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SOLUTIONS OF THE KORTEWEG-DE VRIES EQUATION IN FRACTIONAL ORDER SOBOLEV SPACES

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1. Introduction.

In Bona and Smith [3], it is shown that for k an integer larger than 1, the pure initial-value problem for the Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0, \qquad u(x, 0) = g(x), \qquad x \in \mathbf{R}, \qquad t \geq 0, \tag{1.1}$$

has a unique solution u in $C_b(0, \infty; H^k)$ corresponding to initial data g in the Sobolev space H^k. Recently J.-C. Saut [10] has extended this result to non-integral values of k using a non-linear interpolation theorem of Tartar [11]. He showed that if $r > 3$, $\mu = [r] + 1 - r$ and $g \in H^{r+\frac{1}{2}\mu+\epsilon}$ for some $\epsilon > 0$, then for each $T > 0$, u lies in $L^\infty(0, T; H^r)$. For the case $2 < r < 3$, Saut has the slightly weaker result that if $g \in H^{r+\frac{1}{2}\mu}$, then for each $T > 0$, u lies in $L^\infty(0, T; H^r)$. Thus it would seem that some spatial regularity is lost in solving (1.1) for initial data in non-integral Sobolev classes. The purpose of this note is to show that this is not the case, namely, that for data in H^s, $s \geq 2$, the solution to (1.1) lies in $C(0, T; H^s)$ for all $T > 0$. There is in general no smoothing action in solving other similar model equations for non-linear dispersive waves in the absence of dissipation (cf. Benjamin and Bona [2]), so the results presented here seem likely to be best possible in terms of the relation of the smoothness of data to the smoothness of the solution. In addition to showing that no smoothness is lost in solving (1.1), it is also demonstrated that the solution depends continuously on the data in the sense that, for all $T > 0$ and $s \geq 2$, the mapping $g \mapsto u$ is a continuous map of H^s into $C(0, T; H^s)$. Thus the initial-value problem (1.1) for the Korteweg-de Vries equation is classically well posed in all the Sobolev spaces H^s for $s \geq 2$.

The proof that the solution to (1.1) lies in $C(0, T; H^s)$ for data in H^s relies on a simple extension of the previously mentioned interpolation theorem of Tartar [11]. The continuous dependence also relies on an interpolation theorem for continuous non-linear operators. These preliminaries concerning abstract interpolation theory are presented in section 2. In section 3, we return to the Korteweg-de Vries equation and apply the ideas of section 2 to obtain the results stated above.

2. An interpolation theorem for non-linear operators.

Recall the K-method (or real method) of interpolation (cf. Butzer and Berens [4]). Let B_0 and B_1 be two Banach spaces such that $B_1 \subset B_0$ with the inclusion

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map continuous. Let $f \in B_0$ and define

(2.1) $$K(f, \epsilon) = \inf_{g \in B_1} \{||f - g||_{B_0} + \epsilon\, ||g||_{B_1}\},$$

where $\epsilon > 0$ and $||\ ||_{B_i}$ is the norm on B_i, $i = 0, 1$. For $0 < \theta < 1$ and $1 \leq p \leq \infty$ define

(2.2) $$[B_0, B_1]_{\theta,p} = B_{\theta,p} = \left\{f \in B_0 : ||f||_{B_{\theta,p}} = \left(\int_0^\infty K(f, \epsilon)^p \epsilon^{-\theta p - 1}\, d\epsilon\right)^{1/p} < +\infty\right\}$$

with the usual modification for the case $p = \infty$. Then $B_{\theta,p}$ is a Banach space with norm $||\ ||_{B_{\theta,p}}$. Given two pairs of indices (θ_1, p_1) and (θ_2, p_2) as above, then

(2.3) $$(\theta_1, p_1) < (\theta_2, p_2) \quad \text{means} \quad \begin{cases} \theta_1 < \theta_2 & \text{or} \\ \theta_1 = \theta_2 & \text{and} \quad p_1 > p_2 . \end{cases}$$

If $(\theta_1, p_1) < (\theta_2, p_2)$ then $B_{\theta_1,p_1} \supset B_{\theta_2,p_2}$ with the inclusion map continuous.

PROPOSITION 1. *Let $f \in B_0$, $0 < \theta < 1$ and $1 \leq p \leq \infty$. Suppose that for all $\epsilon > 0$ there are $g_i(\epsilon) \in B_i$ such that $f = g_0(\epsilon) + g_1(\epsilon)$ with $||g_i(\epsilon)||_{B_i} \leq G_i(\epsilon)$ and such that*

$$M_i = \left(\int_0^\infty G_i(\epsilon)^p \epsilon^{(i-\theta)p-1}\, d\epsilon\right)^{1/p} < +\infty$$

for $i = 0, 1$. Then $f \in B_{\theta,p}$ and

$$||f||_{B_{\theta,p}} \leq M_0^{1-\theta} M_1^{\theta}.$$

Proof. This is lemma 3.1, chapter I in Lions and Peetre [6] except that the factor of 2 does not appear because the definition of interpolation spaces used there is slightly different from the definition adopted here. The two definitions are equivalent (cf. Peetre [9]).

