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A Comparison of Solutions of Two Model Equations for Long Waves

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Summary. Considered herein are the partial differential equations,

$$u_t + u_x + uu_x + Lu = 0, \tag{$*$}$$

where L denotes ∂_x^3 (equation A) or $-\partial_x^2\partial_t$ (equation B). In $(*)$ u is a real-valued function defined for all real x and for $t \geq 0$, and interest will be focused on solutions of (A) and (B) that correspond to the initial condition that $u(x, 0)$ is a given function. Equation (A) is the Korteweg-de Vries equation [1895] while (B) is the model studied, for example, in Benjamin, Bona and Mahony [1972]. It has been argued in the last-quoted reference and elsewhere that either (A) or (B) can be used with equal justification to model various physical phenomena.

To establish this claim an exact relation connecting solutions of (A) and (B) is derived, showing that the two models yield predictions whose difference, over significant time scales, is only of the small order that is formally neglected by either model.

Complementing the theoretical study are some numerical experiments based on (B). These experiments suggest that the aforementioned theoretical estimates are sharp, and that they are valid up to the time scale for which either equation formally ceases to be an accurate model of underlying physical phenomena. The experiments also indicate that (B) has the property, which is well known for (A), that certain classes of initial data evolve into a sequence of solitary waves followed by a dispersive wave train.

1. Introduction. This paper is concerned with mathematical models representing the unidirectional propagation of weakly nonlinear dispersive

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long waves. Interest will be directed toward two particular models that were originally studied in the context of surface-wave phenomena in open-channel flows. The rationale behind the derivation of these models, details of their mathematical properties and their applicability to a host of quite disparate physical systems are well documented (see, for example, the review articles of Benjamin [**1974**], Bona [**1980, 1981**], Jeffrey and Kakutani [**1972**], Kruskal [**1975**], Miura [**1974, 1976**], Scott, Chu and McLaughlin [**1973**] and Zabusky [**1981**]).

In attempting to describe open-channel flows the underlying principle in the derivation of these model equations is that their solutions should approximate solutions of the two-dimensional Euler equations, posed with appropriate boundary conditions at the bed of the channel and at the free surface. Within the context of procedures for generating such models it is possible that several different equations may emerge. The choice of which approximation to use will then depend on properties of one model *vis-à-vis* those of another. For long waves on the surface of water, two models have received particular attention. One is the equation of Korteweg and de Vries [**1895**] (equation (A) or the KdV equation hereafter) and the other is an equation first studied theoretically by Benjamin, Bona and Mahony [**1972**] (equation (B) hereafter). The qualitative mathematical properties of solutions of these two models have been studied in detail. This theory is rich and interesting, but appears to offer no definitive reason for preferring one or the other of these models as regards the sort of task for which they were originally derived.

The purpose of the present paper is to make a quantitative comparison between the solutions to the initial-value problem for each of these models. The basic conclusion of the study is that, on a long time scale T naturally related to the underlying physical situation, the equations predict the same outcome to within their implied order of accuracy. In this situation the choice of one of these models over the other to describe a physical problem is apparently immaterial, with factors of incidental convenience probably providing the main criteria in a given situation. It is worth noting that if one is only interested in the evolution over a much shorter time interval than $[0, T]$, then a model simpler than either of the aforementioned (a factored version of the linear wave equation) will suffice. The main analysis leading to the above-stated conclusions is presented in §4. The earlier §§2 and 3 give, respectively, a brief account of the assumptions and formal limitations inherent in the models, and some mathematical definitions and results needed for the analysis of §4.

The question of the relationship between the two model equations has been discussed in general terms by Benjamin *et al.* [**1972**], Kruskal [**1975**] and Whitham [**1974**]. Showalter [**1977**], working carefully through the standard formal scalings and expansions leading to models such as those considered here, derived some alternative systems and conjectured explicitly that the KdV equation and equation (B) will give similar answers on a time scale much smaller than the scale T mentioned earlier. Bona and Smith [**1975**], in their paper on the initial-value problem for the KdV equation, deal with exactly the issue considered herein, but give no attention to the time scales over which their results are valid. For practical purposes, these time scales are crucially important.

In relating the solutions of the two model equations, the present work supplements the studies of Berger [**1974**] and Nishida and Kano [**1983**] in which the relation of the model equations to the Euler equation is examined. It is also complementary to the various studies comparing predictions of these model equations with the outcome of some laboratory experiments (see, for example, Bona, Pritchard and Scott [**1981**], Hammack [**1973**], Hammack and Segur [**1974**], and Zabusky and Galvin [**1971**]).

In addition to the theoretical relation linking the two equations, we have also made numerical experiments designed to afford further comparison between the two models. The first set of numerical experiments relates directly to the results of §4, showing how particular solutions of the two models differ as a function of time. The second experiment appears to confirm that equation (B) shares the property with the KdV equation whereby certain classes of initial data evolve into a sequence of solitary waves followed by a dispersive tail. These results are reported in §5.

2. The model equations. A model often used to describe the unidirectional propagation of irrotational, weakly nonlinear, dispersive waves on the surface of an ideal liquid in a uniform channel is the equation proposed by Korteweg and de Vries [**1895**],

$$\eta_t + \eta_x + \tfrac{3}{2}\,\eta\eta_x + \tfrac{1}{6}\,\eta_{xxx} = 0. \tag{1a}$$

In this equation $\eta = \eta(x, t)$ represents the vertical displacement of the surface of the liquid from its equilibrium position, t is the time and x is the horizontal coordinate (which increases in the direction of propagation of the waves). Equation (1a) is written in dimensionless form, with the length scale taken to be the undisturbed depth h of the liquid and the time scale to be $(h/g)^{1/2}$; g is the gravity constant.

It is assumed in the derivation of (1a) that the maximum amplitude ε of the waves is small and that the waves can be characterized by a horizontal scale δ^{-1}, which is large. In particular, it is crucial that the amplitude scale and the horizontal scale of the waves are such that $\varepsilon\delta^{-2}$ is of order one so that the nonlinear and dispersive corrections to the primary terms η_t and η_x are of comparable importance (see, for example, Meyer [**1979**]). These considerations suggest the introduction of a new dependent variable N and new independent variables ξ and τ such that

$$\eta = \varepsilon N, \qquad x = \varepsilon^{-1/2}\xi, \qquad t = \varepsilon^{-1/2}\tau, \tag{2}$$

in which case N and its derivatives are, by assumption, all order one quantities. In these variables, equation (la) may be rewritten as

$$N_\tau + N_\xi + \tfrac{3}{2}\varepsilon N N_\xi + \tfrac{1}{6}\varepsilon N_{\xi\xi\xi} = O(\varepsilon^2), \tag{1b}$$

where the order of the terms neglected in the derivation of (1) has been indicated explicitly on the right-hand side. It is apparent from (1b) that the nonlinear and dispersive terms, $3\varepsilon NN_\xi/2$ and $\varepsilon N_{\xi\xi\xi}/6$, respectively, constitute corrections of order ε to the order one primary terms. Thus a formal calculation based on the simple equation,

$$N_\tau + N_\xi = \varepsilon,$$

suggests that, on the time scale $\tau_1 = \varepsilon^{-1}$, the nonlinear and dispersis terms may have had a significant influence on the structure of the waves. In the coordinates appearing in (la), τ_1 corresponds to the time scale $t_1 = \varepsilon^{-3/2}$. By the same reasoning the terms neglected in the derivation of (1) could have had a cumulative effect of order one on the time scale $\tau_2 = \varepsilon^{-2}$ (corresponding to the time scale $t_2 = \varepsilon^{-5/2}$). These arguments further suggest that the equation $N_\tau + N_\xi = 0$ would suffice to describe wave evolution on a time scale $\tau_0 = 1$ (or $t_0 = \varepsilon^{-1/2}$).

Because of the relative sizes of the terms in (1), it has been argued (see, for example, Peregrine [**1966**] and Benjamin *et al.* [**1972**]) that the equation

$$N_\tau + N_\xi + \tfrac{3}{2}\varepsilon N N_\xi - \tfrac{1}{6}\varepsilon N_{\xi\xi\tau} = 0, \tag{3a}$$

or equivalently,

$$\eta_t + \eta_x + \tfrac{3}{2}\eta\eta_x - \tfrac{1}{6}\eta_{xxt} = 0, \tag{3b}$$

provides a model comparable with (1) for the physical problem in question. (It is worth remarking that the present discussion applies to situations other than surface waves on an ideal fluid. Indeed, one or the other of these equations has been derived as a model in a wide range of physical contexts. Discussions of certain applications may be found, for example, in the review articles of Jeffrey and Kakutani [**1972**], and Scott, Chu and McLaughlin [**1973**] as well as in the text of Lamb [**1980**]; an account of the principles underlying the frequent appearance of these model equations is given in the article by Benjamin [**1974**].)

In this paper we shall concentrate on the initial-value problems posed by (1) and (3) with

$$\eta(x, 0) = \eta_0(x), \tag{4}$$

for $x \in \mathbf{R}$, the real numbers. This corresponds to the presumption that the wave profile is known everywhere at some given instant of time, and that inquiry is directed to the subsequent evolution of the wave field. Of particular interest will be the comparison of solutions of the two model equations, subject to the same specification (4), over the time scales t_0, t_1 and t_2.

To illustrate the kind of results we have in mind, consider the initial-value problems for the linear versions of (1) and (3),

$$N_\tau + N_\xi + \tfrac{1}{6}\varepsilon N_{\xi\xi\xi} = 0 \tag{5a}$$

and

$$M_\tau + M_\xi - \tfrac{1}{6}\varepsilon M_{\xi\xi\tau} = 0, \tag{5b}$$

together with the initial condition

$$N(\xi, 0) = M(\xi, 0) = F(\xi), \tag{5c}$$

where F is order one. The problems (5a)–(5c) and (5b)–(5c) are easily solved by taking the Fourier transform in the ξ variable. Let m, n and f, respectively, denote the Fourier transform with regard to ξ of the functions M, N and F, the transformed variable being denoted by k. We see at once that

$$n(k, \tau) = \exp(-ik[1 - \varepsilon k^2/6]\tau)f(k)$$

and

$$m(k, \tau) = \exp(-ik[1/(1 + \varepsilon k^2/6)]\tau)f(k).$$

Suppose for simplicity that f is smooth and has bounded support ($f \equiv 0$ outside a bounded region). If the integral with respect to k of the absolute value of the difference between m and n is computed, as a function of $\tau \geq 0$, there is obtained

$$\|m(\cdot, \tau) - n(\cdot, \tau)\|_{L^1(\mathbf{R})} = \int_{-\infty}^{\infty} |m(k, \tau) - n(k, \tau)|\, dk \leq C\varepsilon^2\tau, \tag{6}$$

valid for $0 < \varepsilon \leq 1$, say, where C is a constant depending only on f. It follows, for $\tau \geq 0$ and $1 \geq \varepsilon > 0$, that

$$\begin{aligned} \sup_{-\infty<\xi<\infty} |M(\xi, \tau) - N(\xi, \tau)| &= \sup_{-\infty<\xi<\infty} \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi k}[m(k, \tau) - n(k, \tau)]\, dk \right| \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |m(k, \tau) - n(k, \tau)|\, dk \leq \tilde{C}\varepsilon^2\tau, \end{aligned} \tag{7}$$

where $\tilde{C}$ denotes $C/2\pi$. Remember that M and N are, like F, both of order one and that the neglected order in the variables in (2) is ε. Then (7) shows explicitly at time τ of order ε^{-1}, when dispersive effects may have significantly modified the shape of the initial wave profile, that M and N are still within ε of each other. That is, M and N are the same to the inherent accuracy of either model at time $\tau = \varepsilon^{-1}$.

