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THE EXISTENCE OF INTERNAL SOLITARY WAVES IN A TWO-FLUID SYSTEM NEAR THE KdV LIMIT*

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Two fluid layers of constant density lying one over the other on top of a rigid horizontal lower boundary with either a free upper surface or a rigid upper boundary can support solitary waves. The existence of a unique branch of such waves emanating from the horizontal flow at a critical speed U_* is demonstrated in both cases by use of the Nash–Moser implicit function theorem. These results complement the global results of Amick and Turner (1986) and are analogous to the work of Friedrichs and Hyers (1954) and Beale (1977) for surface waves. It is also noted that the most obvious variational principle which characterizes these waves as constrained extremals (Benjamin, 1984) is of indefinite type, having a Hessian with infinitely many positive and infinitely many negative eigenvalues.

KEY WORDS: Solitary wave, internal waves, bifurcation, Nash–Moser implicit function theorem.

1. INTRODUCTION

Considered here are motions of a system consisting of two layers of inviscid fluid, one over the other and each having finite vertical extent and constant density. These layers are confined to an infinitely long horizontal channel and it is assumed throughout that the fluid velocities, which will depend in a non-trivial way upon both the vertical coordinate and the horizontal coordinate in the unbounded direction, are independent of the spanwise coordinate across the channel. We take it that this system is governed by the two-dimensional Euler equations and treat questions of existence of solitary-wave solutions for both the case of a free surface and that of a rigid upper boundary. Steady flow of uniform, purely horizontal velocity U in which the upper layer has constant depth h_1 and the lower layer has constant depth h_2 will always be a *trivial* solution to the equations of motion (1.2) and (1.4) below. For speeds U near a critical value U_* (which depends upon the

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densities ρ_1 and ρ_2 and the equilibrium depths h_1 and h_2 of the two fluid layers, the gravitational acceleration g, and another side condition to be specified presently), the existence of non-trivial, travelling-wave solutions that correspond to solitary waves will be demonstrated. In the case of a fixed upper boundary, these solutions may represent either waves of depression or elevation depending on whether

$$e=(\rho_2 h_1^2-\rho_1 h_2^2)/(h_1 h_2)^2 \tag{1.1}$$

is positive or negative, respectively. In case the upper fluid surface is free, we obtain solitary waves of elevation. In both cases, these solutions comprise a smooth curve of small-amplitude waves bifurcating from a trivial flow in a space of functions which are analytic except at the interface between the layers. They are all symmetric about a single crest and decay exponentially to the underlying trivial flow away from their crest. The interface and the upper surface are likewise shown to be real-analytic functions.

The Euler equations for a steady flow in the two-fluid system described above with a rigid top boundary are the following system in which ρ_1 is the density of the upper fluid and h_1 is the undisturbed depth of the upper fluid, ρ_2 and h_2 denote the corresponding quantities in the lower fluid and $y=\eta(x)$ is the equation of the interface:

$$\begin{aligned}
&\Delta\psi=0 && \text{in} \quad \Omega_1=\{(x,y): x\in\mathbb{R}, \eta(x)<y<h_1\},\\
&\psi=ch_1 && \text{at} \quad y=h_1,\\
&\psi=0 && \text{at} \quad y=\eta(x),\\
&\Delta\psi=0 && \text{in} \quad \Omega_2=\{(x,y): x\in\mathbb{R}, -h_2<y<\eta(x)\},\\
&\psi=-ch_2 && \text{at} \quad y=-h_2, \quad \text{and}\\
&\rho_1[\tfrac{1}{2}(|\nabla\psi|^2)_{y=\eta^+}+g\eta]-\rho_2[\tfrac{1}{2}(|\nabla\psi|^2)_{y=\eta^-}+g\eta]=\text{constant}\\
&\qquad \text{along} \quad y=\eta(x).
\end{aligned} \tag{1.2}$$

Here, x is the horizontal coordinate along the channel, y is the vertical coordinate taken to be $-h_2$ at the bottom, ψ denotes the stream function, and c refers to the horizontal velocity of the fluid at infinity or, equivalently, the speed of propagation of the progressing wave in a frame of reference moving so that there is no flow at infinity. The system (1.2) is obtained by formulating the time-dependent problem as a Hamiltonian system in terms of η and a conjugate variable $\Phi(x)=\rho_2\phi(x,\eta(x)^-)-\rho_1\phi(x,\eta(x)^+)$ (Benjamin, 1984; Bowman, 1987) where the Hamiltonian is the energy

$$H=\iint_{\Omega_1}\rho_1(\tfrac{1}{2}|\nabla\phi|^2)\,dx\,dy+\iint_{\Omega_2}\rho_2(\tfrac{1}{2}|\nabla\phi|^2)\,dx\,dy+\int_{\mathbb{R}}\tfrac{1}{2}g(\rho_2-\rho_1)\eta^2(x)\,dx. \tag{1.3}$$

Here the velocity potential ϕ is the harmonic function conjugate to the stream function ψ, the Bernoulli condition corresponds to an equation for Φ_t, while the kinematic condition at the interface is given by an equation for η_t. In this formulation, one regards ϕ as determined in Ω_1 and Ω_2 by Φ through a Riemann–Hilbert problem posed along the curve $y=\eta(x)$.

The unknown interface $y=\eta(x)$ is a major source of complication in the formulation (1.2). The Bernoulli condition shows that the velocity must jump across the interface in order to balance the buoyancy force, but this jump is *only* in the tangential velocity since because of the kinematic condition, the free surface must be a streamline which forces the normal velocity of the fluids to match. This state of affairs seems to prohibit use of the complex potential in (1.2) to eliminate the free boundary. Instead, assuming $\psi_y>0$ throughout, one may reformulate the problem with x and ψ as independent variables and $y(x,\psi)$ as the independent variable. Since ψ_y is the constant c for a trivial flow, the condition that ψ_y be positive will be satisfied for nearby flows. This change of variables eliminates the unknown interface, but converts (1.2) into the non-linear, elliptic boundary-value problem (2.1) for y.

In the case where the upper boundary of the two-layer system is left free, both the interface $y=\eta(x)$ between the two layers and the free surface $y=\chi(x)$ are unknown. The analog of the system (1.2), namely

$$\begin{aligned}
&\Delta\psi=0 && \text{in } \Omega_1=\{(x,y): x\in\mathbb{R}, \eta(x)<y<\chi(x)\},\\
&\psi=ch_1 && \text{at } y=\chi(x),\\
&\psi=0 && \text{at } y=\eta(x),\\
&\Delta\psi=0 && \text{in } \Omega_2=\{(x,y): x\in\mathbb{R}, -h_2<y<\eta(x)\},\\
&\psi=-ch_2 && \text{at } y=-h_2,\\
&\rho_1[\tfrac{1}{2}(|\nabla\psi|^2)_{y=\eta^+}+g\eta]-\rho_2[\tfrac{1}{2}(|\nabla\psi|^2)_{y=\eta^-}+g\eta]=\text{constant}\\
&\qquad \text{along } y=\eta(x),\\
&\tfrac{1}{2}(|\nabla\psi|^2)_{y=\chi}+g\eta=\text{constant}, \quad \text{along } y=\chi(x),
\end{aligned} \tag{1.4}$$

now features a second Bernoulli condition imposed at the free surface. The system (1.4) may also be obtained from a Hamiltonian formulation.

Although the second unknown interface complicates matters further, the question of existence of solitary-wage solutions of (1.4) may be addressed by the same techniques that come to the fore in dealing with (1.2). For the system (1.4) there appears to be two distinct branches of solitary waves which bifurcate from distinct trivial solutions. Solutions on the branch bifurcating from the trivial flow with a faster speed are always waves of elevation analogous to the surface solitary wave on a single, constant-density fluid layer of depth $h=h_1+h_2$, while those on the other branch may be either waves of elevation or depression that move at a slower speed and which have their maximum excursion from the trivial flow on the interface between the two fluids, being thus closely analogous to the solitary-wave solutions of (1.2). In the present account, only the former solutions will be addressed.

There is an extensive literature on solitary waves and their implications for theoretical and practical problems in mechanics, dating back to the discovery of the phenomenon by John Scott-Russell more than 150 years ago. The exact theory of such wave motion began with Lavrentiev (1943, 1947) and Friedrichs and Hyers (1954). The present work follows the technical line of development on the problem of surface solitary waves initiated by Friedrichs and Hyers and continued by Beale (1977). Recent work on solitary waves in continuously stratified fluids (Turner, 1981, 1984; Bona *et al.*, 1983; Amick, 1984) has centered on global and variational methods (see however Kirchgässner, 1982). In this vein, Amick and Turner (1986) attack the two-fluid system considered here by realizing it as a limit of continuously stratified systems and thereby prove existence of a global branch of solitary-wave solutions. The results presented herein complement their work by providing a more complete picture of the solutions near the trivial flow with critical velocity. A simple explanation of the role of the parameter e and the critical speed U_* as well as an explicit approximation to the solitary wave are derived from our local analysis. The model problem which emerges (the bifurcation equation) is the KdV equation, which also explains why only supercritical velocities are allowed and gives an approximate relation between the wave speed, amplitude, and the rate at which the wave evanesces at infinity.

A benefit of the present approach is that symmetry of the wave and exponential decay of the solution to the underlying trivial flow follow directly by being incorporated into the function spaces. As an added bonus the solutions obtained are found to be analytic in both variables throughout the flow domain except across the internal fluid interface where the tangential component of the fluid velocity has a jump discontinuity. This in turn implies the analyticity of the interface. In the variational approach, these qualitative properties of the solution are not so readily available.

In a very recent manuscript, Amick and Turner (1989) use a dynamical systems approach as in Kirchgässner (1982) to characterize all small solutions of the problem (1.2). They find, in addition to solitary waves, that there are small-amplitude internal bores and internal cnoidal waves as well as conjugate flows. This elegant work, being less directly concerned with solitary waves *per se*, assumes neither decay nor periodicity in the x-variable and so requires considerable technical dexterity.

