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A Mathematical Theory for Viscous, Free-Surface Flows over a Perturbed Plane

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Abstract

We consider steady, two-dimensional motions of an incompressible, Newtonian fluid flowing under gravity down an inclined channel. If the bottom of the channel is flat, the flow is the classical Poiseuille-Nusselt flow and the free surface is then a plane parallel to the bottom. Motivated by the recent experimental and numerical studies of PRITCHARD, SCOTT & TAVENER, we look at bottom configurations which possess some localized, non-uniform structure. We present an existence theory for steady, highly viscous flow over such configurations. An important consequence of our theory is that the steady flows whose existence is established decay exponentially rapidly to the unperturbed Poiseuille-Nusselt flow away from the local variation in the channel bottom profile.

1. Introduction

The determination of the free surface in fluid flows not all of whose boundaries are constrained is a problem of both scientific and practical interest. Many industrial processes and natural phenomena exhibit fluid motion with steady or evolutionary free surfaces, and this has given a lot of impetus to the study of such flows.

Several years ago, PRITCHARD (1986) concluded a study of the flow of viscous fluid off the end of a finite, inclined channel. In his experiments, which turned up some very interesting phenomena, the flows were dominated by viscosity and surface-tension effects. While the range of both steady and time-dependent flows discovered is fascinating, it appears to be beyond the reach of our present analytical or numerical tools. This prompted another, related experiment which held out more hope for both rigorous analytical treatment and computational analysis. In this experiment, fluid flows at a constant rate under gravity down an inclined channel whose bottom is planar except for a pair of smooth bumps (see Figure 1). The fluid pours off the end of the channel into a reservoir and is then pumped back to the top of the channel. For relatively small flow rates, these motions are also dominated by viscous and surface-tension effects. PRITCHARD observed interesting steady flows in this situation, which are used to check the accuracy of a numerical scheme. This work is reported in a forthcoming paper by PRITCHARD, SCOTT & TAVENER (1991).

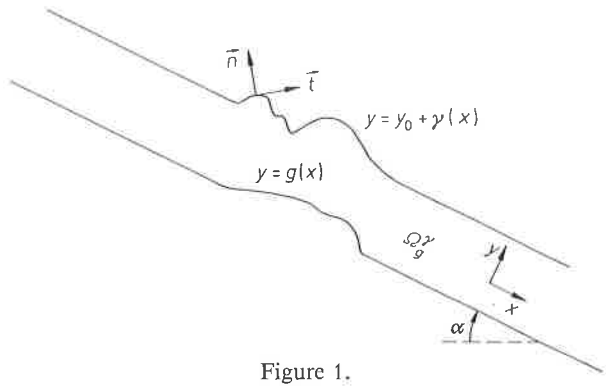

Figure 1.

Our purpose here is to provide an analytical framework modelling these flows and a rigorous existence theory for steady motion corresponding to a wide range of bottom configurations including the one described above. The long, but finite channel used in the experiments is here modelled by an infinitely long channel. An interesting and useful by-product of our theory is that the steady flows corresponding to a bottom perturbation of finite extent decay exponentially rapidly to Poiseuille-Nusselt flow away from the perturbation. This result is helpful in formulating and analyzing the aforementioned numerical scheme. The analytical model assumes an infinitely long channel as a way of getting around the specification of potentially complicated boundary conditions inherent in the consideration of a finite channel. The numerical scheme must be constructed relative to a finite domain, however, and so the issue of boundary conditions cannot be ignored. Because of the rapid decay to Poiseuille-Nusselt flow, it is reasonable to impose exactly the Poiseuille-Nusselt flow conditions at both the upstream and downstream boundaries, provided these are situated a reasonable distance from the part of the channel bottom that has non-uniform (but two-dimensional) structure.

Precursors of the present work include the work of JEAN (1980) and SOLONNIKOV (1980, 1982, 1983), which was concerned with a finite channel. A more recent paper of SOLONNIKOV's (1989) deals with a related, though somewhat different problem in which liquid pours down an inclined plane into an infinite reservoir. In this latter problem, the flow approaches a Poiseuille-Nusselt flow upstream and is matched to a Jeffrey-Hamel flow in a sector of the reservoir.

While the present work was in editorial review, a manuscript by NAZAROV & PILECKAS (1991) came to our attention. While different in a number of technical ways, the underlying physical problem is the same and the general mathematical approach overlaps considerably the present work. A reader interested in our paper will certainly want to consult the NAZAROV & PILECKAS paper as well.

The plan of the paper is as follows. Section 2 lays out the model and the governing equations while Section 3 contains a reformulation of the mathematical issue as a fixed-point problem for a mapping T. A central, technical result, Theorem 3.1, concerning the mapping T is stated at this point. The proof of this result occupies the next three sections. Under the provisional assumption that both the bottom configuration and the free surface are known, the flow domain is mapped to a fixed domain and the velocity field is determined from the stream function in Section 4. In Section 5, the associated pressure field is determined, leading to a short proof of Theorem 3.1 in Section 6. The existence and uniqueness theorem for the free-surface flow is then shown in Section 7 to follow from an implicit-function theorem. Sections 8 and 9 are devoted to establishing the validity of the hypotheses that are needed to invoke the Implicit-Function Theorem. The last section contains the proof of a technical result that arose in Section 4.

