# Intersection cohomology

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January 22, 2025

## 1 Perverse sheaf preliminaries

Let $X$ be an algebraic variety or an analytic space, and denote $\mathrm{D}_c^b$ the full subcategory of $\mathrm{D}^b(\mathrm{Mod}(\mathbb{C}_X))$ of $F^\bullet$ such that $H^j(F^\bullet)$ is constructible on $X$ for all $j \in \mathbb{Z}$.

**Definition 1.1.** We have a $t-$structure ${}^p\mathrm{D}_c^{\leqslant 0}(X), {}^p\mathrm{D}_c^{\geqslant 0}(X)$ as follows:

1. $F^\bullet \in {}^p\mathrm{D}_c^{\leqslant 0}(X)$ iff $\dim \mathrm{supp}\ H^j(F^\bullet) \leqslant -j$ for all $j \in \mathbb{Z}$.
2. $F^\bullet \in {}^p\mathrm{D}_c^{\geqslant 0}(X)$ iff $\dim \mathrm{supp}\ H^j(\mathbb{D}_X F^\bullet) \leqslant -j$ for all $j \in \mathbb{Z}$.

Then we have the abelian category

$$\mathrm{Perv}(\mathbb{C}_X) = {}^p\mathrm{D}_c^{\leqslant 0}(X) \cap {}^p\mathrm{D}_c^{\geqslant 0}(X)$$

We also have truncation functors:

$${}^p\tau^{\leqslant 0} : \mathrm{D}_c^b(X) \to {}^p\mathrm{D}_c^{\leqslant 0}(X), \quad {}^p\tau^{\geqslant 0} : \mathrm{D}_c^b(X) \to {}^p\mathrm{D}_c^{\geqslant 0}(X)$$

and cohomological functor (it gives long exact sequence from distinguised triangle):

$${}^pH^n : \mathrm{D}_c^b(X) \to \mathrm{Perv}(\mathbb{C}_X), \quad F^\bullet \mapsto {}^p\tau^{\leqslant 0 p}\tau^{\geqslant 0}(F^\bullet[n])$$

**Proposition 1.2.** *Let $F^\bullet \in \mathrm{D}_c^b(X)$ and $X = \bigsqcup_{\alpha \in A} X_\alpha$ be a complex stratification of $X$ consisting of connected strata such that $i_{X_\alpha}^{-1}F^\bullet, i_{X_\alpha}^! F^\bullet$ have locally constant cohomology sheaves for any $\alpha \in A$. Then*

1. *$F^\bullet \in {}^p\mathrm{D}_c^{\leqslant 0}(X)$ iff $H^j(i_{X_\alpha}^{-1}F^\bullet) = 0$ for any $\alpha$ and $j > -\dim X_\alpha$.*
2. *$F^\bullet \in {}^p\mathrm{D}_c^{\geqslant 0}(X)$ iff $H^j(i_{X_\alpha}^! F^\bullet) = 0$ for any $\alpha$ and $j < -\dim X_\alpha$*

**Definition 1.3.** Let $\mathrm{D}_1, \mathrm{D}_2$ be two triangulated categories with $t-$structures $(\mathrm{D}_i^{\leqslant 0}, \mathrm{D}_i^{\geqslant 0})$. Suppose we have have a functor $F : \mathrm{D}_1 \to \mathrm{D}_2$, then $F$ is left $t-$exact if $F(\mathrm{D}_1^{\leqslant 0}) \subseteq \mathrm{D}_2^{\leqslant 0}$. Right $t-$exactness is defined similarly. We say that $F$ is $t-$exact if it's both left and right $t-$exact.

Suppose that $\mathrm{D}_i$ has the heart $\mathcal{C}_i$. For any functor $F : \mathrm{D}_1 \to \mathrm{D}_2$, we have an induced addive functor:

$${}^pF : \mathcal{C}_1 \to \mathcal{C}_2, \quad {}^pF \coloneqq {}^pH^0 \circ F \circ (\mathcal{C}_1 \hookrightarrow \mathrm{D}_1)$$

**Proposition 1.4.** *Assume that $F$ is left exact.*

1. *For any $C^\bullet \in \mathrm{D}_1$ we have*

$$\tau^{\leqslant 0} \circ F \circ \tau^{\leqslant 0}(C^\bullet) \simeq \tau^{\leqslant 0} F(C^\bullet)$$

