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SOLITARY-WAVE SOLUTIONS FOR SOME MODEL EQUATIONS FOR WAVES IN NONLINEAR DISPERSIVE MEDIA

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I. Introduction

Solitary waves were first observed and described by Scott Russell in the early 1840's. He witnessed the generation and evolution of a single-crested steadily propagating wave of elevation when a barge came to an abrupt halt in a canal in Scotland. The observation of this phenomenon in nature inspired him to conduct a sequence of careful laboratory experiments on the generation and properties of such waves.

The existence of a solitary wave, as reported by Scott Russell in 1844, could not be explained in terms of the then current theories for surface waves. Boussinesq (1871) was able to give an approximate explanation by means of a system of nonlinear model equations which now bear his name. Rayleigh (1876) also gave an approximate expression for the solitary wave. The matter was elucidated further by Korteweg and de Vries (1895), who derived a model equation for the uni-directional propagation of long surface waves in a uniform rectangular channel. Their equation has an exact solution in the form of a steadily-translating single-crested wave of elevation. For surface waves in a channel, Friedrichs and Hyers (1954) extended the approximation of Boussinesq and Rayleigh and proved existence of small amplitude solitary-wave solutions of the full equations of motion (the Euler equations with nonlinear boundary conditions at the free surface).

In the late 1950's and in the 1960's several other physical systems were shown to manifest solitary waves (e.g. rotating fluids, bubbly liquids, crystalline lattices and density stratified fluids). In the late 1960's a formalism relating to the Korteweg-de Vries equation was developed by Gardner, Greene, Kruskal and Miura (1967, 1974) which indicates that, for a large class of initial wave profiles, solitary waves play an important role in the solution of the pure initial-value problem. Experimental evidence (Hammack 1973 and Hammack and Segur 1974)

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for water waves in a channel are more or less consistent with the general conclusion described by this formalism, that the long-term evolution of an initial wave profile of elevation leads to a sequence of solitary waves. (Firm conclusions on this point appear, however, somewhat premature in light of the evidence so far available.)

It is now generally understood (cf. Benjamin, Bona and Mahony 1972, section 2) that equations of the Korteweg-de Vries type will arise in first-order one-dimensional approximations for uni-directional propagation of waves whenever the dominant physical consideration is, on the one hand, a balance between small nonlinear effects that tend to steepen the wave profile and, on the other, smooth dispersive effects which tend to spread the profile. At this level of approximation many physical systems lead to the Korteweg-de Vries equation or to the alternative model proposed by Benjamin et. al. (1972). However, for long-wave models in which the dispersion relation is not a C^2 function near the origin, a generalized version of these equations obtains, namely

$$u_t + u_x + uu_x + Hu_t = 0, \tag{1}$$

or, with allowance for other forms of nonlinearity,

$$u_t + f(u)_x + Hu_t = 0. \tag{2}$$

Here, $u = u(x,t)$, where x and t are real variables representing space and time respectively and H is a convolution operator determined by the dispersion relation. More precisely,

$$\widehat{Hu}(k) = \alpha(k)\,\hat{u}(k). \tag{3}$$

where the circumflex denotes Fourier transform. Such models were first considered by Benjamin (1967). In the case $\alpha(k) = k^2$, the equation studied by Benjamin et. al. (1972) is recovered from (1).

In two instances, for the class of internal waves treated by Benjamin (1967) and for a model suggested by Whitham (1967) to simulate the peaking and breaking of surface gravity waves, explicit solutions of (1) representing waves propagating steadily without change of form have been given. In various other theoretical discussions (cf. Leibovich 1970, Leibovich and Randall 1972, Smith 1972, and Pritchard 1969, 1970) solitary-wave solutions of models in the form (1) have been assumed to exist. Such waves have been produced in the laboratory by Pritchard (1969) who observed them travelling along the vortex of a rotating fluid, a system for which the symbol $\alpha(k) = k^2(1 + K_0(|k|))$ in suitably scaled variables (here K_0 is the zeroth order modified Bessel function). Hence a significant question naturally arises: under what conditions do equations of the form (1) or (2) have solitary-wave solutions? If such a model does not have solitary-wave solutions, a potentially important aspect of the physical situation is lost and the model may be judged to be inadequate. The question can be formulated for (1) or (2) with respect not only to the solitary-wave problem on the infinite domain $\mathbb{R}$ but also for a spatially periodic version of the same problem. (For the original equation of Korteweg and de Vries, solutions representing permanent periodic wave trains were called cnoidal waves on account of their representation in terms of the Jacobian elliptic function cn.)