2. The Governing Equations

Attention will be given to the situation depicted in Figure 1, namely the flow of a viscous liquid down an infinitely long channel of uniform width, the bed of which is located at a height $g(x)$ above a plane P that slopes at an angle α. It is supposed that the coordinate frame (x, y) is located in this plane and that g is measured in the direction y normal to the plane. In this set of coordinates, we shall further suppose the bottom configuration g of the channel to be smooth and to have compact support, thus representing a perturbation of finite extent of a perfect, planar surface. (In fact, it will only be required that g rapidly approach zero both upstream and downstream.) It is assumed that the fluid motion is steady and two-dimensional. The resulting free surface is taken to be the curve $y = y_0 + \gamma(x)$. The function γ represents the principal unknown in the problem while y_0 is the height of the free surface above P in the limit as $|x|$ becomes unboundedly large. It is expected that the steady flow will tend to the classical Poiseuille-Nusselt flow corresponding to the liquid depth y_0 far upstream and downstream of the portion of the bottom that is not essentially flat. If we let

$$\Omega_g^\gamma = \{(x, y) : -\infty < x < \infty,\ g(x) < y < y_0 + \gamma(x)\}$$

denote the flow domain, then this latter expectation leads to the following mathematical formulation of our problem.

Given a bottom configuration g, find a function $\gamma: \mathbb{R} \to \mathbb{R}$ (the free surface), a vector-valued function u (the velocity field) and a scalar function p (the pressure field) defined on Ω_g^γ such that

$$\begin{gathered}
-\Delta u + \nabla p = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \quad \text{in } \Omega_g^\gamma, \\
\nabla \cdot u = 0 \quad \text{in } \Omega_g^\gamma, \\
u(x, g(x)) = 0, \quad (u \cdot n)\,(x, y_0 + \gamma(x)) = 0, \\
(\nabla_S u \cdot n \cdot t)\,(x, y_0 + \gamma(x)) = 0,
\end{gathered} \tag{2.1}$$

$$\frac{\gamma''(x)}{(1 + (\gamma'(x))^2)^{3/2}} = \tau\{-p(x, y_0 + \gamma(x)) + (\nabla_S u \cdot n \cdot n)\,(x, y_0 + \gamma(x))\}, \tag{2.2}$$

$$\lim_{|x| \to \infty} u(x, y) = u_P(y) = \begin{bmatrix} C_1 (y_0 y - \frac{1}{2} y^2) \\ 0 \end{bmatrix}, \quad \lim_{|x| \to \infty} p(x, y) = p_P(y) = C_2(y - y_0). \tag{2.3}$$

In this formulation, we have taken both the kinematic viscosity and the density to have the value one. Additionally, the flows are taken to be sufficiently slow that inertial effects can be ignored, so the nonlinear term as well as the time derivative in the Navier-Stokes equations has been set to zero. For the equations above, $C_1 = G \sin(\alpha) > 0$, $C_2 = -G \cos(\alpha) < 0$ are the components of the gravitational force field in the chosen coordinate frame, (t, n) is the positively oriented Frenet frame on the boundary with n pointing outward, and G is the gravity constant. Furthermore, $\nabla_S u := \frac{1}{2}(\nabla u + \nabla u^T)$ is the deviatoric part of the stress tensor $\sigma := -pID + \nabla_S u$; the boundary conditions in (2.1) are a non-slip condition at the solid boundary, a contact condition at the free surface, and a zero-shear-stress condition at the free surface. Equation (2.2) is the condition of equilibrium for the free surface, and the positive constant τ is the surface-tension coefficient, while (2.3) assigns a prescribed behavior of the velocity- and pressure-fields at infinity, that of the standard Poiseuille-Nusselt flow with height y_0. The latter, whose velocity and pressure field are represented by u_P and p_P, respectively, corresponds to $g = \gamma \equiv 0$.

3. Formulation as a Fixed-Point Problem

The functional-analytic setting to be used in the analysis of (2.1), (2.2), (2.3) is now introduced and the problem of finding a suitable solution of the equations is recast as the problem of finding a fixed point of a certain operator.

First, the function spaces that are central to our analysis are defined. Given $c > 0$, $m \geqq 0$ an integer and λ with $0 < \lambda < 1$, we define $B_c^{m,\lambda}(\mathbb{R})$ to be the linear space

$$\left\{ f \in C^{m,\lambda}(\mathbb{R}) : \sum_{0 \leqq k \leqq m} \sup_{x \in \mathbb{R}} e^{c|x|} |D_x^k f(x)| < \infty \right\}, \tag{3.1}$$

where $C^{m,\lambda}(\mathbb{R})$ is the usual Hölder class. The space $B_c^{m,\lambda}(\mathbb{R})$ is a Banach algebra with the norm

$$\|f\|_{m,c,\lambda} := \sum_{k=0}^{m} \sup_{x\in\mathbb{R}} e^{c|x|}|D_x^k f(x)| + \sup_{\substack{(x,x')\in\mathbb{R}^2\\ x\neq x'}} \frac{|D_x^m f(x) - D_x^m f(x')|}{|x-x'|^\lambda}, \tag{3.2}$$

or with the equivalent norm

$$|f|_{m,c,\lambda} := \sup_{x\in\mathbb{R}} [e^{c|x|}(|f(x)| + |D_x^m f(x)|)] + \sup_{\substack{(x,x')\in\mathbb{R}^2\\ x\neq x'}} \frac{|D_x^m f(x) - D_x^m f(x')|}{|x-x'|^\lambda}. \tag{3.3}$$

Remark 3.1. In addition to defining a Banach-algebra structure on $B_c^{m,\lambda}(\mathbb{R})$, the bilinear mapping $(f, g) \mapsto f\cdot g$ is continuous as a mapping of $B_c^{m,\lambda}(\mathbb{R})\times C^{n,\lambda}(\mathbb{R})$ into $B_c^{m',\lambda}(\mathbb{R})$ where $m' = \min\{m, n\}$. This easy result will find use later.