PROPOSITION 2. *Let $f \in B_0$ and $f_\epsilon \in B_1$ satisfy the inequality*

(2.4) $$||f - f_\epsilon||_{B_0} + \epsilon\, ||f_\epsilon||_{B_1} \leq 2K(f, \epsilon)$$

for some $\epsilon > 0$. If $f \in B_{\theta,p}$ for some θ and p with $0 < \theta < 1$ and $1 \leq p \leq \infty$, then

(2.5) $$||f_\epsilon||_{B_{\theta,p}} \leq 3\, ||f||_{B_{\theta,p}} .$$

Proof. Let $\delta > 0$. As appears from the choice $g = f_\epsilon$ in the definition (2.1) of K,

$$K(f_\epsilon, \delta) \leq \delta\, ||f_\epsilon||_{B_1} \leq 2 \frac{\delta}{\epsilon} K(f, \epsilon).$$

Now, for any $f \in B_0$, $\epsilon^{-1}K(f, \epsilon)$ is a non-increasing function of ϵ. Hence

$$K(f_\epsilon, \delta) \leq 2K(f, \delta)$$

provided $\delta \leq \epsilon$.

For any $g \in B_1$,

$$K(f_\epsilon, \delta) \leq ||f_\epsilon - g||_{B_0} + \delta ||g||_{B_1}$$
$$\leq ||f_\epsilon - f||_{B_0} + ||f - g||_{B_0} + \delta ||g||_{B_1} .$$

Taking the infimum over $g \in B_1$, it follows that

$$K(f_\epsilon, \delta) \leq ||f_\epsilon - f||_{B_0} + K(f, \delta)$$
$$\leq 2K(f, \epsilon) + K(f, \delta).$$

But for any fixed $f \in B_0$, $K(f, \epsilon)$ is non-decreasing in ϵ, so

$$K(f_\epsilon, \delta) \leq 3K(f, \delta)$$

provided $\delta \geq \epsilon$.

Thus whatever be $\delta > 0$, $K(f_\epsilon, \delta) \leq 3K(f, \delta)$. Since the norm of a function h in $B_{\theta,p}$ is simply the norm of $\delta^{-\theta}K(h, \delta)$ in $L^p(\mathbf{R}^+; d\epsilon/\epsilon)$, (2.5) follows.

With these preliminaries in hand, the abstract result on boundedness of mappings of intermediate spaces can be established.

THEOREM 1. *Let B_0^j and B_1^j be Banach spaces such that $B_0^j \supset B_1^j$ with continuous inclusion mappings, $j = 1, 2$. Let λ and q lie in the ranges $0 < \lambda < 1$ and $1 \leq q \leq \infty$. Suppose* $\mathbf{A}$ *is a mapping such that*

i) $\mathbf{A} : B_{\lambda,q}^1 \to B_0^2$ *and for* $f, g \in B_{\lambda,q}^1$

$$||\mathbf{A}f - \mathbf{A}g||_{B_0^2} \leq c_0(||f||_{B_{\lambda,q}^1} + ||g||_{B_{\lambda,q}^1}) \, ||f - g||_{B_0^1}$$

and

ii) $\mathbf{A} : B_1^1 \to B_1^2$ *and for* $h \in B_1^1$

$$||\mathbf{A}h||_{B_1^2} \leq c_1(||h||_{B_{\lambda,q}^1}) \, ||h||_{B_1^1} ,$$

where $c_i : \mathbf{R}^+ \to \mathbf{R}^+$ *are continuous non-decreasing functions,* $i = 0, 1$. *Then if* $(\theta, p) \geq (\lambda, q)$, $\mathbf{A}$ *maps* $B_{\theta,p}^1$ *into* $B_{\theta,p}^2$ *and for* $f \in B_{\theta,p}^1$

$$||\mathbf{A}f||_{B_{\theta,p}^2} \leq c(||f||_{B_{\lambda,q}^1}) \, ||f||_{B_{\theta,p}^1} ,$$

where, for $\gamma > 0$, $c(\gamma) = 4c_0(4\gamma)^{1-\theta}c_1(3\gamma)^\theta$.

Remark. This result is identical with theorem 2 of Tartar [11] except that Tartar makes the more restrictive assumption that the constants c_0 and c_1 depend only on the B_0^1 norms of the functions in question. The proof given here is modeled on the proof of Tartar, the difference lying in the use of proposition 2 rather than the elementary analog $||f_\epsilon||_{B_0} \leq 3 \, ||f||_{B_0}$ of (2.5) for functions f_ϵ satisfying (2.4).

Proof. Let $f \in B_{\theta,p}{}^1$ and for each $\epsilon > 0$, choose $f_\epsilon \in B_1{}^1$ such that

$$||f - f_\epsilon||_{B_0{}^1} + \epsilon\, ||f_\epsilon||_{B_1{}^1} \leq 2K(f, \epsilon).$$

$(\theta, p) \geq (\lambda, q)$ implies $f \in B_{\lambda,q}{}^1$ and proposition 2 therefore yields

$$||f_\epsilon||_{B_{\lambda,q}{}^1} \leq 3\, ||f||_{B_{\lambda,q}{}^1}\,.$$

Combining this with hypotheses (i) and (ii) yields the following inequalities:

$$\begin{aligned} ||\mathbf{A}f - \mathbf{A}f_\epsilon||_{B_0{}^2} &\leq c_0(||f||_{B_{\lambda,q}{}^1} + ||f_\epsilon||_{B_{\lambda,q}{}^1})\, ||f - f_\epsilon||_{B_0{}^1} \\ &\leq 2c_0(4\, ||f||_{B_{\lambda,q}{}^1})K(f, \epsilon) \end{aligned}$$

and

$$\begin{aligned} \epsilon\, ||\mathbf{A}f_\epsilon||_{B_1{}^2} &\leq c_1(||f_\epsilon||_{B_{\lambda,q}{}^1})\epsilon\, ||f_\epsilon||_{B_1{}^1} \\ &\leq 2c_1(3\, ||f||_{B_{\lambda,q}{}^1})K(f, \epsilon). \end{aligned}$$