Expressing (7) in terms of solutions of the linear versions of (1a) and (3a) the following result emerges. Let u and v be, respectively, solutions of the initial-value problems,

$$u_t + u_x + \tfrac{1}{6}u_{xxx} = 0 \quad \text{and} \quad v_t + v_x - \tfrac{1}{6}v_{xxt} = 0, \tag{8}$$

with $u(x, 0) = v(x, 0) = \varepsilon F(\varepsilon^{1/2}x)$, where $0 < \varepsilon \leq 1$. Then there is a constant C independent of ε and t such that,

$$\sup_{-\infty<x<\infty} |u(x, t) - v(x, t)| \leq C\varepsilon^{7/2}t \tag{9}$$

for all $t \geq 0$. The inequality in (9) ceases to be interesting when t is of order $\varepsilon^{-5/2}$, since both u and v are of order ε in absolute magnitude.

The principal object of the present study is to determine whether (9) holds when nonlinear effects are retained in the model equations. More precisely, suppose g is a given sufficiently smooth function decaying appropriately to zero at $\pm\infty$. Let $\eta = \eta(x, t; \varepsilon)$ and $\zeta = \zeta(x, t; \varepsilon)$ be the solutions of the differential equations,

$$\eta_t + \eta_x + \tfrac{3}{2}\eta\eta_x + \tfrac{1}{6}\eta_{xxx} = 0, \tag{10a}$$

and

$$\zeta_t + \zeta_x + \tfrac{3}{2}\zeta\zeta_x - \tfrac{1}{6}\zeta_{xxt} = 0, \tag{10b}$$

with

$$\eta(x, 0) = \zeta(x, 0) = \varepsilon g(\varepsilon^{1/2}x). \tag{10c}$$

CONJECTURE. *There is a constant C dependent only on g such that for $0 < \varepsilon \leq 1$ and $0 \leq t \leq \varepsilon^{-5/2}$,*

$$\sup_{-\infty<x<\infty} |\eta(x, t; \varepsilon) - \zeta(x, t: \varepsilon)| \leq C\varepsilon^{7/2}t. \tag{11}$$

In §4 this conjecture and similar bounds involving derivatives of η and and ζ will be established for t in the range $[0, \varepsilon^{-3/2}]$. In the next section some notation and useful auxiliary results are set forth in preparation for the analysis in §4.

3. Notation and preliminary results. The standard notation $L_p = L_p(\mathbf{R})$ will be used for the (equivalence classes of) pth power integrable functions $f\colon \mathbf{R} \to \mathbf{R}$, for $1 \leq p < \infty$, with the usual modification for $p = \infty$. The norm of a function f in $L_p(\mathbf{R})$ is

$$\|f\|_{L_p} = \left\{\int_{-\infty}^{\infty} |f(x)|^p \, dx\right\}^{1/p}.$$

For nonnegative integers k, H^k is the Sobolev space of L_2 functions whose first k (generalized) derivatives are also in L_2. The norm in H^k is taken to be

$$\|f\|_k = \{\|f\|_{L_2}^2 + \|f_{(k)}\|_{L_2}^2\}^{1/2},$$

where $f_{(j)}$ denotes the jth derivative of f. Of course, $H^0 = L_2$ and we shall use $\| \;\|$ to mean the same as $\| \;\|_0$. By Plancherel's theorem, the norm in H^k may be expressed as follows:

$$\|f\|_k^2 = \int_{-\infty}^{\infty} (1 + \xi^{2k})\, |\hat{f}(\xi)|^2\, d\xi,$$

where $\hat{f}$ denotes the Fourier transform of f. If k is a negative integer, then H^k is defined to be the dual space of H^{-k}. The space H^k for $k < 0$ may be identified with the class of tempered distributions T whose Fourier transform $\hat{T}$ is a Lebesgue measurable function for which

$$\|T\|_k^2 = \int_{-\infty}^{\infty} \frac{1}{1 + \xi^{-2k}}\, |\hat{T}(\xi)|^2\, d\xi < +\infty.$$

The spaces H^k for $k < 0$ intervene only tangentially in our analysis. By H^∞ we denote $\bigcap_{k\geq 0} H^k$. The elements of H^∞ are infinitely differentiable functions, all of whose derivatives lie in L_2.

If X is an arbitrary Banach space and $T > 0$, the space $C(0, T; X)$ is the collection of continuous functions $u\colon [0, T] \to X$. If $T = +\infty$, it is required in addition that u be bounded for $0 \leq t < T$. This collection is a Banach space with the norm $\sup_{0\leq t\leq T} \|u(t)\|_X$, where $\| \;\|_X$ denotes the norm in X.

In the analysis given in §4 the forms (1a) and (3b) of the two model equations will be used. More precisely, by rescaling η, x and t by the order one constants $3/2$, $(1/6)^{1/2}$ and $(1/6)^{1/2}$, respectively, we may take the model equations in the tidy forms

$$\eta_t + \eta_x + \eta\eta_x + \eta_{xxx} = 0, \qquad \text{(12) or (A)}$$

and

$$\eta_t + \eta_x + \eta\eta_x - \eta_{xxt} = 0. \qquad \text{(13) or (B)}$$

Results pertaining to the initial-value problem for both (A) and (B) will be needed. Also intervening in our analysis is a regularized version of (A), written in moving coordinates, namely,

$$\eta_t + \eta\eta_x + \eta_{xxx} - \varepsilon\eta_{xxt} = 0, \qquad (14)$$

where $\varepsilon > 0$. (In due course the ε appearing in (14) will be identified with the amplitude parameter ε appearing in the last section.) The needed theoretical results relating to (A), (B), and (14) are presented in the following sequence of propositions. We commence with results for the KdV equation (A).

PROPOSITION 1. *Let $g \in H^m$ where $m \geq 2$. Then there exists a unique function u in $C(0, \infty; H^m)$ which is a solution of* (A) *in $\mathbf{R} \times \mathbf{R}^+$ such that $u(\cdot, 0) = g$. Furthermore, $\partial_t^k u \in C(0, \infty; H^{m-3k})$ for k such that $m - 3k \geq -1$. The correspondence $g \mapsto \partial_t^k u$ is, for each $T > 0$, a continuous mapping of H^m into $C(0, T; H^{m-3k})$ for all $k \geq 0$ such that $m - 3k \geq -1$.*

REMARK. By 'solution' we shall always mean a solution u of the differential equation in the sense of distributions for which the initial condition is satisfied in the sense that as $t \to 0$, $u(\cdot, t) - g(\cdot)$ tends to zero in an appropriate function space. Of course if $m > 3$ in the above proposition, the solution will in fact be classical. That is, all the derivatives appearing in the equation exist classically and are continuous, and the equation is verified pointwise by u everywhere in the relevant domain.

Proposition 1 summarizes some of the theory appearing in Kato [**1975**, **1979**] and Bona and Smith [**1975**]. The next result collects together Theorems 2, 4 and 5 of Benjamin *et al.* [**1972**] supplemented by Lemma 2 and Theorem 5 of Bona and Smith [**1975**].

PROPOSITION 2. Let *$g \in H^m$ where $m \geq 1$. Then there exists a unique function u* in *$C(0, \infty; H^1)$ which is a solution of* (B) *in $\mathbf{R} \times \mathbf{R}^+$ such that $u(\cdot, 0) = g$. For each $T > 0$, $u \in C(0, T; H^m)$ and, for each $k > 0$, $\partial_t^k u \in C(0, T; H^{m+1})$. For each $T > 0$, the correspondence $g \mapsto u$ is a continuous mapping of H^m to $C(0, T; H^m)$ while, if $k > 0$, the correspondence $g \mapsto \partial_t^k u$ is a continuous mapping of H^m into $C(0, T; H^{m+1})$.*

In the above proposition, m is an integer. Results of a similar nature are available for both equations in Sobolev spaces with noninteger order (see, for example, Bona and Scott [**1976**], Kato [**1975, 1979**], Saut [**1975**] and Saut and Temam [**1976**] for equation (A) and Benjamin and Bona [**1983**] for equation (B)).

Both (A) and (B) have invariant functionals associated with the solutions described in Propositions 1 and 2. These will play an important role in the subsequent analysis. The simplest of these functionals corresponds to the conservation of mass. More precisely, if u is a solution of either (A) or (B) corresponding to initial data of the kinds indicated in Propositions 1 and 2 (with the restriction for (A) that $m \geq 3$) then the total 'mass'

$$\int_{-\infty}^{\infty} u(x, t)\, dx \tag{15}$$

does not change with time. Thus if the integral in (15) converges as an improper integral at $t = 0$, then it converges for all subsequent times and the value of the integral is independent of t. For (B) there are only two further invariants known, and indeed Olver [**1979**] has established that

(B) has no invariant functionals of the form $\int_{-\infty}^{\infty} p(u, u_x, u_{xx}, \ldots)\, dx$, with p a polynomial, other than those given in (15) and below in (16). (Such invariants will be referred to as polynomial invariants.)

PROPOSITION 3. *Let $g \in H^m$ where $m \geq 1$ and let u be the solution of* (B) *with initial value g, as guaranteed by Proposition 2. Then*

$$\int_{-\infty}^{\infty} [u^2(x,t) + u_x^2(x,t)]\, dx \quad \textit{and} \quad \int_{-\infty}^{\infty} [u_x^2(x,t) - \tfrac{1}{3}u^3(x,t)]\, dx \tag{16}$$

are both independent of t.

For the KdV equation (A), an infinite sequence of polynomial invariants is known. These take the form

$$I_k(u) = \int_{-\infty}^{\infty} [u_{(k)}^2 - c_k u u_{(k-1)}^2 + Q_k(u, u_x, \ldots, u_{(k-2)})]\, dx, \tag{17}$$

where $u_{(r)} = \partial_x^r u$ and, for each $k = 0, 1, 2, \ldots$, Q_k is a polynomial of rank $k + 2$. Here the definition of rank employed by Miura *et al.* [**1968**] is being used. For a monomial, let

$$\operatorname{rank}(u_{(0)}^{a_0} u_{(1)}^{a_1} \cdots u_{(p)}^{a_p}) = \sum_{j=0}^{p} \left(1 + \frac{j}{2}\right) a_j.$$

The rank of a polynomial is then just the maximum of the ranks of its monomial components. In fact, Q_k is composed entirely of monomials of rank $k + 2$. The next proposition is derived from Theorem 1 and Proposition 6 of Bona and Smith [**1975**].

PROPOSITION 4. *Let $g \in H^m$ where $m \geq 2$ and let u be the solution of* (A) *corresponding to the initial value g. Then $I_0(u), \ldots, I_m(u)$ are independent of time. Furthermore, the invariance of these functionals implies that, for $0 \leq k \leq m$,*

$$\|u_{(k)}(\cdot, t)\| \leq q_k(\|g\|_k), \tag{18}$$

independently of $t \geq 0$, where $q_k\colon \mathbf{R}^+ \to \mathbf{R}^+$ may be taken to be the square root of a polynomial with nonnegative coefficients which vanishes at 0.

