1 Recap

Definition 1.1. An algebraic group is reductive if all of its finite dimensional representations are semisimple. An equivalent definition is that it is the product of an algebraic torus and a (Zariski-)connected semi-simple group.

Example 1.2. The 2 examples to keep in mind here is $\mathrm{GL}(H)$ and $\mathbb{S}$.

Definition 1.3. Let L be a finite $\Bbbk$-algebra, and $X = \operatorname{Spec} A$ be an affine algebraic group over L. We have the functor of points

$$h_X : \mathrm{Alg}_L \to \mathrm{Grp}, \quad R \mapsto \mathrm{Hom}_L(A, R)$$

then we can define the Weil restriction $\mathrm{Res}_{L/\Bbbk} X$ to be the algebraic group representing the functor

$$h : \mathrm{Alg}_\Bbbk \to \mathrm{Grp}, \quad R \mapsto h_X(R \otimes_\Bbbk L)$$

Definition 1.4. The Deligne torus $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$.

Note 1.1. We have $\mathbb{S}(\mathbb{R}) = \mathbb{G}_m(\mathbb{R} \otimes_\mathbb{R} \mathbb{C}) = \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$, so $\mathbb{S}$ is just $\mathbb{C}^\times$ thought of as an $\mathbb{R}$-algebraic group. We also have

$$\mathbb{S}(\mathbb{C}) = \mathbb{G}_m(\mathbb{C} \otimes_\mathbb{R} \mathbb{C}) \simeq \mathbb{G}_m(\mathbb{C} \times \mathbb{C}) \simeq \mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C}) \simeq \mathbb{C}^\times \times \mathbb{C}^\times$$

where the second to last isomorphism comes from the fact that covariant representable functors commute with limits (not colimits).

The idea here is that a covariant representable functor looks like $\mathrm{Hom}(A, -)$, and

$$\mathrm{Hom}\big(A, \varprojlim \beta\big) = \varprojlim \mathrm{Hom}(X, \beta)$$

(see Kashiwara's pg. 37).

We have an embedding of $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \hookrightarrow \mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ by $z \mapsto (z, \overline{z})$, and the weight cocharacter $w : \mathbb{G}_m \to \mathbb{S}$, which is just $z \mapsto (z, z)$ at the level of complex points, and $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$ at the level of real points.

Notice that $\mathbb{C}^\times = \mathbb{R}^\times \cdot S^1$, with $S^1 = U(1)(\mathbb{R})$. Then we can extend $S^1 \hookrightarrow \mathbb{C}^\times$ to an embedding $U(1) \hookrightarrow \mathbb{S}$ and get $\mathbb{S} = U(1) \cdot w(\mathbb{G}_m)$.

Let H be a $\mathbb{R}$-vector space, and consider a linear representation $h : \mathbb{S} \to \mathrm{GL}_H$. Equivalently, we have an action of $\mathbb{S}(R)$ on $H \otimes_\mathbb{R} R$ for each $\mathbb{R}-$algebra R. The trick here is that if we look at $\mathbb{S}_\mathbb{C} = \mathbb{S} \times_\mathbb{R} \mathbb{C} \simeq \mathbb{G}_m \times \mathbb{G}_m$, its functor of points is equal to that of $\mathbb{S}$, i.e., for any $\mathbb{C}-$algebra R, $\mathbb{S}_\mathbb{C}(R) = \mathbb{S}(R)$. Let $H_\mathbb{C} = H \otimes_\mathbb{R} \mathbb{C}$. Since the representation $\mathbb{G}_m \times \mathbb{G}_m \to \mathrm{GL}_{H_\mathbb{C}}$ is diagonalizable with characters

$$\chi_{m,n} : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m, \quad (a, b) \mapsto a^{-m} b^{-n}$$

we have a decomposition

$$H_\mathbb{C} = \bigoplus_{p,q} H^{p,q}, \quad H^{p,q} = (H_\mathbb{C})_{\chi_{p,q}} = \Big\{ v \in H_\mathbb{C} \Big| (a, b) \cdot v = a^{-p} b^{-q} v, (a, b) \in \mathbb{S}_\mathbb{C}(\mathbb{C}) = \mathbb{S}(\mathbb{C}) \Big\}$$

