1 Indecomposability and connectedness

Definition 1.1. Let $\mathcal{A}$ be an abelian category, then $\mathrm{Kom}(\mathcal{A})$ is the abelian category of complexes in $\mathcal{A}$. Then there exists a category $\mathrm{D}(\mathcal{A})$, the derived category of $\mathcal{A}$, and a functor

$$Q : \mathrm{Kom}(\mathcal{A}) \to \mathrm{D}(A)$$

such that:

1. If $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism, then $Q(f)$ is an isomorphism in $\mathrm{D}(\mathcal{A})$.
2. Any $F : \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}$ satisfying the above property factors uniquely through Q.

The way we construct $\mathrm{D}(\mathcal{A})$ is by passing through the homotopy category of complexes $\mathrm{K}(\mathcal{A})$, by setting $\mathrm{Ob}(\mathrm{D}(\mathcal{A})) = \mathrm{Ob}(\mathrm{Kom}(\mathcal{A}))$ and $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ as the set of equivalence (in $\mathrm{K}(\mathcal{A})$) classes of diagrams of the form:

$$\begin{array}{ccc} & C^\bullet & \\ {}^{\text{qis}}\swarrow & & \searrow \\ A^\bullet & & B^\bullet \end{array}$$

Definition 1.2. We have the shift operator on $\mathrm{D}(\mathcal{A})$

$$T : \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{A}), \quad A \mapsto A[1]$$

and a triangle, in $\mathrm{D}(\mathcal{A})$, of the form

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

is called distinguished if it is isomorphic, in $\mathrm{D}(\mathcal{A})$, to a triangle of the form

$$A_0^\bullet \xrightarrow{f} B_0^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A_0^\bullet[1]$$

where $C(f)$ is the mapping cone of f, and τ, π are natural morphisms coming from the mapping cone construction. This implies that a distinguished triangle gives a long exact sequence of cohomology

$$... \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to ...$$

Note 1.1. This turns $\mathrm{D}(\mathcal{A})$ into a triangulated category, which is just a category with a shift operator, and a set of distinguished triangles satisfying some conditions.

Note 1.2. One way to make sense of the triangles: if you have a short exact sequence $0 \to A^\bullet \xrightarrow{f} B^\bullet \to C^\bullet \to 0$ then we can show that $C(f)$ is quasi-isomorphic to $C^\bullet$, and then we get a distinguished triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

Definition 1.3. A triangulated category $\mathcal{D}$ is decomposed into triangulated subcategory (subcategory s.t. inclusion is exact as a functor between triangulated categories) $\mathcal{D}_1, \mathcal{D}_2$ if the following 3 conditions are satisfied:

1. Both $\mathcal{D}_1, \mathcal{D}_2$ contain objects non-isomorphic to 0;
2. For all $A \in \mathcal{D}$ there exists a distinguished triangle

$$B_1 \to A \to B_2 \to B_1[1], \quad B_i \in \mathcal{D}_i$$

3. $\mathrm{Hom}(B_1, B_2) = \mathrm{Hom}(B_2, B_1) = 0$ for all $B_1 \in \mathcal{D}_1, B_2 \in \mathcal{D}_2$.

Definition 1.4. Let X be a scheme, its derived category is $\mathrm{D}^b(X) \coloneqq \mathrm{D}^b(\mathbf{Coh}(X))$.

Proposition 1.5. *Let X be a Noetherian scheme. Then X is connected if and only if $\mathrm{D}^b(X)$ is indecomposable.*