Finally, the initial-value problem for the regularized equation (14) will occur at a crucial point in §4. For this problem we have the following result.

PROPOSITION 5. *Let $g \in H^m$ where $m \geq 1$. Then there is a unique function u in $C(0, \infty; H^1)$ which is a solution of* (14) *in $\mathbf{R} \times \mathbf{R}^+$ such that $u(\cdot, 0) = g$. For each $T > 0$ and integer k in $[0, m]$, $\partial_t^k u \in C(0, T; H^{m-k})$ and the mapping*

$g \mapsto \partial_t^k u$ is continuous from H^m to $C(0, T; H^{m-k})$. Furthermore, the functionals

$$\int_{-\infty}^{\infty} [u^2(x, t) + \varepsilon u_x^2(x, t)]\, dx \quad \textit{and} \quad \int_{-\infty}^{\infty} [u_x^2(x, t) - \tfrac{1}{3} u^3(x, t)]\, dx \tag{19}$$

are both independent of $t \geq 0$. If $m \geq 3$, there is a positive constant ε_0, dependent only on $\|g\|_3$, such that for $0 < \varepsilon \leq \varepsilon_0$ and $0 \leq t \leq 1$,

$$\int_{-\infty}^{\infty} u_{xx}^2(x, t)\, dx \leq a_0^2(\|g\|_3), \tag{20}$$

where $a_0\colon \mathbf{R}^+ \to \mathbf{R}^+$ is continuous, monotone increasing and $a_0(0) = 0$.

This proposition is a minor modification of Propositions 2 and 3 in Bona and Smith [**1975**]. Note that the two invariant functionals in (19) are, apart from the term containing an ε, the same as for KdV. It follows as in Proposition 4 that the solutions u provided in Proposition 5 are bounded in H^1, independently of $t \geq 0$ and $\varepsilon > 0$.

COROLLARY 1. *Let $g \in H^m$ where $m \geq 1$ and let u be the solution of* (14) *guaranteed by Proposition 5. Then, for $1 \geq \varepsilon > 0$,*

$$\|u(\cdot, t)\| \leq \|g\|_1 \quad \textit{and} \quad \|u_x(\cdot, t)\| \leq q(\|g\|_1) \tag{21}$$

for all $t \geq 0$, where $q(0) = 0$, and q is continuous and monotone increasing.

PROOF. The first invariant in (19) shows that, for $0 < \varepsilon \leq 1$,

$$\begin{aligned} \|u(\cdot, t)\|^2 &\leq \int_{-\infty}^{\infty} [u^2(x, t) + \varepsilon u_x^2(x, t)]\, dx \\ &= \int_{-\infty}^{\infty} [g^2(x) + \varepsilon g_x^2(x)]\, dx \leq \|g\|_1^2. \end{aligned} \tag{22}$$

Similarly, it is adduced from the second invariant in (19) that,

$$\begin{aligned} \|u_x(\cdot, t)\|^2 &= \|g_x\|^2 + \frac{1}{3}\int_{-\infty}^{\infty} u^3(x, t)\, dx - \frac{1}{3}\int_{-\infty}^{\infty} g^3(x)\, dx \\ &\leq \|g\|_1^2 + \tfrac{1}{3}\|u(\cdot, t)\|_{L^\infty} \|u(\cdot, t)\|^2 + \tfrac{1}{3}\|g\|^2 \|g\|_{L^\infty} \\ &\leq \|g\|_1^2 + \tfrac{1}{3}\|g\|_1^3 + \tfrac{1}{3}\|g\|_1^2 \|u(\cdot, t)\|_1, \end{aligned} \tag{23}$$

where the elementary inequality

$$\|v\|_{L^\infty}^2 \leq \|v\|\, \|v_x\| \leq \tfrac{1}{2}\|v\|_1^2 \tag{24}$$

and (22) have both been used. Adding (22) and (23), it follows that (21) holds, with, for example,

$$q(s) = [\tfrac{12}{5} s^2 + \tfrac{2}{3} s^3 + \tfrac{1}{5} s^4]^{1/2}.$$

4. Analytic comparison of the two model equations. Attention is now focused on the inital-value problems for the model equations (A) and (B). Let g_0 be a given, physically appropriate, initial wave profile. In the scales implied in both (A) and (B), g_0 must therefore be of small amplitude and large wave length, and these must be appropriately related as explained in §2. These assumptions on g_0 may be made precise by introducing the positive parameter ε, which is taken as a measure of the small amplitude of the wave profile, and assuming that g_0 may be represented in the form

$$g_0(x) = \varepsilon g(\varepsilon^{1/2}x). \tag{25}$$

Here g is viewed as fixed and interest lies in the regime $\varepsilon \ll 1$. Thus consideration is given to the initial-value problems

$$\left.\begin{aligned} &\eta_t^\varepsilon + \eta_x^\varepsilon + \eta^\varepsilon\eta_x^\varepsilon + \eta_{xxx}^\varepsilon = 0, \quad \text{for } (x,t) \in \mathbf{R} \times \mathbf{R}^+, \\ \text{with}\qquad & \\ &\eta^\varepsilon(x, 0) = \varepsilon g(\varepsilon^{1/2}x), \qquad\quad \text{for } x \in \mathbf{R}, \end{aligned}\right\} \tag{26a}$$

and

$$\left.\begin{aligned} &\zeta_t^\varepsilon + \zeta_x^\varepsilon + \zeta^\varepsilon\zeta_x^\varepsilon - \zeta_{xxt}^\varepsilon = 0, \quad \text{for } (x,t) \in \mathbf{R} \times \mathbf{R}^+, \\ \text{with}\qquad & \\ &\zeta^\varepsilon(x, 0) = \varepsilon g(\varepsilon^{1/2}x), \qquad\quad \text{for } x \in \mathbf{R}. \end{aligned}\right\} \tag{26b}$$

In what follows, we shall refer to a quantity of the form $q(\|g\|_k)$, where $q\colon \mathbf{R}^+ \to \mathbf{R}^+$ is a C^1 function with $q(0)=0$, as an *order-one quantity*. Thus an order-one quantity is a function of a Sobolev norm of g. The principle result of our investigation may now be stated.

THEOREM 1. *Let $g \in H^{k+5}$ where $k \geq 0$. Let $\varepsilon > 0$ and let η^ε and ζ^ε be the unique solutions, guaranteed by Propositions* 1 *and* 2, *of the initial-value problems in* (26). *Then there is an $\varepsilon_0 > 0$ and order-one constants M_j such that if $0 < \varepsilon \leq \varepsilon_0$, then*

$$\|\eta_{(j)}^\varepsilon(\cdot, t) - \zeta_{(j)}^\varepsilon(\cdot, t)\| \leq M_j \varepsilon^{j/2+7/4}(\varepsilon^{3/2}t) \tag{27}$$

at least for $0 \leq t \leq \varepsilon^{-3/2}$, where $0 \leq j \leq k$.

REMARK. We continue to use the symbol $u_{(r)}$ introduced in (17) to denote the rth derivative of u with respect to the spatial variable x, $\partial_x^r u$.

A comparison over a short time interval. Before giving the proof of the just-stated theorem, a related issue will be addressed. This somewhat simpler point is of interest in its own right, and its resolution suggests an analysis of the more complex situation reflected in Theorem 1.

As explained in §2 either model equation (A) or (B) may be viewed as a small perturbation of the basic one-way propagator $u_t + u_x = 0$. Moreover, in the formal derivation of these models from more complete sets of equations, assumptions are made concerning the sizes of various combinations of the dependent variable and its derivatives. These assumptions play a crucial role in the derivation of the model equation, for they allow certain terms in a formal expansion to be retained while others are dropped. As already explained in §2 it is this procedure that leads ultimately to equations such as (A) and (B). Moreover, at a cruder level of approximation than that anticipated for in (A) and (B), this procedure would yield exactly the factored form of the one-dimensional linear wave equation $u_t + u_x = 0$. It is our purpose here to establish rigorously the natural suppositions concerning the size of the dependent variable and its derivatives. It will also be shown that either model (A) or (B) may be replaced by the equation $u_t + u_x = 0$ without loss of order of accuracy provided time scales no longer than $\varepsilon^{-1/2}$ are in question.

In view of the theorem stated above concerning the relation between the two models, it suffices to consider only one of these models at the present stage of discussion. Results established for one model will apply to the other model by virtue of Theorem 1.

Consideration is therefore given to solutions of (26a) and their relationship to solutions of the initial-value problem,

$$\sigma^\varepsilon_t + \sigma^\varepsilon_x = 0, \quad \text{for } (x, t) \in \mathbf{R} \times \mathbf{R}^+, \tag{28}$$

with $\sigma^\varepsilon(x, 0) = \varepsilon g(\varepsilon^{1/2}x)$, for $x \in \mathbf{R}$, where $0 < \varepsilon \ll 1$. The solution of (28) is

$$\sigma^\varepsilon(x, t) = \varepsilon g(\varepsilon^{1/2}(x - t)). \tag{29}$$

Define

$$\left.\begin{aligned} u(x, t) &= \varepsilon^{-1}\eta^\varepsilon(\varepsilon^{-1/2}x + \varepsilon^{-3/2}t, \varepsilon^{-3/2}t) \\ \text{and} \qquad & \\ w(x, t) &= \varepsilon^{-1}\sigma^\varepsilon(\varepsilon^{-1/2}x + \varepsilon^{-3/2}t, \varepsilon^{-3/2}t). \end{aligned}\right\} \tag{30}$$

Then u and w satisfy the initial-value problems,

$$\left.\begin{aligned} u_t + uu_x + u_{xxx} &= 0 \\ w_t &= 0 \end{aligned}\right\}, \quad \text{for } (x, t) \in \mathbf{R} \times \mathbf{R}^+, \tag{31}$$

with

$$u(x, 0) = w(x, 0) = g(x), \quad \text{for } x \in \mathbf{R}.$$

Let $h = u - w$. Then, h is a solution of the initial-value problem

$$\begin{aligned} h_t &= -uu_x - u_{xxx}, \quad \text{for } (x, t) \in \mathbf{R} \times \mathbf{R}^+, \\ h(x, 0) &= 0, \quad \text{for } x \in \mathbf{R}. \end{aligned} \tag{32}$$

If (32) is integrated over the temporal interval $[0, t]$, there appears the formula,

$$h(x, t) = -\int_0^t [u(x, \tau)u_x(x, \tau) + u_{xxx}(x, \tau)]\, d\tau.$$

It then follows that for any nonnegative integer k

$$\|h(\cdot, t)\|_k \leq \int_0^t (\|u(\cdot, \tau)u_x(\cdot, \tau)\|_k + \|u_{xxx}(\cdot, \tau)\|_k)\, d\tau. \tag{33}$$

This inequality is the key to the proof of the following result.