Sufficient conditions for the existence of solitary-wave solutions of (1) have been derived by Bona and Bose (1974), using an extension of the positive operator methods of Krasnosel'skii. A feature of the analysis, which recurs below, is that there are two trivial solutions to the problem and the object is to establish a third, non-trivial solution. Here a different approach to the problem is considered. The question for periodic waves is settled first by means of a variational argument, and then it is shown that as the wavelength becomes large, the periodic wave-train tends to a solitary wave in an appropriate metric.

One possible advantage of developing the proof as outlined below is that stability to perturbations periodic of the same period of the waves in question may be inferred. For it is shown that solutions to (1) or (2) that are the counterparts of cnoidal waves realize a maximum of a certain functional, subject to constraints on another functional. Such a situation has been exploited by Benjamin (1972) and Bona (1975) in a proof of the stability of the solitary-wave solution of the Korteweg-de Vries equation. Of course it is a long way from the variational principle to a complete proof of stability, but nevertheless such an approach seems to be the best technique available in this type of problem.

In section II the periodic problem is discussed. Section III is devoted to the solitary-wave problem. To keep the technical details at a minimum, attention is restricted to equation (1).

II. The problem of steady periodic waves

The question of existence of solutions $u(x,t) = \phi(x-\lambda t)$ to (1) such that ϕ is a periodic function of period 2ℓ is now considered. A solution of this problem has already been sketched by Benjamin (1974). Benjamin's method and results along with a few extensions, will be briefly recalled.

When the desired form of solution is substituted into (1) and the equation is then integrated once, there results the 'ordinary' pseudo-differential equation

$$\lambda(\phi + H\phi) = \phi + \tfrac{1}{2}\phi^2. \tag{4}$$

By inversion of the operator $I + H$ in (4), a Hammerstein integral equation is obtained.

$$\lambda\phi = K * (\phi + \tfrac{1}{2}\phi^2) \tag{5}$$

where $\hat{K}(k) = \{1 + \alpha(k)\}^{-1}$, α being the symbol of H, and $*$ denotes convolution over the entire real axis. Henceforth it is assumed, in keeping with most of the applications in view, that $\alpha(k)$ is even, non-negative, increasing on $\mathbb{R}^+$, that $\alpha(0) = 0$ and that $\{1 + \alpha(k)\}^{-1} = O(|k|^{-\beta})$ with $\beta > 1$ as $|k| \to +\infty$.

Equation (5) is appropriate for steady-wave solutions of arbitrary period 2ℓ and also for solitary-wave solutions of (1) (since the Green function obtained by inverting $I + H$ subject to periodic boundary conditions of period 2ℓ is simply the periodic function $K_\ell(x) = \sum_{m=-\infty}^{\infty} K(x + 2m\ell)$). Subsequently it will be desirable to fix ℓ and view (5) only on the fundamental period interval $[-\ell,\ell]$. If g is a periodic function of period 2ℓ, then $K*g(x) = \int_{-\infty}^{\infty} K(x-y)g(y)\,dy = \int_{-\ell}^{\ell} K_\ell(x-y)g(y)\,dy$, hence (5) is equivalent to

$$\lambda\phi = K_\ell * (\phi + \tfrac{1}{2}\phi^2) \tag{6}$$

where the convolution is now understood to be over the interval $[-\ell,\ell]$.