The operator T central to the subsequent analysis is now introduced. Suppose that (g, γ) lies in $B_c^{4,\lambda}(\mathbb{R})\times B_c^{4,\lambda}(\mathbb{R})$ and suppose that g and γ are restricted in size in a way to be made precise in the next section. Then the domain Ω_g^γ is well defined, and one can solve in principle equations (2.1) with the asymptotic conditions (2.3) for the velocity field u and the pressure p. Taking these as determined, we can then solve the differential equation

$$\begin{aligned}\mathscr{L}T &:= D_x^3 T - \tau C_1 y_0 D_x^2 T + \tau C_2 D_x T + 3\tau \frac{C_1}{y_0} T\\ &= \tau D_x\{(1+(\gamma'(x))^2)^{3/2}[-p(x, y_0+\gamma(x)) + (\nabla_S u\cdot n\cdot n)(x, y_0+\gamma(x))]\}\\ &\quad - \tau C_1 y_0 D_x^2\gamma + \tau C_2 D_x\gamma + 3\tau\frac{C_1}{y_0}\gamma \end{aligned}\tag{3.4}$$

for $x\in\mathbb{R}$, with zero boundary conditions at $\pm\infty$. The correspondence $(g, \gamma)\mapsto T$ will be viewed as an operator which happens to map $B_c^{4,\lambda}(\mathbb{R})\times B_c^{4,\lambda}(\mathbb{R})$ into $B_c^{4,\lambda}(\mathbb{R})$. The solution $T = T(g, \gamma)$ of (3.4) is not in general a solution of the original surface-tension equation (2.2). Indeed, equation (2.2) has been considerably modified for technical reasons. However, it happens that if $T(g, \gamma) = \gamma$, then (3.4) and (2.2) are indeed equivalent.

The idea behind our theory is to apply the Implicit-Function Theorem to the operator $\gamma - T(g, \gamma)$. This approach seems natural since $T(0, 0) = 0$, a relation that corresponds to the fact that if the bottom of the channel is perfectly planar, then the Poiseuille-Nusselt flow $(u, p) = (u_P, p_P)$ with a planar free surface at height y_0 above the channel bed is a solution of the flow problem (2.1)–(2.3). The seemingly awkward choice for T in (3.4) will appear as quite natural in the calculations to appear subsequently.

Here is the principal qualitative result concerning the operator T. The proof of the result is somewhat lengthy, and comprises most of the content of Sections 4, 5 and 6.

Theorem 3.1. *Let a value τ of the surface-tension parameter and a Hölder exponent $\lambda \in (0, 1)$ be given, and let $y_0 > 0$ be specified. Then there exists a value $\varepsilon_1 > 0$ and $\rho_0 > 0$ such that if $0 < \varepsilon = \sin(\alpha) < \varepsilon_1$ and $0 < c < \min\{\tilde{c}, r_2(\varepsilon)\}$, where $\tilde{c}$ and $r_2(\varepsilon)$ are defined in Proposition 4.2 and Lemma 4.1, respectively, then* (i) *T is well defined from the open ball B_0 of radius ρ_0 centered at the origin in $B_c^{4,\lambda} \times B_c^{4,\lambda}$ into $B_c^{4,\lambda}$, and* (ii) *T is continuously Fréchet differentiable on B_0.*

As mentioned before, the proof of this qualitative theorem will require some effort. In the next section, for sufficiently small g and γ, we will map Ω_g^γ onto a fixed, infinite strip. The velocity field may then be determined as a function of (g, γ). In Section 5, a similar determination of the pressure is made. In Section 6, the differential equation (3.4) is analyzed, and this puts the finishing touches on the proof of Theorem 3.1.

4. The Dependence of the Velocity on (g, γ)

Because the velocity field is divergence-free, we can write u as $\nabla \times \Psi$. Taking account of the asymptotic conditions (2.3) we expect the flow to satisfy, we find it prudent to write $\Psi = \Phi + \psi$ where Φ is a stream function in Ω_g^γ closely associated to Poiseuille-Nusselt flow, chosen to satisfy as many boundary conditions as possible. One then envisions working with the perturbation stream function ψ as a primary dependent variable, and expects that it will decay to zero at infinity. An appropriate choice for Φ, which will be used throughout, is

$$\Phi(x, y) = C_1\left\{\frac{y_0}{2}\left(\frac{y_0(y - g(x))}{y_0 + \gamma(x) - g(x)}\right)^2 - \frac{1}{6}\left(\frac{y_0(y - g(x))}{y_0 + \gamma(x) - g(x)}\right)^3\right\}. \tag{4.1}$$