The plan of the paper is as follows. In Section 2 the aforementioned change of variables is employed for the system with a rigid upper boundary to convert (1.2) into a more tractable system. A stretching in the horizontal variable is then introduced that leads to a nonlinear elliptic system involving a small parameter $\varepsilon>0$. The Banach spaces of analytic functions used in our analysis are then introduced and the existence problem for solitary waves reformulated as a search for zeroes of an appropriate mapping in the context of these spaces. The bifurcation point is then determined from the linearization of this map at $\varepsilon=0$. It turns out that the problem at $\varepsilon=0$ is degenerate, so the usual implicit function theorem does not seem to apply. Instead, a Nash–Moser type theory is used to infer the existence of a branch of solutions emanating from the bifurcation point. Section 3, which is the heart of the paper, begins with a statement of the principal result concerning solitary waves for the case of a fixed upper boundary. The remainder of the section is devoted to verifying technical conditions needed in justifying the application of the Nash–Moser implicit function theorem to obtain a smooth branch of zeroes of the mapping, and thus a branch of solitary-wave solutions of the original problem. The modifications of this program that are needed for the case of a free upper boundary are described in Section 4. Section 5 is concerned with the smoothness of the solution branches whose existence is guaranteed in the earlier sections, whilst in Section 6 a brief comment is made on a variational characterization of internal solitary waves. It turns out that the relevant quadratic form is indefinite with infinitely many positive and negative eigenvalues. This aspect poses a severe difficulty as regards the prospects for developing a stability theory of these solitary waves along the lines of the current theories that are applicable to simplified models such as the *KdV* equation.

2. REFORMULATION OF THE PROBLEM AND DETERMINATION OF THE BIFURCATION POINT

The solitary-wave problem in the case of a rigid upper boundary is reformulated in the way indicated in Section 1 and an associated bifurcation problem relating to the existence of small-amplitude travelling waves is derived.

Starting from the Euler equations for steady propagation given in (1.2), it is a straightforward calculation to reinterpret this system by viewing y as a function of x and ψ. Assuming that $\psi_y \neq 0$ in $\Omega_1 \cup \Omega_2$, the new system is

$$\begin{aligned}
&-\left(\frac{y_x}{y_\psi}\right)_x + \frac{1}{2}\left(\frac{1+y_x^2}{y_\psi^2}\right)_\psi = 0 \quad \text{in} \quad \Omega_1' = \{(x,\psi): x \in \mathbb{R}, 0<\psi<ch_1\},\\
&\qquad y = h_1 \quad \text{at} \quad \psi = ch_1,\\
&-\left(\frac{y_x}{y_\psi}\right)_x + \frac{1}{2}\left(\frac{1+y_x^2}{y_\psi^2}\right)_\psi = 0 \quad \text{in} \quad \Omega_2' = \{(x,\psi): x \in \mathbb{R}, -ch_2<\psi<0\},\\
&\qquad y = -h_2 \quad \text{at} \quad \psi = -ch_2,\\
&\qquad y \text{ is continuous at } \psi = 0,\\
&\rho_2\left[\left(\frac{1}{2}\left(\frac{y_x}{y_\psi}\right)^2 + \left(\frac{1}{y_\psi}\right)^2\right)\Bigg|_{\psi=0^-} + gy(x,0)\right] - \rho_1\left[\frac{1}{2}\left(\frac{y_x^2+1}{y_\psi^2}\right)\Bigg|_{\psi=0^+} + gy(x,0)\right]\\
&\qquad = \text{constant.}
\end{aligned} \tag{2.1}$$

The (unknown) interface is located at $\{(x,\psi): \psi=0\}$ in the new variables, and is determined by requiring $y(x,0)$ to be continuous while the other boundary conditions in (1.2) are simply carried over to (2.1).

A linear analysis of (1.2) or (2.1) near the trivial flow $\psi = cy$ shows that there is a critical velocity $c = U_*$ at which the linearized equations support waves. As worked out in Lamb (1932, Section 231), the dispersion relation for frequency ω as a function of wavenumber k for the system obtained by linearizing (2.1) about a trivial flow is

$$\omega^2 = \frac{gk(\rho_2 - \rho_1)}{\rho_2 \coth(kh_2) + \rho_1 \coth(kh_1)}.$$

The long-wave limit wherein $k \to 0$ then yields the critical velocity U_* whose value is found to be

$$U_*^2 = g(\rho_2 - \rho_1) \Big/ \left(\frac{\rho_1}{h_1} + \frac{\rho_2}{h_2}\right).$$

Following Friedrichs and Hyers (1954) and Beale (1977), a scaling of the spatial variable x corresponding to a hypothesis that the waves are long and a scaling of the amplitude relative to the trivial flow corresponding to a hypothesis of small amplitude may be introduced along with the relationship

$$c^2 = U_*^2 e^{\alpha \varepsilon}, \tag{2.2}$$

where the constant α will be chosen presently and the small parameter ε will measure the supercriticality of c. For a given $\varepsilon > 0$, the scaling is

$$\begin{aligned} x' &= \varepsilon^{1/2} x, \\ \xi &= \psi / c, \\ y(x, \xi) &= \xi + \varepsilon w(x', \xi). \end{aligned} \tag{2.3}$$

The prime will be dropped immediately. The system (2.1) in the new variables is, after cancellation of common factors of ε,

$$\begin{aligned} &-\varepsilon\left(\frac{w_x}{1+\varepsilon w_\xi}\right)_x + \frac{1}{2}\left(\frac{-2w_\xi - \varepsilon w_\xi^2 + \varepsilon^2 w_x^2}{(1+\varepsilon w_\xi)^2}\right)_\xi = 0 \quad \text{in} \quad \{-h_2 < \xi < 0\} \cup \{0 < \xi < h_1\}, \\ &w = 0 \text{ at } \xi = h_1, \\ &w \text{ is continuous at } \xi = 0, \\ &w = 0 \text{ at } \xi = -h_2, \\ &\rho_2\left[\left(\frac{-2w_\xi - \varepsilon w_\xi^2 + \varepsilon^2 w_x^2}{(1+\varepsilon w_\xi)^2}\right)_{\xi=0^-}\right] - \rho_1\left[\left(\frac{-2w_\xi - \varepsilon w_\xi^2 + \varepsilon^2 w_x^2}{(1+\varepsilon w_\xi)^2}\right)_{\xi=0^+}\right] \\ &\quad + \frac{2g(\rho_2 - \rho_1)}{c^2} w(x, 0) = 0. \end{aligned} \tag{2.4}$$

At $\varepsilon=0$, (2.4) reduces to the equations

$$\begin{aligned} &-w_{\xi\xi}=0 \quad \text{in} \quad \{-h_2<\xi<0\}\cup\{0<\xi<h_1\}, \\ &w=0 \quad \text{at} \quad h_1, -h_2, \\ &w \quad \text{continuous at} \quad \xi=0, \\ &-2\rho_2 w_\xi|_{\xi=0^-}+2\rho_1 w_\xi|_{\xi=0^+}+\frac{2g(\rho_2-\rho_1)}{U_*^2}w(x,0)=0. \end{aligned} \tag{2.5}$$

The most general solution to (2.5) is

$$w_0(x,\xi)=\begin{cases} W(x)(\xi+h_2)/h_2 & \text{for } -h_2<\xi<0, \\ W(x)(h_1-\xi)/h_1 & \text{for } 0<\xi<h_1, \end{cases} \tag{2.6}$$

where $W(x)=w_0(x,0)$ is arbitrary. (Note that since $w_{0\xi}(x,0^-)=W(x)/h_2$ and $w_{0\xi}(x,0^+)=-W(x)/h_1$, the Bernoulli condition follows from the definition of U_*^2). If one seeks $w(x,\xi)$ in the form of a perturbation expansion, the unknown function $W(x)$ is determined at the next order from substitution into (2.4). This equation will be exhibited shortly and admits a unique symmetric, exponentially decaying solution $W(x)=A\,\mathrm{sech}^2(Bx)$ where the constants A and B depend upon ρ_1, ρ_2, h_1, h_2, and α.

Rather than pursuing the perturbation–expansion approach, the system (2.4) will be formulated as a mapping problem $F(w,\varepsilon)=0$ in the present section and the solution $w_0(x,\xi)$ at $\varepsilon=0$ will be viewed as a point from which the desired travelling wave bifurcates. In Section 3 this mapping problem will be shown to have a solution which is close to w_0 for small, positive ε by an application of a generalized implicit-function theorem of Nash–Moser type.

To begin, appropriate function spaces are defined. For a fixed σ^* and any σ with $0<\sigma\leqq\sigma^*$ let X_σ be the Banach space of functions $u(x)$ which are continuous, even functions of x on the complex strip $|\mathcal{I}m\, x|\leqq\sigma$, analytic in $|\mathcal{I}m\, x|<\sigma$, real for real x, with norm

$$\|u\|_\sigma=\sup_x\{\exp(\mu|\mathcal{R}e\, x|)|u(x)|\}$$

for some constant $\mu>0$ to be chosen presently. The Banach space Y_σ is the linear space of functions $u(x,\xi)$ defined for $|\mathcal{I}m\, x|\leqq\sigma$ and $-h_2\leqq\xi\leqq h_1$ such that $u(\cdot,\xi)$ is a continuous mapping of $-h_2\leqq\xi\leqq h_1$ into X_σ for which $u(x,h_1)=u(x,-h_2)=0$ for all x, equipped with the norm $\sup_\xi\|u(\cdot,\xi)\|_\sigma$. Use will also be made of the spaces $X_{\sigma,j}$ defined for positive integers j to consist of those functions in X_σ for which all derivatives of order at most j exist and lie also in X_σ. Similarly $Y_{\sigma,j}$ is the subspace of Y_σ of functions whose derivatives up to order j exist in the subdomains.

$$D_1=\{(x,\xi):|\mathscr{Im}\,x|<\sigma, 0<\xi<h_1\},$$

and

$$D_2=\{(x,\xi):|\mathscr{Im}\,x|<\sigma, -h_2<\xi<0\},$$

having continuous extensions to each closed subdomain $\bar{D}_1$ and $\bar{D}_2$. It is not required that the derivatives are continuous across the curve $\{(x,\xi):\xi=0\}$, however. Both the spaces $X_{\sigma,j}$ and $Y_{\sigma,j}$ are given the obvious normed structure induced by that of X_σ and Y_σ, respectively. It will occasionally be convenient to consider the restriction of $w\in Y_\sigma$ to D_1 and D_2; the set of all such restrictions are denoted Y_σ^1 and Y_σ^2, respectively, and these spaces possess a natural Banach-space structure of their own.

