1 Recap

We have the following locuses

$$C^r_d(C) = \left\{ D \in C^{(d)} \middle| h^0(D) \geq r+1 \right\}$$
$$W^r_d(C) = \left\{ L \in \operatorname{Pic}^d(C) \middle| h^0(L) \geq r+1 \right\}$$

Theorem 1.1 (Brill-Noether)**.** *Let C be a curve of genus g, and let $d \geq 1, r \geq 0$ be integers. Consider the Brill-Noether number:*

$$\rho = g - (r+1)(g-d+r)$$

If $\rho \geq 0$ then $W^r_d(C)$ is non empty. Now if C is a general curve, and $\rho < 0$, then $W^r_d(C)$ is empty.

Henry had also told us that there is always a $D = g^2_6$ on a smooth curve of genus 6. If D has base points, then we have the following cases:

- If D has 2 base points, then (mapping using D is the same as mapping using D minus base points) it gives a map $\phi_D : C \to \mathbb{P}^2$ which is either a birational map to a quadric, in which case the genus doesn't match, a 2-1 map to a conic, in which case C is hyperelliptic, or a 4-1 map to a line. The last case doesn't happen because $\phi_D(C)$ is nondegenerate.
- If D has 1 base point, then ϕ_D embeds C as a smooth quintic curve (5-1 map can't happen because nondegeneracy).
- Can use a similar argument to rule out the case of ≥ 3 base points.

If D has no base point, then $6 = \deg \phi_D \cdot \deg \phi_D(C)$, and we have the following cases:

- ϕ_D maps C in a 3-1 manner onto a conic. Here C is trigonal.
- ϕ_D maps C in a 2-1 manner onto a smooth plane cubic. Here C is bi-elliptic.
- ϕ_D maps C in a 2-1 manner onto a singular plane cubic. Here C is hyperelliptic.
- ϕ_D maps C birationally to a plane sextic curve C_0. In this case C_0 cannot have a point of multiplicity ≥ 4 (genus drops too much). If C_0 has a C_0 has a triple point, then C is trigonal.

Note 1.1. The genus formula for a plane curve $C \subset \mathbb{P}^2$ of degree d with singularities of multiplicities m_i is

$$g = \binom{d-1}{2} - \sum \binom{m_i}{2}$$

2 General curve of genus 6

We want to know what a general curve of genus 6 looks like. Brill-Noether gives us this for free, but for genus 6, we can do some hands on inspections instead. Recall we have the moduli $\mathcal{M}_g$, and the Hurwitz space:

$$\mathcal{H}_{d,g} = \left\{ (C, f) \middle| C \in \mathcal{M}_g, f : C \to \mathbb{P}^1 \text{ a simply branched cover of degree } d \right\}$$

which has dimension $2d + 2g - 2$ by Riemann-Hurwitz (once we know the branched points, monodromy, which is finite, gives the cover). We have the projection map $\pi : \mathcal{H}_{d,g} \to \mathcal{M}_g$. Say $C \in \operatorname{im} \pi$, i.e., C has a g^1_d, then we expect

$$\dim \pi^{-1}(C) \geq \dim \mathrm{PGL}(2) + \dim W^r_d(C) = 3 + \dim W^r_d(C)$$

where $W^r_d(C)$ is the locus of line bundles $\mathcal{L}$ of degree d with $h^0(\mathcal{L}) \geq r + 1$. In our case, $g = 6$, and for $d = 2, 3$, we have

$$\dim \mathcal{H}_{d,g} = 2d + 10 < 18 = \dim \mathcal{M}_g + 3$$

so a general curve of genus 6 cannot have a g^1_2 or g^1_3. For $d = 4$, we have that a general curve of genus 6 can only have finitely many g^1_4.

If C is bi-elliptic or a smooth plane quintic, then C has at least 1-dimension worth of g^1_4 (since we get a g^1_2 from any point on elliptic curve), so C cannot be general either. It follows that a general curve of genus 6 is birational to a plane sextic with 4 simple nodes.