Naturally, the bottom profile g and the putative free surface γ are not unrestricted in size here. However, if

$$\max\left\{\sup_{x\in\mathbb{R}} |g(x)|, \sup_{x\in\mathbb{R}} |\gamma(x)|\right\} < \frac{y_0}{2}, \tag{4.2}$$

then Ω_g^γ is indeed a well-defined, connected domain and Φ is a smooth function of g and γ in this domain. The assumption (4.2) will be in force throughout our analysis. If one takes the curl of the equation in (2.1) and rewrites everything in terms of ψ, one obtains the following boundary-value problem for ψ:

$$\Delta^2\psi = -\Delta^2\Phi \quad \text{in } \Omega_g^\gamma,$$

$$\psi(x, g(x)) = \frac{\partial\psi}{\partial\nu}(x, g(x)) = 0 \quad \text{for } x \in \mathbb{R},$$

$$\psi(x, y_0 + \gamma(x)) = 0 \quad \text{for } x \in \mathbb{R},$$

$$\tfrac{1}{2}\,(1-\gamma'^2(x))\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)(x, y_0+\gamma(x)) - 2\gamma'(x)\,\frac{\partial^2\psi}{\partial x\,\partial y}\,(x, y_0+\gamma(x))$$
$$= -\tfrac{1}{2}\,(1-\gamma'^2(x))\left(\frac{\partial^2\Phi}{\partial y^2}-\frac{\partial^2\Phi}{\partial x^2}\right)(x, y_0+\gamma(x)) + 2\gamma'(x)\,\frac{\partial^2\Phi}{\partial x\,\partial y}\,(x, y_0+\gamma(x))$$
$$\text{for } x\in\mathbb{R} \tag{4.3}$$

where $\Delta^2 = \dfrac{\partial^4}{\partial x^4} + 2\,\dfrac{\partial^4}{\partial x^2\,\partial y^2} + \dfrac{\partial^4}{\partial y^4}$ is the bi-Laplacian. The boundary conditions follow from those imposed on u, except that in principle one only has $\psi(x, g(x)) = \psi(x, y_0+\gamma(x)) = \text{constant}$, but we have chosen this constant to be zero.

Consider the change of variables

$$\hat{x} = x, \quad \hat{y} = \frac{y_0(y-g(x))}{y_0+\gamma(x)-g(x)}, \tag{4.4}$$

which maps Ω_g^γ onto the strip $\Sigma = \mathbb{R}\times(0, y_0)$. Let $\hat{\psi}$ be defined in Σ by $\hat{\psi}(\hat{x}, \hat{y}) = \psi(x, y)$ and introduce the space

$$B_c^{m,\lambda}(\bar{\Sigma}) := \left\{\hat{\psi}\in C^{m,\lambda}(\bar{\Sigma}) : \sup_{k+l\le m}\ \sup_{(\hat{x},\hat{y})\in\bar{\Sigma}} e^{c|\hat{x}|}\,|D_{\hat{x}}^k D_{\hat{y}}^l \hat{\varphi}(\hat{x}, \hat{y})| < +\infty\right\}$$

modelled after $B_c^{m,\lambda}(\mathbb{R})$. The norm of a function $\hat{\varphi}$ in $B_c^{m,\lambda}(\bar{\Sigma})$ is defined to be

$$\|\hat{\varphi}\|_{m,c,\lambda} = \sum_{k+l\le m}\ \sup_{(\hat{x},\hat{y})\in(\bar{\Sigma})^2} e^{c|\hat{x}|}\,|D_{\hat{x}}^k D_{\hat{y}}^l\hat{\varphi}(\hat{x}, \hat{y})|$$
$$+ \sup_{k+l=m}\ \sup_{\substack{((\hat{x},\hat{y}),(\hat{x}',\hat{y}'))\in(\bar{\Sigma})^2\\(\hat{x},\hat{y})\neq(\hat{x}',\hat{y}')}} \frac{|D_{\hat{x}}^k D_{\hat{y}}^l\hat{\varphi}(\hat{x}, \hat{y}) - D_{\hat{x}}^k D_{\hat{y}}^l\hat{\varphi}(\hat{x}', \hat{y}')|}{((\hat{x}-\hat{x}')^2+(\hat{y}-\hat{y}')^2)^{\lambda/2}}. \tag{4.5}$$

The space $B_c^{m,\lambda}(\bar{\Sigma})$ equipped with this norm is a Banach algebra. One can also define $|\hat{\varphi}|_{m,c,\lambda}$, the analogue of (3.3); furthermore, Remark 3.1 is easily extended to the case of functions defined on Σ.

With these preliminaries in hand, we now state the main result of this section:

Theorem 4.1. *There exists an open ball $\mathscr{B}$ of radius $r_0 > 0$, centered at the origin in $B_c^{4,\lambda}(\mathbb{R})\times B_c^{4,\lambda}(\mathbb{R})$ such that whenever $(g, \gamma)\in\mathscr{B}$, the following statements hold:*
(i) *Problem* (4.3) *has a unique solution ψ.*
(ii) *$\hat{\psi}$, the image of ψ through the change of variables* (4.4), *is in $B_c^{4,\lambda}(\bar{\Sigma})$.*
(iii) *The mapping $S: (g, \gamma)\to\hat{\psi}$ is continuously differentiable from $\mathscr{B}$ into $B_c^{4,\lambda}(\bar{\Sigma})$.*

The following proposition will aid materially in the proof of Theorem 4.1. Its somewhat technical proof is postponed until Section 10.