Following Benjamin (1973), the operator K is now split into positive square roots. More precisely, let M be defined by $\hat{M}(k) = \{1 + \alpha(k)\}^{-\frac{1}{2}}$. Then if $B\psi = M*\psi$ (convolution over $\mathbb{R}$) and $B_\ell\psi = M_\ell * \psi$ (convolution over $[-\ell,\ell]$) where M_ℓ is defined from M as K_ℓ is from K, (5) and (6) become respectively

$$\lambda\phi = B^2(\phi + \tfrac{1}{2}\phi^2) \quad \text{and} \quad \lambda\phi = B_\ell^2(\phi + \tfrac{1}{2}\phi^2) \tag{7}$$

where B^2 means the operator B applied twice. In (7) make the substitution $\psi = B_\ell\phi$. The corresponding equation for periodic steady waves becomes

$$\lambda\psi = B_\ell\left(B_\ell\psi + \tfrac{1}{2}(B_\ell\psi)^2\right), \tag{8}$$

and similarly for the equation for solitary waves. The functionals

$$V(u) = \int_{-\ell}^{\ell} u^2(x)\,dx \quad \text{and} \quad W(u) = \int_{-\ell}^{\ell} \{(B_\ell u)^2 + \tfrac{1}{3}(B_\ell u)^3\}\,dx$$

are well-defined on the Hilbert space $\mathcal{H}$ of periodic square-integrable function on $\mathbb{R}$ with period 2ℓ. Moreover, both these functionals possess a gradient at any point of $\mathcal{H}$ and (8) is identical with

$$\lambda G_V(\psi) = G_W(\psi) \tag{9}$$

where the operators G_V and G_W are the respective gradients of V and W. Because of the assumed growth condition on α, B_ℓ maps L_2 continuously into the Sobolev space $H^{\beta/2}$. Since $\beta > 1$, both L_2 and L_3 are compactly imbedded in $H^{\beta/2}$ and it follows that W is a weakly continuous functional on $\mathcal{H}$.

Hence if the constrained maximazation problem

$$\text{maximize } W(u), \ \text{subject to } V(u) \le R^2, \tag{10}$$

is posed, standard results insure that this problem has a solution, say ψ_ℓ. It is straightforward to check that ψ_ℓ cannot lie in the interior of the ball $\{u : V(u) \le R^2\}$, and hence the usual theory (Vainberg, 1964, chapter IV) implies the existence of a constant λ_ℓ such that

$$\lambda_\ell G_V(\psi_\ell) = G_W(\psi_\ell). \tag{11}$$

There are two trivial solutions of (11). One is the function identically zero, which is excluded since it lies in the interior of the ball $\{u : V(u) \le R^2\}$. The other, representing a so-called conjugate flow (cf. Benjamin 1971), is the constant function $\psi_0 \equiv R/\sqrt{2\ell}$, corresponding to which $\lambda_\ell = 1 + \frac{1}{2}\psi_0$. By considering the second derivative of W at ψ_0, Benjamin showed that for ℓ larger than a certain critical value ℓ_c, dependent only on α, ψ_0 does not achieve a maximum of W on $\{u : V(u) \le R^2\}$. Hence ψ_ℓ is a non-constant 2ℓ-periodic solution of (8) and accordingly $\phi_\ell = B_\ell \psi_\ell$ is a non-constant 2ℓ-periodic solution of (6).

Various additional properties of ψ_ℓ, and so of ϕ_ℓ, may be established by use of the extremal property of ψ_ℓ. On the additional assumption that the kernel M of the operator B is a non-negative even function, non-increasing on $[0,\infty)$, so that similar properties accrue for M_ℓ on the period $[-\ell,\ell]$, it may be inferred that the following conditions can be satisfied by a maximizing function.

(a) $\psi_\ell \ge 0$ and ψ_ℓ may be normalized against translations in x so that it is even and monotone non-increasing on $[0,\ell]$. (Other possible solutions of (8) whose fundamental period is a fraction of 2ℓ, which will realize only a conditional stationary value of W, will not have this property.) Because of the assumptions made concerning the operator B, it follows that $\phi_\ell = B_\ell \psi_\ell$ may be chosen with the same properties.