and since this is an $\mathbb{R}-$action, we must have $\overline{H^{p,q}} = H^{q,p}$. We can also look at the restricted action $h\Big|_{\mathbb{G}_m} = h \circ w : \mathbb{G}_m \to \mathrm{GL}_H$. The characters here are $\chi_k : \mathbb{G}_m \to \mathbb{G}_m$ mapping $a \mapsto a^{-k}$, so we have a decomposition

$$H = \bigoplus_k H^k, \quad H^k = H_{\chi_k} = \left\{ v \in V \Big| a \cdot v = a^{-k} v, a \in \mathbb{G}_m(\mathbb{R}) \right\}$$

and over $\mathbb{C}$ this action is $z \mapsto (z, z) \mapsto (v \mapsto (z, z) \cdot v)$. If $v \in H^{p,q}$, then $h \circ w(z)(v) = z^{-p} z^{-q} v = z^{-(p+q)} v$. Thus we have a decomposition $H^k \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$.

Theorem 1.5. *Let $H_{\mathbb{R}}$ be a real vector space. A $\mathbb{R}$-Hodge structure on $H_{\mathbb{R}}$ is equivalent to a linear representation $h : \mathbb{S} \to \mathrm{GL}(H_{\mathbb{R}})$. Furthermore, if $H_{\mathbb{R}} = H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$, and*

$$h \circ w : \mathbb{G}_m \to \mathrm{GL}(H \otimes_{\mathbb{Q}} \mathbb{R})$$

is defined over $\mathbb{Q}$, then we have a $\mathbb{Q}-$Hodge structure on $H_{\mathbb{Q}}$. Notice that the Weil operator here is just $C = h(i)$.

Note 1.2. Here we have an equivalence between the category of $\mathbb{R}$-Hodge structures and the category of representations of $\mathbb{S}$. This is an example of Tannakian duality, where the category of $\mathbb{R}$-Hodge structure is Tannakian, hence has a Tannakian dual which is $\mathbb{S}$.

Definition 1.6. Let $h : \mathbb{S} \to \mathrm{GL}(H_{\mathbb{R}})$ be a $\mathbb{Q}$-Hodge structure of pure weight k. A polarization for h is a morphism of Hodge structures

$$S : H_{\mathbb{Q}} \otimes_{\mathbb{Q}} H_{\mathbb{Q}} \to \mathbb{Q}(-k)$$

such that $Q(u, v) = (2\pi i)^k S(C(u) \otimes v)$ is symmetric and positive definite on $H_{\mathbb{R}}$.

Just a sanity check, let q be the intersection form, then $S(u \otimes v) = (2\pi i)^{-k} q(u, v)$. For $u \in H^{p,q}$ and $v \in H^{r,s}$ then $u \otimes v$ is in the $(p + r, q + s)$-piece. Thus $S(u \otimes v) = 0$ which implies $q(u, v) = 0$ if $(p + r, q + s) \neq (k, k)$.

We also have $\overline{u} \in H^{q,p}$ so $S(C(u) \otimes \overline{u}) > 0$ for $u \neq 0$, which implies $i^{p-q} q(u, \overline{u}) = q(C(u), \overline{u}) > 0$.

2 Mumford-Tate group

Consider a $\mathbb{Q}-$Hodge structure $(H_{\mathbb{Q}}, F^\bullet)$ of weight k. This is equivalent to a linear representation $h : \mathbb{S} \to \mathrm{GL}(H_{\mathbb{R}})$ such that $h \circ w : \mathbb{G}_m \to \mathrm{GL}(H_{\mathbb{R}})$ is mapping $t \mapsto t^{-k} \cdot \mathrm{Id}_{H_{\mathbb{R}}}$.