THEOREM 2. *Let $g \in H^m$ where $m \geq 3$. Let η^ε and σ^ε be the solutions of* (26a) *and* (28) *corresponding to the initial data $\varepsilon g(\varepsilon^{1/2}x)$, where $\varepsilon > 0$.* Then *there are order-one constants C_j such that for $0 < \varepsilon \leq 1$ and $0 \leq t$,*

$$\|\eta^\varepsilon_{(j)}(\cdot, t) - \sigma^\varepsilon_{(j)}(\cdot, t)\| \leq C_j \varepsilon^{j/2+9/4} t, \tag{34}$$

for $0 \leq j \leq m - 3$.

PROOF. Fix ε in the range $(0, 1]$ and perform the change of variables indicated in (30). Let u, w and h be as defined in (30) and just below (31). For any $T > 0$, $\eta^\varepsilon \in C(0, T; H^m)$. Hence, for any $T > 0$, $u \in C(0, T; H^m)$. From (29) it is plain that $\sigma^\varepsilon \in C(0, T; H^m)$ for any $T > 0$. Hence $h \in C(0, T; H^m)$ for any $T > 0$. Thus, provided $k \leq m - 3$, the right-hand side of (33) is finite, and the formal calculations leading to this inequality are easily justified.

But, (33) immediately implies the estimate,

$$\|h(\cdot, t)\|_k \leq t\{\|uu_x\|_{C(0,t;H^k)} + \|u_{xxx}\|_{C(0,t;H^k)}\},$$

for $0 \leq t$ and $0 \leq k \leq m - 3$. Elementary considerations involving the Sobolev norms then yield

$$\|h(\cdot, t)\|_k \leq M_k t\{\|u\|^2_{C(0,t;H^{k+1})} + \|u\|_{C(0,t;H^{k+3})}\}, \tag{35}$$

holding for all $t \geq 0$ and $0 \leq k \leq m - 3$, where the constants M_k depend only on k. Proposition 4, which applies equally to the KdV equation (A) and to the KdV equation (31) written in coordinates moving with speed one, allows the right-hand side of (35) to be bounded in terms of the data g as follows:

$$\begin{aligned}\|h(\cdot, t)\|_k &\leq M_k t\{q_0(\|g\|)^2 + q_{k+1}(\|g\|_{k+1})^2 + [q_0(\|g\|)^2 + q_{k+3}(\|g\|_{k+3})^2]^{1/2}\} \\ &= C_k t,\end{aligned}$$

for $0 \leq k \leq m - 3$, where the q_k are defined in (18). According to the earlier definition, C_k is an order-one quantity. It follows that

$$\|h_{(k)}(\cdot, t)\| \leq C_k t, \tag{36}$$

for $0 \leq k \leq m - 3$. It remains simply to express η^ε and σ^ε in terms of u and w, so inverting the change of variables (30). This gives

$$\eta^\varepsilon(x, t) = \varepsilon u(\varepsilon^{1/2}(x - t), \varepsilon^{3/2}t), \tag{37}$$

and similarly for σ^ε in terms of w. It then follows from (36) and (37) that, for $0 \leq k \leq m - 3$,

$$\|\eta^\varepsilon_{(k)}(\cdot, t) - \sigma^\varepsilon_{(k)}(\cdot, t)\| \leq C_k \varepsilon^{k/2+9/4} t,$$

where C_k is an order-one constant. This is just what we set out to prove.

COROLLARY 2. *Let g, η^ε and σ^ε be as in the statement of Theorem 2 and suppose $m \geq 4$. Then, there are order-one constants B_k such that, for all $t \geq 0$,*

$$\sup_{x \in \mathbf{R}} |\eta^\varepsilon_{(k)}(x, t) - \sigma^\varepsilon_{(k)}(x, t)| \leq B_k t \varepsilon^{(k+5)/2} \tag{38}$$

provided $0 \leq k \leq m - 4$.

PROOF. First note that if $f \in H^1$, then according to (24),

$$|f(x)| \leq \|f\|^{1/2} \|f'\|^{1/2}, \tag{39}$$

for all $x \in \mathbf{R}$. Apply this with $f = \eta^\varepsilon_{(k)} - \sigma^\varepsilon_{(k)}$ and then use (34) to bound the resulting right-hand side.

It is instructive to consider the case $k = 0$ in (38). The result then reads

$$\sup_{x \in \mathbf{R}} |\eta^\varepsilon(x, t) - \sigma^\varepsilon(x, t)| \leq B_0 \varepsilon^2 (\varepsilon^{1/2} t). \tag{40}$$

Recall that both η^ε and σ^ε are of order ε in maximum magnitude, and that ε^2 is the neglected order in the scaling appearing in (26a) and (28). The inequality in (40) states therefore that η and σ agree to the neglected order, at least over the time interval $[0, \varepsilon^{-1/2}]$. In other words, if interest lies in the evolution of the wave profile over a time $t \leq \varepsilon^{-1/2}$, then one might as well employ (28) (that is, translate the initial wave profile at speed one without change of shape) rather than one of the more complicated models in (26). Over longer time scales, this may not be true. Indeed, examples are presented in §5 showing clearly that the estimates in (34) and (38) cannot be improved in general. So nonlinear and dispersive effects are increasingly felt for times in the range $t > \varepsilon^{-1/2}$. When t is of the order $\varepsilon^{-3/2}$, the difference between η and σ is of order ε, the basic size of each term separately. The models in (26) and the simple model (28) have thus diverged in their predictions by this time. This accords with the formalism explained in §2.

Scaling and longer term comparisons. We now show that solutions η^ε of the initial-value problem (26a) scale with respect to ε in the way that is expected and used in formal studies of this equation.

THEOREM 3. *Let $g \in H^m$ where $m \geq 1$ and let $\varepsilon > 0$. Let η^ε be the solution of* (26a) *corresponding to the intitial data $\varepsilon g(\varepsilon^{1/2}x)$. Then there are order-one constants D_j such that for all $t \geq 0$,*

$$\|\eta^\varepsilon_{(j)}(\cdot, t)\| \leq D_j \varepsilon^{j/2+3/4}, \tag{41}$$

whenever $0 \leq j \leq m$.

PROOF. This is a straightforward consequence of Proposition 4. First let u be defined by (30) as before. Then u satisfies the initial-value problem (31). Proposition 4 thus implies that for all $t \geq 0$,

$$\|u_{(j)}(\cdot, t)\| \leq q_j(\|g\|_j) = D_j, \tag{42}$$

for $0 \leq j \leq m$, as in (18). The inequality (41) now follows upon using the expression (37) for η^ε in terms of u in (42).

COROLLARY 3. *Let g, m, ε, and η^ε be as in Theorem 3. Then, there exist order-one constants F_j such that, for all $t \geq 0$,*

$$\sup_{x \in \mathbf{R}} |\eta^\varepsilon_{(j)}(x, t)| \leq F_j \varepsilon^{1+j/2} \tag{43}$$

for $0 \leq j \leq m - 1$.

PROOF. This follows immediately from (39) and (41) if we define $F_j = (D_j D_{j+1})^{1/2}$.

Theorem 3 and its corollary show that solutions of (A) corresponding to initial data of the form $\varepsilon g(\varepsilon^{1/2}x)$ scale with respect to the parameter ε just as the data does. In particular, (43) yields

$$|\eta^\varepsilon| = O(\varepsilon), \quad |\eta^\varepsilon_x| = O(\varepsilon^{3/2}), \quad \text{and} \quad |\eta^\varepsilon_{xxx}| = O(\varepsilon^{5/2}), \tag{44}$$

as $\varepsilon \downarrow 0$, so that $|\eta^\varepsilon \eta^\varepsilon_x| = O(\varepsilon^{5/2})$, as $\varepsilon \downarrow 0$. Solving equation (A) for η^ε_t, and taking account of (44), it follows that $|\eta^\varepsilon_t| = O(\varepsilon^{3/2})$, as $\varepsilon \downarrow 0$. These results emphasize again that the nonlinear and dispersive terms in the KdV equation represent small corrections to the basic wave equation $\eta_t + \eta_x = 0$. We turn now to the more exacting and technical task of proving Theorem 1.

PROOF (OF THEOREM 1). Suppose at the outset that $g \in H^\infty$, that is, g is a C^∞ function all of whose derivatives lie in L_2. As in (30), define

$$\left.\begin{aligned} u(x, t) &= \varepsilon^{-1}\eta^\varepsilon(\varepsilon^{-1/2}x + \varepsilon^{-3/2}t, \varepsilon^{-3/2}t), \\ v(x, t) &= \varepsilon^{-1}\zeta^\varepsilon(\varepsilon^{-1/2}x + \varepsilon^{-3/2}t, \varepsilon^{-3/2}t). \end{aligned}\right\} \tag{45}$$

A short calculation shows that u and v satisfy the initial-value problems

$$\left.\begin{aligned} u_t + uu_x + u_{xxx} &= 0, \\ v_t + vv_x + v_{xxx} - \varepsilon v_{xxt} &= 0, \\ u(x, 0) = v(x, 0) &= g(x). \end{aligned}\right\} \tag{46}$$

Let $w = v - u$, so that $v = w + u$. Then w is seen to satisfy

$$w_t + ww_x + w_{xxx} - \varepsilon w_{xxt} = \varepsilon u_{xxt} - (uw)_x, \qquad w(x, 0) \equiv 0. \tag{47}$$

We continue to use the notation $f_{(j)}$ to denote $\partial_x^j f$.

The task to be accomplished now is the estimation of $\|w_{(j)}\|$, for $j = 0, 1, 2, \ldots$. Because $g \in H^\infty$, both η^ε and ζ^ε, and hence u and v, are C^∞ functions of x and t all of whose derivatives are in L_2 with respect to the spatial variable. This fact justifies the following computations.

Multiply (47) by $w_{(2j)}$ and integrate the result over $\mathbf{R}$ and over $[0, t]$. After a few integrations by parts, and taking account of the fact that $w(x, 0) \equiv 0$, the following identity is seen to hold, as in Bona and Smith [**1975**, equation 7.9],

$$\begin{aligned} &\int_{-\infty}^{\infty} \{w_{(j)}^2(x, t) + \varepsilon w_{(j+1)}^2(x, t)\}\, dx \\ &\qquad = 2\int_0^t \int_{-\infty}^{\infty} w_{(j)} \{\varepsilon u_{\tau,\,(j+2)} - (uw + \tfrac{1}{2} w^2)_{(j+1)}\}\, dx\, d\tau, \end{aligned} \tag{48}$$

for $j = 0, 1, 2, \ldots$. This relation will be used repeatedly.