*and in particular, for $C^\bullet \in \mathrm{D}_1^{\geqslant}0$, ${}^pF({}^pH^0(C^\bullet)) \simeq {}^pH^0(F(C^\bullet))$.*

2. ${}^pF$ *is a left exact functor between abelian categories.*

**Proposition 1.5.** *The Verdier duality functor* $\mathbb{D}_X : \mathrm{D}^b_c(X) \to \mathrm{D}^b_c(X)^{\mathrm{op}}$ *is* $t-$*exact and induces an exact functor*

$$\mathbb{D}_X : \mathrm{Perv}(\mathbb{C}_X) \to \mathrm{Perv}(\mathbb{C}_X)^{\mathrm{op}}$$

**Proposition 1.6.** *Let* $Z$ *be a locally closed subvariety of* $X$ *and* $i : Z \hookrightarrow X$ *be the embedding.*

1. $i^{-1} : \mathrm{D}^b_c(X) \to \mathrm{D}^b_c(Z)$ *is right* $t-$*exact.*
2. $i^! : \mathrm{D}^b_c(X) \to \mathrm{D}^b_c(Z)$ *is left* $t-$*exact.*
3. *For any* $G^\bullet \in {}^p\mathrm{D}^{\geqslant 0}_c$ *such that* $\mathrm{R}i_*G^\bullet \in \mathrm{D}^b_c(X)$ *we have* $\mathrm{R}i_*G^\bullet \in {}^p\mathrm{D}^{\geqslant 0}_c(X)$.
4. *For any* $G^\bullet \in {}^p\mathrm{D}^{\leqslant 0}_c$ *such that* $i_!G^\bullet \in \mathrm{D}^b_c(X)$ *we have* $i_!G^\bullet \in {}^p\mathrm{D}^{\leqslant 0}_c(X)$.

## Minimal extensions

Let $X$ be an irreducible algebraic variety or an irreducible analytic space, and $U$ a Zariski open dense subset of $X$. Let $Z = X - U$ and denote $i : Z \hookrightarrow X$, $j : U \hookrightarrow X$ be the closed and open embeddings.

**Definition 1.7.** We say that a stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ is compatible with $F^\bullet \in \mathrm{D}^b_c(U)$ if $U = \bigsqcup_{\alpha\in B} X_\alpha$ for $B \subseteq A$, and $F^\bullet\big|_{X_\alpha}, \mathbb{D}_U F^\bullet\big|_{X_\alpha}$ both have locally constant cohomology sheaves for any $\alpha \in B$. Such a stratification always exists if $X$ is an algebraic variety.

By taking refinement we might assume that we have a Whitney stratification as well. Then $\mathrm{R}j_*F^\bullet, j_!F^\bullet \in \mathrm{D}^b_c(X)$. Now assume that $F^\bullet$ is a perverse sheaf on $U$. Recall that we have an inclusion morphism of functor $j_! \hookrightarrow j_*$ which induces a morphism of derived functors, i.e., we have

$$j_!F^\bullet \to \mathrm{R}j_*F^\bullet$$

**Definition 1.8.** Denote ${}^pj_{!*}F^\bullet$ to be the image of the canonical morphism ${}^pj_!F^\bullet \to {}^pj_*F^\bullet$ in $\mathrm{Perv}(\mathbb{C}_X)$, i.e., we have a factorization

$${}^pj_!F^\bullet \twoheadrightarrow {}^pj_{!*}F^\bullet \hookrightarrow {}^pj_*F^\bullet$$

in $\mathrm{Perv}(\mathbb{C}_X)$. We call this the minimal extension of $F^\bullet \in \mathrm{Perv}(\mathbb{C}_X)$.

**Proposition 1.9.** *The minimal extension* $G^\bullet = {}^pj_{!*}F^\bullet$ *of* $F^\bullet \in \mathrm{Perv}(\mathbb{C}_U)$ *is the unique perverse sheaf on* $X$ *which satisfies:*

1. $G^\bullet\big|_U \simeq F^\bullet$;
2. $i^{-1}G^\bullet \in {}^p\mathrm{D}^{\leqslant -1}_c(Z)$;
3. $i^!G^\bullet \in {}^p\mathrm{D}^{\geqslant 1}_c(Z)$.