Proposition 4.2. *Consider the boundary-value problem*

$$\Delta^2 v = b_1 \text{ in } \Sigma; \quad v(\hat{x}, 0) = b_2(\hat{x}), \ \hat{x} \in \mathbb{R};$$

$$\frac{\partial v}{\partial \hat{y}}(\hat{x}, 0) = b_3(\hat{x}), \ \hat{x} \in \mathbb{R}; \quad \hat{v}(\hat{x}, y_0) = b_4(\hat{x}), \ \hat{x} \in \mathbb{R}; \tag{4.6}$$

$$\frac{1}{2}\left(\frac{\partial^2 v}{\partial \hat{y}^2} - \frac{\partial^2 v}{\partial \hat{x}^2}\right)(\hat{x}, y_0) = b_5(\hat{x}), \quad \hat{x} \in \mathbb{R};$$

with the assumptions that $(b_1, b_2, b_3, b_4, b_5) \in B_c^{0,\lambda}(\bar{\Sigma}) \times B_c^{4,\lambda}(\mathbb{R}) \times B_c^{3,\lambda}(\mathbb{R}) \times B_c^{4,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R})$. *There exists* $\tilde{c} > 0$ *depending only on* y_0 *such that whenever* $0 < c < \tilde{c}$, *Problem* (4.6) *has a unique solution* $v \in B_c^{4,\lambda}(\bar{\Sigma})$. *Furthermore, the solution map is a topological isomorphism between the corresponding spaces. Finally, the norm of the solution map is bounded on any compact subinterval of* $[0, \tilde{c})$.

An immediate corollary of the proof of Proposition 4.2 is the following result. Commentary about this result appears in Section 10 after the proof of Proposition 4.2.

Corollary 4.3. *The conclusions of Proposition 4.2 hold when*

$$(f_i)_{1 \leq i \leq 5} \in B_c^{1,\lambda}(\bar{\Sigma}) \times B_c^{5,\lambda}(\mathbb{R}) \times B_c^{4,\lambda}(\mathbb{R}) \times B_c^{5,\lambda}(\mathbb{R}) \times B_c^{3,\lambda}(\mathbb{R}),$$

provided that the norm on v *is that of* $B_c^{5,\lambda}(\bar{\Sigma})$ *throughout.*

Proof of Theorem 4.1. The boundary-value problem (4.3) for ψ transforms under the change of variables (4.4) into a boundary-value problem for $\hat{\psi}$ on Σ. Consider for instance the differential operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$; they transform, respectively, into

$$\frac{\partial}{\partial \hat{x}} + \frac{(-g' y_0^2 - g' y_0 \gamma - y_0 y \gamma' + y_0 y g' + y_0 g \gamma')}{(y_0 + \gamma - g)^2} \frac{\partial}{\partial \hat{y}} \quad \text{and} \quad \frac{y_0}{y_0 + \gamma - g} \frac{\partial}{\partial \hat{y}}.$$

Iterating this result, one sees that the differential operator $\frac{\partial^k}{\partial x^p \, \partial y^{k-p}}$ transforms into

$$\frac{\partial^k}{\partial x^p \, \partial y^{k-p}} + \frac{1}{(y_0 + \gamma - g)^{n(k)}} \sum_{q_1 + q_2 \leq k} P_{q_1, q_2} \frac{\partial^{q_1 + q_2}}{\partial x^{q_1} \, \partial y^{q_2}}, \tag{4.7}$$

where P_{q_1, q_2} is a polynomial in the $(2k + 2)$ functions $(g, g', \ldots, D_x^k g, \gamma, \gamma', \ldots, D_x^k \gamma)$ such that $P_{q_1, q_2}(0) = 0$, and where $n(k)$ is an integer depending on k. It follows that the boundary-value problem for $\hat{\psi}$ is a perturbation of Problem (4.6) on Σ. More precisely, denote by $\mathscr{A}$ the linear operator defined by

$$\mathscr{A}: B_c^{4,\lambda}(\bar{\Sigma}) \to B_c^{0,\lambda}(\bar{\Sigma}) \times B_c^{4,\lambda}(\mathbb{R}) \times B_c^{4,\lambda}(\mathbb{R}) \times B_c^{3,\lambda}(\mathbb{R}) \times B_c^{2,\lambda}(\mathbb{R}), \tag{4.8}$$

$$v \mapsto \left(\Delta^2 v,\ v(\cdot, 0),\ v(\cdot, y_0),\ \frac{\partial v}{\partial y}(\cdot, 0),\ \tfrac{1}{2}\left(\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 v}{\partial x^2}\right)(\cdot, y_0)\right),$$

and let $\mathscr{A}_g^\gamma$ be the corresponding operator, for g, γ non-zero, resulting from the change of variables (4.4). Denoting by $\mathscr{Y}$ the target space of $\mathscr{A}$ and using the expressions in (4.7), we easily verify that

$$\|(\mathscr{A} - \mathscr{A}_g^\gamma)v\|_{\mathscr{Y}} \leqq L(\|g\|_{4,c,\lambda}, \|\gamma\|_{4,c,\lambda})\|v\|_{4,c,\lambda}, \tag{4.9}$$

where L is a continuous function with $L(0, 0) = 0$. In fact, because of Remark 3.1, one can replace the norms of g and γ in $B_c^{4,\lambda}(\mathbb{R})$ by their norms in the (unweighted) Hölder space $C^{4,\lambda}(\mathbb{R})$, and (4.9) remains valid; indeed, L is independent of the value of c. Now, we use the fact established in Proposition 4.2 that $\mathscr{A}$ is a continuous isomorphism to assert the existence of an r_0 such that $\mathscr{A}_g^\gamma$ is also an isomorphism provided that

$$\|g\|_{4,c,\lambda} + \|\gamma\|_{4,c,\lambda} \leqq r_0. \tag{4.10}$$

In order to prove (ii) (and, by the same token, (i)), one just needs to verify that the right-hand sides for the boundary-value problem for $\hat{\psi}$ belong to $\mathscr{Y}$, a fact that is established by straightforward calculations.