Thus the decomposition $\mathbf{A}f = (\mathbf{A}f - \mathbf{A}f_\epsilon) + \mathbf{A}f_\epsilon$ satisfies the hypotheses of proposition 1 (with $g_0(\epsilon) = \mathbf{A}f - \mathbf{A}f_\epsilon$ and $g_1(\epsilon) = \mathbf{A}f_\epsilon$) since

$$\begin{aligned} M_i &\equiv 2c_i((4 - i)\, ||f||_{B_{\lambda,q}{}^1})\left(\int_0^\infty K(f, \delta)^p \delta^{-\theta p - 1}\, d\delta\right)^{1/p} \\ &= 2c_i((4 - i)\, ||f||_{B_{\lambda,q}{}^1})\, ||f||_{B_{\theta,p}{}^1}\,, \end{aligned}$$

$i = 0, 1$. An application of proposition 1 establishes the stated conclusion.

Attention is now focused on the question of continuity of $\mathbf{A}$ as a mapping of intermediate spaces, assuming $\mathbf{A}$ is known to be continuous as a mapping of the initial spaces. For this the following notion is useful.

Definition. Let B_0 and B_1 be Banach spaces with B_1 continuously included in B_0. Let θ and p be as usual. We say the pair B_0, B_1 has a (θ, p) *approximate identity* if there is a family of continuous mappings $\mathbf{S}_\epsilon : B_{\theta,p} \to B_1$, for $0 < \epsilon \leq 1$, such that

1) $||\mathbf{S}_\epsilon f||_{B_{\theta,p}} + \epsilon^{1-\theta}\, ||\mathbf{S}_\epsilon f||_{B_1} \leq c\, ||f||_{B_{\theta,p}}$ for all $f \in B_{\theta,p}$ and ϵ in $(0, 1]$, and
2) $||\mathbf{S}_\epsilon f - f||_{B_{\theta,p}} + \epsilon^{-\theta}\, ||\mathbf{S}_\epsilon f - f\,||_{B_0} \to 0$ as $\epsilon \downarrow 0$ for $f \in B_{\theta,p}$, and uniformly on compact subsets of $B_{\theta,p}$.

Example. Take $B_0 = L^2 = L^2(\mathbf{R})$ and, for k a positive integer, $B_1 = H^k = H^k(\mathbf{R})$, the Sobolev space of L^2 functions whose first k derivatives lie in L^2. It is well known that $[L^2, H^k]_{\theta,2} \cong H^s$ with $s = \theta k$ (cf. Magenes [7]). Hence there are constants M_s and N_s such that

$$M_s\, ||u||_{[L^2,H^k]_{\theta,2}} \leq ||u||_{H^s} \equiv \left(\int_{-\infty}^{\infty} (1 + \xi^2)^s\, |\hat{u}(\xi)|^2\, d\xi\right)^{\frac{1}{2}} \leq N_s\, ||u||_{[L^2,H^k]_{\theta,2}}\,, \tag{2.6}$$

where $\hat{u}$ denotes the Fourier transform of u. Let ϕ be a C^∞ function such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $[-1, 1]$ and $\phi \equiv 0$ outside of $(-2, 2)$. Define $\mathbf{S}_\epsilon$ by

$$\widehat{\mathbf{S}_\epsilon u}(\xi) = \phi(\epsilon^{1/k}\xi)\hat{u}(\xi).$$

Clearly $||\mathbf{S}_\epsilon u||_{H^s} \leq ||u||_{H^s}$ since $|\phi| \leq 1$. Also,

$$\begin{aligned} ||\mathbf{S}_\epsilon u||_{H^k}^2 &= \int_{-\infty}^{\infty} (1 + \xi^2)^k \phi^2(\epsilon^{1/k}\xi)\, |\hat{u}(\xi)|^2\, d\xi \\ &\leq \sup_{\xi \in \mathbf{R}} \{\phi^2(\epsilon^{1/k}\xi)(1 + \xi^2)^{k-s}\} \int_{-\infty}^{\infty} (1 + \xi^2)^s\, |\hat{u}(\xi)|^2\, d\xi \\ &\leq c\epsilon^{2(\theta-1)}\, ||u||_{H^s}^2 . \end{aligned}$$

This shows that $\mathbf{S}_\epsilon$ is a continuous mapping of H^s to H^k and establishes property 1. For any real $r \leq s$,

$$\begin{aligned} ||u - \mathbf{S}_\epsilon u||_{H^r}^2 &= \int_{-\infty}^{\infty} (1 + \xi^2)^r (1 - \phi(\epsilon^{1/k}\xi))^2\, |\hat{u}(\xi)|^2\, d\xi \\ &\leq \int_{|\xi| \geq \epsilon^{-1/k}} (1 + \xi^2)^r\, |\hat{u}(\xi)|^2\, d\xi \\ &\leq \sup_{|\xi| \geq \epsilon^{-1/k}} \{(1 + \xi^2)^{r-s}\} \int_{|\xi| \geq \epsilon^{-1/k}} (1 + \xi^2)^s |\hat{u}(\xi)|^2\, d\xi \\ &\leq c\epsilon^{-2(r-s)/k} \int_{|\xi| \geq \epsilon^{-1/k}} (1 + \xi^2)^s\, |\hat{u}(\xi)|^2\, d\xi. \end{aligned}$$

