30 - 4

A POSTERIORI ERROR ESTIMATES FOR EXACT AND APPROXIMATE SOLUTIONS OF TIME-DEPENDENT PROBLEMS

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INTRODUCTION

In the numerical solution of complicated nonlinear equations in mechanics, one often assumes the existence of a smooth solution in order to derive error estimates for the approximate solution. Sometimes rigorous bounds for the derivatives of the solution exist but are pessimistic in comparison to observed behavior of numerical solutions; in more extreme cases, for example the Navier-Stokes equations, no such bound is known at all (globally in time for large data). The philosophy often put forward is that the error estimates for the numerical scheme simply show that the approximate solution will be close to the exact solution if there is one which is sufficiently regular. The point of this lecture is to observe that, in some cases, more information can be derived from the error estimates when coupled with a posteriori knowledge of the discrete solution. In fact, one can sometimes conclude more regularity of the exact solution than was known prior to determination of the discrete solution. This is effected using properties of the discrete solution, the a priori error estimates for the discrete solution, and some auxiliary estimates for the exact solution. This process will be described in complete detail for the BBM equation [1] , and then implications concerning global existence in time of smooth solutions of the Navier-Stokes equations will be discussed.

1. A priori estimates for the BBM equation

To begin with, we review the known results (and present some new ones) concerning the equation [1]

(1.1) $u_t + u_x + u\,u_x - u_{xxt} = 0$

If the pure-initial-value problem is posed, i.e., (1.1) is to hold for $x \in \mathbb{R}$ and $t \geq 0$ with data $u(x,0) = g(x)$, then the (Sobolev) H^1-norm of u is invariant in time. This follows by multiplying (1.1) by u, integrating over $\mathbb{R}$, and integrating by parts. Thus Sobolev's inequality implies that

(1.2) $$||u||_{L^\infty}(t) \leq ||u||_{H^1}(t) = ||g||_{H^1}, \text{ for } t \geq 0.$$

However, when the initial-and-boundary-value problem , $u(x,0) = g(x)$ for $x \geq 0$ and $u(0,t) = h(t)$ for $t \geq 0$, with $g(0) = h(0)$, is posed for (1.1) (with u defined for $x,t \geq 0$) , the situation is different. Multiplying by u in (1.1) now yields

(1.3) $$\frac{1}{2}\frac{d}{dt}\int_0^\infty u^2+u_x^2\,dx = -h(t)u_{xt}(0,t) + \frac{1}{2}h(t)^2 + \frac{1}{3}h(t)^3,$$

and one must bound the non-data term $u_{xt}(0,t)$. Bona and Bryant [2] were able to do this in a way that yielded

(1.4) $$||u||_{L^\infty}(t) < c_1 e^{c_2 t} \quad \text{for } t \geq 0,$$

where c_1 and c_2 are constants depending on the data g and h. We can improve on this bound as follows.

Multiply (1) by $(2u_{xt}-u^2)$ and apply the same operations as before to get

$$(1.5)\qquad \frac{d}{dt}\int_0^\infty u_x^2 - \tfrac{1}{3}u^3 dx + u_{xt}^2(0,t) = h(t)^2 u_{xt}(0,t) + \phi_1(t),$$

where, here and below, ϕ_i denotes a function of t depending only on the data g and h. Integrating (1.3) and (1.5) in time yields, after simplification,

$$(1.6)\qquad ||u||^2_{H^1}(t) \leq \left(\int_0^t h(s)^2 ds\right)^{\frac{1}{2}} \left(\int_0^t u_{xt}^2(0,s)ds\right)^{\frac{1}{2}} + \phi_2(t)$$

$$(1.7)\qquad \int_0^\infty u_x^2(x,t)dx + \frac{1}{2}\int_0^t u_{xt}^2(0,s)ds \leq \frac{1}{3}\int_0^\infty u^3(x,t)dx + \phi_3(t).$$

Applying Sobolev's inequality and (1.6) to bound $\int u^3$ in (1.7) gives

$$(1.8)\qquad \int_0^\infty u_x^2(x,t)dx + \frac{1}{2}\int_0^t u_{xt}^2(0,s)ds \leq \phi_4(t)\left(\int_0^t u_{xt}^2(0,s)ds\right)^{3/4} + \phi_5(t).$$

Applying the arithmetic-geometric-mean inequality shows that

$$(1.9)\qquad \int_0^\infty u_x^2(x,t)dx + \frac{1}{4}\int_0^t u_{xt}^2(0,s)ds \leq \phi_6(t).$$

If $g \in H^1$ and $h \in C^1$, then $\phi_6(t) \leq c_3 + c_4 t^3$ where c_3 and c_4 are constants depending only on $||g||_{H^1}$ and $||h||_{C^1}$. Applying (1.9) in (1.6) gives

$$(1.10)\qquad ||u||_{L^\infty}(t) \leq ||u||_{H^1}(t) \leq \phi_7(t),$$

where, under the above assumptions, $\phi_7(t) \leq c_5 + c_6 t$.