Definition 2.1. The Mumford-Tate group $\mathrm{MT}(h)$ is the $\mathbb{Q}-$Zariski closure of $h(\mathbb{S})$ in $\mathrm{GL}(H_{\mathbb{R}})$, i.e., the smallest $\mathbb{Q}-$algebraic subgroup G of $\mathrm{GL}(H_{\mathbb{R}})$ such that $G(\mathbb{R})$ contains $h(\mathbb{S})$ (or, equivalently, $G(\mathbb{C})$ contains $h(\mathbb{S}(\mathbb{C}))$).

Proposition 2.2. *Let T be a finite direct sum of spaces of the form $T^{m,n} = H_{\mathbb{R}}^{\otimes m} \otimes (H_{\mathbb{R}}^{\vee})^{\otimes n}$. We can view T as a $\mathbb{Q}$-Hodge structure, and there is an action of $\mathrm{MT}(h)$ on T (induced by the action on $H_{\mathbb{R}}$).*

Then for any $\mathbb{Q}$-subspace $W \subseteq T$, W is a $\mathbb{Q}$-Hodge substructure if and only if it is a $\mathrm{MT}(h)$-submodule.

Proof. ($\Leftarrow$) : This is saying that we have a rational subrepresentation $\mathrm{MT}(h) \to \mathrm{GL}(W)$. Then $\mathbb{S} \hookrightarrow \mathrm{MT}(V, F^\bullet) \to \mathrm{GL}(W)$ gives a rational Hodge structure on W.
($\Rightarrow$) : Suppose W is a $\mathbb{Q}-$Hodge substructure. Consider the $\mathbb{Q}-$Zariski closure $M \subset \mathrm{GL}(H_\mathbb{R})$ of $\{g \in \mathrm{GL}(H_\mathbb{R}) \big| g \cdot W \subseteq W\}$. Since W is a $\mathbb{Q}$-Hodge substructure, W is stable under the induced action of $\mathbb{S}$, i.e., $h(\mathbb{S}) \subseteq M$. It follows that $\mathrm{MT}(h) \subseteq M$, and W is stable under the action of $\mathrm{MT}(h)$. □

Note 2.1. Another way to phrase this result: The subcategory $\langle H_\mathbb{R} \rangle^\otimes$ (of $\mathbb{Q}-$Hodge structure) generated by $H_\mathbb{R}$ (under tensor product, direct sum, subquotients) is Tannakian, and its dual is $\mathrm{MT}(h)$.

Definition 2.3 (Hodge tensors)**.** Hodge tensors in T are rational tensors of pure Hodge type, i.e., elements of $T_\mathbb{Q} \cap (T_\mathbb{C})^{p,p}$ for some p.

Note 2.2. Let $t \in T$ and consider L a line spanned by t. Then L is a $\mathbb{Q}-$Hodge substructure iff t is a Hodge tensor.

Proposition 2.4. *A vector $t \in T$ is a weight 0 Hodge tensor if and only if t is fixed by* $\mathrm{MT}(h)$.

Proof. Suppose $\mathrm{MT}(h)$ fixes t, then it fixes the line L spanned by t hence t is a Hodge tensor of some weight (p, p). But then $h \circ w(\mathbb{G}_m) \subset \mathrm{MT}(h)$ acts on t by $g \cdot v = g^{-2p} v$ so it can only be fixed if $p = 0$.

Now suppose t is a weight 0 Hodge tensor. Consider $T^{0,0}$ the one-dimensional trivial representation. Then $1 \in T^{0,0}$ is a Hodge tensor. Then $(1, t)$ is a weight 0 Hodge tensor, thus the line spanned by $(1, t) \in T^{0,0} \oplus L$ is a $\mathbb{Q}-$Hodge substructure. Let $g \in \mathrm{MT}(h)$ then $g \cdot (1, t) = (g(1), g(t)) = (1, g(t))$, so $(1, g(t))$ on the line $\mathbb{R} \cdot (1, t)$. This is only possible if $g(t) = t$, i.e., t is fixed by $\mathrm{MT}(h)$. □

Note 2.3. I was confused for a while about why the second paragraph doesn't apply to Hodge tensor of any weight. If t has weight (p, p) then $T^{0,0} \oplus L$ is a 2-dimensional $\mathbb{Q}$-Hodge structure of weight $(0, 0) + (p, p)$. Then for the line spanned by $(1, t)$ to be a $\mathbb{Q}-$Hodge substructure, we must have the line being contained entirely in either $T^{0,0}$ or L. This is clearly not true.