Proof. $(\Leftarrow)$: Suppose for the sake of contradiction that X is not connected, i.e., $X = X_1 \sqcup X_2$, with X_i closed. The idea is that for any $\mathcal{F}^\bullet \in \mathrm{D}^b(X)$, we have decomposition

$$\mathcal{F}^\bullet = \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet, \quad \mathrm{Supp}(\mathcal{F}_i^\bullet) \subseteq X_i$$

where the support of a complex $\mathcal{F}^\bullet$ is the union of the supports of all its cohomology sheaves. We will prove using induction, the base case for shifts of arbitrary coherent sheaves is clear. Let $\mathcal{F}^\bullet$ be a complex of length at least 2. Let m be the minimum number such that $\mathcal{H} = H^m(\mathcal{F}^\bullet) \neq 0$. We then have a distinguished triangle

$$\mathcal{H}[-m] \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}[1-m]$$

such that $H^q(\mathcal{F}^\bullet) = H^q(\mathcal{G}^\bullet)$ for $q > m$ and $H^q(\mathcal{G}^\bullet) = 0$ for $q \leq m$. This implies that $\mathcal{G}^\bullet$ is quasi-isomorphic to a complex $\mathcal{N}^\bullet$ with $\mathcal{N}^q = 0$ for $q \leq m$, hence in $\mathrm{D}^b(X)$, $\mathcal{G}^\bullet$ must have length 1 smaller than the length of $\mathcal{F}^\bullet$. By induction, we then have

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \mathcal{G}^\bullet = \mathcal{G}_1^\bullet \oplus \mathcal{G}_2^\bullet$$

and then define $\mathcal{F}_i^\bullet$ to be the complexes completing the 2 triangles

$$\mathcal{F}_i^\bullet \to \mathcal{G}_i^\bullet \to \mathcal{H}_i[1-m] \to \mathcal{F}_i^\bullet[1]$$

We can then show that

$$\mathrm{Hom}(\mathcal{G}_1^\bullet, \mathcal{H}_2[1-m]) = \mathrm{Hom}(\mathcal{G}_2^\bullet, \mathcal{H}_1[1-m]) = 0$$

which implies $\mathcal{F}^\bullet = \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet$.

$(\Rightarrow)$: Suppose that $\mathrm{D}^b(X)$ is decomposable by $\mathcal{D}_1, \mathcal{D}_2$. Then $\mathcal{O}_X = \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet$ (write down the distinguished triangle, and notice the last map is trivial hence the triangle is split). Since (co)kernel commute with direct sum, we have that $H^q(\mathcal{F}_i^\bullet) = 0$ for $i > 0$, hence $\mathcal{F}_i^\bullet$ is quasi-isomorphic to a coherent sheaf $\mathcal{F}_i$.

This direct sum is a $\mathcal{O}_X-$module decomposition, so we must have $\mathcal{F}_i \simeq \mathcal{I}_{X_i}$, an ideal sheaf which cuts out a subscheme X_i. Then

$$\mathcal{O}_X = \mathcal{I}_{X_1} + \mathcal{I}_{X_2} \subseteq \mathcal{I}_{X_1 \cap X_2}, \quad \mathcal{I}_{X_1 \cup X_2} \subseteq \mathcal{I}_{X_1} \cap \mathcal{I}_{X_2} = 0$$

thus $X = X_1 \sqcup X_2$. By our assumption we must have $X_1 = \emptyset$ or $X_2 = \emptyset$. WLOG assume X_2 is empty, then $\mathcal{O}_X \in \mathcal{D}_2$. Now, for any point $x \in X$, its structure sheaf is $\Bbbk(x) \simeq \mathcal{O}_X/\mathfrak{m}_x \in \mathrm{D}^b(X)$. A decomposition of $\Bbbk(x)$ must be trivial (since it's a direct sum of $\mathcal{O}_X$-modules), and there exists a non-trivial $\mathcal{O}_X \to \Bbbk(x)$ hence $\Bbbk(x) \in \mathcal{D}_1$ for all $x \in X$.