As $\epsilon \downarrow 0$,

$$\int_{|\xi| \geq \epsilon^{-1/k}} (1 + \xi^2)^s\, |\hat{u}(\xi)|^2\, d\xi \to 0$$

uniformly on compact subsets of H^s. Thus

$$||u - \mathbf{S}_\epsilon u||_{H^r} = o(\epsilon^{(s-r)/k})$$

as $\epsilon \downarrow 0$, uniformly on compact subsets of H^s. Specializing to the cases $r = 0$ and $r = s$ yields

$$||u - \mathbf{S}_\epsilon u||_{L^2} = o(\epsilon^\theta) \quad \text{and} \quad ||u - \mathbf{S}_\epsilon u||_{H^s} = o(1)$$

as $\epsilon \downarrow 0$, uniformly on compact subsets of H^s. This establishes 2, and shows that $\{\mathbf{S}_\epsilon\}$ is a $(\theta, 2)$ approximate identity for the pair L^2, H^k for any θ in $(0, 1)$.

THEOREM 2. *Let B_0^1, B_1^1, B_0^2, B_1^2, λ, q and* **A** *be as in theorem* 1. *Assume additionally that the pair B_0^1, B_1^1 has a (θ, p) approximate identity $\{\mathbf{S}_\epsilon\}$ for some $(\theta, p) \geq (\lambda, q)$ and that*

iii) **A** *is continuous as a map of B_1^1 to B_1^2. Then* **A** *is a continuous map from $B_{\theta,p}^1$ to $B_{\theta,p}^2$.*

Proof. It is first demonstrated that $\mathbf{AS}_\epsilon f \to \mathbf{A}f$ in $B_{\theta,p}^2$ as $\epsilon \downarrow 0$ uniformly for f in compact subsets of $B_{\theta,p}^1$. Proposition 1 will be used to obtain an estimate of $||\mathbf{AS}_\epsilon f - \mathbf{A}f||_{B_{\theta,p}^2}$. First for each $\delta > 0$, let $f_\delta \in B_1^1$ be such that

$$||f - f_\delta||_{B_0^1} + \delta\, ||f_\delta||_{B_1^1} \leq 2K^1(f, \delta).$$

(K^1 refers to the pair B_0^1, B_1^1.) From proposition 2,

$$||f_\delta||_{B_{\lambda,q}^1} \leq 3\, ||f||_{B_{\lambda,q}^1},$$

for all $\delta > 0$. Fix ϵ in (0, 1]. Define $g_1(\delta)$ by

$$g_1(\delta) = \begin{cases} \mathbf{AS}_\epsilon f - \mathbf{A}f_\delta & \text{if } \delta \leq \epsilon, \text{ and} \\ 0 & \text{if } \delta > \epsilon. \end{cases}$$

Then $g_1(\delta) \in B_1^2$ for each $\delta > 0$. Set $g_0(\delta) = \mathbf{AS}_\epsilon f - \mathbf{A}f - g_1(\delta)$. For $\delta \leq \epsilon$,

$$\begin{aligned} ||g_0(\delta)||_{B_0^2} &= ||\mathbf{A}f_\delta - \mathbf{A}f||_{B_0^2} \\ &\leq c_0(4\, ||f||_{B_{\lambda,q}^1})\, ||f_\delta - f||_{B_0^1} \\ &\leq 2c_0(4\, ||f||_{B_{\lambda,q}^1})K^1(f, \delta). \equiv G_0(\delta). \end{aligned}$$

For $\delta > \epsilon$,

$$\begin{aligned} ||g_0(\delta)||_{B_0^2} &= ||\mathbf{AS}_\epsilon f - \mathbf{A}f||_{B_0^2} \\ &\leq c_0(||\mathbf{S}_\epsilon f||_{B_{\lambda,q}^1} + ||f||_{B_{\lambda,q}^1})\, ||\mathbf{S}_\epsilon f - f||_{B_0^1} \\ &\leq c_0((1 + c)c'||f||_{B_{\theta,p}^1})\, ||\mathbf{S}_\epsilon f - f||_{B_0^1} \equiv G_0(\delta), \end{aligned}$$

where c is the constant in condition 1 of the definition of an approximate identity and c' is the norm of the inclusion $B_{\theta,p}^1 \subset B_{\lambda_q}^1$. Thus

$$\begin{aligned} M_0(\epsilon)^p &\equiv \int_0^\infty G_0(\delta)^p \delta^{-\theta p - 1}\, d\delta \\ &= 2^p c_0(4\, ||f||_{B_{\lambda,q}^1})^p \int_0^\epsilon K^1(f, \delta)^p \delta^{-\theta p - 1}\, d\delta \qquad (2.7) \\ &\quad + c_0((1 + c)c'\, ||f||_{B_{\theta,p}^1})^p\, ||\mathbf{S}_\epsilon f - f||_{B_0^1}^p \frac{\epsilon^{-\theta p}}{\theta p}. \end{aligned}$$