**Proposition 1.10.** *Let* $F^\bullet \in \mathrm{Perv}(\mathbb{C}_U)$. *Then*

1. ${}^pj_*F^\bullet$ *has no non-trivial subobject whose support is contained in* $Z$.
2. ${}^pj_!F^\bullet$ *has no non-trivial quotient object whose support is contained in* $Z$.

*Proof.* For part 1, let $G^\bullet \subset {}^p j_* F^\bullet$ be a subobject such that supp $G^\bullet \subseteq Z$. Notice that $i^! G^\bullet \simeq i^{-1} G^\bullet$.

**Note 1.1.** $i^! G^\bullet \simeq i^{-1} G^\bullet$ should be because $G^\bullet$ is supported in $Z$. I (incorrect) thought that it's due to $i_! = i_*$, but $i^{-1}$ is a left adjoint and $i^!$ is a right adjoint. ▲

Now $i^{-1}$ is right $t-$exact and $i^!$ is left $t-$exact so $i^! G^\bullet \in \mathrm{Perv}(\mathbb{C}_U)$ and thus $i^! G^\bullet = {}^p i^! G^\bullet$.

We have that $G^* \simeq i_* i^! G^\bullet$ ($i^! G^\bullet \simeq i^{-1} G^\bullet$, and $i_*$ is just extension by 0 while supp $G^\bullet \subset Z$), so it suffices to show ${}^p i^! G^\bullet \simeq 0$. We have an exact sequence

$$0 \to G^\bullet \to {}^p j_* F^\bullet$$

and applying the left exact functor ${}^p i^!$ we get

$$0 \to {}^p i^! G^\bullet \to {}^p i^! {}^p j_* F^\bullet$$

Next we have $i^! \mathrm{R} j_* F^\bullet \simeq 0$ (extending from $U$ to $X$ then pullback to $Z$), and since $i^!$ is left exact we have, by proposition 1.4.1,

$$0 = {}^p H^0(i^! \mathrm{R} j_* F^\bullet) \simeq {}^p i^! \circ {}^p H^0(\mathrm{R} j_* F^\bullet) = {}^p i^! {}^p j_* F^\bullet$$

which implies ${}^p i^! G^\bullet = 0$ as desired. Part 2 is similar. ■

**Corollary 1.11.** *The minimal extension ${}^p j_{!*} F^\bullet$ has neither non-trivial subobject nor non-trivial quotient object whose support is contained in $Z$.*

*Proof.* A subobject of ${}^p j_{!*} F^\bullet$ is a subobject of ${}^p j_* F^\bullet$. A quotient object of ${}^p j_{!*} F^\bullet$ is a quotient object of ${}^p j_! F^\bullet$. ■

**Corollary 1.12.** *We have the following:*

1. *Let $0 \to F^\bullet \to G^\bullet$ be an exact sequence in* $\mathrm{Perv}(\mathbb{C}_U)$. *Then $0 \to {}^p j_{!*} F^\bullet \to {}^p j_{!*} G^\bullet$ is also exact.*
2. *Let $F^\bullet \to G^\bullet \to 0$ be an exact sequence in* $\mathrm{Perv}(\mathbb{C}_U)$. *Then ${}^p j_{!*} F^\bullet \to {}^p j_{!*} G^\bullet \to 0$ is also exact.*

*Proof.* The idea here is that the kernel and cokernel are both supported in $Z$ (since they are 0 on $U$ by exactness and ${}^p j_{!*} F^\bullet\big|_U \simeq F^\bullet$). ■

**Corollary 1.13.** *Let $F^\bullet$ be a simple object in* $\mathrm{Perv}(\mathbb{C}_U)$. *Then the minimal extension ${}^p j_{!*} F^\bullet$ is also a simple object in* $\mathrm{Perv}(\mathbb{C}_X)$.