First, for $j = 0$, there appears, after two more integrations by parts,

$$\int_{-\infty}^{\infty} (w^2 + \varepsilon w_x^2)\, dx = 2\int_0^t \int_{-\infty}^{\infty} \varepsilon(wu_{xx\tau})\, dx\, d\tau - \int_0^t \int_{-\infty}^{\infty} (u_x w^2)\, dx\, d\tau. \tag{49}$$

From this the following inequality is derived:

$$\|w\|^2 \le \int_0^t (2\varepsilon\|w\|\ \|u_{xx\tau}\| + \|u_x\|_\infty\ \|w\|^2)\, d\tau,$$

where, as before, $\|\ \|$ denotes the norm in L_2 and, throughout this proof, $\|\ \|_\infty$ denotes the norm in L_∞. By a variant of Gronwall's lemma it follows that

$$\|w\| \le \varepsilon(C_2/C_1)\,(e^{C_1 t} - 1) \le \varepsilon t C_2 e^{C_1} = M_0 \varepsilon t, \tag{50}$$

where C_1 and C_2 are bounds for $\frac{1}{2}\|u_x\|_\infty$ and $\|u_{xxt}\|$ respectively and t is restricted to the range $[0, 1]$. Using the results of Proposition 4 in §3, and using the differential equation, the following estimate can be made:

$$\begin{aligned} \sup_{t \ge 0} \|u_{xxt}\| &= \sup_{t \ge 0} \|\partial_x^2(-u_{xxx} - uu_x)\| \\ &\le \sup_{t \ge 0} \{\|u_{(5)}\| + \|u\|_\infty \|u_{(3)}\| + 3\|u_x\|_\infty \|u_{xx}\|\}. \end{aligned}$$

As remarked in (24), $\|f\|_\infty^2 \leq \|f\| \; \|f_x\|$. Hence, in the notation of (18), with the q_i referred to g of course, C_2 may be defined by

$$\sup_{t\geq 0} \|u_{xxt}\| \leq q_5 + (q_0q_1)^{1/2}q_3 + 3(q_1q_2)^{1/2}q_2 = C_2. \tag{51}$$

It is even easier to estimate a value for C_1;

$$\sup_{t\geq 0} \|u_x\|_\infty \leq \sup_{t\geq 0} (\|u_x\| \; \|u_{xx}\|)^{1/2} \leq (q_1q_2)^{1/2} = C_1. \tag{52}$$

Since both C_1 and C_2 are order-one quantities, so too is M_0. This result is already interesting, as will appear shortly. Further bounds lead to a better overall picture, and to L_∞ estimates.

Integrating (48) by parts, in the case $j = 1$, the following relation is derived.

$$\begin{aligned}\int_{-\infty}^{\infty} (w_x^2 + \varepsilon w_{xx}^2)\, dx &= 2\varepsilon \int_0^t \int_{-\infty}^{\infty} (w_x u_{xxx\tau})\, dx\, d\tau \\ &\quad - \int_0^t \int_{-\infty}^{\infty} (w_x^3 + 3u_x w_x^2 + 2u_{xx} w w_x)\, dx\, d\tau.\end{aligned} \tag{53}$$

The integrand on the right-hand side of the latter equation may be bounded above by

$$\begin{aligned}&2\varepsilon \|u_{xxx\tau}\| \; \|w_x\| + \|w_x + 3u_x\|_\infty \|w_x\|^2 + 2\|u_{xx}\|_\infty \|w\| \; \|w_x\| \\ &\qquad \leq (4 \|u_x\|_\infty + \|v_x\|_\infty)\|w_x\|^2 + \varepsilon(2 \|u_{xxx\tau}\| + 2(\|u_{xx}\| \; \|u_{xxx}\|)^{1/2}M_0\tau)\|w_x\|.\end{aligned}$$

Now using the equation satisfied by u and the results of Proposition 4, we may derive the following estimate, valid for $0 \leq \tau \leq 1$.

$$\begin{aligned}&\sup_{0\leq\tau\leq 1} \{2\|u_{xxx\tau}\| + 2(\|u_{xx}\| \; \|u_{xxx}\|)^{1/2}M_0\tau\} \\ &\quad \leq \sup_{0\leq\tau\leq 1} \{2\|\partial_x^3(-uu_x - u_{xxx})\| + 2(q_2q_3)^{1/2}M_0\tau\} \\ &\quad \leq 2q_6 + 2(q_0q_1)^{1/2}q_4 + 8(q_1q_2)^{1/2}q_3 + 6(q_2q_3)^{1/2}q_2 + 2(q_2q_3)^{1/2}M_0 = C_4.\end{aligned}$$

Since M_0 is an order-one quantity, C_4 is an order-one quantity. Also, using Proposition 4 as before and relying on Proposition 5 and its corollary for the bounding of $\|v_x\|_\infty$, there appears

$$\begin{aligned}\sup_{0\leq\tau\leq 1} \{4\|u_x\|_\infty + \|v_x\|_\infty\} &\leq 4(q_1q_2)^{1/2} + \sup_{0\leq\tau\leq 1} (\|v_x\| \, \|v_{xx}\|)^{1/2} \\ &\leq 4(q_1q_2)^{1/2} + (qa_0(\|g\|_{H^3}))^{1/2} = C_3.\end{aligned}$$

This latter quantity is order-one also. Thus at least over the interval $0 \leq t \leq 1$, the inequality

$$\|w_x\|^2 \leq \int_0^t (C_3\|w_x\|^2 + \varepsilon C_4\|w_x\|)\, d\tau$$

is implied. It follows immediately that

$$\|w_x\| \leq \varepsilon(C_4/C_3)(e^{C_3 t} - 1) \leq \varepsilon t C_4 e^{C_3} = \varepsilon t M_1, \tag{54}$$

for $0 \leq t \leq 1$. Since C_3 and C_4 are order-one quantities, M_1 is also an order-one quantity.

For the case of a general j, the procedure for obtaining a bound on $\|w_{(j)}\|$ is similar to that followed above in the cases $j = 0$ and $j = 1$. Suppose inductively that for $j < k$, where $k > 1$, there have been established bounds of the form,

$$\|w_{(j)}\| \leq \varepsilon(C_{2j}/C_{2j-1})\,(e^{C_{2j-1}t} - 1) \leq \varepsilon t C_{2j} e^{C_{2j-1}} = \varepsilon t M_j, \tag{55}$$

for $t \in [0, 1]$, where C_{2j-1} and C_{2j} are both order-one quantities. The goal now is to establish the same type of bound for $j = k$. To this end, consider the equation (48) in the case $j = k$. Using Leibnitz' rule, (48) may be written as

$$\int_{-\infty}^{\infty} (w_{(k)}^2 + \varepsilon w_{(k+1)}^2)\, dx = 2\int_0^t \int_{-\infty}^{\infty} \varepsilon w_{(k)} u_{\tau,(k+2)}\, dx\, d\tau$$
$$- 2\int_0^t \int_{-\infty}^{\infty} \sum_{j=0}^{k+1} \alpha_j \{\tfrac{1}{2} w_{(k+1-j)}\, w_{(j)} + w_{(k+1-j)} u_{(j)}\} w_{(k)}\, dx\, d\tau.$$

Here the α_j are the constants that appear in Leibnitz' rule. Separating the top-order derivatives and estimating the rest directly, we have

$$\begin{aligned}\int_{-\infty}^{\infty} (w_{(k)}^2 + \varepsilon w_{(k+1)}^2)\, dx \leq 2\varepsilon \int_0^t \|w_{(k)}\|\; \|u_{\tau,(k+2)}\|\, d\tau \\ - 2\int_0^t \int_{-\infty}^{\infty} (w w_{(k)} w_{(k+1)} + u w_{(k)} w_{(k+1)})\, dx\, d\tau \\ + \int_0^t \int_{-\infty}^{\infty} \sum_{j=1}^{k} \alpha_j |w_{(k)} w_{(k+1-j)} w_{(j)}|\, dx\, d\tau \\ + 2\int_0^t \int_{-\infty}^{\infty} \sum_{j=1}^{k+1} \alpha_j |w_{(k)} u_{(j)} w_{(k+1-j)}|\, dx\, d\tau.\end{aligned} \tag{56}$$

The induction hypothesis (55) assures us that on the time interval [0, 1] $\|w_{(j)}\|$ and $\|w_{(i)}\|_\infty$ are bounded by order-one constants if $0 \leq j \leq k - 1$ and $0 \leq i \leq k - 2$. By Proposition 4, $\|u_{(j)}\|$, $\|u_{(j)}\|_\infty$ and, using the differential equation, $\|u_{\tau,(k+2)}\|$ are all bounded by order-one constants, for $0 \leq j \leq k + 1$, independently of $\tau \geq 0$. Also, by Proposition 5 and its corollary, for $\tau \in [0, 1]$, $\|w\|$, $\|w_x\|$ and $\|w_{xx}\|$ are bounded by order-one quantities. Because of (24), $\|w_x\|_\infty$ and $\|w\|_\infty$ are similarly bounded. The second integral on the right-hand side of (56) is equal to

$$\frac{1}{2}\int_0^t \int_{-\infty}^{\infty} (w_x + u_x) w_{(k)}^2\, dx\, d\tau \leq \frac{1}{2} C \int_0^t \|w_{(k)}\|^2\, d\tau,$$

where, provided $0 \leq t \leq 1$, C may be inferred to be an order-one quantity. The third and fourth integrals on the right side of (56) are both estimated similarly:

$$\begin{aligned}\int_0^t \int_{-\infty}^{\infty} \sum_{j=1}^{k} \alpha_j |w_{(k)} w_{(k+1-j)} w_{(j)}| \, dx \, d\tau \\ \leq C \int_0^t \|w_x\|_\infty \|w_{(k)}\|^2 \, d\tau + C\varepsilon \int_0^t \left[\|w_{(k)}\| \sum_{j=2}^{k-1} M_j M_{k+1-j} \tau^2 \right] d\tau \\ \leq C \int_0^t \|w_{(k)}\|^2 \, d\tau + \varepsilon C \int_0^t \|w_{(k)}\| \, d\tau,\end{aligned}$$

valid for $0 \leq t \leq 1$ at least. The constants appearing in this inequality are order-one. The same estimate holds for the fourth term on the right side of (56). Hence in sum, for $0 \leq t \leq 1$,

$$\int_{-\infty}^{\infty} (w_{(k)}^2 + \varepsilon w_{(k+1)}^2) \, dx \leq C_{2k-1} \int_0^t \|w_{(k)}\|^2 \, d\tau + \varepsilon C_{2k} \int_0^t \|w_{(k)}\| \, d\tau,$$

where C_{2k-1} and C_{2k} are order-one quantities. The result (55) for $j = k$ now follows and the inductive step is completed.

It is worth noting that the constants C_{2k-1} and C_{2k} depend only on $\|g_{(j)}\|$ for $0 \leq j \leq k + 5$. In fact, this consideration is dominated by the term $u_{t,(k+2)}$, appearing on the right-hand side of (56) which, by use of the differential equation and Proposition 4, is bounded in terms of $q_0, \ldots, q_{k+5}$, and so in terms of $\|g\|, \ldots, \|g_{k+5}\|$.

This last remark, coupled with the continuous-dependence results in Propositions 1 and 2, allows the weakening of our initial assumption that $g \in H^\infty$. By approximating $g \in H^{k+5}$ by a sequence $\{g_n\}_{n=1}^{\infty} \subset H^\infty$, we may conclude that

$$\|w_{(j)}\| \leq \varepsilon t M_j, \tag{57}$$

for $0 \leq t \leq 1$ and $0 \leq j \leq k$, where $w = u - v$ and u and v are as in (46) with initial data g.

Now it is only necessary to translate the result (57), a relation concerning u and v, into a result relating η^ε and ζ^ε. This simply involves inverting the transformation involved in (45). It appears immediately that,

$$\eta(x, t) = \varepsilon u(\varepsilon^{1/2}(x - t), \varepsilon^{3/2} t), \tag{58}$$

and similarly for ζ and v.