(b) The 'Lagrange multiplier' λ_ℓ satisfies $1 < \mu_0 \leq \lambda_\ell \leq \mu_1 < +\infty$ for all $\ell \geq \ell_c$. The constants μ_0 and μ_1 depend only on α and not on ℓ. (These bounds on λ_ℓ are obtained by evaluating W for particular functions in the ball $\{u : V(u) < R^2\}$.)

Finally a standard 'bootstrap' argument shows that ψ_ℓ must be an H^∞ function on $[-\ell,\ell]$ (i.e. an $L_2(-\ell,\ell)$ function with derivatives of all orders which are also in $L_2(-\ell,\ell)$).

III. Existence of solitary waves

The facts outlined in section II will be used to show that, as the period of the steady periodic waves tends to infinity, the wave profile converges, in a sense to be described below, to a solitary-wave solution of (1). That is, there is a finite constant $\lambda > 1$ and a non-negative even C^∞ function ϕ, defined on $\mathbb{R}$, which is monotone decreasing to 0 as $x \to +\infty$ and satisfies equation (5).

Let $\mathfrak{C}(\mathbb{R})$ denote the class of continuous real-valued functions defined on $\mathbb{R}$. $\mathfrak{C}(\mathbb{R})$ is given the structure of a Fréchet space by introducing the semi-norms

$$p_j(u) = \sup_{-j \leq x \leq j} |u(x)| . \qquad (11)$$

The corresponding metric may be taken to be, for example,

$$d(u,v) = \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{p_j(u-v)}{1 + p_j(u-v)} . \qquad (12)$$

Thus the statement $u_n \to u$ in the metric d means that $\{u_n\}$ converges to u pointwise, and uniformly on compact subsets of $\mathbb{R}$. The notation B_r will be used for the ball $\{u \in \mathfrak{C}(\mathbb{R}) : d(u,0) \leq r\}$. Note that $B_1 = \mathfrak{C}(\mathbb{R})$.

Now $\mathfrak{C}(\mathbb{R})$ has two properties of particular importance in the present context. First, the periodic permanent-wave solutions of (6) and solitary-wave solutions of (5) are all members of $\mathfrak{C}(\mathbb{R})$. Secondly, the operation of convolution with the kernel K is a compact mapping of certain convex subsets of $\mathfrak{C}(\mathbb{R})$ which will be defined below. For $\ell > 0$, let

$C_\ell = \{u \in \mathfrak{C}(\mathbb{R})$: u is non-negative, 2ℓ-periodic, even and monotone non-increasing on $[0,\ell]\}$, and let

$C = \{u \in \mathcal{C}(\mathbb{R}) :$ u is non-negative, even and monotone non-increasing on $[0,\infty)\}$.

The sets C_ℓ and C are closed and convex in $\mathcal{C}(\mathbb{R})$. In fact, they are closed cones. Let $Au = K*(u + \frac{1}{2}u^2)$, where the convolution is over the entirety of $\mathbb{R}$. Of course, A cannot be defined on the whole of $\mathcal{C}(\mathbb{R})$, which includes functions unbounded at infinity, but, since $K \in L^1(\mathbb{R})$, it can be considered as a mapping of C or of C_ℓ for any $\ell > 0$. As a mapping of these cones, A has some useful properties summarized in a preparatory lemma.

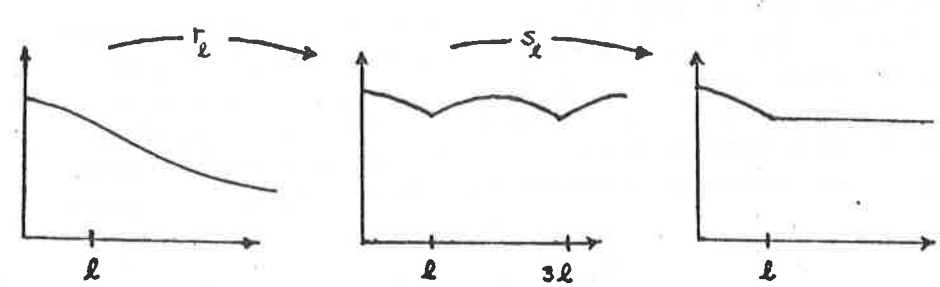
LEMMA 1. A is a continuous map of C into itself and, for each $\ell > 0$, a continuous map of C_ℓ into itself. Moreover, for fixed $\ell > 0$ and r in (0,1), $A(C_\ell \cap B_r)$ (respectively $A(C \cap B_r)$) is a relatively compact subset of C_ℓ (respectively C).