The system (2.4) may be viewed as an equation $F=0$ where the mapping $F=(F^1,F^2,F^3)$ of $O_\sigma\times[0,1)\to Y_\sigma^1\times Y_\sigma^2\times X_\sigma$ is given by

$$\begin{aligned}
F^1(w,\varepsilon)&=-\varepsilon\left(\frac{w_x}{1+\varepsilon w_\xi}\right)_x+\frac{1}{2}\left(\frac{-2w_\xi-\varepsilon w_\xi^2+\varepsilon^2 w_x^2}{(1+\varepsilon w_\xi)^2}\right)_\xi \quad \text{in } D_1,\\
F^2(w,\varepsilon)&=-\varepsilon\left(\frac{w_x}{1+\varepsilon w_\xi}\right)_x+\frac{1}{2}\left(\frac{-2w_\xi-\varepsilon w_\xi^2+\varepsilon^2 w_x^2}{(1+\varepsilon w_\xi)^2}\right)_\xi \quad \text{in } D_2,\\
F^3(w,\varepsilon)&=\rho_2\left[\frac{-2w_\xi-\varepsilon w_\xi^2+\varepsilon^2 w_x^2}{(1+\varepsilon w_\xi)^2}\bigg|_{\xi=0^-}\right]-\rho_1\left[\frac{-2w_\xi-\varepsilon w_\xi^2+\varepsilon^2 w_x^2}{(1+\varepsilon w_\xi)^2}\bigg|_{\xi=0^+}\right]\\
&\quad+\frac{2g(\rho_2-\rho_1)}{c^2(\varepsilon)}w(x,0).
\end{aligned}\tag{2.7}$$

Here $O_\sigma\subset Y_{\sigma,2}$ consists of those $w\in Y_{\sigma,2}$ such that $1+\varepsilon w_\xi\neq 0$ for all (x,y) such that $|\mathscr{Im}\,x|\leqq\sigma$ and $\{-h_2<y<0\}\cup\{0<y<h_1\}$ and all ε in some interval $[0,\varepsilon_0)$ where ε_0 is sufficiently small. Seeking a solution branch to $F(w,\varepsilon)=0$ emanating from $(w_0,0)$ will necessitate some additional considerations because of the degeneracy of the equation $F(w,0)=0$ that was noted in (2.5) above. As in Beale's paper, a modified mapping $\tilde{F}(w,\varepsilon)$ is employed that gets around this degeneracy.

A computation at $\varepsilon=0$ shows that

$$F^1(w,0)=-w_{\xi\xi} \quad \text{in } D_1; \quad F^2(w,0)=-w_{\xi\xi} \quad \text{in } D_2,$$

and

$$F^3(w,0)=-2\rho_2 w_\xi(x,0^-)+2\rho_1 w_\xi(x,0^+)+\frac{2g(\rho_2-\rho_1)}{U_*^2}w(x,0).$$

These three expressions are linear in w and are not independent because F_3 may be derived from F^1 and F^2 as follows:

$$\frac{2\rho_2}{h_2}\int_{-h_2}^{0} F^2(w,0)(\xi+h_2)\,d\xi-\frac{2\rho_1}{h_1}\int_{0}^{h_1} F^1(w,0)(\xi-h_1)\,d\xi$$

$$=\frac{2\rho_2}{h_2}(-h_2 w_\xi(x,0^-)+w(x,0))-\frac{2\rho_1}{h_1}(-h_1 w_\xi(x,0^+)-w(x,0)),$$

$$=F^3(w,0),$$

since

$$\frac{g(\rho_2-\rho_1)}{U_*^2}=\frac{\rho_2}{h_2}+\frac{\rho_1}{h_1}.$$

These formulas suggest the introduction of a projection Q defined on the range of F, namely

$$Q(\theta^{(1)}(x,\xi),\theta^{(2)}(x,\xi),\theta^{(3)}(x)):=(0,0,\theta^{(3)}(x)+q(x)),$$

where (2.8)

$$q(x)=-\frac{2\rho_2}{h_2}\int_{-h_2}^{0}\theta^{(2)}(x,\xi)(\xi+h_2)\,d\xi+\frac{2\rho_1}{h_1}\int_{0}^{h_1}\theta^{(1)}(x,\xi)(\xi-h_1)\,d\xi.$$

Notice the projection Q has the property that $QF(w,0)\equiv 0$. One may utilize Q to introduce a modified mapping $\tilde{F}(w,\varepsilon)$ defined by

$$F\tilde{F}(w,\varepsilon):=\begin{cases}\dfrac{1}{\varepsilon}QF(w,\varepsilon)+(I-Q)F(w,\varepsilon) & \text{for } \varepsilon>0,\\[2ex] QF_\varepsilon(w,0)+(I-Q)F(w,0) & \text{for } \varepsilon=0,\end{cases} \tag{2.9}$$

where

$$F_\varepsilon(w,0)=\frac{d}{d\varepsilon}F(w,\varepsilon)\big|_{\varepsilon=0}.$$

Since $QF(w,0)\equiv 0$, $\tilde{F}$ is a smooth mapping from $O_\sigma\times[0,1)\to Y_\sigma^1\times Y_\sigma^2\times X_\sigma$. Moreover, for $\varepsilon>0$, $\tilde{F}(w,\varepsilon)=0$ if and only if $F(w,\varepsilon)=0$.

The equation $\tilde{F}(w,0)=0$ has a unique, non-trivial solution $w_0\in O_\sigma$. First, since $QF(w,0)\equiv 0$, the equation $\tilde{F}(w,0)=0$ implies $F(w,0)=0$, which has a general solution

$$w(x,\xi)=\begin{cases}W(x)(\xi+h_2)/h_2 & \text{in } D_2,\\ W(x)(h_1-\xi)/h_1 & \text{in } D_1,\end{cases}$$

with $W \in X_{\sigma,2}$. Since $F(w,0)=0$, the relation $\tilde{F}(w,0)=0$ further entails that $QF_\varepsilon(w,0)=0$. But $F_\varepsilon(w,0)$ has components

$$\theta_1(x,\xi) = -w_{xx} + 3w_\xi w_{\xi\xi} \quad \text{in } D_1,$$

$$\theta_2(x,\xi) = -w_{xx} + 3w_\xi w_{\xi\xi} \quad \text{in } D_2,$$

$$\theta_3(x,\xi) = 3\rho_2[w_\xi(x,0^-)]^2 - 3\rho_1[w_\xi(x,0^+)]^2 - w(x,0)\frac{2g(\rho_2-\rho_1)}{U_*^4}\frac{d(c^2)}{d\varepsilon}\bigg|_{\varepsilon=0},$$

so one may compute the equation $QF_\varepsilon(w,0)=0$ for w of the above form. Recalling that $c^2 = U_*^2 e^{\alpha\varepsilon}$, the result is

$$\frac{2}{3}(\rho_2 h_2 + \rho_1 h_1)w_{xx}(x,0) + 3\left(\frac{\rho_2}{h_2^2} - \frac{\rho_1}{h_1^2}\right)[w(x,0)]^2 - 2\alpha\left(\frac{\rho_1}{h_1} + \frac{\rho_2}{h_2}\right)w(x,0) = 0. \quad (2.10)$$

This is the equation for a solitary-wave solution of the Korteweg–de Vries equation provided the coefficient of the quadratic term is non-zero. (Note that, according to (1.1) this coefficient is simply $3e$. At the critical depth where $e=0$, our theory does not predict solitary waves, and indeed there appears to be no such waves in this case (see Amick and Turner, 1989)). The unique, real solution of (2.10) which is symmetric about the origin is

$$W(x) = A\,\mathrm{sech}^2(Bx) \quad (2.11)$$

where

$$A = \frac{2\alpha}{3e}\left(\frac{\rho_1}{h_1} + \frac{\rho_2}{h_2}\right) \quad \text{and} \quad B^2 = \frac{3\alpha(\rho_1/h_1 + \rho_2/h_2)}{4(\rho_1 h_1 + \rho_2 h_2)}.$$

This demonstrates simultaneously the supercriticality of the solitary-wave velocity ($\alpha > 0$) and the agreement between the signs of the amplitude A and the parameter e.

With $w_0(x,\xi)$ in hand, the conclusion that there is a solution which lies near to $w_0(w,\xi)$ of $\tilde{F}(w,\varepsilon)=0$ for positive, sufficiently small ε will follow by appeal to a Nash–Moser-type implicit-function theorem (Moser 1966; Zehnder, 1975). This is elucidated in the next section.

While the values of A and B are dependent upon α, reference to (2.3) shows that the approximate solution of the original set of equations (2.1) depends upon α only in the form $\alpha\varepsilon$, and so only on the difference $c^2 - U_*^2$. Nevertheless, in the scaled variables the choice of α determines the x variable. These in turn strongly influence the choices of σ^* and μ that appear in the discussion of the function spaces given earlier. To make this aspect more concrete one could, for example, choose α so that $B=1/2$. The resulting function $A\,\mathrm{sech}^2(x/2)$ is analytic in the complex strip

$|\mathscr{Im}\, x| < \pi$ and has finite norm in Y_σ provided $\sigma < \pi$ and $\mu < 1$. The specification may be completed by choosing $\mu = 1/2$ and $\sigma^* = 3$, say.

3. EXISTENCE OF A BRANCH OF SOLITARY WAVES

In this section we prove existence of a branch of symmetric, solitary-wave solutions associated with the two-fluid system with a fixed upper boundary. The main result is the following.

THEOREM 1. *Suppose the speed of propagation c to exceed the critical velocity*

$$U_* = \left[\frac{g(\rho_2 - \rho_1)}{(\rho_1/h_1 + \rho_2/h_2)}\right]^{1/2}, \tag{3.1}$$

where ρ_1 and h_1 connote the density and undisturbed depth of the upper layer of fluid in a two-fluid system and similarly for ρ_2 and h_2 relative to the lower fluid layer. Suppose also the parameter e defined in (1.1) to be non-zero. Then if c is sufficiently close to U_* there exists a unique, non-trivial, symmetric, exponentially decaying, piecewise analytic solitary-wave solution $y = y(x, \psi)$ of (2.1) with $y - \psi/c$ positive if $e > 0$ and $y - \psi/c$ negative if $e < 0$. The solution y is given approximately by $\tilde{y} = \psi/c + \varepsilon w_0(\varepsilon^{1/2} x) l(\psi)$ where

$$l(\psi) = \begin{cases} 1 - \dfrac{\psi}{ch_1} & \text{for } 0 < \psi < ch_1 \text{ and} \\[2ex] 1 + \dfrac{\psi}{ch_2} & \text{for } -ch_2 < \psi < 0 \end{cases}$$

and $w_0(z) = A\,\mathrm{sech}^2(Bz)$ with A and B given explicitly in (2.11). The interface $y(x, 0)$ is an analytic function of x.

The crux of the existence theory as propounded here is the invertibility of the mapping $\tilde{F}_w(w, \varepsilon)$ introduced in Section 2, for (w, ε) near $(w_0, 0)$. A technical result in this direction that is suitable for our purposes is the subject of the next proposition.