Attention is now given to the differentiability of S. Define two mappings S_1 and S_2 as follows:

$$S_1:\ \mathscr{B} \to \mathscr{L}(B_c^{4,\lambda}(\bar{\Sigma}), \mathscr{Y}),$$
$$(g, \gamma) \mapsto \mathscr{A}_g^\gamma,$$
$$S_2:\ \text{Isom}\,(B_c^{4,\lambda}(\bar{\Sigma}), \mathscr{Y}) \to \text{Isom}\,(\mathscr{Y}, \mathscr{B}_c^{4,\lambda}(\bar{\Sigma}))$$
$$L \mapsto L^{-1}.$$

Then it follows that

$$S(g, \gamma) = (S_2 \circ S_1(g, \gamma))\,(\mathscr{F}(g, \gamma)),$$

where $\mathscr{F}(g, \gamma)$ is the right-hand side of the boundary value problem for $\hat{\psi}$; in fact, $\mathscr{F}(g, \gamma) = -\mathscr{A}_g^\gamma(\hat{\Phi}) + (0, 0, \frac{1}{3}C_1 y_0^3, 0, 0)$ where $\hat{\Phi}$ is the image of Φ defined in (4.1) under the change of variables (4.4), i.e., $\hat{\Phi}(\hat{x}, \hat{y}) = C_1(\frac{1}{2}y_0\hat{y}^2 - \frac{1}{6}\hat{y}^3)$. As it is well known that S_2 is a continuously differentiable mapping, it is simply a matter of combining the chain rule, the product rule and Lemma 4.4 below together with the fact that $\hat{\Phi}$ is independent of (g, γ) to conclude the validity of Theorem 4.1. □

Lemma 4.4. *Let P be a rational function of k variables which is devoid of poles in a neighborhood of the origin in $\mathbb{R}^k$ (k a positive integer) such that $P(0) = 0$. Then the mapping*

$$\mathscr{P}: (g_1, \ldots, g_k) \mapsto P(g_1, \ldots, g_k)$$

maps $\Pi_{i=1}^{k} B_c^{n_i,\lambda}(\mathbb{R})$ *into* $B_c^{n_0,\lambda}(\mathbb{R})$, *where* $n_i \geqq 0$ *are integers, and* $n_0 = \min\{n_i : 1 \leqq i \leqq k\}$; *moreover,* $\mathscr{P}$ *is continuously differentiable in a neighborhood* $\mathscr{N}$ *of the origin in* $\Pi_{i=1}^{k} B_c^{n_i,\lambda}(\mathbb{R})$.

Remark 4.1. The elementary proof of this result is omitted. When applying Lemma 4.4 in the proof of Theorem 4.1, we consider expressions of the form $P(g, \ldots, D_x^k g, \gamma, \ldots, D_x^k \gamma) \dfrac{\partial^p}{\partial \bar{x}^q \, \partial \bar{y}^{p-q}}$; the differentiability of such expressions follows from Lemma 4.3, Remark 2.1, the product rule and the linearity of the mapping $(g, \gamma) \mapsto (g, \ldots, D_x^k g, \gamma, \ldots, D_x^k \gamma)$.

Remark 4.2. In what follows, use will be made of the actual expression of the derivative of S with respect to γ in a neighbourhood of the origin. However, the formula for this derivative is not needed for arbitrary g, γ.

Remark 4.3. Lemma 4.4 is easily extended to the case of a real analytic function.

Remark 4.4. The real number r_0 introduced in (4.10) is bounded away from zero provided c runs over a compact subinterval of $[0, \bar{c})$, a fact determined by reference to Proposition 4.2.

5. The Dependence of the Pressure on (g, γ)

For a given bottom profile g and a free surface γ, the pressure field p in (2.1) is obtained as a standard by-product of the results in Section 4. Classical regularity results (cf. SOLONNIKOV 1980, BEMELMANS 1981) show that p lies in $C^{2,\lambda}_{\mathrm{loc}}(\Omega_g^\gamma)$ and that ∇p is in $C^{1,\lambda}(\bar{\Omega}_g^\gamma) \times C^{1,\lambda}(\bar{\Omega}_g^\gamma)$. For the problem under consideration here, the values of p along the curve $y = y_0 + \gamma(x)$ are particulary interesting. Recalling that $u(x, y) = \nabla \times (\Phi + \psi)(x, y)$ and denoting $p(x, y_0 + \gamma(x))$ by $q(x)$, one obtains that

$$q'(x) = \left(\frac{\partial p}{\partial x} + \gamma'(x) \frac{\partial p}{\partial y}\right)(x, y_0 + \gamma(x)) \tag{5.1}$$

in the original variables (x, y). Using the first equation in (2.1), we can express $q'(x)$ in a more complicated way as

$$q'(x) = C_1 + C_2 \gamma'(x) + \Delta\left(\frac{\partial}{\partial y}(\Phi + \psi)\right) - \gamma'(x)\, \Delta\left(\frac{\partial}{\partial x}(\Phi + \psi)\right)$$

where the functions of two variables in this last formula are all evaluated at $(x, y_0 + \gamma(x))$.