A relationship between the cones C_ℓ and C is now needed. Define mappings $r_\ell : C \to C_\ell$ and $s_\ell : C_\ell \to C$ as follows. For $u \in C$ and $v \in C_\ell$,

$$(r_\ell u)(x) = \begin{cases} u(x) & \text{for } -\ell \le x \le \ell, \\ u(n\ell - |x|) & \text{for } (n-1)\ell \le |x| \le (n+1)\ell, \\ & \quad n = 2,3,\cdots. \end{cases}$$

$$(s_\ell v)(x) = \begin{cases} v(x) & \text{for } -\ell \le x \le \ell, \\ v(\ell) & \text{for } \ell \le |x|. \end{cases}$$

These maps are pictured in the accompanying sketch.



LEMMA 2. Let $\ell > 0$. Then $r_\ell : C \to C_\ell$ and $s_\ell : C_\ell \to C$ are continuous with respect to the relative topology induced by $\mathcal{C}(\mathbb{R})$. The composition $r_\ell \circ s_\ell$ equals id_{C_ℓ}, the identity mapping of C_ℓ. If γ is a positive constant and $f \in C_\ell$, then $s_\ell(\gamma f) = \gamma s_\ell(f)$.

In topological language, the mappings r_ℓ and s_ℓ are an r-domination of the cone C_ℓ by the cone C (cf. Granas 1972).

Let A_ℓ denote the restriction of A to C_ℓ. If $u \in C_\ell$, then $A_\ell u = K*(u + \frac{1}{2}u^2) = K_\ell*(u + \frac{1}{2}u^2)$, where the first convolution is over $\mathbb{R}$ and the second over $[-\ell,\ell]$. The composition $s_\ell A_\ell r_\ell$ maps C to itself. Moreover, $s_\ell A_\ell r_\ell \to A$ on C as $\ell \to +\infty$. More precisely, we have:

LEMMA 3. Let $r \in (0,1)$ and $\varepsilon > 0$ be given. There exists an $\ell_o = \ell_o(\varepsilon, r)$ such that if $\ell \geq \ell_o$,

$$d(Au, s_\ell A_\ell r_\ell u) < \varepsilon$$

for all u in $C \cap B_r$.

A few additional pieces of information are needed concerning the periodic permanent-wave solutions ϕ_ℓ determined in section II. Fix the parameter R, namely the $L^2(-\ell,\ell)$ norm of ϕ_ℓ, and let $\ell \geq \ell_c$. Then ϕ_ℓ is a member of C_ℓ and

$$\lambda_\ell \phi_\ell = K_\ell * (\phi_\ell + \tfrac{1}{2}\phi_\ell^2) = A_\ell \phi_\ell . \tag{13}$$

When this relation is evaluated at 0, and account is taken of the facts that

$$1 = \hat{K}(0) = \int_{-\infty}^{\infty} K(y)dy = \int_{-\ell}^{\ell} K_\ell(y)dy$$

and that $0 \leq \phi_\ell(x) \leq \phi_\ell(0)$ for all x, there appears the lower bound $\lambda_\ell - 1 \leq \phi_\ell(0)$. It is also easily confirmed that

$$\phi_\ell(0) \leq (\lambda_\ell - 1)^{-1} K_\ell(0) \, \|\phi_\ell\|^2_{L^2(-\ell,\ell)},$$

and the right-hand side is bounded above by a constant N which is independent of $\ell \geq \ell_c$. [The only term requiring further comment is $K_\ell(0)$. Because of the growth conditions assumed for α, we have $K(0) < +\infty$ and $K \in L^1(\mathbb{R})$. Combined with the positivity and monotonicity of K, these two properties imply $K_\ell(0)$ is finite and that $K_\ell(0)$ descreases to $K(0)$ as $\ell \to +\infty$.] These results are summarized in the next lemma. As before,

ϕ_ℓ denotes a periodic steady-wave solution of (6) of fundamental period 2ℓ and with $\|\phi_\ell\|_{L^2(-\ell,\ell)} = R$, determined as in section II.