Estimate (1.10) seems optimal concerning the growth of the energy $||u||_{H^1}(t)$, but overly pessimistic concerning $||u||_{L^\infty}$. Numerical computations indicate that (for suitably bounded h) $||u||_{L^\infty}$ stays bounded. In the next section it is shown how to prove this by making use of the numerical computations.

2. A posteriori error estimates for the BBM equation

The authors [3] have developed a numerical integration scheme for (1.1) that is 4-th order accurate with respect to the spatial and temporal discretization parameters, Δx and Δt, respectively. Let $\tilde{u}$ denote the discrete approximation. Error estimates for the scheme have been derived of the form

$$(2.1) \qquad ||u-\tilde{u}||_{L^\infty}(t) \leq c(D(t),t)(\Delta t^4 + \Delta x^4 + e^{-X}),$$

where X is a measure of the length of the spatial domain used and D(t) is bound for the spatial and temporal derivatives of u (of order 4 and 5, respectively, on $\mathbb{R}^+ \times [0,t]$). Here, it is assumed that the initial data g decays exponentially (e.g., $g \equiv 0$); the proof of (2.1) involves showing that u(x,t) decays exponentially in x for t > 0 as well.

It is further shown in [3] that D(t) can be bounded in terms of $\sigma(t) \equiv \max\{|u(x,s)|: x \geq 0,\ 0 \leq s \leq t\}$. For example, it is shown in [2] that

$$(2.2) \qquad u_t(x,t) = h'(t)e^{-x} + \int_0^\infty K(x,y)(u+\tfrac{1}{2}u^2)(y,t)\,dy,$$

where $K(x,y) = \frac{1}{2}\left[e^{-(x+y)} + \operatorname{sgn}(x-y)e^{-(x-y)}\right]$. Thus Hölder's inequality implies that

$$\|u_t\|_{L^\infty}(t) \leq |h'(t)| + \sigma(t) + \frac{1}{2}\sigma(t)^2 .$$

Differentiating (2.2) with respect to t and x similarly yields (inductively) bounds on higher derivatives of u in terms of $\sigma(t)$:

(2.3) $\quad D(t) \leq f(\sigma(t),h,g,t),$

where the form of f is given explicitly (it is a polynomial in σ and t). Thus (2.1) becomes

(2.4) $\quad \|u-\tilde{u}\|_{L^\infty}(t) \leq c(\sigma(t),t)\,\theta$

where $\theta \equiv \Delta t^4 + \Delta x^4 + e^{-X}$. Therefore

(2.5) $$\|u\|_{L^\infty}(t) \leq \|u-\tilde{u}\|_{L^\infty}(t) + \|\tilde{u}\|_{L^\infty}(t)$$
$$\leq c(\sigma(t),t)\,\theta + \|\tilde{u}\|_{L^\infty}(t).$$

The function c in (2.4) is such that $c(\cdot,0) = 0$ and $c(\sigma,t)$ is non-decreasing in σ and strictly increasing in t. Therefore the a posteriori bound

(2.6) $$\sigma(T) = \max_{t\in[0,T]} \|u\|_{L^\infty}(t) \leq \max_{t\in[0,T]} \|\tilde{u}\|_{L^\infty}(t) + 1$$

holds provided T is such that

(2.7) $$c(\max_{t\in[0,T]} \|\tilde{u}\|_{L^\infty} + 1,\ T) \leq \frac{1}{\theta} .$$

Note that we can take $T \to \infty$ as $\theta \to 0$. One reason that it is desirable to have a good bound on $\sigma(t)$ is that it appears exponentially in the error constant c, i.e., c contains a factor of the form $e^{\sigma(t)t}$. If $\sigma(t)$ were to grow with t, the error estimate (2.1) would become useless for t of moderate size.