PROPOSITION 2. *There exist positive values ε_0 and δ such that for $0 \leqq \varepsilon < \varepsilon_0$ and $\|w - w_0\|_\sigma < \delta$, the mapping $\tilde{F}_w(w, \varepsilon)$ has an unbounded, linear, right inverse $R(w, \varepsilon)$ with the properties*

$$R(w, \varepsilon)\colon Y_\sigma^1 \times Y_\sigma^2 \times X_\sigma \to \{v \in O_{\sigma'} : \|v - w_0\|_{\sigma'} < \delta\} \times [0, \varepsilon_0] \equiv V_{\sigma'},$$

for any $\sigma' \in (0, \sigma)$ and for any $f \in Y_\sigma^1 \times Y_\sigma^2 \times X_\sigma$, $\tilde{F}_w(w, \varepsilon) R(w, \varepsilon) f = f$ at least in the space $Y_{\sigma'}^1 \times Y_{\sigma'}^2 \times X_{\sigma'}$, and, moreover, there exists a constant M such that for all $f \in Y_\sigma^1 \times Y_\sigma^2 \times X_\sigma$

$$\|R(w, \varepsilon) f\|_{Y_{\sigma', 2}} \leqq \frac{M}{(\sigma - \sigma')^3} \|f\|_{Y_\sigma^1 \times Y_\sigma^2 \times X_\sigma}.$$

Proof The proof is accomplished by first considering the case A where $\varepsilon=0$ and then the case B where ε is small, but positive. The first case is exactly where the elliptic operator becomes degenerate, and it is just this degeneracy that necessitates the use of the Nash–Moser technique.

A. Invertibility at $\varepsilon=0$

For $\varepsilon=0$, we have $\tilde{F}(w,0)=QF_\varepsilon(w,0)+(I-Q)F(w,0)$ and hence $\tilde{F}_w(w,0)=F_{\varepsilon w}(w,0)+(I-Q)F_w(w,0)$. Also, the mapping $F(w,0)$ is linear in w, so $F_w(w,0)v=F(v,0)$. Moreover, $F(v,0)=0$ if v has the form

$$v(x,\xi)=\begin{cases} v(x,0)(\xi+h_2)/h_2 & \text{in } D_2, \\ v(x,0)(h_1-\xi)/h_1 & \text{in } D_1. \end{cases}$$

It is thus natural to define another projection P by the formula

$$(Pv)(x,\xi)=\begin{cases} v(x,0)(\xi+h_2)/h_2 & \text{in } D_2, \\ v(x,0)(h_1-\xi)/h_1 & \text{in } D_1. \end{cases}$$

for any function $v=v(x,\xi)$ defined and continuous on $\{(x,\xi): |\mathscr{Im}\, x|<\sigma$ and $-h_2<\xi<h_1\}$. Using both P and Q the mapping $\tilde{F}_w$ may be split into a 2×2 matrix of mappings, each of which may be analysed separately, namely

$$\begin{bmatrix} Q\tilde{F}_w P & (I-Q)\tilde{F}_w P \\ Q\tilde{F}_w(I-P) & (I-Q)\tilde{F}_w(I-P) \end{bmatrix}. \tag{3.2}$$

Since $\tilde{F}_w(w,0)P\equiv 0$ according to the definition of P, it follows that the upper right-hand entry in the matrix is the zero operator. In consequence, the entire operator will be invertible if the two diagonal entries are invertible and the lower left-hand entry is bounded. The latter point being clear from the definitions, interest is focussed on the two diagonal entries in (3.2). Since Q maps only onto the third component of the range and P is determined by $w(x,0)\in X_\sigma$, one may view $Q\tilde{F}_w(w,0)P=QF_{\varepsilon w}(w,0)P$ as a mapping from $X_{\sigma,2}$ to X_σ. Looked at in this manner, its invertibility follows readily from existing theory.

LEMMA 3. *$Q\tilde{F}_w(w,0)P\colon X_{\sigma,2}\to X_\sigma$ has a bounded inverse at $(w_0,0)$.*

Proof A direct computation of $F_{\varepsilon w}(w_0,0)$ yields the formula

$$(Q\tilde{F}_w(w_0,0)P)v=\tfrac{2}{3}(\rho_2 h_2+\rho_1 h_1)V_{xx}(x)$$
$$+6(\rho_2/h_2^2-\rho_1/h_1^2)V(x)W(x)-2\alpha(\rho_1/h_1+\rho_2/h_2)V(x), \tag{3.3}$$

where $V(x)=v(x,0)$ and $W(x)=w_0(x,0)$. The right-hand side of (3.3) is just the linearized form of (2.10), and it follows from results of Friedrichs and Hyers (1954, lemma 7.1) and Beale (1977, lemma 1) that this operator has a classical, bounded inverse mapping X_σ into $X_{\sigma,2}$. The idea of the argument is that a Green's function may be constructed using $W_x(x)$, which is odd in x, and a second solution, which must grow exponentially, neither of which is in X_σ. ■

Since the inverse just obtained is bounded, the next result follows as an immediate corollary.

LEMMA 4. *There exist positive constants δ_0 and ε_0 which are independent of σ such that $Q\tilde{F}_w(w,\varepsilon)P$ is invertible for all w with $\|w-w_0\|_\sigma<\delta_0$ and all ε such that $0\leqq\varepsilon<\varepsilon_0$.*

To conclude the proof of invertibility of $\tilde{F}_w(w,0)$, consider the lower diagonal entry $(I-Q)\tilde{F}_w(w,0)(I-P)$ in the matrix (3.2). Since $\tilde{F}(w,0)$ is linear in $w, \tilde{F}_w(w,0)v=\tilde{F}(v,0)$, which has components (f^1,f^2,f^3) given by

$$f^1=-v_{\xi\xi}(x,\xi)\quad \text{for } 0<\xi<h_1,$$

$$f^2=-v_{\xi\xi}(x,\xi)\quad \text{for } -h_2<\xi<0,$$

$$f^3=-2\rho_2 v_\xi(x,0^-)+2\rho_1 v_\xi(x,0^+)+\frac{2g(\rho_2-\rho_1)}{U_*^2}v(x,0).$$

Also, $(I-P)v(x,\xi)$ has the property that it vanishes at $\xi=0$ in addition to vanishing at $\xi=h_1$ and $\xi=-h_2$. The inversion of $(I-Q)\tilde{F}_w(w,0)(I-P)$ amounts to solving the equation $(I-Q)\tilde{F}_w(w,0)(I-P)v=f$ uniquely for v given $f\in(I-Q)(Y_\sigma^1\times Y_\sigma^2\times X_\sigma)$. The relation $(I-Q)f=f$ implies

$$f^3(x)=\frac{2\rho_2}{h_2}\int_{-h_2}^{0} f^2(x,\xi)(\xi+h_2)\,d\xi-\frac{2\rho_1}{h_1}\int_0^{h_1} f^1(x,\xi)(\xi-h_1)\,d\xi,$$

and consequently it is enough to solve the system

$$\begin{aligned} &-v_{\xi\xi}=f^1(x,\xi)\quad \text{for } 0<\xi<h_1,\\ &-v_{\xi\xi}=f^2(x,\xi)\quad \text{for } -h_2<\xi<0,\\ &v(x,0)=0,\quad v(x,h_1)=0,\quad v(x,-h_2)=0. \end{aligned}\tag{3.4}$$

The homogeneous problem corresponding to (3.4) has no non-trivial solutions and one readily constructs $v(x,\xi)$ using an integral representation involving linear functions of ξ. This gives a unique solution v such that v, v_ξ and $v_{\xi\xi}$ may be estimated in the Y_σ-norm in terms of the Y_σ^1-norm of f^1 and the Y_σ^2-norm of f^2. To obtain differentiability in x, use is made of Cauchy estimates on the domain obtained by shrinking σ to $\sigma'<\sigma$.

Thus $\tilde{F}_w(w,0)$ is invertible, albeit with loss of regularity, for w sufficiently close to w_0. Moreover, since there are no nontrivial solutions to $\tilde{F}_w(w,0)v=0$, the inverse is two-sided. ■

B. Invertibility for $\varepsilon > 0$

From the definition of F in (2.9), it is deduced by direct calculation that $F_w(w,\varepsilon)v$ has components

$$f^1 = -\varepsilon\left(\frac{v_x}{1+\varepsilon w_\xi} - \frac{\varepsilon w_x v_\xi}{(1+\varepsilon w_\xi)^2}\right)_x - \left(\frac{v_\xi(1+\varepsilon^3 w_x^2)}{(1+\varepsilon w_\xi)^3} - \frac{\varepsilon^2 w_x v_x}{(1+\varepsilon w_\xi)^2}\right)_\xi \quad \text{in } 0<\xi<h_1,$$

$$f^2 = -\varepsilon\left(\frac{v_x}{1+\varepsilon w_\xi} - \frac{\varepsilon w_x v_x}{(1+\varepsilon w_\xi)^2}\right)_x - \left(\frac{v_\xi(1+\varepsilon^3 w_x^2)}{(1+\varepsilon w_\xi)^3} - \frac{\varepsilon^2 w_x v_x}{(1+\varepsilon w_\xi)^2}\right)_\xi \quad \text{in } -h_2<\xi<0,$$

$$f^3 = \rho_2\left(\frac{-2v_\xi(x,0^-)(1+\varepsilon^3(w_x(x,0^-))^2)}{(1+\varepsilon w_\xi(x,0^-))^3} + \frac{2\varepsilon^2 w_x(x,0^-)v_x(x,0^-)}{(1+\varepsilon w_\xi(x,0^-))^2}\right) \tag{3.5}$$

$$-\rho_1\left(\frac{-2v_\xi(x,0^+)(1+\varepsilon^3(w_x(x,0^+))^2)}{(1+\varepsilon w_\xi(x,0^+))^3} + \frac{2\varepsilon^2 w_x(x,0^+)v_x(x,0^+)}{(1+\varepsilon w_\xi(x,0^+))^2}\right)$$

$$+\frac{2g(\rho_2-\rho_1)}{c^2}v(x,0).$$

To solve the equation $\tilde{F}_w(w,\varepsilon)v=f$, the problem is again split into four pieces using the projections P and Q introduced previously

If the matrix of operators in (3.2) is written symbolically in the form

$$\begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{pmatrix},$$

Then the reasoning just concluded regarding invertibility of this matrix of operators at $\varepsilon=0$ was that $\mathscr{B}=0$, $\mathscr{C}$ is bounded, $\mathscr{A}$ has a bounded inverse, and $\mathscr{D}$ has an unbounded inverse. For ε positive but small, this argument is modified to establish that $\mathscr{C}$ is bounded, $\mathscr{B}$ has a small norm, $\mathscr{A}$ is boundedly invertible and $\mathscr{D}$ possesses a generalized inverse, all of which still implies the entire matrix of operators to possess an unbounded right inverse.