Our goal is to investigate the locus $W^1_4(C)$ of g^1_4 on a general genus 6 curve. Such a divisor looks like $D = q_1 + q_2 + q_3 + q_4$; we can always assume these are distinct points. For this to be a g^1_4, we need $h^0(D) \geq 2$. Recall Riemann-Roch

$$h^0(D) - h^0(K_C - D) = 4 - 6 + 1$$

so we want $h^0(K_C - D) \geq 3$. But $h^0(K_C) = 6$, so D moves in a pencil iff it fails to impose independent conditions on K_C.

Back to our plane model, let's first consider $C_0 \subset \mathbb{P}^2$ with 4 nodes $\{p_1, p_2, p_3, p_4\}$, no 3 of which are collinear. We can blow up these 4 points to get the normalization $\widetilde{C_0}$, and by universal property C is isomorphic to $\widetilde{C_0}$.

$$\begin{array}{ccc} C & \longrightarrow & S = \mathrm{Bl}_{p_1,\ldots,p_4}\mathbb{P}^2 \\ \downarrow & & \downarrow \pi \\ C_0 & \longrightarrow & \mathbb{P}^2 \end{array}$$

Here S is a quintic del Pezzo surface. Let $\mathrm{Pic}(S) = \langle H, E_1, E_2, E_3, E_4 \rangle$ where H is the pullback of a line in $\mathbb{P}^2$, and E_i are exceptional divisors. We have

$$E_i^2 = -1, \quad H^2 = 1, \quad H \cdot E_i = 0, \quad E_i \cdot E_j = 0$$

and

$$K_S = -3H + E_1 + E_2 + E_3 + E_4$$

$$C = \pi^* C_0 - \sum_1^4 \text{mult}(C, p_i) \cdot E_i = 6H - 2\sum_1^4 E_i$$

$$K_C = (K_S + C)\big|_C = 3H - \sum_1^4 E_i$$

Note 2.1. Notice that $K_C = -K_S\big|_C$, so this shows that if we embeds

$$S \xrightarrow{-K_S} \mathbb{P}^5$$

then the image of C coincides with $\phi_K(C)$ of the canonical embedding.

In other words, the canonical divisor K_C is cut out by cubics through 4 points Γ. Then $D = \sum_1^4 q_i$ failing to impose independent conditions on K_C is equivalent to $\Gamma = \{p_1, ..., p_4, q_1, ..., q_4\}$ failing to impose independent conditions on cubics.

Proposition 2.1. *Let Γ be a set of 8 points in $\mathbb{P}^2$, then Γ fails to impose independent conditions on cubics iff*

- *Γ contains 5 collinear points, or*
- *Γ is contained in a conic.*

Applying the proposition, we get that either Γ is contained in a conic, or 5 of them are collinear. By our choice, no 3 of $\{p_i\}$ are collinear, so either $\{p_1, p_2, q_1, q_2, q_3\}$ are collinear or $\{p_1, q_i\}$. In the first case, let l be the line, then

$$l \cap C_0 = \{p_1, p_2, q_1, q_2, q_3\}$$

so the intersection number is 7, which contradicts that C_0 is a sextic. It follows that a g_4^1 is cut out by either conics through $\{p_i\}$ or lines through each p_i. That gives us all 5 g_4^1 on a general curve of genus 6.

3 Scheme structure on $W_4^1(C)$

Now let p_1, p_2, p_3 be collinear. We can still only have the previous 2 cases because of intersection number. A g_4^1 is, once again, cut out by conics through $\{p_i\}$ or lines through each p_i. There is an issue here, which is that each conic through $\{p_i\}$ contains a line through p_4, and vice versa. As a result we only have four g_4^1.

This has to do with the scheme structure on $W_4^1(C)$. In the previous case of 4 general points, $W_4^1(C)$ has 5 points each of which is reduced. In this collinear case, we have 4 points but the one corresponding to conics is nonreduced. The usual way of checking whether a point is smooth is by looking at the tangent space.