Theorem 2.5. $\mathrm{MT}(h)$ *is the largest algebraic subgroup of* $\mathrm{GL}(H_\mathbb{R})$ *which fixes weight 0 Hodge tensors in any finite direct sum T of tensor representations $T^{m,n}$.*

Proof. We will first prove the following proposition:

Proposition 2.6. *Let $M \subseteq \mathrm{GL}(H_\mathbb{R})$ be a closed subgroup such that every character of M is the restriction of a character of* $\mathrm{GL}(H_\mathbb{R})$*. Then there exists a finite direct sum T of $T^{m,n}$, and $t \in T$ such that $M = \mathrm{Stab}_{\mathrm{GL}(H_\mathbb{R})}(t)$.*

By Chevalley's theorem, M is the stabilizer of a line L in a finite dimensional representation V of $\mathrm{GL}(H_\mathbb{R})$. We also have that any such representation can be built out of V through tensor products, duals, direct sums, and subquotients. Thus there exists T such that $M = \mathrm{Stab}_{\mathrm{GL}(H_\mathbb{R})}(L)$ with $L \subseteq T$.

Note 2.4. It's quite important that $\mathrm{GL}(H_\mathbb{R})$ is reductive here, otherwise subquotient doesn't imply containment.

Now, a line L fixed by M corresponds to a character of M (pick a generator $l \in L$, for each $g \in M$, $g \cdot L$ is a multiple of l, i.e. $g \cdot L = \chi(g)l$ which gives our character). Since characters on M extend to $\mathrm{GL}(H_{\mathbb{R}})$, we have that L is invariant under $\mathrm{GL}(H_{\mathbb{R}})$ as well.

Then look at the line $L \otimes L^\vee$ inside the $\mathrm{GL}(H_{\mathbb{R}})-$module $T \otimes L^\vee$, and let t be a generator. The claim is that $g \in \mathrm{GL}(H)$ fixes t iff g fixes L.

Note 2.5. We need to prove that L is a $\mathrm{GL}(H_{\mathbb{R}})-$module so that we have a rep $\rho : \mathrm{GL}(H_{\mathbb{R}}) \to \mathrm{GL}(L)$. Then $\phi : \mathrm{GL}(H_{\mathbb{R}}) \to \mathrm{GL}(L \otimes L^\vee)$ is defined to be

$$\phi(g)(l \otimes f) = \rho(g)(l) \otimes \rho(g^{-1})^\vee(f)$$

If g fixes L then $\rho(g)(l) = r_0 \cdot l$ which implies $\rho(g^{-1})(l) = r_0^{-1} \cdot l$. For any $l_1 \in L$, we have $l_1 = r_1 \cdot l$ and

$$\rho(g^{-1})^\vee(f)(l_1) = f(\rho(g^{-1})(l_1)) = r_1 \cdot f(\rho(g^{-1})(l)) = \frac{r_1}{r_0} f(l) = r_0^{-1} f(l_1)$$

so $\phi(g)(l \otimes f) = l \otimes f$. The other direction is similar, using the fact that since L is a line, any element of $L \otimes L^\vee$ looks like $l \otimes f$.

Back to the theorem, we need to check that every character $\chi : \mathrm{MT}(h) \to \mathbb{G}_m$ extends to all of $\mathrm{GL}(H_{\mathbb{R}})$. Such a character corresponds to a rational line L fixed by $\mathrm{MT}(h)$, and since $\mathrm{MT}(h)$ is the $\mathbb{Q}$-Zariski closure of $h(\mathbb{S})$, the $\mathrm{MT}(h)$-action on L is determined by the $\mathbb{S}-$action.