It is straightforward to check that $\Delta\left(\dfrac{\partial \Phi}{\partial y}\right)(x, y_0 + \gamma(x)) + C_1$ belongs to $B_c^{1,\lambda}(\mathbb{R})$, and therefore it follows that q' belongs to $B_c^{1,\lambda}(\mathbb{R})$. The function q is determined from q' by setting it equal to zero at $-\infty$, so that

$$q(x) = C_2\gamma(x) + \int_{-\infty}^{x} \left[\triangle\left(\frac{\partial}{\partial y}(\psi + \Phi)\right) - \gamma'(s)\triangle\left(\frac{\partial}{\partial x}(\psi+\Phi)\right)(s, y_0 + \gamma(s)) + C_1\right] ds. \tag{5.2}$$

This step is legitimate thanks to the exponential decay of the integrand at $-\infty$. It is not necessarily true that q itself decays exponentially at $+\infty$, and this is the principal reason why equation (2.2) has been so severely modified in the definition of T given in (3.4).

The main result of this section is the following proposition.

Proposition 5.1. *Let* (g, γ) *be in the open ball* $\mathscr{B}$ *whose existence was established in Theorem 4.1 and let* R *be the mapping* $(g, \gamma) \mapsto q$, *where* q *is defined in* (5.2). *Then it follows that*

(i) R *is continuously differentiable from* $\mathscr{B}$ *into* $C^{2,\lambda}(\mathbb{R})$.

(ii) *The mapping* $(g, \gamma) \mapsto D_x q$ *is continuously differentiable from* $\mathscr{B}$ *into* $B_c^{1,\lambda}(\mathbb{R})$.

Proof. This is a straightforward consequence of Theorem 4.1. One transforms (5.2) into the corresponding expression with $\hat{\psi}$ and $\hat{\Phi}$ replacing ψ and Φ, respectively, and uses the differentiability of S. The only new ingredient is the obvious fact that the mapping $f(x) \mapsto \int_{-\infty}^{x} f(s)\, dx$ is linear and continuous, and therefore continuously differentiable, from $B_c^{m,\lambda}(\mathbb{R})$ into $C^{m+1,\lambda}(\mathbb{R})$, for every $m \geqq 0$. ☐

Remark 5.1. The function q converges exponentially fast to zero at $-\infty$ and exponentially fast to a constant at $+\infty$. This observation stems from the following simple lemma.

Denote by $B_c^{m,\lambda}(I)$ the set of restrictions of functions in $B_c^{m,\lambda}(\mathbb{R})$ to the interval I, equipped with the obvious norm. Of course, $B_c^{m,\lambda}(I) = C^{m,\lambda}(I)$ unless I is unbounded.

Lemma 5.2. *The correspondences*

$$f(x) \mapsto \int_{-\infty}^{x} f(s)\, ds, \quad f(x) \mapsto \int_{x}^{\infty} f(s)\, ds$$

are linear and continuous from $B_c^{m,\lambda}((-\infty, 0])$ *into* $B_c^{m+1,\lambda}((-\infty, 0])$ *and from* $B_c^{m,\lambda}([0, \infty))$ *into* $B_c^{m+1,\lambda}([0, \infty))$, *respectively.*

Proof. The proof is similar in both cases. For the second case, say, define $F(x) = \int_x^\infty f(s)\,ds$. Then one has immediately that

$$\begin{aligned}\sup_{x\in[0,\infty)} e^{cx}|F(x)| &\leqq \sup_{x\in[0,\infty)} e^{cx}\int_x^\infty |f(s)|\,ds\\ &\leqq \sup_{x\in[0,\infty)} e^{cx}\int_x^\infty e^{-cs}\|f\|_{m,c,\lambda}\,ds\\ &= \frac{1}{c}\|f\|_{m,c,\lambda},\end{aligned}$$

and the conclusion now follows. □

6. Proof of Theorem 3.1

The following elementary fact will be needed to conclude the proof of Theorem 3.1.

Lemma 6.1. *Let a function $f \in B_c^{1,\lambda}(\mathbb{R})$ be fixed. Then there is a unique $T \in B_c^{4,\lambda}(\mathbb{R})$ which is a solution of the non-homogeneous ordinary differential equation*

$$\mathscr{L}T = D_x^3T - \tau C_1 y_0 D_x^2 T + \tau C_2 D_x T + 3\tau\frac{C_1}{y_0}T = f. \tag{6.1}$$

The correspondence $f \mapsto T$ is linear and continuous from $B_c^{1,\lambda}(\mathbb{R})$ into $B_c^{4,\lambda}(\mathbb{R})$ provided that the angle of inclination $\alpha > 0$ of the plane P is small enough and provided that, once α is fixed, the decay rate $c > 0$ is small enough.

Remark 6.1. The proof of this lemma follows from that of the slightly more elaborate result in Lemma 9.1. It is worth noting that the result in Lemma 6.1 remains valid if it is only assumed that the polynomial $z^3 - \tau C_1 y_0 z^2 + \tau C_2 z + 3\tau C_1/y_0$ has real roots. As will appear later, small values of α ensure this property. As other restrictions on α intervene in the subsequent analysis, we have eschewed stating a more general result. The issue of exactly how small α and c need to be is dealt with in Lemma 9.1.

Lemma 6.1 is the last major component to be used in the proof of Theorem 3.1.