We have $\text{Pic}^d(C) \simeq \text{Pic}^0(C) \simeq J(C)$, and we know the tangent space to $J(C)$, thus for any $L \in \text{Pic}^d(C)$ we have

$$T_L\text{Pic}^d(C) \simeq H^1(C, \mathcal{O}_C)$$

and on the other hand we have the identification

$$H^1(C,\mathcal{O}_C) \simeq \left\{ \mathcal{L} \in \operatorname{Pic}\left(\operatorname{Spec}\frac{\mathbb{C}[\epsilon]}{(\epsilon^2)} \times C\right) \middle| \mathcal{L}\big|_C \simeq L \right\}$$
$$\{\phi_{\alpha\beta}\} \leftrightarrow \tau_{\alpha\beta}(1+\epsilon\phi_{\alpha\beta})$$

which is the space of first order deformations of L. Then if $L \in W^r_d(C)\backslash W^{r+1}_d(C)$, i.e., $r(L) = r$, we have

$$T_L W^r_d(C) \simeq \left\{ \mathcal{L} \in \operatorname{Pic}\left(\operatorname{Spec}\frac{\mathbb{C}[\epsilon]}{(\epsilon^2)} \times C\right) \middle| \mathcal{L}\big|_C \simeq L, \text{ every } \sigma \in H^0(L) \text{ extends to } H^0(\mathcal{L}) \right\}$$

Let $v \in H^1(C,\mathcal{O}_C)$ corresponding to some $\mathcal{L}$, one can check that a section $\sigma \in H^0(L)$ extends to a section of $\mathcal{L}$ iff

$$H^1(C,\mathcal{O}_C) \otimes H^0(C,L) \to H^1(C,L)$$
$$v \smile \sigma \mapsto 0$$

so all sections extend iff $v \smile H^0(C,L) = 0$. By Serre Duality we have

$$H^0(C,K_C)^\vee \otimes H^0(C,L) \to H^0(C,K_C \otimes L^\vee)^\vee$$

and dualizing again to get multiplication map

$$H^0(C,K_C \otimes L^\vee) \otimes H^0(C,L) \xrightarrow{\mu} H^0(C,K_C)$$

and we can identify $T_L W^r_d(C) \simeq \operatorname{Ann}(\operatorname{im}\mu)$.

Back to our case of interest, $W^1_4(C)$ is 0 dimensional. Let D be the g^1_4 cut out by conics through $\{p_i\}$, then $D = 2H - \sum E_i$ then look at the multiplication map

$$H^0(C,H) \otimes H^0\Big(C, 2H - \sum E_i\Big) \to H^0\Big(C, 3H - \sum E_i\Big)$$

If p_1, p_2, p_3 are collinear, on a line l, then any conic through those p_i has to contain l. On the other hand, cubics through 4 points p_i don't have to contain l (take product of 3 lines $\overline{p_4 p_i}$), so this map cannot be surjective. It follows that $\operatorname{Ann}(\operatorname{im}\mu)$ has positive dimension, and D is not a smooth point.

We claimed that D is reduced if $\{p_i\}$ are in general position. This seemingly implies that the map

$$H^0(C,H) \otimes H^0\Big(C, 2H - \sum E_i\Big) \to H^0\Big(C, 3H - \sum E_i\Big)$$

is surjective, i.e., any cubic through 4 general points can be written as a linear combination of reducible cubics, which is quite surprising. This is actually just a straightforward consequence of $AF + BG$ theorem; here the key point is that 4 general points is a complete intersection of 2 conics.

Note 3.1. The version we are using is that for $f, g \in \mathbb{C}[x,y,z]$ two homogenous polynomials and $V(f) \cap V(g) = \Gamma$ transversely, i.e., in a finite number of reduced points, then any curve $V(h)$ containing Γ has $h \in (f,g)$. This generalizes to transverse intersection in higher dimensions as well.