If $L \simeq \mathbb{R}(0)$, then $\mathrm{MT}(h)$ acts trivially on it, and we can extend this to the trivial $\mathrm{GL}(H_{\mathbb{R}})-$action. Otherwise, $L \simeq \mathbb{R}(-p)$, and $\mathbb{S}$ acts on it by $z^{-p}(\overline{z})^{-p}$. If we look at the matrix model of $\mathbb{S}$, this is acting by multiplying with $(\det(z))^{-p}$. This supposedly extend to all of $\mathrm{GL}(H_{\mathbb{R}})$.

Note 2.6. There is an issue here: Let $h : \mathbb{S} \to \mathrm{GL}(H_{\mathbb{R}})$ be the representation, look at the matrix model of $z \in \mathbb{S}$, it's not true that $\det(z) = \det h(z)$.

We have the decomposition $H = \bigoplus_{i=0}^k H^{i,k-i}$. Take a basis $\left\langle e_j^{i,k-i} \right\rangle$ corresponding to the decomposition. Then w.r.t. this basis, $h(z)$ looks like a diagonal matrix with diagonal entries $z^{-i}(\overline{z})^{i-k}$ and each of those appears $\dim H^{i,k-i}$ times. So $\det h(z) = \prod_{i=0}^{k} (z^{-i}(\overline{z})^{i-k})^{\dim H^{i,k-i}}$. Since $\dim H^{i,k-i} = \dim H^{k-i,i}$, $\det h(z) = (z \cdot \overline{z})^N$ for some $N \in \mathbb{Z}$, so

$$\det h(z) = \det(z)^N \quad \Rightarrow \quad \det(z)^{-p} = \det h(z)^{-p/N}$$

but $\det^{-p/N}$ is not a character on $\mathrm{GL}(H_{\mathbb{R}})$.

One possible fix here is to look at $\det H_{\mathbb{R}} = \bigwedge^{\dim H_{\mathbb{R}}} H_{\mathbb{R}}$ which has weight $N \neq 0$ (if $H_{\mathbb{R}}$ has weight 0 then everything has weight 0). Then the line

$$L^{\otimes N} \otimes (\det H_{\mathbb{R}})^{\otimes -2p} \simeq \mathbb{R}(-pN) \otimes (\det H_{\mathbb{R}})^{\otimes -2p}$$

has weight $2pN - 2pN = 0$. Then the $\mathbb{S}-$action on this new line is trivial, which extends to the trivial action on $\mathrm{GL}(H_{\mathbb{R}})$. Back to the theorem, we then get that $\mathrm{MT}(h) = \mathrm{Stab}_{\mathrm{GL}(H_{\mathbb{R}})}(t)$ for some weight 0 Hodge tensor t, and we are done. □

Proposition 2.7. *The Mumford-Tate group of a polarizable $\mathbb{Q}-$Hodge structure is reductive.*

Proof. Let $h : \mathbb{S} \to \mathrm{GL}(H_{\mathbb{R}})$ be the polarizable $\mathbb{Q}$-Hodge structure. The key observation here is that $H_{\mathbb{R}}$ is semi-simple (for any subrep we can look at the orthogonal complement), hence the subcategory $\langle H_{\mathbb{R}} \rangle^{\otimes}$ generated by $H_{\mathbb{R}}$ is semisimple. This implies all finite dimensional reps of $\mathrm{MT}(h)$ is semisimple, thus G is reductive. □