LEMMA 4. There are constants μ_0 and N, independent of $\ell \geq \ell_c$, such that

$$\mu_0 \leq \phi_\ell(0) \leq N \tag{14}$$

for all $\ell \geq \ell_c$.

Armed with these facts, we are ready to consider the existence problem for solitary waves. Define

$$\rho_\ell(x) = \begin{cases} \phi_\ell(x) & \text{for } -\ell \leq x \leq \ell, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Then $\rho_\ell \in L^2(\mathbb{R})$ for $\ell \geq \ell_c$ and $\|\rho_\ell\|_{L^2(\mathbb{R})} = R$. Here is the main result.

THEOREM. There is a non-constant $H^\infty(\mathbb{R})$ function ϕ in the cone C and a finite constant $\lambda > 1$ such that $\lambda\phi = A\phi$. This function is the limit, uniformly on compact subsets of $\mathbb{R}$, of a sequence $\{\phi_{\ell_m}\}_{m=1}^{\infty}$ of periodic functions satisfying (6) for $\ell_m \to +\infty$. Moreover, the associated cut-off functions $\{\rho_{\ell_m}\}_{m=1}^{\infty}$ defined in (15) converge to ϕ in $L^2(\mathbb{R})$ and so $\|\phi\|_{L^2(\mathbb{R})} = R$.

<u>Proof</u>. The conclusion (14) of lemma 4 may be interpreted to mean that $\mu_0 \leq p_1(\phi_\ell) \leq N$ for all $\ell \geq \ell_c$. Hence also $\mu_0 \leq p_1(s_\ell\phi_\ell) \leq N$. Referring to the definition (12) of the metric on $\mathbb{C}(\mathbb{R})$, it is concluded that there are constants δ and Δ with $0 < \delta < \Delta < 1$ such that $\delta < d(s_\ell\phi_\ell, 0) < \Delta$, provided $\ell \geq \ell_c$. In particular, $s_\ell\phi_\ell \in C \cap B_\Delta$ for $\ell \geq \ell_c$.

Let $\epsilon > 0$ be given. Lemma 3 implies that there is an ℓ_0 such that if $\ell \geq \ell_0$,

$$\epsilon > d(Au, s_\ell A_\ell r_\ell u) \quad \text{for all } u \text{ in } C \cap B_\Delta.$$

Now, $s_\ell A_\ell r_\ell(s_\ell\phi_\ell) = s_\ell A_\ell \phi_\ell = s_\ell(\lambda_\ell\phi_\ell) = \lambda_\ell(s_\ell\phi_\ell)$. Thus for $\ell \geq \ell_1 = \max(\ell_0, \ell_c)$,

$$\epsilon > d(As_\ell\phi_\ell, s_\ell A_\ell r_\ell s_\ell\phi_\ell) = d(As_\ell\phi_\ell, \lambda_\ell s_\ell\phi_\ell). \tag{16}$$

Since ε is arbitrary, it can be concluded that $As_\ell\phi_\ell - \lambda_\ell s_\ell\phi_\ell \to 0$ with respect to the metric d as $\ell \to +\infty$.