Since

$$||f||_{B_{\theta,p}^1} = \left(\int_0^\infty K^1(f, \delta)^p \delta^{-\theta p - 1}\, d\delta\right)^{1/p},$$

the tail integral

$$\int_0^\epsilon K^1(f, \delta)^p \delta^{-\theta p - 1}\, d\delta$$

tends to 0, as $\epsilon \downarrow 0$, uniformly on compact subsets of $B_{\theta,p}^1$. Condition 2 of the definition of approximate identities assures that the second term on the right-hand side of (2.7) tends to zero uniformly on compact subsets of $B_{\theta,p}^1$. Thus

$$M_0(\epsilon) \to 0,$$

as $\epsilon \downarrow 0$, uniformly on compact subsets of $B_{\theta,p}^1$. Now consider $g_1(\delta)$. For $\delta \leq \epsilon$,

$$\begin{aligned} \|g_1(\delta)\|_{B_1^2} &= \|\mathbf{AS}_\epsilon f - \mathbf{A}f_\delta\|_{B_1^2} \\ &\leq \|\mathbf{AS}_\epsilon f\|_{B_1^2} + \|\mathbf{A}f_\delta\|_{B_1^2} \\ &\leq c_1(c'\mathbf{c}\|f\|_{B_{\theta,p}^1}) \|\mathbf{S}_\epsilon f\|_{B_1^1} + c_1(3\|f\|_{B_{\lambda,q}^1}) \|f_\delta\|_{B_1^1} \\ &\leq c_1(c'\mathbf{c}\|f\|_{B_{\theta,p}^1}) c\epsilon^{\theta-1} \|f\|_{B_{\theta,p}^1} + 2c_1(3\|f\|_{B_{\lambda,q}^1}) K^1(f, \delta)\, \delta^{-1} \\ &\equiv G_1(\delta). \end{aligned}$$

Since $g_1(\delta) = 0$ for $\delta > \epsilon$, let $G_1(\delta) \equiv 0$ for $\delta > \epsilon$. Then

$$\begin{aligned} M_1(\epsilon) &\equiv \left(\int_0^\infty G_1(\delta)^p \delta^{(1-\theta)p-1}\, d\delta\right)^{1/p} \\ &= c_1(c\mathbf{c}\|f\|_{B_{\theta,p}^1}) \frac{c}{((1-\theta)p)^{1/p}} \|f\|_{B_{\theta,p}^1} \\ &\quad + 2c_1(3\|f\|_{B_{\lambda,q}^1}) \left(\int_0^\epsilon K^1(f, \delta)^p \delta^{-\theta p-1}\, d\delta\right)^{1/p}. \end{aligned}$$

Thus $M_1(\epsilon)$ is seen to be bounded on bounded sets in $B_{\theta,p}^1$, hence certainly on compact subsets of $B_{\theta,p}^1$.

Proposition 1 allows the conclusion

$$\|\mathbf{AS}_\epsilon f - \mathbf{A}f\|_{B_{\theta p}^2} \leq 2\, M_0(\epsilon)^{1-\theta} M_1(\epsilon)^\theta.$$

Thus $\mathbf{AS}_\epsilon f \to \mathbf{A}f$ in $B_{\theta,p}^2$, uniformly on compact subsets.

With the last piece of information in hand, it is easy to show $\mathbf{A}$ is continuous. Let $\{f_n\}_{n=1}^\infty$ be a sequence in $B_{\theta,p}^1$ and suppose $f_n \to f$ in $B_{\theta,p}^1$. Then if $\epsilon > 0$,

$$\|\mathbf{A}f - \mathbf{A}f_n\|_{B_{\theta,p}^2} \leq \|\mathbf{A}f - \mathbf{AS}_\epsilon f\|_{B_{\theta,p}^2} + \|\mathbf{AS}_\epsilon f - \mathbf{AS}_\epsilon f_n\|_{B_{\theta,p}^2} + \|\mathbf{AS}_\epsilon f_n - \mathbf{A}f_n\|_{B_{\theta,p}^2}.$$

Let $\gamma > 0$ be given. Since the set $\{f\} \cup \{f_n : n = 1, 2, \cdots\}$ is compact in $B_{\theta,p}^1$, there is an $\epsilon_0 > 0$ such that

$$\|\mathbf{A}f - \mathbf{AS}_{\epsilon_0} f\|_{B_{\theta,p}^2} \leq \tfrac{1}{3}\gamma \quad \text{and} \quad \|\mathbf{A}f_n - \mathbf{AS}_{\epsilon_0} f_n\|_{B_{\theta,p}^2} \leq \tfrac{1}{3}\gamma$$

for $n = 1, 2, \cdots$. Condition iii assures that $\mathbf{A}$ is continuous from B_1^1 to B_1^2. Since $\mathbf{S}_{\epsilon_0}$ is continuous from $B_{\theta,p}^1$ to B_1^1, the composition $\mathbf{AS}_{\epsilon_0}$ is continuous from $B_{\theta,p}^1$ to B_1^2. Hence there is an N so that $n \geq N$ implies

$$\|\mathbf{AS}_{\epsilon_0} f - \mathbf{AS}_{\epsilon_0} f_n\|_{B_{\theta,p}^2} \leq c'' \|\mathbf{AS}_{\epsilon_0} f - \mathbf{AS}_{\epsilon_0} f_n\|_{B_1^2} \leq \tfrac{1}{3}\gamma.$$

where c'' is the norm of $B_1^2 \subset B_{\theta,p}^2$. Thus if $n \geq N$,

$$\|\mathbf{A}f - \mathbf{A}f_n\|_{B_{\theta,p}^2} \leq \gamma,$$

and so $\mathbf{A}$ is continuous as a mapping of $B_{\theta,p}^1$ to $B_{\theta,p}^2$.