3 Variations of Hodge structures

Suppose we have a polarized (in the sense of morphism of local systems) variation of Hodge structure with quasi-projective smooth base S, giving a period map:

$$\mathcal{P} : S \to \Gamma \backslash D$$

The key observation to defining the Mumford-Tate group of a variation is that a Hodge tensor stays Hodge over a closed subvariety of S. Consider the local system $T_s^{m,n} = H_s^{\otimes m} \otimes (H_s^{\vee})^{\otimes n}$. Consider t a section $T^{m,n}$ and let

$$Z(t) = \{ s \in S \big| t_s \text{ is Hodge} \}$$

This is an analytic subvariety of S (the proof is similar to Voisin's vol 2, pg. 144, the idea is that consider the holomorphic bundle $\mathcal{T}^{m,n} = T^{m,n} \otimes \mathcal{O}_S$, then this set is just the zero set of the projection $\mathcal{T}^{m,n} \to \mathcal{T}^{m,n}/F^p\mathcal{T}^{m,n}$ which is holomorphic). Next define

$$Z = \bigcup_{Z(t) \neq S} Z(t)$$

which is a countable union of proper subvariety. Then on $S_{\text{gen}} = S - Z$ we have no unexpected Hodge tensor popping up, hence the Mumford-Tate groups $\mathrm{MT}(h_s)$ are the same.

Definition 3.1. Informally speaking, we can define the Mumford-Tate group of a variation $\mathrm{MT}(\mathcal{P})$ to be the Mumford-Tate group over a very general point.

Fix a very general point $s \in S$, then the local system H corresponds to a monodromy representation

$$\rho : \pi_1(S, s) \to \mathrm{GL}(H_s)$$

Definition 3.2. The algebraic monodromy group $\mathrm{Mon}(\mathcal{P})$ is the connected component of the identity of the $\mathbb{Q}$-Zariski closure of $\pi_1(S, s)$ in $\mathrm{GL}(H_s)$.

Proposition 3.3. $\mathrm{Mon}(\mathcal{P})$ *is a subgroup of* $\mathrm{MT}(\mathcal{P})$.

Proof. The idea here is that the monodromy action preserves polarization

Note 3.1. In the geometric case, it's a topological invariant hence preserves cup product and hyperplane class hence the polarization by Lefschetz's hard theorem.

For the abstract case, look at Voisin's vol 2 pg. 72, and see that she is building a monodromy representation out of a local system by picking natural/unique isomorphism, which should preserve the polarization.

We know that $\mathrm{MT}(\mathcal{P}) = \mathrm{Stab}_{\mathrm{GL}(H_s)}(t)$ for some weight 0 Hodge tensor $t \in T^{m,n}$. The space of weight 0 Hodge classes of $T^{m,n}$ is preserved under the monodromy action (since we are picking a very general $s \in S$, Hodge tensors on s are Hodge everywhere). The polarization is definite on the $(0,0)$ piece of $T^{m,n}$ and thus the monodromy action is taking values in the orthogonal group of a lattice with definite form, which is finite.

Note 3.2. It's the orthogonal group of a lattice, since the underlying local system is that of $\mathbb{Z}-$modules. Next, why is $\mathrm{Aut}(V_{\mathbb{Z}}, q)$ finite if q is definite? We can think of $V_{\mathbb{Z}}$ as sitting inside $V_{\mathbb{R}}$, and so $\mathrm{Aut}(V_{\mathbb{Z}}, q) \subset \mathrm{Aut}(V_{\mathbb{R}}, q) \subset \mathrm{Aut}(V_{\mathbb{R}}) \subset \mathbb{R}^{n \times n}$. If q is definite, then $\mathrm{Aut}(V_{\mathbb{R}}, q)$ is compact (closed and bounded, see here). $\mathrm{Aut}(V_{\mathbb{Z}}, q)$ is a lattice inside a bounded set in $\mathbb{R}^{n \times n}$ so it has only finitely many points.

Back to the proposition, the monodromy action is finite, so there is a finite index subgroup Γ' fixing the Hodge tensor t (by Orbit-Stabilizer or something). It follows that there is a finite index subgroup of $\overline{\rho(\pi_1(S))}^{\mathrm{Zar}}$ fixing t. Now the connected component of the identity is sitting inside every finite index subgroup (orbits are disjoint) hence $\mathrm{Mon}(\mathcal{P})$ fixes t. It follows that $\mathrm{Mon}(\mathcal{P})$ is a subgroup of $\mathrm{MT}(\mathcal{P})$. □

Proposition 3.4.