As determined already in Lemma 4, the upper diagonal entry $\mathscr{A}=Q\tilde{F}_wP$ is certainly invertible at (w,ε) provided w is near enough w_0 and ε is sufficiently small. As before, $\mathscr{C}=Q\tilde{F}_w(I-P)$ and $\mathscr{B}=(I-Q)\tilde{F}_wP$ are bounded operators. Moreover, $\mathscr{B}$ is small as the following result attests.

LEMMA 5. *For w near w_0 in $Y_{\sigma,2}$ and ε near 0, there is a constant C independent of w and ε such that*

$$\|\mathscr{B}\| \leq \varepsilon C.$$

Proof Because $(I-Q)\tilde{F}_w(w_0,0)P=0$, it follows that

$$(I-Q)\tilde{F}_w(w,\varepsilon)Pv=(I-Q)[\tilde{F}_w(w,\varepsilon)-\tilde{F}_w(w_0,0)]Pv,$$

so (3.5) shows $[\tilde{F}_w(w,\varepsilon)-\tilde{F}_w(w_0,0)]P$ to be of order ε if w is near w_0 and ε is small. The same is true after applying the projection $I-Q$. ■

Thus it remains only to show that $(I-Q)\tilde{F}_w(w,\varepsilon)(I-P)$ is suitably invertible. Since $\tilde{F}=(1/\varepsilon)QF+(I-Q)F$, it suffices to consider only $(I-Q)F_w(I-P)$. Here is the result in view.

LEMMA 6. *For positive, small values of ε and w near to w_0 in Y_σ, the equation $(I-Q)F_w(I-P)v=(I-Q)f$ has a unique solution $v\in Y_{\sigma,2}$ for each $f\in Y_\sigma$. Moreover, there exist constants C_1, C_2, C_3 such that*

(i) $\|v\|_\sigma \leqq C_1\|f\|_\sigma$,
(ii) $\|\varepsilon^{1/2}v_x\|_\sigma, \|v_\xi\|_\sigma \leqq C_2\|f\|_\sigma$,
(iii) $\|\varepsilon v_{xx}\|_\sigma, \|\varepsilon^{1/2}v_{x\xi}\|_\sigma, \|v_{\xi\xi}\|_\sigma \leqq C_3\|f\|_\sigma$.

The proof of Lemma 6 is complicated by the linear operator's dependence upon w. This problem is circumvented in a standard way, by considering the full equation as a perturbation of a scaled Laplacian and then iterating. The conclusion regarding the scaled equation is stated in the next lemma.

LEMMA 7. *If v solves the boundary-value problem*

$$\left.\begin{aligned} &-\varepsilon v_{xx}-v_{\xi\xi}=f \quad \text{in } D_1\cup D_2,\\ &\left.\begin{aligned} v(x,-h_2)&=0\\ v(x,0)&=0\\ v(x,h_1)&=0 \end{aligned}\right\} \quad x\in\mathbb{R}, \end{aligned}\right\} \tag{3.5}$$

then v is unique and for $\varepsilon>0$ sufficiently small, we have

$$\max\{\|v\|_\sigma, \|\varepsilon^{1/2}v_x\|_\sigma, \|v_\xi\|_\sigma, \|\varepsilon v_{xx}\|_\sigma, \|\varepsilon^{1/2}v_{x\xi}\|_\sigma, \|v_{\xi\xi}\|_\sigma\} \leqq \text{const.}\,\|f\|_\sigma.$$

In fact, "sufficiently small" means that

$$\varepsilon^{1/2}<\min\left\{\frac{1}{h_1},\frac{1}{h_2}\right\}\frac{\pi}{\sigma^*}.$$

Taking the validity of Lemma 7 as granted, we now address the proof of Lemma 6. Let $v=\sum_{j=0}^{\infty}v^{(j)}$ with $v^{(j)}$ the solution of

$$\left.\begin{aligned} &-\varepsilon v^{(j)}_{xx}-v^{(j)}_{\xi\xi}=Lv^{(j-1)} \quad \text{in } D_1\cup D_2,\\ &\left.\begin{aligned} v^{(j)}(x,-h_2)&=0\\ v^{(j)}(x,0)&=0\\ v^{(j)}(x,h_1)&=0 \end{aligned}\right\} \quad \text{for } x\in\mathbb{R}, \end{aligned}\right\} \tag{3.6a}$$

for $j=1,2,\ldots$, where

$$Lv=\left(-\varepsilon^2\frac{v_x w_\xi}{1+\varepsilon w_\xi}-\varepsilon^2\frac{w_x v_\xi}{(1+\varepsilon w_\xi)^2}\right)_x$$

$$+\left(-\varepsilon^2\frac{w_x v_x}{(1+\varepsilon w_\xi)^2}+\varepsilon\frac{3w_\xi v_\xi-3\varepsilon w_\xi^2 v_\xi-\varepsilon^2 w_\xi^3 v_\xi+\varepsilon^2 w_x^2 v_\xi}{(1+\varepsilon w_\xi)^3}\right)_\xi, \tag{3.6b}$$

and

$$-\varepsilon v_{xx}^{(0)}-v_{\xi\xi}^{(0)}=f \quad \text{in } D_1\cup D_2,$$

$$\left.\begin{aligned} v^{(0)}(x,-h_2)&=0\\ v^{(0)}(x,0)&=0\\ v^{(0)}(x,h_1)&=0\end{aligned}\right\}\quad \text{for } x\in\mathbb{R}.$$

Formally, the function v then solves $(I-Q)F_w(I-P)v=f$. From Lemma 7, there follows estimates on $v^{(0)}$ in terms of f and on $v^{(j)}$ in terms of $Lv^{(j-1)}$. Since $w\in Y_{\sigma,2}$, one easily checks that

$$\|Lv^{(j)}\|_{Y_\sigma}\leqq\varepsilon C\|v^{(j)}\|_{Y_{\sigma,2}},$$

where C is a constant depending only upon $\|w\|_{Y_{\sigma,2}}$. Hence, for ε sufficiently small the series $\sum_{j=0}^{\infty}v^{(j)}$ converges in $Y_{\sigma,2}$. Uniqueness follows from the ellipticity of F_w and thus Lemma 6 is established. ■

Now consider Eq. (3.5). First, notice that uniqueness of a solution corresponding to a given $f\in Y_\sigma$ follows by multiplying the equation by v and integrating over $D_1\cup D_2$. The solution v can be constructed explicitly by use of the Fourier transform. If $\hat{v}(k,\xi)$ denotes the transform of v with respect to the variable x, then

$$\hat{v}(k,\xi)=\begin{cases}\displaystyle\int_{-h_2}^{0} g_2(k;\xi,\eta)\hat{f}(k,\eta)\,d\eta, & \text{for } -h_2\leqq\xi\leqq 0,\\[2ex] \displaystyle\int_{0}^{h_1} g_1(k;\xi,\eta)\hat{f}(k,\eta)\,d\eta, & \text{for } 0\leqq\xi\leqq h_1,\end{cases} \tag{3.7a}$$

where

$$g_2(k;\xi,\eta)=\begin{cases}\dfrac{\sinh(\varepsilon^{1/2}k(\xi+h_2))\sinh(-\varepsilon^{1/2}k\eta)}{\varepsilon^{1/2}k\sinh(\varepsilon^{1/2}kh_2)} & -h_2\leqq\xi<\eta\leqq 0,\\[3ex] \dfrac{\sinh(\varepsilon^{1/2}k(\eta+h_2))\sinh(-\varepsilon^{1/2}k\xi)}{\varepsilon^{1/2}k\sinh(\varepsilon^{1/2}kh_2)} & -h_2\leqq\eta<\xi\leqq 0,\end{cases} \tag{3.7b}$$

and

$$g_1(k;\xi,\eta)=\begin{cases}\dfrac{\sinh(\varepsilon^{1/2}k(h_1-\eta))\sinh(\varepsilon^{1/2}k\xi)}{\varepsilon^{1/2}k\sinh(\varepsilon^{1/2}kh_1)} & 0\leqq\xi<\eta\leqq h_1.\\[2ex] \dfrac{\sinh(\varepsilon^{1/2}k(h_1-\xi))\sinh(\varepsilon^{1/2}k\eta)}{\varepsilon^{1/2}k\sinh(\varepsilon^{1/2}kh_1)} & 0\leqq\eta<\xi\leqq h_1.\end{cases} \tag{3.7c}$$

Except across the line $\{(\xi,\eta):\xi=\eta\}$, these kernels are analytic in k and decay exponentially to zero as $|\mathcal{R}e(k)|\to+\infty$ in the complex strip $|\mathcal{I}m(k)|<\varepsilon^{-1/2}\pi\min(1/h_1,1/h_2)$. Since $f\in Y_\sigma$, $\hat{f}$ is analytic in a fixed complex strip in the variable k and also decays exponentially to zero as $|\mathcal{R}e(k)|\to\infty$. These facts imply that $\hat{v}(k,\xi)$ enjoys the same properties as long as ε is sufficiently small. Thus $\hat{v}$ is seen to lie in the class $Y_{\sigma,2}$. The various norm estimates follow from the explicit formulas above and elementary facts about the Fourier transform. Thus Lemma 7 is established. ∎

Combining all these results, Proposition 2 has now been proved. An application of the following generalized implicit-function theorem (Moser, 1966; Zehnder, 1975) will now lead to a proof of Theorem 1.

THEOREM. *Let $\{W_\sigma\}$ and $\{Z_\sigma\}$ be families of Banach spaces for $0<\sigma\leqq 1$ such that for $\sigma'\leqq\sigma$, $W_{\sigma'}\supset W_\sigma$ and $\|u\|_{\sigma'}\leqq\|u\|_\sigma$ for $u\in W_\sigma$, and similarly for Z_σ. Suppose that $\tilde{F}:O_\sigma\to Z_\sigma$ is smooth and commutes with the inclusions and that $\tilde{F}(w_0,0)=0$ for some $w_0\in W_1$, where the neighborhoods O_σ are given by $\{(w,\varepsilon)\in W_\sigma\times\mathbb{R}:\|w-w_0\|_\sigma<\delta,$ $0\leqq\varepsilon<\varepsilon_1\}$ for some fixed positive constants δ and ε_1. Assume moreover that there exists an unbounded right inverse R for $\tilde{F}_w$ such that for all $z\in Z_\sigma$, $R:Z_\sigma\to W_{\sigma'}$ with $R\tilde{F}_w(w,\varepsilon)z=z$ in $W_{\sigma'}$, for every $\sigma'<\sigma$ and*

$$\|R(w,\varepsilon)z\|_{W_{\sigma'}}\leqq\frac{C\|z\|_{Z\sigma}}{(\sigma-\sigma')^\beta},$$

for some values of C and β. Then for each $\varepsilon>0$ sufficiently small, there is a unique solution in $W_{1/2}$ of $\tilde{F}(w,\varepsilon)=0$ such that $(w,\varepsilon)\in O_{1/2}$. Moreover, if $R(w,\varepsilon)$ is also a left inverse for $\tilde{F}_w$ in the sense that for all $(w,\varepsilon)\in O_\sigma$ and all $v\in W_\sigma$ we have $R(w,\varepsilon)\tilde{F}_w(w,\varepsilon)v=v$ in $W_{\sigma'}$ for any $\sigma'<\sigma$, and if in addition the inclusion $W_\sigma\subset W_{\sigma'}$ is injective, then the solution $w=w(\varepsilon)$ is unique in a W_σ-neighborhood of w_0 for any $\sigma\leqq 1/2$, and is Lipschitz-continuous in ε in the $W_{\sigma'}$-norm for any $\sigma'<1/2$.