Thus, supposing now that $g \in H^{k+5}$ and that $j \leq k$, we deduce that

$$\begin{aligned}
&\|\eta^\varepsilon_{(j)}(\cdot, t) - \zeta^\varepsilon_{(j)}(\cdot, t)\|^2 \\
&\quad = \int_{-\infty}^{\infty} \{\varepsilon^{j/2+1}[u_{(j)}(\varepsilon^{1/2}(x-t), \varepsilon^{3/2}t) - v_{(j)}(\varepsilon^{1/2}(x-t), \varepsilon^{3/2}t)]\}^2\,dx \\
&\quad = \varepsilon^{j+2}\int_{-\infty}^{\infty} \{u_{(j)}(z, \varepsilon^{3/2}t) - v_{(j)}(z, \varepsilon^{3/2}t)\}^2\,\varepsilon^{-1/2}\,dz \\
&\quad = \varepsilon^{j+3/2}\|w_{(j)}(\cdot, \varepsilon^{3/2}t)\|^2 \le \varepsilon^{j+7/2}M_j^2(\varepsilon^{3/2}t)^2,
\end{aligned}$$

and this is valid as long as $0 \le \varepsilon^{3/2}t \le 1$. Hence if $0 \le t \le \varepsilon^{-3/2}$,

$$\|\eta^\varepsilon_{(j)}(\cdot, t) - \zeta^\varepsilon_{(j)}(\cdot, t)\| \le \varepsilon^{j/2+7/4}M_j(\varepsilon^{3/2}t), \tag{59}$$

where M_j is an order-one quantity. This finishes the proof of the theorem.

COROLLARY 4. *Assume the hypotheses and notation of Theorem* 1. *Then there is an* $\varepsilon_0 > 0$ *and order-one constants* N_j *such that for* $0 < \varepsilon \le \varepsilon_0$ *and for* $0 \le t \le \varepsilon^{-3/2}$,

$$\sup_{x\in\mathbf{R}} |\eta^\varepsilon_{(j)}(x, t) - \zeta^\varepsilon_{(j)}(x, t)| \le N_j\varepsilon^{2+j/2}(\varepsilon^{3/2}t), \tag{60}$$

for $0 \le j \le k - 1$.

PROOF. This follows instantly from (27) and the inequality (24) if N_j is defined to be $(M_jM_{j+1})^{1/2}$.

COROLLARY 5. *Let* $g \in H^{k+5}$, *where* $k \ge 0$. *Let* $\varepsilon > 0$ *and let* ζ^ε *be the solution of* (26b) *corresponding to the initial data* $\varepsilon g(\varepsilon^{1/2}x)$. *Let* σ^ε *be the solution of* (28) *corresponding to the same initial data. Then, there is an* $\varepsilon_0 > 0$ *and order-one constants* E_j, $0 \le j \le k$, *such that for* $0 < \varepsilon \le \varepsilon_0$ *and* $0 \le t \le \varepsilon^{-3/2}$,

$$\|\zeta^\varepsilon_{(j)}(\cdot, t) - \sigma^\varepsilon_{(j)}(\cdot, t)\| \le E_jt\varepsilon^{j/2+9/4}, \tag{61}$$

for $0 \le j \le k$, *and, if* $k \ge 1$,

$$\sup_{x\in\mathbf{R}} |\zeta^\varepsilon_{(j)}(x, t) - \sigma^\varepsilon_{(j)}(x, t)| \le E_jt\varepsilon^{(j+5)/2}, \tag{62}$$

for $0 \le j \le k - 1$.

COROLLARY 6. *Assume the hypotheses and notation of Theorem* 1. *Then there is an* $\varepsilon_0 > 0$ *and order-one constants* F_j, $0 \le j \le k$, *such that for* $0 < \varepsilon \le \varepsilon_0$ *and* $0 \le t \le \varepsilon^{-3/2}$,

$$\|\zeta^\varepsilon_{(j)}(\cdot, t)\| \le F_j\varepsilon^{j/2+3/4}, \tag{63}$$

for $0 \le j \le k$, *and, if* $k \ge 1$,

$$\sup_{x\in\mathbf{R}} |\zeta^\varepsilon_{(j)}(\cdot, t)| \le F_j\varepsilon^{j/2+1}. \tag{64}$$

REMARKS. Corollaries 5 and 6, which are concerned with the model equation (13), are immediate consequences of Theorems 2 and 3, and their corollaries, once Theorem 1 is established. For example, to obtain (63), proceed as follows. For $\varepsilon \leq \varepsilon_0$, we have from (41) and (27),

$$\begin{aligned}\|\zeta^\varepsilon_{(j)}(\cdot, t)\| &\leq \|\zeta^\varepsilon_{(j)}(\cdot, t) - \eta^\varepsilon_{(j)}(\cdot, t)\| + \|\eta^\varepsilon_{(j)}(\cdot, t)\| \\ &\leq M_j\varepsilon^{j/2+7/4}(\varepsilon^{3/2}t) + D_j\varepsilon^{j/2+3/4},\end{aligned}$$

at least for $0 \leq t \leq \varepsilon^{-3/2}$. Take $E_j = D_j + M_j$, which is an order-one quantity since D_j and M_j both have this property. The result then follows.

Corollaries 5 and 6 can be improved somewhat in the sense that weaker hypotheses concerning g suffice for the stated conclusions. The conclusions themselves cannot be improved in general, as is shown in the next section.

An interesting case in Corollary 4 is $j = 0$, when (60) yields the estimate

$$\sup_{x\in\mathbf{R}} |\eta^\varepsilon(x, t) - \zeta^\varepsilon(x, t)| \leq N_0\varepsilon^{7/2}t, \tag{65}$$

holding for $0 \leq t \leq \varepsilon^{-3/2}$. Inequality (65) is exactly that obtained in (9) when comparing solutions of the two linearized model equations. Note in particular that at $t = t_1 = \varepsilon^{-3/2}$,

$$\sup_{x\in\mathbf{R}} |\eta^\varepsilon(x, t_1) - \zeta^\varepsilon(x, t_1)| \leq N_0\varepsilon^2,$$

showing explicitly that the two solutions are the same to the formal order of accuracy ε^2 achieved by either model. Recall from §2 that t_1 is the temporal scale over which it is expected that significant modifications of the wave profile will occur, due to the accumulation of nonlinear and dispersive effects.

Whilst t_1 is indeed a long time scale for the present considerations, it is nevertheless expected that the estimates (27) and (60) will continue to hold for t in the range $[\varepsilon^{-3/2}, \varepsilon^{-5/2}]$. So far this kind of result has proved elusive to analytical methods for the nonlinear case. Such results are easily established if the nonlinear terms in the two model equations are neglected, as in the sample calculation in §2. Numerically obtained evidence supports the validity of the estimates (27) and (60) on the longer time scale $t_2 = \varepsilon^{-5/2}$. This and some other results are presented in the next section.

5. Further comparisons of the model equations. Additional amplification and interpretation of the theory developed in §4 is provided here. This is accomplished principally by way of some specific examples. In the present discussion, we retain the notation and scaling appropriate to (26) in §4. In particular, ε continues to be used as the measure of the small amplitude of the initial wave profiles. According to the presentation in §2, $\varepsilon^{-1/2}$ is therefore a measure of the length scale characterizing these profiles.

It is worth reiterating that $\varepsilon^{-3/2}$ is the smallest time scale over which the nonlinear and dispersive terms in either model equation, written in the forms (26), act effectively to alter the shape of the initial wave profile. On time scales significantly smaller than $\varepsilon^{-3/2}$, Theorem 2 and its corollary show that the dominant effect experienced by the initial profile with the passage of time is induced by the top-order portion, $\sigma_t + \sigma_x = 0$, of the equations, and is therefore simply translational. As pointed out in (65), solutions of the two model equations corresponding to the same initial data, as in (26), differ by order at most ε^2 over the entire temporal interval $[0, \varepsilon^{-3/2}]$. For the models written in the form (26), both η^ε and ζ^ε are quantities of order ε. Thus their difference is seen to be a factor of ε smaller than the functions themselves. This is exactly the formal order that the terms, neglected in the derivation of these equations, would be expected to contribute over this time scale. Put another way, both η^ε and ζ^ε have formal resolution of order ε over the time interval $[0, \varepsilon^{-3/2}]$. Hence our results show unambiguously that on this time scale the solutions of the two equations are the *same* to the formal order of approximation afforded by either equation.

Two issues arise naturally upon further consideration of Theorems 1, 2, and 3. First, the sharpness of the results deserves consideration. Second, it seems probable that the estimates expressed in (27) and (60) are valid on the longer time scale $t_2 = \varepsilon^{-5/2}$, as asserted in our conjecture at the end of §2. For the linearized model equations it may be readily demonstrated that (27) and (60) are sharply valid for t in the range $[0, \varepsilon^{-5/2}]$. Moreover, as pointed out in §2, formal arguments indicate that (27) and (60) are sharply valid for t in $[0, \varepsilon^{-5/2}]$ for the nonlinear problem as well. Both of these issues are addressed below, though neither has been conclusively resolved.

To fix ideas in the present context, let us agree to call an inequality of the general form $\|\theta^\varepsilon_{(j)}\| \le C_j\varepsilon^r$, where C_j is an order-one constant, *sharp* if there is another order-one constant $\tilde{C}_j$ such that

$$\tilde{C}_j\varepsilon^r < \|\theta^\varepsilon_{(j)}\|.$$

Here θ^ε is a solution, or difference of two solutions, of equations (26) or (30), r is some fixed real number and $\| \ \|$ denotes some norm, not necessarily that of L_2.

One way to test the sharpness of the results obtained thus far is by resort to examples. A particularly simple class of examples is obtained by choosing the initial data $\varepsilon g(\varepsilon^{1/2}x)$ to be a small-amplitude solitary-wave solution of (26a), namely

$$g(x) = 3\,\mathrm{sech}^2(\tfrac{1}{2}x). \tag{66}$$

This function is an element of H^∞, so it certainly satisfies the hypotheses in Theorems 1, 2, and 3 for any value of k or m. For $\varepsilon > 0$ given, the exact solution of (26a) for this choice of g is

$$S_\varepsilon(x, t) = 3\varepsilon \operatorname{sech}^2\left(\frac{\varepsilon^{1/2}}{2}[x - (1 + \varepsilon)t]\right). \tag{67}$$

For this special similarity solution, it is obvious that the norms appearing in Theorem 3 and Corollary 3 do not vary with t. Hence, for all $j \geq 0$,

$$\|\partial_x^j S_\varepsilon(\cdot, t)\| = \|\partial_x^j[\varepsilon g(\varepsilon^{1/2}\,\cdot)]\| = d_j \varepsilon^{j/2+3/4},$$

where d_j is a positive constant determined by g. Similarly, for all $j \geq 0$,

$$\sup_{x\in\mathbb{R}} |\partial_x^j S_\varepsilon(x, t)| = \sup_{x\in\mathbb{R}} |\partial_x^j[\varepsilon g(\varepsilon^{1/2}x)]| = f_j \varepsilon^{j/2+1},$$

where again the f_j are positive constants determined by g. These simple calculations show that, in general, the bounds obtained in (41) and (43) cannot be improved. We have not proved a theorem of genericity, that all, or, more likely, nearly all sufficiently smooth choices of g result in solutions that sharply obey (41) and (43).