Let $\mathcal{F}^\bullet$ be nontrivial in $\mathcal{D}_2$. Then there is a maximal m such that $\mathcal{H} = H^m(\mathcal{F}^\bullet) \neq 0$. Then there exists a point x in the support of $\mathcal{H}$ and we have a surjection $\mathcal{H} \to \Bbbk(x)$. We have a quasi-isomorphism (the one identifying complex with zero cohomology past m with a complex which is 0 past m)

$$\mathcal{F}^\bullet \to \left(... \to \mathcal{F}^{m-1} \to \ker(d^m) \to 0 \to 0 \to ...\right)$$

and the nontrivial map

$$\left(... \to \mathcal{F}^{m-1} \to \ker(d^m) \to 0 \to 0 \to ...\right) \to \mathcal{H}[-m] \to \Bbbk(x)[-m]$$

and taking composition we get a nontrivial map $\mathcal{F}^\bullet \to \Bbbk(x)[-m]$ which is a contradiction since $\Bbbk(x) \in \mathcal{D}_1$. □

2 Semi-orthogonal decomposition

So our previous definition of decomposition is too strong, hence we need a weaker notation, called the semi-orthogonal decomposition.

Definition 2.1. A full triangulated subcategory $\mathcal{D}_0 \subseteq \mathcal{D}$ is called admissible if the inclusion has a right adjoint $\pi : \mathcal{D} \to \mathcal{D}_0$, i.e., $\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{D}_0}(A, \pi(B))$ for all $A \in \mathcal{D}_0$ and $B \in \mathcal{D}$.

The orthogonal complement of $\mathcal{D}_0$ is the full subcategory $\mathcal{D}_0^\perp$ of all objects $C \in \mathcal{D}$ such that $\mathrm{Hom}(A, C) = 0$ for all $A \in \mathcal{D}_0$.

Definition 2.2. A sequence of full admissible triangulated subcategories $\mathcal{D}_1, ..., \mathcal{D}_n \subset \mathcal{D}$ is semi-orthogonal if $D_i \subset \mathcal{D}_j^\perp$ for all $i < j$.

Such a sequence defines a semi-orthogonal decomposition if $\mathcal{D}$ is equivalent, via inclusion, to the smallest full triangulated subcategory of $\mathcal{D}$ containing all $\mathcal{D}_i$.

Proposition 2.3. *A semi-orthogonal sequence defines a decomposition if the intersection of all $\mathcal{D}_i^\perp$ is trivial.*

Example 2.4. In a saturated category, the orthogonal complement of an admissible subcategory is admissible, hence if $\mathcal{D}_0 \subset \mathcal{D}$ is orthogonal we get a semi-orthogonal decomposition $\mathcal{D} = \left\langle \mathcal{D}_0^\perp, \mathcal{D}_0 \right\rangle$.

The other way to get a semi-orthogonal decomposition is through exceptional sequences:

Definition 2.5. An exceptional sequence is a sequence of objects $E_1, ..., E_n \in \mathcal{D}$ such that

$$\mathrm{Hom}(E_j, E_i[l]) = \begin{cases} \Bbbk & \text{if } l = 0, i = j \\ 0 & \text{if } i < j \text{ or } l \neq 0, i = j \end{cases}$$

Such a sequence is full if any full triangulated subcategory containing all objects E_i is equivalent to $\mathcal{D}$.

Example 2.6. The point is that the full subcategory $\langle E_i \rangle$, whose objects are of the form $\bigoplus E_i[l]^{\alpha_l}$, is an admissible triangulated subcategory. Thus if $E_1, ..., E_n$ is a full exceptional sequence then

$$\langle E_1 \rangle, \quad \langle E_2 \rangle, \quad ..., \quad \langle E_n \rangle$$

form a semi-orthogonal decomposition.

Next we want to show that $\mathcal{O}(a), \mathcal{O}(a+1), ..., \mathcal{O}(a+n)$ forms a full exceptional sequence in $\mathrm{D}^b(\mathbb{P}^n)$. For this we need the following 2 propositions. Let $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ be the diagonal, and π_1, π_2 be the two projections $\mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$.