3. Global existence of smooth solutions of the Navier-Stokes equations

Consider the initial-and-boundary-value problem

$$(3.1) \qquad \left.\begin{aligned} u_t + u.\nabla u - \nabla^2 u &= -\nabla p \\ \nabla\cdot u &= 0 \end{aligned}\right\} \text{ in } \Omega \times [0,T]$$

$$u(x,0) = u_o(x) \text{ for } x \in \Omega,\ u(x,t)=0 \quad \text{for } x \in \partial\Omega,$$

where $\Omega \subset \mathbb{R}^3$. It is not known whether (3.1) has a smooth solution for arbitrarily large $T > 0$, unless u_o is suitably restricted. However, it is possible to derive bounds analogous to (2.3) for derivatives of u in terms of bounds for u, as is well known [4]. For example, let $\sigma(t) = \max_{s \in [0,t]} \|u\|_{L^p(\Omega)}(s)$. Then multiplying (3.1) by $\nabla^2 u$, integrating over Ω, integrating by parts, etc., leads to

$$(3.2) \qquad \|u\|_{H^1}(t) \leq \sqrt{t}\ \Psi_1(\sigma(t),p) + \|u_o\|_{H^1}$$

provided $p > 3$. Here, Ψ_1 is a function, depending only on Ω and $\|u_o\|_{L^2}$, that is increasing in σ, decreasing in p, and $\Psi_1(0,p)=0$. Thus, it should be possible to apply the ideas of the previous section in this context as well.

Suppose that we have a discrete approximation $\tilde{u}$ to u such that

$$(3.3)\qquad ||u-\tilde{u}||_{L^2}(t) < \theta \max_{s\in[0,t]} ||u||_{H^1}(s),$$

where θ represents a mesh parameter. Further, suppose that $\tilde{u}$ lies in a linear space S having the approximation property

$$(3.4)\qquad \inf_{v\in S}||u-v||_{L^p} \leq \delta^{3/p-1/2} ||u||_{H^1}$$

for $1 \leq p \leq 6$ and the inverse property

$$(3.5)\qquad ||v||_{L^p} \leq \delta^{3/p-3/2} ||v||_{L^2} \quad \text{for } v \in S \text{ and } p \geq 2.$$

Then, standard arguments using (3.3-5) show that

$$(3.6)\qquad ||u-\tilde{u}||_{L^p}(t) \leq \delta^{3/p-1/2} \left[2 + \frac{\theta}{\delta}\right] \max_{s\in[0,t]} ||u||_{H^1}(s).$$

From (3.2), (3.6) and the triangle inequality it follows that

$$(3.7)\qquad ||u||_{L^p}(t) \leq ||\tilde{u}||_{L^p}(t) + \delta^{3/p-1/2}\left[2 + \frac{\theta}{\delta}\right]\Psi_2(t,\sigma(t)),$$

where $\Psi_2(t,\sigma) \equiv \sqrt{t}\ \ \Psi_1(\sigma,p) + ||u_0||_{H^1}$. Thus we have the <u>a posteriori</u> estimate

$$(3.8)\qquad \sigma(T) = \max_{t\in[0,T]} ||u||_{L^p}(t) \leq \max_{t\in[0,T]} ||\tilde{u}||_{L^p}(t) + 1$$

as long as T satisfies

(3.9) $$\delta^{3/p-1/2}\left[2+\frac{\epsilon}{\delta}\right]\Psi_2(T, \max_{t\in[0,T]} \|\tilde{u}\|_{L^p(t)} + 1) \leq 1.$$

Note that taking equality in (3.9) gives $T \sim \delta^{1-6/p}$ provided $\epsilon = c\delta$ (recall that $p > 3$ is required). Combining (3.2) and (3.8) yields

(3.10) $$\max_{t\in[0,T]} \|u\|_{H^1}(t) \leq \Psi_2(T, \max_{t\in[0,T]} \|\tilde{u}\|_{L^p(t)} + 1)$$

provided T satisfies (3.9). Note that, as long as $\epsilon \leq c\,\delta^r$ for some $r > \frac{1}{2}$, T can be taken arbitrarily large by letting δ tend to zero. Bounds for higher derivatives of u also follow on $[0,T]$ by standard techniques. Recent results of John Heywood show the singularities of a weak solution of (3.1) must be restricted to a finite interval of time, so if T can be taken large enough in (3.9), the question of global existence of a smooth solution to (3.1) can be completely resolved.

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