Proof of Theorem 1 To check the applicability of the theorem, use the spaces $Y_{\sigma,2}$ as W_σ and Y_σ as Z_σ. These certainly have the correct relations between the σ-norm and the σ'-norm. The inclusions are injective by analytic continuation and the mapping $\tilde{F}$ is smooth and commutes with inclusions. The desired estimate for the norm, namely

$$\|R(w,\varepsilon)f\|_{Y_{\sigma',2}}\leqq\frac{C\|f\|_{Y\sigma}}{(\sigma-\sigma')^\beta},$$

was established for w near w_0 and $\varepsilon=0$ earlier. For $\varepsilon>0$, an estimate for $R(w,\varepsilon)f$

in $Y_{\sigma,2}$ appears in which x-derivative estimates blow up as $\varepsilon \to 0^+$. Note however that in Lemma 6 bounds for $R(w,\varepsilon)f$ in Y_σ along with ξ-derivatives were found which are independent of ε. Using Cauchy estimates one can thereby find bounds for the x-derivatives which have the desired form in all strips of width $|\mathcal{I}m\, x| \leq \sigma' < \sigma$. Analyticity of the interface $\{(x, y(x,0)) : x \in \mathbb{R}\}$ follows because $y - \psi/c$ lies in Y_σ and hence $y(x,0) \in X_\sigma$ since $\psi = 0$ at the interface. ■

4. THE CASE OF A FREE SURFACE

When suitably modified, the argument presented in Section 3 may also be used to establish the existence of a smooth branch of solitary waves when the upper surface is left free. The Bernoulli condition imposed on the free surface involves the parameter g/c^2 and the critical value of c is determined from the roots of a quadratic equation as in Peters and Stoker (1960). This quadratic has two real roots and a bifurcation equation can be worked out corresponding to each of these. There are technical difficulties associated with the operator equation for waves corresponding to the slower speed, and attention is henceforth restricted to the faster speed which corresponds to waves of elevation whose maximum amplitude occurs at the free surface. Such waves would appear to be more similar to the classical surface solitary waves (see Beale, 1977; Friedrichs and Hyers, 1954) than to the interfacial waves discussed in the preceding sections. Because the arguments parallel those given in detail in Section 3 for the case of a fixed upper boundary, they will only be sketched in the case of a free surface considered now.

After changing variables and rescaling as in (2.3), the system (1.4) to be solved takes the form

$$
\begin{aligned}
&-\varepsilon\left(\frac{w_x}{1+\varepsilon w_\xi}\right)_x + \frac{1}{2}\left(\frac{-2w_\xi - \varepsilon w_\xi^2 + \varepsilon^2 w_x^2}{(1+\varepsilon w_\xi)^2}\right)_\xi = 0 \quad \text{in } \mathbb{R}\times(-h_2,0) \cup \mathbb{R}\times(0,h_1),\\
&w = 0 \quad \text{at } \xi = -h_2,\\
&w \quad \text{continuous across } \xi = 0,\\
&\rho_2\left(\frac{-2w_\xi - \varepsilon w_\xi^2 + \varepsilon^2 w_x^2}{(1+\varepsilon w_\xi)^2}\right)_{\xi=0^-} - \rho_1\left(\frac{-2w_\xi - \varepsilon w_\xi^2 + \varepsilon^2 w_x^2}{(1+\varepsilon w_\xi)^2}\right)_{\xi=0^+}\\
&\qquad + 2\frac{g(\rho_2-\rho_1)}{c^2} w(x,0) = 0,\\
&\rho_1\left(\frac{-2w_\xi - \varepsilon w_\xi^2 + \varepsilon^2 w_x^2}{(1+\varepsilon w_\xi)^2}\right)_{\xi=h_1} + 2\frac{\rho_1 g}{c^2} w(x,h_1) = 0,
\end{aligned}
\tag{4.1}
$$

analogous to (2.4). As before, the speed c depends upon ε in a way to be made precise presently. At $\varepsilon = 0$, the system (4.1) becomes

$$\begin{aligned}
&-w_{\xi\xi}=0 \qquad \text{in } D_1 \cup D_2,\\
&w=0 \qquad \text{at } \xi=-h_2,\\
&w \quad \text{continuous at } \xi=0,\\
&-2\rho_2 w_\xi|_{\xi=0^-}+2\rho_1 w_\xi|_{\xi=0^+}+2\frac{g(\rho_2-\rho_1)}{c_0^2}w(x,0)=0,\\
&-2\rho_1 w_\xi|_{\xi=h_1}+2\frac{g\rho_1}{c_0^2}w(x,h_1)=0.
\end{aligned} \tag{4.2}$$

The most general solution to the system (4.2) has the form $w(x,\xi)=W(x)l(\xi)$ where $l(\xi)$ is the piecewise linear function

$$l(\xi)=\begin{cases} \dfrac{(\xi+h_2)}{h_2}, & -h_2\leqq\xi\leqq 0,\\[2ex] \dfrac{U_*^2/gh_1-(h_1-\xi)/h_1}{U_*^2/gh_1-1}, & 0\leqq\xi\leqq h_1, \end{cases} \tag{4.3}$$

normalized so that $l(0)=1$. The critical velocity U_* satisfies an equation arising from the imposition of Bernoulli's law on the interface, namely

$$-\frac{\rho_2}{h_2}+\frac{\rho_1}{h_1}\left(\frac{1}{\mu-1}\right)+\frac{(\rho_2-\rho_1)}{h_1\mu}=0,$$

where $\mu=U_*^2/gh_1$. This latter equation is equivalent to the quadratic equation

$$\mu^2-\left(1+\frac{h_2}{h_1}\right)\mu+\left(1-\frac{\rho_1}{\rho_2}\right)\frac{h_2}{h_1}=0, \tag{4.4}$$

mentioned above. This equation always has two real roots μ_+ and μ_- with $0<\mu_-<1<\mu_+$ since static stability requires $\rho_1/\rho_2<1$. In case $\mu=\mu_-$, it follows from (4.3) that $l(\xi)$ changes sign in the interval $(0,h_1)$. Moreover, for practical values of the parameters ($\mu_-<1/2$), one has $|l(h_1)|<1$ and the maximum displacement then occurs along the interface $\{\xi=0\}$. Henceforth, interest will focus on the larger root μ_+ and the associated critical velocity

$$U_*^2=\frac{1}{2}g\left[h_1+h_2+\sqrt{\left\{(h_1-h_2)^2+4h_1h_2\frac{\rho_2}{\rho_1}\right\}}\right]. \tag{4.5}$$

As in (2.2), set $c^2=U_*^2e^{\alpha\varepsilon}$ where α is to be determined.

To establish the desired existence theory, regard the system (4.1) as a four-component mapping G defined on an appropriate scale of Banach spaces. The components of the mapping G are

$$
\begin{aligned}
G^1(w,\varepsilon) &= -\varepsilon\left(\frac{w_x}{1+\varepsilon w_\xi}\right)_x + \frac{1}{2}\left(\frac{-2w_\xi-\varepsilon w_\xi^2+\varepsilon^2 w_x^2}{(1+\varepsilon w_\xi)^2}\right)_\xi, \quad \text{in } D_1,\\
G^2(w,\varepsilon) &= -\varepsilon\left(\frac{w_x}{1+\varepsilon w_\xi}\right)_x + \frac{1}{2}\left(\frac{-2w_\xi-\varepsilon w_\xi^2+\varepsilon^2 w_x^2}{(1+\varepsilon w_\xi)^2}\right)_\xi, \quad \text{in } D_2,\\
G^3(w,\varepsilon) &= -2\rho_2 w_\xi|_{\xi=0^-} + 2\rho_1 w_\xi|_{\xi=0^+} + \frac{2g(\rho_2-\rho_1)}{c^2(\varepsilon)} w(x,0),\\
G^4(w,\varepsilon) &= -2\rho_1 w_\xi|_{\xi=h_1} + \frac{2g\rho_1}{c^2(\varepsilon)} w(x,h_1).
\end{aligned}
\tag{4.6}
$$

The function G maps $O_\sigma\times[0,\varepsilon_0)$ into $\tilde{Y}_\sigma^1\times Y_\sigma^2\times X_\sigma\times X_\sigma$ where the spaces Y_σ^2, X_σ are as defined previously in Section 2 and $\tilde{Y}_\sigma^1$ is the Banach space consisting of all functions $u(x,\xi)$ defined for $|\mathcal{Im}\, x|\leqq\sigma$ and $0\leqq\xi\leqq h_1$ such that $u(\cdot,\xi)\in X_\sigma$ for $0\leqq\xi\leqq h_1$, with norm $\sup_{0\leqq\xi\leqq h_1}\|u(\cdot,\xi)\|_{X_\sigma}$. (Recall that the previous space Y_σ^1 incorporates $u(x,h_1)=0$ which is not relevant to the free surface case.) The neighborhood $O_\sigma\subset\tilde{Y}_{\sigma,2}$ is defined in complete analogy with our earlier discussion.