Let B be a Banach space and $T > 0$. Denote by $C(0, T; B)$ the Banach space of continuous functions from $[0, T]$ to B with norm given by

$$\|f\|_{C(0,T:B)} = \sup_{0 \le t \le T} \|f(t)\|_B .$$

When T is understood, the notation will be shortened to simply $C(B)$.

In the application of the above theory to the Korteweg-de Vries equation, the following simple fact about interpolation between spaces of the form $C(B)$ will be used.

Proposition 3. *Let B_0 and B_1 be Banach spaces with B_1 included continuously in B_0. Let θ and p lie in the ranges $0 < \theta < 1$ and $1 \le p < \infty$. Then for any $T > 0$,*

$$[C(B_0), C(B_1)]_{\theta,p} \subset C([B_0, B_1]_{\theta,p}),$$

with the inclusion mapping continuous.

Proof. Let K^c be the interpolation function for $C(B_0)$, $C(B_1)$. That is, for f in $C(B_0)$ and $\epsilon > 0$,

$$K^c(f, \epsilon) = \inf_{g \in C(B_1)} \{\|f - g\|_{C(B_0)} + \epsilon \|g\|_{C(B_1)}\}.$$

Let K denote the interpolation function for B_0, B_1 as before. Then certainly for $f \in C(B_0)$,

$$K^c(f, \epsilon) \ge \sup_{0 \le t \le T} K(f(t), \epsilon)$$

for all $\epsilon > 0$. Hence

$$\|f\|_{[C(B_0),C(B_1)]_{\theta,p}} \ge \sup_{0 \le t < T} \|f(t)\|_{[B_0,B_1]_{\theta,p}} = \|f\|_{L^\infty([B_0,B_1]_{\theta,p})} .$$

Thus

$$[C(B_0), C(B_1)]_{\theta,p} \subset L^\infty([B_0, B_1]_{\theta,p}),$$

with the inclusion mapping continuous.

But theorem 2.1, chapter III, in Lions and Peetre [6] implies that $C(B_1)$ is a dense subspace of $[C(B_0), C(B_1)]_{\theta,p}$. Since $C(B_1) \subset C([B_0, B_1]_{\theta,p})$, then certainly $C([B_0, B_1]_{\theta,p}) \cap [C(B_0), C(B_1)]_{\theta,p}$ is a dense subspace of $[C(B_0), C(B_1)]_{\theta,p}$ as well. Since $C([B_0, B_1]_{\theta,p})$ is a closed subspace of $L^\infty([B_0, B_1]_{\theta,p})$, the result follows.

3. Application to the Korteweg-de Vries equation.

We turn now to the application of the results of section 2 to the initial-value problem for the Korteweg-de Vries equation. Let $g \in H^s$ where $s \ge 2$, and let u denote the unique solution of the initial-value problem (1.1) posed with initial data g. The results of Bona and Smith [3] establish that if $s = k$ is an integer, then for any finite $T > 0$, the mapping $g \mapsto u$ is a continuous mapping of H^k to $C(0, T; H^k)$. This conclusion is now extended to non-integer values of s.

Theorem 3. *Let $T > 0$ and $s \geq 2$. Then the mapping $g \mapsto u$ is continuous from H^s to $C(0, T; H^s)$. Moreover, there is a continuous non-decreasing function $c_{s,T} : \mathbf{R}^+ \to \mathbf{R}^+$ such that if $g \in H^s$, then*

$$(3.1) \qquad \|u\|_{C(0,T;H^s)} \leq c_{s,T}(\|g\|_{H^{[s]}}) \, \|g\|_{H^s} ,$$

where $[s]$ is the largest integer smaller than s.

Proof. Denote by $\mathbf{A}g$ the unique solution u of (1.1) corresponding to initial data g. From [3] it is known that if k is an integer greater than 1 and T is a finite positive number, then $\mathbf{A}$ is a continuous mapping of H^k to $C(0, T; H^k)$. Further, as will be shown subsequently,

$$(3.2) \qquad \|\mathbf{A}g\|_{C(0,T;H^k)} \leq c_k(\|g\|_{H^{k-1}}) \, \|g\|_{H^k} ,$$

where $c_k : \mathbf{R}^+ \to \mathbf{R}^+$ is a continuous non-decreasing function independent of T. Assuming (3.2) is valid for the moment, the proof of theorem 3 is complete for s an integer larger than 1.