Proposition 2.7. *There exists a locally free resolution of $\mathcal{O}_\Delta$ of the form*

$$0 \to \bigwedge^n (\mathcal{O}(-1) \boxtimes \Omega(1)) \to \bigwedge^{n-1} (\mathcal{O}(-1) \boxtimes \Omega(1)) \to ... \to \mathcal{O}(-1) \boxtimes \Omega(1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0$$

Proof. We can think of $\mathbb{P}^n = \mathbb{P}(V)$, the space of lines $l \subset V$. Then $\mathcal{O}(-1)$, the tautological line bundle, has fiber l above a point $l \in \mathbb{P}(V)$. On the other hand, look at the Euler sequence

$$0 \to \Omega(1) \to V^\vee \otimes \mathcal{O} \to \mathcal{O}(1) \simeq \mathcal{O}(-1)^\vee \to 0$$

we get that the fiber of $\Omega(1)$ above $l \in \mathbb{P}(V)$ is the space of linear functional $\varphi : V \to \mathbb{k}$ that is trivial on $l \subset V$. Look at the evaluation map:

$$s : \mathcal{O}(-1) \boxtimes \Omega(1) \to \mathcal{O}_{\mathbb{P}(V) \times \mathbb{P}(V)}$$

which at a point $(l_1, l_2) \in \mathbb{P}(V) \times \mathbb{P}(V)$ is given by $x \otimes \varphi \mapsto \varphi(x)$ with $x \in l_1$ and $\varphi : V \to \mathbb{k}$ vanishing on l_2. Clearly then Δ is the zero locus of $s \in H^0\big(\mathbb{P}(V) \times \mathbb{P}(V), (\mathcal{O}(-1) \boxtimes \Omega(1))^\vee\big)$. Take the Koszul resolution coming from this section (of a vector bundle), we get the desired resolution. □

Proposition 2.8. *For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ there exists two natural spectral sequence*

$$E_1^{p,q} := H^q(\mathbb{P}^n, \mathcal{F}(p)) \otimes \Omega^{-p}(-p) \Rightarrow E^{p+q} = \begin{cases} \mathcal{F} & \text{if } p+q=0 \\ 0 & \text{otherwise} \end{cases}$$

$$E_1^{p,q} := H^q(\mathbb{P}^n, \mathcal{F} \otimes \Omega^{-p}(-p)) \otimes \mathcal{O}(p) \Rightarrow E^{p+q} = \begin{cases} \mathcal{F} & \text{if } p+q=0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Both of these come from the spectral sequence of filtered complex $A^\bullet$,

$$E_1^{p,q} = \mathrm{R}^q F(A^p) \Rightarrow \mathrm{R}^{p+q} F(A^\bullet)$$

Note 2.1. If $F = \Gamma$ and $A^\bullet = \Omega^\bullet$ then this gives you the Hodge-to-de Rham spectral sequence. The proof of this general sequence is the same, the point is that we have a double complex resolution (Cartan-Eilenberg resolution) of $A^\bullet$.

Here we let $A^\bullet = \pi_1^* \mathcal{F} \otimes \mathcal{L}^\bullet$, where $\mathcal{L} = \mathcal{O}(-1) \boxtimes \Omega(1)$ and $F = \pi_{2*}$. We have

$$\begin{aligned} \mathrm{R}^q F(A^{-p}) &= \mathrm{R}^q \pi_{2*}\left(\pi_1^*\mathcal{F} \otimes \bigwedge^p \pi_1^*\mathcal{O}(-1) \otimes \pi_2^*\Omega(1)\right) \\ &= \mathrm{R}^q \pi_{2*}(\pi_1^*\mathcal{F} \otimes \pi_1^*\mathcal{O}(-p) \otimes \pi_2^*\Omega^p(p)) \\ &= \mathrm{R}^q \pi_{2*}(\pi_1^*\mathcal{F}(-p) \otimes \pi_2^*\Omega^p(p)) \\ &= (\mathrm{R}^q \pi_{2*}\pi_1^*\mathcal{F}(-p)) \otimes \Omega^p(p) \\ &= H^q(\mathbb{P}^n, \mathcal{F}(-p)) \otimes \Omega^p(p) \end{aligned}$$