*Proof.* Suppose we have a short exact sequence

$$0 \to G^\bullet \to {}^p j_{!*} F^\bullet \to H^\bullet \to 0$$

then applying the $t-$exact functor ${}^p j^! = j_! = j^{-1}$ we get

$$0 \to j^{-1} G^\bullet \to F^\bullet \to j^{-1} H^\bullet \to 0$$

and since $F^\bullet$ is simple, either $G^\bullet$ or $H^\bullet$ is supported in $Z$. It then follows from 1.11 that either must be zero. ■

# 2 Intersection cohomology

Set $d = \dim X$ and $X_k = \bigsqcup_{\dim X_\alpha \leqslant k} X_\alpha$, then we get a filtration

$$X = X_d \supset X_{d-1} \supset \cdots \supset X_0 \supset X_{-1} = \varnothing$$

Denote $U_k = X - X_{k-1} = \bigsqcup_{\dim X_\alpha \geqslant k} X_\alpha$, with

$$U = U_d \overset{j_d}{\hookrightarrow} U_{d-1} \overset{j_{d-1}}{\longrightarrow} \ldots \overset{j_2}{\hookrightarrow} U_1 \overset{j_1}{\hookrightarrow} U_0 = X$$

a sequence of open embeddings.

**Proposition 2.1.** *Let $L$ be a local system on $U$. We have an isomorphism*

$${}^p j_{!*} L[\dim X] \simeq \left(\tau^{\leqslant -1} \mathrm{R} j_{1*}\right) \circ \cdots \circ \left(\tau^{\leqslant -d} \mathrm{R} j_{d*}\right)(L[\dim X])$$

*Proof.* Since minimal extension plays well with composition of open inclusions, it suffices to show that

$${}^p j_{k!*} F^\bullet \simeq \tau^{\leqslant -k} \mathrm{R} j_{k*} F^\bullet$$

for any perverse sheaf $F^\bullet$ on $U_k$ whose restriction to each $X_\alpha \subset U_k$ has locally constant cohomology. To do this, we check that the RHS satisfies all 3 conditions in proposition 1.9. ∎

**Corollary 2.2.** *We have a canonical morphism $(j_* L)[\dim X] \to {}^p j_{!*}(L[\dim X])$.*

*Proof.* From the previous proposition we get

$$\tau^{\leqslant -\dim X}\, {}^p j_{!*}(L[\dim X]) \simeq (j_{1*} \circ \cdots \circ j_{d*})(L)[\dim X] \simeq (j_* L)[\dim X]$$

since we have $\tau^{\leqslant -k} = \tau^{\leqslant 0}[k]$. ∎

**Definition 2.3.** For an irreducible algebraic variety $X$, we define its intersection cohomology complex by

$$\mathrm{IC}_X^\bullet = {}^p j_{!*}\left(\mathbb{C}_{X_{\mathrm{reg}}^{\mathrm{an}}}[\dim X]\right) \in \mathrm{Perv}(\mathbb{C}_X)$$

where $X_{\mathrm{reg}}$ is the regular part and $j : X_{\mathrm{reg}} \hookrightarrow X$ is the embedding.

**Theorem 2.4.** *We have $\mathbb{D}_X(\mathrm{IC}_X^\bullet) = \mathrm{IC}_X^\bullet$.*

*Proof.* We have $\mathbb{D}_X({}^p j_{!*} F^\bullet) \simeq {}^p j_{!*}(\mathbb{D}_U F^\bullet)$, since $\mathbb{D}$ is a $t-$exact functor. ∎

**Definition 2.5.** Let $X$ be an irreducible analytic space. For $i \in \mathbb{Z}$ we set

$$\mathrm{IH}^i_{(c)}(X) \coloneqq H^i\left(\mathrm{R}\Gamma_{(c)}(X, \mathrm{IC}_X^\bullet)[-\dim X]\right)$$

called the $i - th$ intersection cohomology group (with compact supports).

**Theorem 2.6** (Generalized Poincaré duality)**.** *Let $X$ be an irreducible analytic space of dimension $d$. Then we have*

$$\mathrm{IH}^i(X) \simeq \mathrm{IH}_c^{2d-i}(X)^\vee$$

*for any $0 \leqslant i \leqslant 2d$.*

*Proof.* By Verdier duality we get

$$\mathrm{R}\mathcal{H}om_{\mathbb{C}}(\mathrm{R}a_{X!}\mathrm{IC}_X^\bullet, \mathbb{C}) \simeq \mathrm{R}a_{X*}\mathrm{R}\mathcal{H}om_{\mathbb{C}_X}(\mathrm{IC}_X^\bullet, \omega_X^\bullet) = \mathrm{R}a_{X*}\mathbb{D}_X(\mathrm{IC}_X^\bullet) \simeq \mathrm{R}a_{X*}\mathrm{IC}_X^\bullet$$

where $a_X : X \to \{\mathrm{pt}\}$. Next, notice that

$$\mathrm{R}a_{X!}\mathrm{IC}_X^\bullet \simeq \mathrm{R}\Gamma_c(X, \mathrm{IC}_X^\bullet)$$

so we get

$$\mathrm{R}\Gamma_c(X, \mathrm{IC}_X^\bullet)^\vee \simeq \mathrm{R}\Gamma(X, \mathrm{IC}_X^\bullet)$$

and taking $(i-d)$-th cohomology groups of both sides gives the desired isomorphism. ∎