Now suppose that $k - 1 < s < k$ where k is an integer greater than 2. Theorems 1 and 2 will be applied with

$$B_0^1 = L^2, \qquad B_0^2 = C(0, T; L^2), \qquad B_1^1 = H^k, \qquad B_1^2 = C(0, T; H^k),$$

$$\lambda = \frac{k-1}{k}, \qquad q = 2, \qquad \theta = \frac{s}{k} \quad \text{and} \quad p = 2.$$

The example in section 2 shows that the pair L^2, H^k admits a $(\theta, 2)$ approximate identity. Hence to draw the conclusions of theorems 1 and 2 in this situation, it is sufficient to confirm conditions i, ii and iii of these theorems. Condition iii is known already from [3]. Condition ii is simply (3.2), whose validity will be established below. Condition i states, in this case, that $\mathbf{A}$ maps H^{k-1} to $C(0, T; L^2)$ and that if $f, g \in H^{k-1}$, then

$$(3.3) \qquad \|\mathbf{A}f - \mathbf{A}g\|_{C(0,T;L^2)} \leq c_{k,T}(\|f\|_{H^{k-1}} + \|g\|_{H^{k-1}}) \, \|f - g\|_{L^2} .$$

Since $k > 2$, it has already been observed that $\mathbf{A}$ maps H^{k-1} continuously to $C(0, T; H^{k-1})$ and so *a fortiori* $\mathbf{A}$ maps H^{k-1} to $C(0, T; L^2)$. Since $\mathbf{A}$ is continuous, it suffices to prove (3.3) for f and g in some dense subset of H^{k-1}. So suppose $f, g \in H^\infty$ and let $u = \mathbf{A}f$, $v = \mathbf{A}g$ and $w = u - v$. Then w satisfies the initial-value problem

$$w_t + \tfrac{1}{2}[(u + v)w]_x + w_{xxx} = 0,$$

$$w(x, 0) = f(x) - g(x).$$

Furthermore u, v and therefore w are C^∞ functions all of whose partial derivatives are in L^2 in the spatial variable. Multiply the differential equation by w and integrate over $\mathbf{R}$. Because of the smoothness properties of w,

$$\int_{\mathbf{R}} ww_{xxx}\,dx = -\int_{\mathbf{R}} w_x w_{xx}\,dx = -\frac{1}{2}\int_{\mathbf{R}} \partial_x w^2\,dx = 0,$$

and

$$\int_{\mathbf{R}} \frac{1}{2}\,[(u+v)w]_x w = -\int_{\mathbf{R}} \frac{1}{2}\,[(u+v)w]w_x\,dx = -\frac{1}{4}\int_{\mathbf{R}} (u+v)\partial_x w^2\,dx$$

$$= \frac{1}{4}\int_{\mathbf{R}} (u+v)_x w^2\,dx.$$

Using the elementary Sobolev inequality $||h||_{L^\infty} \le ||h||_{H^1}$, there appears

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}} w^2\,dx = \frac{1}{4}\int_{\mathbf{R}} (u+v)_x w^2\,dx$$

$$\le \frac{1}{4}\,(||u||_{H^2} + ||v||_{H^2})\int_{\mathbf{R}} w^2\,dx.$$

Applying (3.2), it follows that

$$\frac{d}{dt}\,||w||_{L^2}^2 \le c(||f||_{H^2} + ||g||_{H^2})\,||w||_{L^2}^2\,,$$

where $c(\lambda) = \lambda c_2(\lambda)$. By Gronwall's lemma,

$$||w(\cdot,t)||_{L^2}^2 \le ||w(\cdot,0)||_{L^2}^2 e^{ct}.$$

Writing $c_{2,T}(\lambda) = e^{c(\lambda)T/2}$, taking the square root and taking the supremum over t in $[0, T]$ of the last inequality, it is verified that

$$||\mathbf{A}f - \mathbf{A}g||_{C(0,T;L^2)} \le c_{2,T}(||f||_{H^2} + ||g||_{H^2})\,||f-g||_{L^2},$$

and this certainly implies (3.3).

Finally (3.2) is established. Recall that there is a sequence of functionals $\{I_j\}_{j=0}^\infty$ for which $I_j(u)$ is independent of $t \ge 0$ provided u is a solution of the Korteweg-de Vries equation. These have the form

$$I_0(u) = \int_{\mathbf{R}} u^2\,dx, \qquad I_1(u) = \int_{\mathbf{R}} (u_x^2 - \tfrac{1}{3}u^3)\,dx,$$

and in general for $j \ge 2$,

$$I_j(u) = \int_{\mathbf{R}} \{u_{(j)}^2 + \alpha_j u u_{(j-1)}^2 + Q_j(u, \cdots, u_{(j-2)})\}\,dx,$$

where $u_{(m)}$ is shorthand notation for $\partial_x^m u$, α_j is a constant and Q_j is a polynomial in $j-1$ variables with $Q_j(0, \cdots, 0) = 0$. Each monomial term in Q_j has 'rank' $j+2$, where the rank of the monomial $u_{(0)}^{a_0}u_{(1)}^{a_1}\cdots u_{(r)}^{a_r}$ is defined to be $\sum_{i=0}^r (1+i/2)a_i$. (These invariants were first discovered by Miura, Gardner and Kruskal [8]. See also Kruskal *et al.* [5].) More precisely, $I_j(u)$ is independent of $t \ge 0$ for u in $C(0, T; H^r)$ provided that $r \ge 2$, u is a solution of the Korteweg-de Vries equation and $0 \leq j \leq r$ (cf. theorem 2 of [3]). From the invariance of I_0, for all $t \geq 0$,

$$\|u\|_{L^2} = \|g\|_{L^2}. \tag{3.5}$$

From the first invariant and the afore mentioned Sobolev inequality,

$$\int_{\mathbf{R}} u_x^2\,dx = \frac{1}{3}\int_{\mathbf{R}} u^3\,dx + \int_{\mathbf{R}} (g')^2\,dx - \frac{1}{3}\int_{\mathbf{R}} g^3\,dx$$
$$\leq \tfrac{1}{3}\|u\|_{H^1}\|u\|_{L^2}^2 + \|g'\|_{L^2}^2 + \tfrac{1}{3}\|g\|_{H^1}\|g\|_{L^2}^2.$$