Having observed already that the system $G(w,0)=0$ is degenerate, a modified mapping $\tilde{G}$ may be introduced in exactly the same fashion as in (2.9) above, except that the projection Q which annihilates the range of $G(w,0)$ is given by

$$
Q(\theta^{(1)}(x,\xi),\theta^{(2)}(x,\xi),\theta^{(3)}(x),\theta^{(4)}(x)) := (0,0,q(x),0), \tag{4.7}
$$

where

$$
q(x)=\theta^{(3)}(x)+\frac{U_*^2/gh_1}{U_*^2/gh_1-1}\theta^{(4)}(x)-2\rho_2\int_{-h_2}^{0}\theta^{(2)}(x,\xi)l(\xi)\,d\xi-2\rho_1\int_0^{h_1}\theta^{(1)}(x,\xi)l(\xi)\,d\xi,
$$

with $l(\xi)$ given in (4.3). The modified equation $\tilde{G}(w,0)=0$ has a unique solution given by $w_0(x,\xi)=W(x)l(\xi)$ where $W(x)$ satisfies the ordinary differential equation analogous to (2.10), namely

$$
W''\left[\rho_1\int_0^{h_1} l^2(\xi)\,d\xi+\rho_2\int_{-h_2}^{0} l^2(\xi)\,d\xi\right]-\alpha W\left[\frac{(\rho_2-\rho_1)}{\mu_+h_1}+\frac{\rho_1}{(\mu_+-1)h_1}\right]
$$

$$
+\frac{3}{2}W^2\left[\frac{\rho_2}{h_2^2}-\frac{\rho_1}{h_1^2(\mu_+-1)^2}+\frac{\rho_1\mu_+}{h_1^2(\mu_+-1)^3}\right]=0. \tag{4.8}
$$

The expressions within brackets in (4.8) are seen to be positive, and so positive solutions that decay to zero at infinity exist for $\alpha>0$, that is, for supercritical speeds $c>U_*$. The solution $W(x)$ has the form $W(x)=A\,\mathrm{sech}^2(Bx)$ with A and B^2 linear in α. As in the discussion at the end of Section 2, once the value of α is decided, a suitable exponential decay rate and domain of analyticity is then obtained and the scale of Banach spaces determined. As before, a convenient choice of α is that which makes $B=1/2$.

To verify the applicability of the Nash–Moser-type implicit-function theorem, the invertibility at the cost of loss of regularity of $\tilde{G}_w(w,\varepsilon)$ for (w,ε) near $(w_0,0)$ must be checked. This proceeds very much like the corresponding argument for the fixed-boundary case. At $\varepsilon=0$, we use the projection Q defined earlier and another projection P given by $(Pw)(x,\xi)=w(x,0)l(\xi)$ to write $\tilde{G}_w(w,0)v=f$ in the form of a 2×2 matrix of operators as in (3.2). As before, one off-diagonal entry is the zero operator and the invertibility reduces to being able to solve the linearized form of the KdV equation (4.8) as in Lemma 3 and to solving a simple ordinary differential equation in ξ, as in (3.4) above. This equation now reads:

$$
\begin{aligned}
&-v_{\xi\xi}=f^1(x,\xi) \quad \text{for } 0<\xi<h_1,\\
&-v_{\xi\xi}=f^2(x,\xi) \quad \text{for } -h_2<\xi<0,\\
&-2\rho_1 v_\xi\big|_{\xi=h_1}+2\frac{g\rho_1}{U_*^2}v(x,h_1)=f^{(4)}(x),\\
&v(x,0)=0,\\
&v(x,-h_2)=0.
\end{aligned}
\tag{4.9}
$$

One again finds a unique solution with good estimates on ξ-derivatives, but no estimates on x-derivatives. Using Cauchy estimates on subdomains, one establishes the invertibility for $(w,0)$ near $(w_0,0)$.

For $\varepsilon>0$, the classical invertibility of $Q\tilde{G}_w(w,\varepsilon)P$ is combined with the solution of the approximate problem analogous to (3.5), namely

$$
\begin{aligned}
&-\varepsilon v_{xx}-v_{\xi\xi}=\theta_1 \quad \text{in } D_1,\\
&-\varepsilon v_{xx}-v_{\xi\xi}=\theta_2 \quad \text{in } D_2,\\
&-2\rho_2 v_\xi(x,0^-)+2\rho_1 v_\xi(x,0^+)+2\frac{g(\rho_2-\rho_1)}{U_*^2}v(x,0)=\theta_3,\\
&-2\rho_1 v_\xi(x,h_1)+2\frac{g\rho_1}{U_*^2}v(x,h_1)=\theta_4,\\
&v(x,-h_2)=0,
\end{aligned}
\tag{4.10}
$$

with $\theta=(\theta_1,\theta_2,\theta_3,\theta_4)$ and $\theta=(I-Q)\theta$. The weak formulation of (4.10) leads to a coercive problem, a claim that is verified by proving the absence of negative eigenvalues. This follows by solving $-v_{\xi\xi}=-\alpha^2 v$ with the Bernoulli conditions. Using the condition at $\xi=h_1$ along with the normalization $v(0)=1$, one readily finds v to be given by

$$v(x,\xi;\alpha)=\begin{cases}\dfrac{\sinh(\alpha(\xi+h_2))}{\sinh(\alpha h_2)}, & -h_2\leqq\xi\leqq 0,\\[2ex] \cosh(\alpha\xi)+\sinh(\alpha\xi)\left[\dfrac{\coth(\alpha h_1)-\mu_+\alpha h_1}{\mu_+\alpha h_1\coth(\alpha h_1)-1}\right], & 0\leqq\xi\leqq h_1.\end{cases}$$

Imposing the Bernoulli condition at $\xi=0$ leads to the equation

$$\frac{\rho_2}{h_2}[\alpha h_2\coth(\alpha h_2)-1]=\frac{\rho_1}{h_1}\left[\alpha h_1\left(\frac{\coth(\alpha h_1)-\mu_+\alpha h_1}{\mu_+\alpha h_1\coth(\alpha h_1)-1}\right)-\frac{1}{\mu_+-1}\right]. \tag{4.11}$$

The left-hand side of (4.11) is positive for $\alpha\neq 0$ whilst the right-hand side is negative, hence there are no negative eigenvalues. (This argument uses essentially the fact that $\mu_+-1>0$, and fails if μ_- replaces μ_+). Solving (4.10) by taking Fourier transforms in x establishes the analogue of Lemma 7.

The linearized problem $(I-Q)G_w(I-P)v=(I-Q)f$ will again be solvable by iteration using (4.10), as in Lemma 6 above. This yields an inverse in $\tilde{Y}_{\sigma,2}$ of the desired form. One may therefore apply again the Nash–Moser implicit-function theorem to establish the following result.

THEOREM 8. *For the two-layer system with free upper surface, there exists a unique branch of non-trivial, positive, symmetric, exponentially decaying, piecewise analytic solitary waves for speeds $c>U_*$ with $c-U_*$ sufficiently small, where*

$$U_*=\left[\frac{1}{2}g\left(h_1+h_2+\sqrt{\left\{(h_1-h_2)^2+4h_1h_2\frac{\rho_2}{\rho_1}\right\}}\right)\right]^{1/2}.$$

The solution y is given approximately by $\tilde{y}=1\psi/c+\varepsilon w_0(\varepsilon^{1/2}x)l(\psi/c)$ where $l(\psi)$ is given by (4.3) and $w_0(z)$ is $A\,\mathrm{sech}^2(z/2)$ where A is

$$A=\frac{1}{3}\left[\frac{\rho_1h_1(\mu_+^3-1)}{(\mu_+-1)^2}+\rho_2h_2\right]\Bigg/\left[\frac{\rho_2}{h_2^2}-\frac{\rho_1}{h_1^2(\mu_+-1)^2}+\frac{\rho_1\mu_+}{h_1^2(\mu_+-1)^3}\right],$$

with $c^2=U_*^2e^{\alpha\varepsilon}$ and

$$\alpha=\frac{1}{3}\left[\frac{\rho_1h_1(\mu_+^3-1)}{(\mu_+-1)^2}+\rho_2h_2\right]\Bigg/\left[\frac{(\rho_2-\rho_1)}{\mu_+h_1}+\frac{\rho_1}{(\mu_+-1)h_1}\right].$$

The free surface $y(x,h_1)$ and the interface $y(x,0)$ are analytic functions of x.

5. SMOOTHNESS OF THE SOLUTION BRANCHES

Considered here is the smoothness in the parameter ε of the solitary-wave branches whose existence was the object of the theory heretofore. The outcome of the analysis is that for any $\sigma<1/2$, $w=w(\varepsilon)$ is a C^∞-function when viewed as a mapping of $[0,\varepsilon_0)$ into $Y_{\sigma,2}$ or $\tilde{Y}_{\sigma,2}$. This result is valid for both the fixed and the free upper boundary. As the proofs are quite similar for both cases, we present them for the case of a fixed upper boundary only and state the results for the other case. The proofs rely substantially on our previous calculations. The notation from the earlier sections is taken over entirely here.

The first step in establishing the advertised smoothness of the solution branch to (2.4), the fixed boundary case, is the demonstration that the unbounded inverse $R=R(w,\varepsilon)$ is C^∞ in ε when viewed as a mapping of $O_\sigma\times[0,\varepsilon_0)$ into the Banach space of bounded linear operators from $Y_\sigma^1\times Y_\sigma^2\times X_\sigma$ to $Y_{\sigma'}^1\times Y_{\sigma'}^2\times X_{\sigma'}$ where $\sigma'<\sigma$.

LEMMA 9. *Let ε_0 and the inverse mapping $R=R(w,\varepsilon)$ of $\tilde{F}_w$ be as determined in Proposition 2. Suppose also that $\sigma'<\sigma\leqq\sigma^*$. Then the correspondence $(w,\varepsilon)\mapsto R(w,\varepsilon)$ mapping $O_\sigma\times[0,\varepsilon_0)$ into $\mathscr{L}(Y_\sigma^1\times Y_\sigma^2\times X_\sigma; Y_{\sigma'}^1\times Y_{\sigma'}^2\times X_{\sigma'})$ is infinitely differentiable in ε.*

Remark The symbol $\mathscr{L}(A;B)$ where A and B are Banach spaces connotes the collection of bounded linear operators mapping A to B.

Proof The mapping R was obtained in Section 3 using separate arguments for $\varepsilon>0$ and $\varepsilon=0$. For $\varepsilon>0$, the constant coefficient problem (3.5) and an interation were used to construct solutions to the linearized equation. It is a standard result in functional analysis that the inverse of a mapping depending smoothly on an external parameter ε is also smooth in ε. Thus the smoothness of R for $\varepsilon>0$ is assured and it remains to check the special point $\varepsilon=0$.

Recalling the splitting based on the projection Q and its complement $I-Q$, it follows from Lemma 3 that the piece of R related to the range of Q, namely the linearized ordinary differential equation (3.3), is smooth even at $\varepsilon=0$. Therefore the remaining task is to analyze the behavior of the solution of $(I-Q)F_w(w,\varepsilon)v=(I-Q)f$ as $\varepsilon\to0^+$. This will be accomplished by checking continuity and then applying a bootstrapping argument to obtain higher differentiability in ε.