**Definition 2.7.** By Verdier duality, we get that $\mathbb{H}^{-i}(X, \omega_X^\bullet) \simeq H_c^i(X)^\vee$. We call this the $i-$th Borel-Moore homology group, denoted $H_i^{RmBM}(X)$.

**Proposition 2.8.** *There exist canonical morphisms:*

$$H^i(X) \to \mathrm{IH}^i(X) \to H_{2d-i}^{\mathrm{BM}}(X)$$

*where the composition is the same as the cup product with the fundamental class* $[X]$.

*Proof.* We have canonical morphisms:

$$\mathbb{C}_X \to \mathrm{IC}_X^\bullet[-\dim X] \to \omega_X^\bullet[-2\dim X]$$

where the first one comes from 2.2, and the second comes from taking Verdier dual. ∎

**Definition 2.9.** Let $X$ be an algebraic variety or analytic space. We say that $X$ is rationally smooth if for any $x \in X$ we have

$$H_{\{x\}}^i(X, \mathbb{C}_X) = \mathbb{C} \quad i = 2\dim X$$

and 0 for other $i$.

**Proposition 2.10.** *Let $X$ be a rationally smooth irreducible analytic space. Then the canonical morphisms $\mathbb{C}_X \to \omega_X^\bullet[-2\dim X]$ and $\mathbb{C}_X \to \mathrm{IC}_X^\bullet[-\dim X]$ are isomorphisms.*

*Proof.* Let $i_{\{x\}} : \{x\} \to X$ be the embedding, then

$$i_{\{x\}}^{-1}\omega_X^\bullet = i_{\{x\}}^{-1}\mathbb{D}_X(\mathbb{C}_X) \simeq \mathbb{D}_{\{x\}}i_{\{x\}}^!(\mathbb{C}_X) \simeq \mathrm{RHom}_{\mathbb{C}}\big(\mathrm{R}\Gamma_{\{x\}}(X, \mathbb{C}_X), \mathbb{C}\big)$$

thus we get

$$H^{j-2d}\left(i_{\{x\}}^{-1}\omega_X^\bullet\right) \simeq H_{\{x\}}^{2\dim X - j}(X, \mathbb{C}_X)^\vee$$

and by rationally smoothness we get $\mathbb{C}_X \simeq \omega_X^\bullet[-2\dim X]$. For the other isomorphism, let $F^\bullet \coloneqq \mathbb{C}_X[\dim X]$ we have $\mathbb{D}_X F^\bullet \simeq F^\bullet$. We can then show that $F^\bullet \simeq \mathrm{IC}_X^\bullet$ by showing that it is a minimal extension. ∎

**Corollary 2.11.** *Let $X$ be a rationally smooth irreducible analytic space. Then $H^i(X) \simeq \mathrm{IH}^i(X)$ for any $i \in \mathbb{Z}$.*

**Theorem 2.12** (Decomposition theorem)**.** *Let $f : X \to Y$ be a proper morphism of algebraic variety, then*

$$\mathrm{R}f_*(\mathrm{IC}_X^\bullet) \simeq \bigoplus_k i_{k*}\mathrm{IC}_{Y_k}(L_k)^\bullet[l_k]$$

*where, for each $k$, $Y_k$ is an irreducible closed subvariety of $Y$, $i_k : Y_k \hookrightarrow Y$ the embedding, $L_k$ is a local system on some smooth Zariski open subset of $Y_k$.*

**Corollary 2.13.** *Let $\pi : \widetilde{X} \to X$ be a resolution of singularities of $X$. Then $\mathrm{IH}^i(X)$ is a direct summand of $H^i\left(\widetilde{X}\right)$ for any $i \in \mathbb{Z}$.*

# References

[HTT07] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-Modules, Perverse Sheaves, and Representation Theory*. Birkhäuser Boston, MA, 2007.