Adding $\|u\|_{L^2}^2$ to both sides, and using (3.5), the last inequality comes to

$$\|u\|_{H^1}^2 \leq \tfrac{1}{3}\|u\|_{H^1}\|g\|_{L^2}^2 + \|g\|_{H^1}^2 + \tfrac{1}{3}\|g\|_{H^1}\|g\|_{L^2}^2$$
$$\leq \tfrac{1}{6}\|u\|_{H^1}^2 + \tfrac{1}{6}\|g\|_{L^2}^4 + \|g\|_{H^1}^2 + \tfrac{1}{6}\|g\|_{H^1}^2 + \tfrac{1}{6}\|g\|_{L^2}^4.$$

Subtracting $\frac{1}{6}\|u\|_{H^1}^2$, there appears

$$\|u\|_{H^1}^2 \leq \tfrac{2}{5}\|g\|_{L^2}^4 + \tfrac{7}{5}\|g\|_{H^1}^2$$
$$\leq c_1(\|g\|_{L^2})^2\|g\|_{H^1}^2, \tag{3.6}$$

where $c_1(\lambda) = [\frac{1}{5}(2\lambda^2 + 7)]^{\frac{1}{2}}$. The general case is handled by induction. Suppose (3.2) is valid for $k < m$ where $m \geq 2$ and that $I_m(u)$ is invariant. First the integral of Q_m is estimated. This is a sum of integrals of monomials of the form $u_{(0)}^{a_0} \cdots u_{(m-2)}^{a_{m-2}}$. Since the monomial has rank $m + 2$, $2 \leq \sum_{i=0}^{m-2} a_i \leq m + 2$. Hence by an elementary use of the Sobolev inequality mentioned earlier,

$$\left|\int_{\mathbf{R}} u_{(0)}^{a_0} \cdots u_{(m-2)}^{a_{m-2}})\,dx\right| \leq (\|u\|_{H^{m-1}})^{\sum_{i=0}^{m-2} a_i}.$$

Hence for some numerical constant β_m,

$$\left|\int_{\mathbf{R}} Q_m(u_{(0)}, \cdots, u_{(m-2)})\,dx\right| \leq \beta_m(1 + \|u\|_{H^{m-1}}^m)\|u\|_{H^{m-1}}^2.$$

A similar estimate holds for the monomial $\alpha_m u u_{(m-1)}^2$. Thus utilizing the induction hypothesis,

$$\left|I_m(u) - \int_{\mathbf{R}} u_{(m)}^2\,dx\right| \leq \beta_m'(1 + \|u\|_{H^{m-1}}^m)\|u\|_{H^{m-1}}^2$$
$$\leq c_m'(\|g\|_{H^{m-1}})\|g\|_{H^{m-1}}^2.$$

This estimate holds in particular at $t = 0$. Combining this with the invariance of $I_m(u)$,

$$\left|\int_{\mathbf{R}} u_{(m)}^2\,dx - \int_{\mathbf{R}} (g^{(m)})^2\,dx\right| = \left|\int_{\mathbf{R}} u_{(m)}^2\,dx - I_m(u) + I_m(g) - \int_{\mathbf{R}} (g^{(m)})^2\,dx\right|$$
$$\leq 2c_m'(\|g\|_{H^{m-1}})\|g\|_{H^{m-1}}^2.$$

Hence if $c_m(\lambda) = (1 + 2c_m'(\lambda) + c_{m-1}(\lambda)^2)^{\frac{1}{2}}$,

$$||u||_{H^m} \leq c_m(||g||_{H^{m-1}})\, ||g||_{H^m},$$

proving (3.2).

Thus all the hypotheses of theorems 1 and 2 are valid in the context under consideration here. It follows that

$$\mathbf{A} : [L^2, H^k]_{\theta,2} \to [C(0, T; L^2), C(0, T; H^k)]_{\theta,2}$$

continuously. Now $[L^2, H^k]_{\theta,2} \cong H^s$ and, by proposition 3,

$$[C(0, T; L^2), C(0, T; H^k)]_{\theta,2} \subset C(0, T; [L^2, H^k]_{\theta,2}) \cong C(0, T; H^s).$$

Hence $\mathbf{A} : H^s \to C(0, T; H^s)$ continuously. Finally, by theorem 1,

$$||\mathbf{A}g||_{C(0,T;H^s)} \leq \frac{N_s}{M_s} c_{s,T}\left(\frac{1}{M_{k-1}} ||g||_{H^{k-1}}\right) ||g||_{H^s},$$

where $c_{s,T}(\lambda) = 2c_{2,T}(4\lambda)^{1-\theta}c_m(3\lambda)^\theta$ and M_s, M_{k-1} and N_s are defined in (2.6).

Remark. It seems likely that the last inequality should hold independently of T, as in the case $s = k$ is an integer. Our proof does not lead to this conclusion since the Lipschitz constant $c_{2,T}$ depends on T. Indeed, it is known that $\mathbf{A}$ is not Lipschitz continuous in L^2 with a Lipschitz constant which is independent of time (cf. [1, section 4] or [3, section 6]), so the method of proof adopted here cannot establish such a time-independent estimate.

More recently, in collaboration with R. Teman, Saut has obtained an existence theorem similar to the one presented here, by using a fractional order Leibnitz rule. This work will appear in the Israel Journal of Mathematics.

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