From Lemma 6 and Lemma 7, it follows that the difference between the solution operators for $(I-Q)F_w(w,\varepsilon)v$ and the constant coefficient operator (3.5) studied in Lemma 7 tends to 0 in operator norm as $\varepsilon\to0^+$. Thus the issue of continuity is reduced to studying the constant coefficient case as $\varepsilon\to0^+$. Now recall that the solution of this problem was achieved using Fourier transforms. Indeed, the formula for v is precisely

$$v(x,\xi;\varepsilon)=\int_{-\infty}^{\infty}\int_{-h_2}^{h_1} e^{ikx}g(k;\xi,\eta)\hat{f}(k,\eta)\,d\eta\,dk\equiv G(\varepsilon)f, \tag{5.1}$$

where

$$g(k;\xi,\eta)=\begin{cases} g_2(k;\xi,\eta) & \text{for} \quad -h_2<\xi<0 \\ g_1(k;\xi,\eta) & \text{for} \quad 0<\xi<h_1 \end{cases}$$

and g_1 and g_2 are given by the formula (3.7) above. Since $\hat{f}$ decays exponentially, the integration with respect to k may be split into the integral over $\{k: |k| \geqq \varepsilon^{-1/4}\}$, which is negligible as $\varepsilon \to 0^+$, and the integral over the interval $\{k: |k| \leqq \varepsilon^{-1/4}\}$. A direct calculation using (5.1) shows that as $\varepsilon \to 0^+$ with $|k| \leqq \varepsilon^{-1/4}$, the solution $v(x, \xi; \varepsilon)$ approaches the solution of the x-independent problem

$$\begin{aligned} &-v_{\xi\xi} = f_1 \quad \text{in } 0 < \xi < h_1, \\ &-v_{\xi\xi} = f_2 \quad \text{in } -h_2 < \xi < 0, \\ &-2\rho_2 v_\xi(x, 0^-) + 2\rho_1 v_\xi(x, 0^+) + 2\frac{g(\rho_2 - \rho_1)}{U_*^2} v(x, 0) = f_3, \end{aligned} \tag{5.2}$$

that satisfies $Qv = 0$, where $f = (I - Q)f$.

It is now shown that G is smooth in ε via a bootstrap argument. Fix an ε in the interval $[0, \varepsilon_0)$ and let h be such that $\varepsilon + h$ also lies in this interval. The function $\chi = [G(\varepsilon + h) - G(\varepsilon)]f$ satisfies the equation

$$\begin{aligned} &-\varepsilon\chi_{xx} - \chi_{\xi\xi} = h(G(\varepsilon + h)f)_{xx} \quad \text{in } \{0 < \xi < h_1\} \cup \{-h_2 < \xi < 0\}, \\ &-2\rho_2 \chi(x, 0^-) + 2\rho_1 \chi(x, 0^+) + 2\frac{g(\rho_2 - \rho_1)}{U_*^2} \chi(x, 0) = 0, \end{aligned}$$

i.e. $\chi = hG(\varepsilon)LG(\varepsilon + h)f$ where the matrix L of differential operators is $L = \operatorname{diag}(\partial_x^2, \partial_x^2, 0)$. Thus the difference quotient satisfies

$$\frac{1}{h}[G(\varepsilon + h) - G(\varepsilon)]f = G(\varepsilon)LG(\varepsilon + h)f,$$

and so by the continuity of G as a function of ε, we obtain that G is differentiable as a function of ε and that

$$G'(\varepsilon)f = [G(\varepsilon)LG(\varepsilon)]f. \tag{5.3}$$

Since the right-hand side of this relationship is a continuous function of ε, G is seen to be a C^1-function of ε. Now that G is known to be continuously differentiable in ε, we may argue inductively from (5.3) that G is C^∞ as a function of ε. Thus R is C^∞ in ε as well. ∎

Theorem 10. *The solitary-wave solution $w = w(\varepsilon)$ of (2.4) corresponding to the two-fluid system with a fixed upper boundary is a C^∞ mapping of $[0, \varepsilon_0)$ into $Y_\sigma^1 \times Y_\sigma^2 \times X_\sigma$ for any $\sigma < 1/2$.*

Proof Since R is known to be a two-sided inverse, it follows from the Nash–Moser implicit-function theorem that w is Lipschitz. Hence there is a constant C independent of ε such that

$$\|w(\varepsilon)-w(\varepsilon')\|_{Y^1_\sigma\times Y^2_\sigma\times X_\sigma}\leqq C|\varepsilon-\varepsilon'|, \tag{5.4}$$

for all $\varepsilon, \varepsilon'$ in $[0, \varepsilon_0)$ and any $\sigma<1/2$. Now, by Taylor's theorem, we have

$$0=\tilde{F}(w(\varepsilon'),\varepsilon')-\tilde{F}(w(\varepsilon),\varepsilon)$$

$$=\tilde{F}_w(w(\varepsilon),\varepsilon)(w(\varepsilon')-w(\varepsilon))+\tilde{F}_\varepsilon(w(\varepsilon),\varepsilon)(\varepsilon'-\varepsilon)+0(\|w(\varepsilon')-w(\varepsilon)\|^2_{Y^1{}_\sigma\times Y^2{}_\sigma\times X_\sigma}+|\varepsilon'-\varepsilon|^2).$$

Applying $R(w(\varepsilon),\varepsilon)$ to this relationship yields

$$w(\varepsilon')-w(\varepsilon)+R(w(\varepsilon),\varepsilon)\tilde{F}_\varepsilon(w(\varepsilon),\varepsilon)(\varepsilon'-\varepsilon)=0(|\varepsilon'-\varepsilon|^2),$$

where (5.4) was used to bound the norm of $w(\varepsilon')-w(\varepsilon)$ in the remainder term. Thus $w'(\varepsilon)$ is seen to exist, and, moreover,

$$w'(\varepsilon)=-R(w(\varepsilon),\varepsilon)\tilde{F}_\varepsilon(w(\varepsilon),\varepsilon). \tag{5.5}$$

Again, one deduces that $w(\varepsilon)$ is a C^1 function of ε because the right-hand side of (5.5) is a continuous function of ε with values in $Y^1_\sigma\times Y^2_\sigma\times X_\sigma$. The relation (5.5) may now be used to obtain the desired result. ■

The case of the free upper surface proceeds in a completely analogous fashion. The precise kernel for the constant-coefficient approximation to $(I-Q)\tilde{G}_w(w,\varepsilon)v=f$ is of course different, but its limit as $\varepsilon\to 0^+$ is again determined by a direct calculation based on Fourier transforms. The rest of the argument is unchanged. Thus the following result obtains in this case.

THEOREM 11. *The solitary-wave solution* $w=w(\varepsilon)$ *of (4.1) corresponding to the two-fluid system with free upper boundary is a* C^∞ *mapping of* $[0,\varepsilon_0)$ *into* $\tilde{Y}^1_\sigma\times Y^2_\sigma\times X_\sigma\times X_\sigma$ *for any* $\sigma<1/2$.

6. REMARKS ON A VARIATIONAL CHARACTERIZATION OF THE SOLITARY WAVE

The original system of Euler equations for the full time-dependent problem has a Hamiltonian structure (Bowman, 1987) similar to that discovered by Zakharov (1968) for the analogous free-surface problem. In this formulation, there are two variables, which correspond to the deviation of the interface $\eta(x,t)$ from its rest position and a potential variable

$$V(x,t)=\rho_2(\phi(x,\eta(x,t),t))_x-\rho_1(\phi(x,\eta(x,t),t))_x$$

where ϕ is the velocity potential. The variable V may also be expressed in terms of ψ, the stream function. The total energy of the system is given by the functional

$$H(\eta, V)=\iint_{\Omega_1} \frac{1}{2}\rho_1|\nabla\psi|^2\,dx\,dy+\iint_{\Omega_2} \frac{1}{2}\rho_2|\nabla\psi|^2\,dx\,dy+\int_{\mathbf{R}} \frac{1}{2}(\rho_2-\rho_1)\eta^2\,dx, \tag{6.1}$$

where $\Omega_1=\{\eta(x,t)<y<h_1\}$, $\Omega_2=\{-h_2<y<\eta(x,t)\}$, and V determines ψ via a Riemann–Hilbert problem. The evolution equations have the form

$$\begin{pmatrix}\eta\\ V\end{pmatrix}_t=\begin{pmatrix}0 & -\partial_x\\ -\partial_x & 0\end{pmatrix}\begin{pmatrix}\delta H/\delta\eta\\ \delta H/\delta V\end{pmatrix},$$

and traveling waves with velocity c thus satisfy

$$\begin{pmatrix}0 & -\partial_x\\ -\partial_x & 0\end{pmatrix}\begin{pmatrix}\delta H/\delta\eta\\ \delta H/\delta V\end{pmatrix}=\begin{pmatrix}-c\eta_x\\ -cV_x\end{pmatrix}=c\begin{pmatrix}0 & -\partial_x\\ -\partial_x & 0\end{pmatrix}\begin{pmatrix}\delta I/\delta\eta\\ \delta I/\delta V\end{pmatrix},$$

where $I=\int_{\mathbf{R}} V\eta\,dx$. Thus the solitary wave satisfies

$$\begin{pmatrix}0 & -\partial_x\\ -\partial_x & 0\end{pmatrix}\begin{pmatrix}\delta(H-cI)/\delta\eta\\ \delta(H-cI)/\delta V\end{pmatrix}=0.$$

With appropriate conditions at ∞, this leads to the requirement that $H'-cI'=0$ when evaluated at a solitary wave.

Benjamin (1984) proposes an approach to the existence of solitary waves based on this variational problem, which may be regarded as searching for an extremal of H for a given value of I. It is here remarked that this attractive idea presents severe technical difficulties because the functional in question is not bounded above or below. More precisely, note that H involves the L_2-norm of η and essentially the H^1-norm of ψ or ϕ over the full domain. By standard Sobolev trace theory, this is equivalent to the $H^{-1/2}$-norm of the value of ϕ on $y=\eta$ (assumed smooth), which is equivalent to the $H^{1/2}$ norm of V. Thus $I=\int V\eta\,dx$, which is clearly indefinite, cannot be dominated by H and therefore $H-cI$ is indefinite. The Hessian $H''-cI''$ evaluated at the solitary wave is a symmetric operator with infinitely many positive and infinitely many negative eigenvalues. Besides complicating any existence theory, an analysis of the stability of solitary-wave solutions of this system cannot be carried out using the known Lyapunov techniques (Arnol'd, 1966; Benjamin, 1972; Bona, 1975; Bona *et al.*, 1987 and the references contained therein). These theories all rely upon $H''-cI''$ having one negative and one zero eigenvalue.

The same difficulty arises in the case of a single homogeneous layer with a free surface.

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