' = \beta + \beta' = \gamma + \gamma'$$

simultaneously for the three pairs of points with which V is collinear. Let us write:

$$\rho = \beta - \gamma = \gamma' - \beta',\quad \sigma = \gamma - \alpha = \alpha' - \gamma',\quad \tau = \alpha - \beta = \beta' - \alpha'.$$

Then the point represented by ρ is linearly dependent on B and C, and therefore lies on the line BC, and it is also linearly dependent on B' and C', and therefore also lies