The examples in (67) may also be used to show that, in general, Theorem 2 and Corollary 2 are sharp. For the exact solution of (28) subject to the initial condition $\varepsilon g(\varepsilon^{1/2}x)$ is

$$\sigma^\varepsilon(x, t) = 3\varepsilon \operatorname{sech}^2((\tfrac{1}{2}\varepsilon^{1/2})\,[x - t]). \tag{68}$$

The functions in (67) and (68) are identical except that they propagate at slightly different speeds. Because of this, they draw apart and the norm of their difference grows. A straightforward calculation shows that (34) and (38) are sharply verified in this particular instance. Thus both Theorem 2 and Corollary 2 are sharp in general. Again, a generic result along these lines has eluded us.

Finally, we try our example (67) in the context of the initial-value problems for (A) and (B) posed in (26). There is not available a closed-form solution of (26b) corresponding to the initial data $\varepsilon g(\varepsilon^{1/2}x)$, with g is as in (66). Consequently, we have had to rely upon a numerical integration of (26b). The numerical scheme used for these experiments has been described in Bona, Pritchard and Scott [**1980**]. It results essentially from discretizing an integral equation that is equivalent to (26b). The resulting scheme is quite efficient, and has been proved to be unconditionally stable and fourth-order accurate in both the spatial and temporal mesh size. Extensive convergence tests for this scheme have been carried out using the solitary-wave,

$$s_C(x, t) = 3C \operatorname{sech}^2\left\{\left[\frac{1}{2}\frac{C}{1 + C}\right]^{1/2}[x - (C + 1)t]\right\}. \tag{69}$$

For any $C > 0$, this is an exact solution of (26b), and the convergence, or lack thereof, of a numerical scheme may be conveniently tested on it.

In Tables 1 and 2, we show the errors arising when our scheme was used to integrate (26b) (in the variables appurtenant to (10b)) with the initial condition

$$h(x) = 2C\,\mathrm{sech}^2\left\{\left[\frac{3C}{2(1+C)}\right]^{1/2} x\right\}.$$

The errors recorded are with regard to the discrete L_2 norm of the difference between the exact solution

$$U(x, t) = 2C\,\mathrm{sech}^2\left\{\left[\frac{3C}{2(1+C)}\right]^{1/2}[x - (C+1)t]\right\}, \tag{70}$$

and that predicted by our scheme. If u_{ij} is the value given by the numerical scheme at the point $(i\Delta x, j\Delta t)$, where Δx and Δt are, respectively, the spatial and temporal mesh size, then the error at the jth time step is defined to be

$$E_j = \{\sum_i [u_{ij} - U(i\Delta x, j\Delta t)]^2 \Delta x\}^{1/2}, \tag{71}$$

and the relative error at the jth time step is taken to be

$$e_j = \frac{E_j}{\{\sum_i U(i\Delta x, j\Delta t)^2 \Delta x\}^{1/2}}. \tag{72}$$

It is the relative errors that are tabulated. (Note that the summation in the above formulas must be truncated. Because of the rapid spatial decay of the solution in question, such a truncation can be made without sensibly affecting the approximation to the L_2 norm. In all cases, spatially-truncated values for $U_{ij}/2C$ were less than 10^{-8} in our computations.) Preliminary experiments had shown that the choice $\Delta t = \Delta x$ gave the best results (in the sense of accuracy achieved for the work expended) and for these computations we have taken $\Delta x = \Delta t = \Delta$, say.

Having determined the sort of accuracy inherent in the numerical

Δ	Time 0.640	9.600	30.720	72.320	120.320
0.16	0.147E−3	0.688E−3	0.249E−2	0.141E−1	0.400E−1
ratio	15.7	17.2	28.4	31.3	31.3
0.08	0.937E−5	0.400E−4	0.876E−4	0.451E−3	0.128E−2
ratio	15.9	16.7	26.5	32.2	32.4
0.04	0.588E−6	0.240E−5	0.330E−5	0.140E−4	0.395E−4
ratio	15.7	16.1	21.4	24.4	28.8
0.02	0.374E−7	0.149E−6	0.154E−6	0.574E−6	0.137E−5

TABLE 1. The relative error e_j induced in integrating a solitary wave (70) of amplitude $2C = 1.0$. ($\Delta x = \Delta t = \Delta$; an entry in a row labelled 'ratio' is the ratio of the numbers above and below that entry.)

Δ	Time	0.640/9.600		30.720	51.840	72.320	93.440	120.320
0.32		0.227E−3	0.334E−2	0.108E−1	0.183E−1	0.256E−1	0.332E−1	0.429E−1
ratio		16.0	15.7	15.8	15.8	15.7	15.7	15.7
0.16		0.142E−4	0.213E−3	0.685E−3	0.116E−2	0.163E−2	0.212E−2	0.274E−2
ratio		16.0	15.9	15.9	15.9	15.8	15.9	15.9
0.08		0.890E−6	0.134E−4	0.430E−4	0.731E−4	0.103E−3	0.133E−3	0.172E−3
ratio		16.0	16.0	16.0	16.0	16.1	16.0	15.9
0.04		0.557E−7	0.837E−6	0.269E−5	0.457E−5	0.641E−5	0.832E−5	0.108E−4

TABLE 2. The relative error e_j induced in integrating a solitary wave (70) of amplitude $2C = 0.1$. ($\Delta x = \Delta t = \Delta$; an entry in a row labelled 'ratio' is the ratio of the number above and below that entry.)

scheme, we then introduced the initial data $\varepsilon g(\varepsilon^{1/2}x)$, where $g(x) = 2\text{sech}^2([3/2]^{1/2}x)$ (corresponding to (66), except for the equation in the form (10a)). Equation (10b) was integrated numerically, with the just-mentioned form of initial data, for a range of values of ε. At time t, the difference between the numerically computed solution of (10b) and the exact solution of (10a) was formed. Let $M_\varepsilon(t)$ denote the maximum value of this difference, and consider the function $\log(M_\varepsilon(t))$. Because of the estimate (65), it is expected that

$$\log(M_\varepsilon(t)) \cong \text{constant} + (7/2)\log\varepsilon + \log t.$$

Attention is fixed on the particular times, $t_0 = \varepsilon^{-1/2}$, $t_1 = \varepsilon^{-3/2}$, $t_2 = \varepsilon^{-5/2}$, and $t_3 = \varepsilon^{-7/2}$ considered earlier. For these times, we expect

$$\log(M_\varepsilon(t_j)) \cong \text{constant} + (7/2 - 1/2 - j)\log\varepsilon = \text{constant} + (3 - j)\log\varepsilon.$$

An idea of how well this relation is obeyed may therefore be obtained by plotting $\log M_\varepsilon(t_j)$ versus $\log\varepsilon$ for various small values of ε. This is shown in Figure 1. The general pattern appears to confirm that (65) is sharply valid over the entire temporal range $[0, \varepsilon^{-5/2}]$, at least for this particular example. In Table 3, the results plotted in Figure 1 are tabulated, along with the numerical values of the slopes determined therefrom.

REMARK. A direct comparison of (A) and (B) could also be made by posing as initial data a small-amplitude solitary-wave solution of (B). One could then attempt to use the inverse-scattering theory to infer properties of the resulting solution of the KdV equation (A). We have investigated this possibility and have found the fruits of our labour to be suggestive, but not conclusive. For the expert, it is worth adding that we were not able to gain sufficient control of the dispersive tail to effect the needed estimates.

A final and more delicate way in which these two model equations were compared is now described.

It is an interesting fact that certain classes of solutions of the KdV

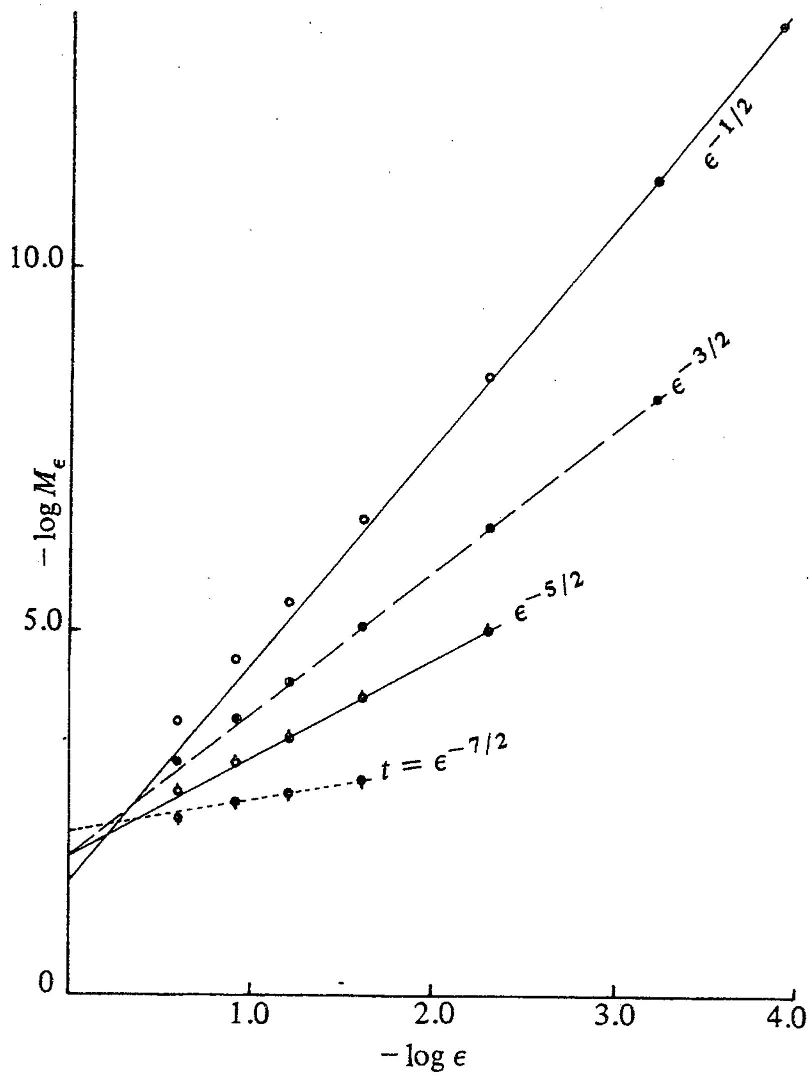

FIGURE 1. The logarithm of the maximum difference between solutions of the model equations (10a) and (10b) (with a solitary wave initial-data of amplitude) ε plotted against-log ε.

equation (A) have a rather simple structure for large values of t. Basically, the asymptotic form of these solutions comprises a widely-spaced sequence of independently propagating solitary waves followed by a small-amplitude dispersive tail. This behaviour was first noted numerically, and was then established analytically by use of the so-called inverse-scattering theory for KdV (see, for example, Miura **[1974, 1976]**).