Lemma 1 implies that $A(C\cap B_\Delta)$ is a relatively compact subset of C. A subsequence $\{\ell_m\}_{m=1}^\infty$ of wavelengths can therefore be found, with $\ell_m < \ell_{m+1}$, $\ell_m \to +\infty$ as $m \to +\infty$, together with an element ψ in C, such that if $\psi_m = s_{\ell_m}\phi_{\ell_m}$, then $A\psi_m \to \psi$ in the metric d. Since $1 < \mu_0 \leq \lambda_\ell \leq \mu_1$, it may be assumed there exists a λ such that $\{\lambda_m\}_{m=1}^\infty$, with $\lambda_m = \lambda_{\ell_m}$, has $\lambda_m \to \lambda$ as $m \to +\infty$. Obviously $1 < \mu_0 \leq \lambda \leq \mu_1$. In consequence of the conclusion (16), $A\psi_m - \lambda_m\psi_m \to 0$ with respect to d as $m \to +\infty$. Hence $\lambda_m\psi_m \to \psi$, with respect to d as $m \to +\infty$, or, since $\lambda_m \to \lambda$ in $\mathbb{R}$, $\psi_m \to \lambda^{-1}\psi = \phi$, say. As A is continuous on C, $A\psi_m \to A\phi$ for the metric d. But $A\psi_m \to \psi = \lambda\phi$. Hence $A\phi = \lambda\phi$. Note that ϕ is non-zero since $\delta \leq d(\psi_m,0)$ for all m. Further, the convergence of the sequence $\{\psi_m\}$, with $\psi_m = s_m\phi_m$, to ϕ uniformly on compact subsets of $\mathbb{R}$ implies that the sequence $\{\phi_m\}$ also converges to ϕ uniformly on compact subsets of $\mathbb{R}$, for s_m alters ϕ_m only outside the fundamental period $[-\ell_m,\ell_m]$ of ϕ_m. Thus it is proved that there exists a non-zero solution ϕ of (5) in C which is the limit of a sequence of periodic steady-wave solutions of (6).

Let $\rho_m = \rho_{\ell_m}$ be the cut-off functions associated with ϕ_m as in (15). View $\{\rho_m\}$ as a sequence in $L^2(\mathbb{R})$. Then $\|\rho_m\|_{L^2(\mathbb{R})} = R$ for all m. Moreover, $\rho_m \to \phi$ uniformly on compact subsets of $\mathbb{R}$, hence certainly pointwise. Fatou's lemma implies that $\phi \in L^2(\mathbb{R})$ and that $\rho_m \to \phi$ in $L^2(\mathbb{R})$. It follows that $\|\phi\|_{L^2(\mathbb{R})} = R$. This incidentally shows that ϕ is not the trivial (conjugate flow) solution $\psi_0(x) \equiv 2(\lambda-1)$. Since ϕ is not the zero-function or the constant function ψ_0, ϕ cannot be a constant function.

Finally, since $\phi \in L^2(\mathbb{R})$ and ϕ is bounded, it follows that $\phi^2 \in L^2(\mathbb{R})$. Therefore $A\phi = K*(\phi+\frac{1}{2}\phi^2) \in H^1(\mathbb{R})$, whence $\phi \in H^1(\mathbb{R})$. Continuing this argument shows that $\phi \in H^\infty(\mathbb{R})$. This concludes the proof.

A computation using the Fourier transform shows that ϕ is a solution of the pseudo-differential equation (4), and hence ϕ provides a permanent-wave solution u_s of the evolution equation (1) by setting $u_s(x,t) = \phi(x-\lambda t)$.

The approach to the problem presented here is to specify the total 'energy' of the wave in question (the L^2 norm of the wave) and to then determine a wave-speed for the resulting solution. The approach followed earlier by Bona and Bose (1974) was to specify the wave speed. The view taken here seems to be the right one from the experimental standpoint. The possibility of establishing a stability result by means of the variational method is also inviting. Note, incidentally, that in the special case where the symbol α of H is homogeneous of degree $\sigma > 0$, a change of variables of the form $\psi(x) = a\phi(bx)$, where a and b are positive and satisfy $a(\lambda-1)+1 = a\lambda b^{\sigma}$, converts the solution ϕ of (5) to a solution ψ of (5) with λ replaced by $a(\lambda-1)+1$.

In conclusion, it deserves remark that the approach presented here can be carried over to certain two-dimensional problems, notably internal waves in heterogeneous fluid flows along a channel and rotating flows down a pipe. The details are naturally different, but the main outline and general conclusions are the same.

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