where the last equality comes from flat base change

$$\begin{array}{ccc} \mathbb{P}^n \times \mathbb{P}^n & \xrightarrow{\pi_1} & \mathbb{P}^n \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \phi_1} \\ \mathbb{P}^n & \xrightarrow[\phi_2]{} & \operatorname{Spec} \Bbbk \end{array}$$

which gives

$$\mathrm{R}^q \pi_{2*} \pi_1^* \mathcal{F}(-p) = \phi_2^* \mathrm{R}^q \phi_{1*} \mathcal{F}(-p) \simeq H^q(\mathbb{P}^n, \mathcal{F}(-p))$$

since $\mathrm{R}^q \phi_{1*} \mathcal{F}(-p)$ is just the sheaf of $H^q(\mathbb{P}^n, \mathcal{F}(-p))$ over a single point $\operatorname{Spec} \Bbbk$. This gives the desired

$$\mathrm{R}^q F(A^p) = H^q(\mathbb{P}^n, \mathcal{F}(p)) \otimes \Omega^{-p}(-p)$$

On the other hand, we have

$$\begin{aligned} \mathrm{R}^{p+q} F(A^\bullet) &= \mathrm{R}^{p+q} \pi_{2*}(\pi_1 \mathcal{F} \otimes \mathcal{L}^\bullet) \\ &= H^{p+q}(\Phi_{\mathcal{L}^\bullet}(\mathcal{F})) \\ &= H^{p+q}(\Phi_{\mathcal{O}_\Delta}(\mathcal{F})) = H^{p+q}(\mathcal{F}) \end{aligned}$$

which is $\mathcal{F}$ if $p+q=0$, and 0 otherwise. □

Proposition 2.9. *Any sequence of line bundles of the form*

$$\mathcal{O}(a), \quad \mathcal{O}(a+1), \quad ..., \quad \mathcal{O}(a+n)$$

forms a full exceptional sequence in $\mathrm{D}^b(\mathbb{P}^n)$.

Proof. We have

$$\operatorname{Hom}_{\mathrm{D}^b(\mathbb{P}^n)}(\mathcal{O}(i), \mathcal{O}(j)[q]) \simeq \operatorname{Hom}(\mathcal{O}, \mathcal{O}(j-i)[q]) \simeq \mathrm{R}^q\Gamma(\mathbb{P}^n, \mathcal{O}(j-i)) \simeq H^q(\mathbb{P}^n, \mathcal{O}(j-i))$$

which tells us that this is an exceptional sequence. It remains to show that the sequence is full.

We want to show that the orthogonal complement $\langle \mathcal{O}(a), ..., \mathcal{O}(a+n) \rangle^\perp$ is trivial. We will only do the case where we have a genuine sheaf $\mathcal{F}$. Suppose $\mathcal{F}$ is orthogonal to all $\mathcal{O}(i)$, then

$$\operatorname{Hom}(\mathcal{O}(i), \mathcal{F}[q]) = 0 \quad \forall\, s, \forall\, i = a, ..., a+n$$

then apply the Beilinson spectral sequence to $\mathcal{F}(-a)$ we get

$$E_1^{p,q} = H^q(\mathbb{P}^n, \mathcal{F}(p-a)) \otimes \Omega^{-p}(-p) \simeq \operatorname{Hom}(\mathcal{O}(a-p), \mathcal{F}[q]) \otimes \Omega^{-p}(-p)$$

The point here is that $\Omega^{-p}(-p) = 0$ unless $0 \le -p \le n$, in which case $\operatorname{Hom}(\mathcal{O}(a-p), \mathcal{F}[q]) = 0$. So this sequence converges to the zero sheaf, i.e., $\mathcal{F}(-a) = 0$ which implies $\mathcal{F} = 0$. □

Note 2.2. This sequence is actually stro‘ng, so we get an equivalence between $\mathrm{D}^b(\mathbb{P}^n)$ and the derived category of right $A-$modules for $A = \operatorname{End}(\bigoplus E_i)$.