It is natural to inquire whether or not the same property holds for

ε	$\log \varepsilon$	$\log M_\varepsilon(\varepsilon^{-1/2})$	$\frac{\Delta \log M_\varepsilon(\varepsilon^{-1/2})}{\Delta \log \varepsilon}$	$\log M_\varepsilon(\varepsilon^{-3/2})$	$\frac{\Delta \log M_\varepsilon(\varepsilon^{-3/2})}{\Delta \log \varepsilon}$	$\log M_\varepsilon(\varepsilon^{-5/2})$	$\frac{\Delta \log M_\varepsilon(\varepsilon^{-5/2})}{\Delta \log \varepsilon}$	$\log M_\varepsilon(\varepsilon^{-7/2})$	$\frac{\Delta \log M_\varepsilon(\varepsilon^{-7/2})}{\Delta \log \varepsilon}$
0.02	−3.9120	−13.2583							
			(2.979)						
0.04	−3.2189	−11.1939		−8.208					
			(2.935)		(1.948)				
0.1	−2.3026	−8.5043		−6.424		−5.020			
			(2.867)		(1.928)		(1.343)		
0.2	−1.6094	−6.5144		−5.086		−4.088		−2.96	
			(2.794)		(1.975)		(1.383)		(0.42)
0.3	−1.2040	−5.3829		−4.286		−3.528		−2.79	
			(2.729)		(1.774)		(1.208)		(0.45)
0.4	−0.9163	−4.5970		−3.775		−3.180		−2.66	
			(2.676)		(1.762)		(1.244)		(0.71)
0.55	−0.5978	−3.7447		−3.214		−2.784		−2.43	
Estimate (65)			(3.0)		(2.0)		(1.0)		(0.0)

TABLE 3. Comparison of solutions of (10a) and (10b) with initial data a solitary-wave solution of (10a) with amplitude 2ε. (Here $M_\varepsilon(t)$ represents the maximum difference at time t between the two solutions, corresponding to the solitary-wave initial data with amplitude 2ε.)

equation (B). A related property of the KdV equation, whereby solitary waves recover their exact form upon emerging from an interaction, appears to be false for (B) (see Bona *et al.* **[1980]** and the references included therein). This latter result does not preclude the possibility that solutions of (B) exhibit the aforementioned asymptotic form. The issue seems to be beyond the reach of the analytical tools currently at our disposal. Consequently, it has been investigated using the numerical integration procedure for (B) discussed earlier.

We present here an example of the kind of outcome observed in a number of numerical experiments. In this example, the initial data was taken to be

$$\zeta(x, 0) = \exp(-x^2/10). \tag{73}$$

The calculations were performed with $\Delta x = \Delta t = 0.16$, and a check was made by comparing with the same integration where $\Delta x = \Delta t = 0.08$. The procedure appeared to have converged, for all practical considerations. As always, the calculations were run on a finite spatial grid which was chosen to be large enough that values of ζ truncated by this limitation did not exceed 10^{-8}. A feature of our numerical procedure was a peak-finding subroutine which, at each discrete time step, located local maxima in the bulk of the solution and compared the wave profile near such a peak with a solitary-wave solution of (10b) having the corresponding amplitude and positioned so that its peak coincided with the local maximum in question. (This routine ignored peaks of amplitude less than 0.01.) The routine also computed the local speed of the wave based on the movement of its crest. The result of our calculation is pictured in Figure 2, which is now described in some detail. In the ensuing discussion the terms 'speed error' and 'shape error' are used, respectively, to mean the difference in speed and shape between the computed waveform and a solitary wave (70) having the same amplitude $2C$ as that of the computed wave. All differences between the computed profile and various solitary-wave solutions of (10b) are *relative* discrete L_2 norms, as defined in (72).

(1) The initial profile differed from a solitary wave of the same amplitude by 0.963.

(2) Two peaks were visible by time 6.4, one of amplitude 1.307 and another of amplitude 0.4442.

(3) By time 16.0, three peaks had resolved themselves, with respective amplitudes 1.358, 0.4633 and 0.0883.

(4) By time 35.2, four peaks were discernible with amplitudes 1.357, 0.4738, 0.07206, and 0.01072.

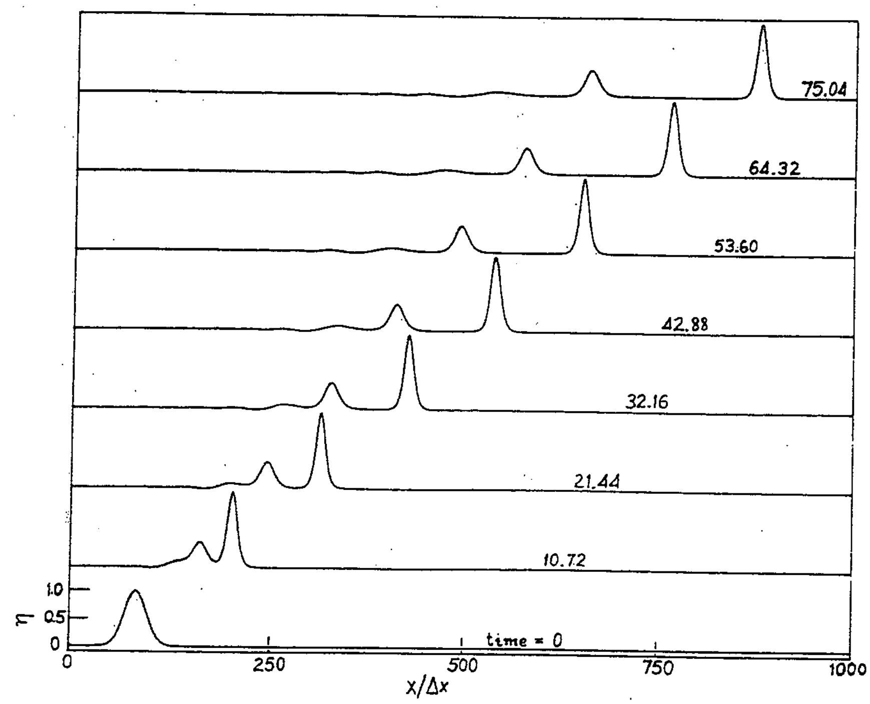

FIGURE 2. The development over time of the numerical approximation to the solution of (10b) with initial data $\exp(-x^2/10)$.

(5) A fifth peak had emerged by time 105.6. The amplitudes of the peaks at this time were 1.351, 0.4748, 0.0595, 0.0161, 0.0101.

Now the finer structure of the evolution of the various peaks is described.

Peak no 1. The first peak grew steadily in amplitude to a maximum value of 1.358 at about $t = 13$. Thereafter its amplitude decreased very slowly to a value of 1.348 at $t = 144$. This latter decrease is believed to result from numerical errors; such slow attrition was consistent with our integration of solitary-wave solutions of (10b).

At $t = 12.8$, when the wave had virtually reached its ultimate height, its speed of propagation was 1.679, which differed by 0.3E−2 from that of a solitary-wave solution of (10b) with the same amplitude. The difference between the profile near this first crest and a solitary wave of the same height was .067. At $t = 32.0$, the speed differed from that of the 'appropriate' solitary wave by only 0.35E−3 and the error in shape was 0.89E−3. At $t = 48.0$ and 144.0 the speed error was .35E−3 and .34E−3, respectively, and the shape error was .89E−3 and .89E−3, respectively.

Peak no. 2. This crest emerged fairly quickly, beginning its independent existence with an amplitude of 0.4480, and growing steadily in amplitude to a maximum of 0.4748 at about time 55. It held this latter value thenceforth. At $t = 9.6$, the speed of this wave as determined by the movement of the peak was 1.208, which is 0.015 slower than a solitary wave of its amplitude. The difference in shape was 2.476.

Both the error in speed and shape decreased rapidly as this peak became more isolated. At $t = 16.0$ and 32.0 the speed errors were .36E−2 and .96E−3, respectively, and the shape errors were .72 and .072, respectively.

When the amplitude had stabilized, at about $t = 55$, the speed error was .42E−3 whilst the shape error was .68E−2. At $t = 144.0$, the speed difference was .37E−3 and the shape difference .14E−2.

Peak no. 3. This peak first emerged at time about 16 with an amplitude of 0.08828. Its speed at this time was 0.9477 and it differed markedly from a solitary wave (shape error 5.73). The amplitude decreased monotonically taking values 0.07339, 0.06458, 0.06038 and 0.05695 at $t = 32.0$, 64.0, 96.0 and 144.0, respectively.

The speed of the crest increased monotonically, surpassing 1 by $t = 28.8$. At this time its speed differed from that of a solitary wave by 0.0345 and the shape difference was 4.13. These two measures continued their development follows.

t	speed error	shape error
48.0	0.165E−1	4.51
64.0	0.117E−1	4.66
96.0	0.708E−2	0.50
128.0	0.483E−2	0.46
144.0	0.417E−2	0.44

Peak no. 4. This peak first emerged at $t \cong 35$ with an amplitude of 0.01072. Its initial speed and shape were quite different from those of the appropriate solitary wave; the relevant errors were 0.1129 and 43.8, respectively. The amplitude of the crest initially increased to a value of 0.01421 at $t = 51.2$, to 0.01540 at $t = 64.0$ and reached a maximum of 0.01615 at $t = 96.0$. The amplitude thereupon decreased slowly, taking values 0.01586 at $t = 128.0$ and 0.01560 at $t = 144.0$.

The speed of this crest increased with time, taking a value of 0.9091 at $t = 51.2$, a value of 0.9221 at $t = 64.0$, and a value of 0.9417 at $t = 96.0$.

For t in excess of 96.0 the speed of the crest continued to increase, even though the amplitude was now decreasing, taking the values 0.9527 and 0.9564 at $t = 128.0$ and 144.0, respectively. At $t = 144.0$, this wavelet differed in shape from that of a solitary wave by 2.96.

Peak no. 5. This peak first emerged near $t = 105.6$ with an amplitude of 0.01008. It seemed to have grown from zero. Its amplitude then increased with time to a value of 0.01097 at $t = 144.0$.

The initial speed of the crest was 0.9135 at $t = 108.8$. This rose steadily to the value 0.9291 at $t = 144.0$.

As the reader will easily discern from the foregoing description, it would be optimistic to claim that the full picture of the evolution of the initial data (73) under the action of (10b) is captured by our numerical calculations. Fully realizing the need for some caution here, we nevertheless feel the following summary is an accurate description of what really happens to this initial data.

It appears that three solitary waves have emerged from this initial profile, though the third solitary wave still had considerable evolution to undergo before it could reasonably be said to have established its asymptotic form. The fourth and fifth waves were surely part of a 'dispersive tail'. They showed no signs of settling down to a uniform amplitude. More importantly their speeds were significantly less than 1. By consulting the formula (70), one determines that a solitary-wave solution of (10b) propagates with a speed exceeding 1. Moreover, the other details of these last two waves fit within the general structure observed for dispersive tails arising in the integration of (10b) (see Bona *et al.* [**1980**]).

If one takes the view that this, and other like calculations, do point to (10b) having the property that certain initial profiles resolve themselves into a sequence of solitary waves and a dispersive tail, then a host of questions present themselves. These questions will not be addressed here. But it is worth pointing out that if indeed our surmise is correct, then solutions of the two model equations corresponding to physically relevant initial data may agree qualitatively over indefinitely large time scales. This would be the case, for example, if by the time $t_2 = \varepsilon^{-3/2}$, both solutions had already sorted themselves into more or less independently propagating solitary waves. For then, by Theorem 1, these solitary waves must correspond to each other in number and be very close in amplitude. The further evolution of these solutions will then be quite simple; an observer far downstream would report of both, seeing like sequences of solitary waves trailed by a small-amplitude dispersing disturbance.

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