Proof of Theorem 3.1. The operator $T = T(g, \gamma)$ is defined to be the solution of equation (3.4), which is written with new notation as

$$\mathscr{L}T = D_x[(1+(\gamma')^2)^{3/2}(-q+\chi)] - \tau C_1 y_0 D_x^2\gamma + \tau C_2 D_x\gamma + 3\tau\frac{C_1}{y_0}\gamma \tag{6.2}$$

where $\mathscr{L}$ is defined in (6.1), $\chi(x) := (\nabla_S u\cdot n\cdot n)\,(x, y_0+\gamma(x))$ and q is defined in (5.2). First, it is readily deduced from Theorem 4.1 that the mapping $(g, \gamma) \mapsto \chi$ is continuously differentiable from the ball $\mathscr{B}$ defined in (4.10) having radius r_0 and centered at the origin in $B_c^{4,\lambda}(\mathbb{R}) \times B_c^{4,\lambda}(\mathbb{R})$ into $B_c^{2,\lambda}(\mathbb{R})$. In consequence of Remark 4.3 and Remark 3.1, the correspondence $(g, \gamma) \mapsto D_x[(1 + (\gamma')^2)^{3/2}\chi]$ is therefore continuously differentiable when considered as a mapping of $\mathscr{B}_0$ into $B_c^{1,\lambda}(\mathbb{R})$, where $\mathscr{B}_0$ is the ball of radius ρ_0 centered at the origin in $B_c^{4,\lambda}(\mathbb{R}) \times B_c^{4,\lambda}(\mathbb{R})$ and $\rho_0 > 0$ is such that, simultaneously,

$$\rho_0 \leqq r_0 \tag{6.3}$$

and

$$\text{the mappings } z \mapsto (1 + z^2)^{1/2} \text{ and } z \mapsto (1 + z^2)^{3/2} \text{ are analytic in } \{z : |z| \leqq \rho_0\}. \tag{6.4}$$

Conditions (6.3) and (6.4) may be achieved simply by choosing $\rho_0 < \min\{r_0, 1\}$.

An argument similar to that given for the velocity term which uses Theorem 5.1 and Remarks 4.3 and 3.1 shows that the correspondence associated with the pressure term $-D_x[(1 + (\gamma')^2)^{3/2} q]$ is a continuously differentiable mapping of $\mathscr{B}_0$ into $B_c^{1,\lambda}(\mathbb{R})$. Composing these two results with the fact that

$$\gamma \mapsto -\tau C_1 D_x^2 \gamma + \tau C_2 D_x \gamma + 3\tau \frac{C_1}{y_0} \gamma$$

is obviously a continuously differentiable mapping of $B_c^{4,\lambda}(\mathbb{R})$ into $B_c^{2,\lambda}(\mathbb{R})$, and applying Lemma 6.1 leads to the desired conclusion. □

7. Solution of the Free-Surface Problem

The main result, namely, the existence and uniqueness of the solution to equations (2.1)–(2.3), is presented and established here when the angle of inclination α is small enough. First, it is shown that fixed points of T are solutions of these equations.

Proposition 7.1. *Let T be the mapping defined in* (3.4) *and suppose that $T(g, \gamma) = \gamma$ for some pair $(g, \gamma) \in B_c^{4,\lambda}(\mathbb{R}) \times B_c^{4,\lambda}(\mathbb{R})$ that satisfies* (4.2). *Then for the bottom configuration g and the free-surface γ there exists a vector-valued function u and a scalar function p such that (u, p) defines a classical solution of* (2.1)–(2.3) *in Ω_g^γ.*

Proof. Clearly all that needs to be shown is that the normal-stress condition in equation (2.2) holds. From the definition of T, it follows from the relation $\gamma = T(g, \gamma)$ that

$$\gamma'' = \tau[[1 + (\gamma')^2]^{3/2}(-q + \chi)] + \text{constant}$$

for some unknown constant. However, the behavior at $-\infty$ of all the functions appearing in the last formula necessitates that this constant be zero, and therefore (2.2) holds. □

The way is paved to state the principal result that emerges from our study.

Theorem 7.2. *There exists an $\varepsilon_0 > 0$ such that whenever $0 < \sin(\alpha) < \varepsilon_0$, then there exists a positive real number $c = c(\sin(\alpha))$, an open neighborhood $\mathcal{N}_\alpha$ of the origin in $B_c^{4,\lambda}(\mathbb{R})$ and a continuously differentiable mapping $g \mapsto \gamma(g)$ from $\mathcal{N}_\alpha$ into $B_c^{4,\lambda}(\mathbb{R})$ such that $T(g, \gamma(g)) = \gamma(g)$ for all $g \in \mathcal{N}_\alpha$.*

The proof of Theorem 7.2 presented here rests upon the following technical result.

Proposition 7.3. *Let $T_\gamma(0, 0)$ be the Fréchet derivative of T with respect to γ evaluated at $(g, \gamma) = (0, 0)$. Then the mapping $I - T_\gamma(0, 0)$ is a continuous linear isomorphism from $B_c^{4,\lambda}(\mathbb{R})$ into itself provided that α is near enough to zero and $c = c(\sin(\alpha))$ is small enough.*

Remark 7.1. The proof of this result is given in Section 9, where the smallness requirements on α and c are made precise.

Proof of Theorem 7.2. Let $T_1(g, \gamma) = \gamma - T(g, \gamma)$. Then T_1 is continuously differentiable in $\mathcal{B}_0$ by Theorem 3.1. Moreover, as noted earlier, we have $T_1(0, 0) = 0$. The conclusions of Theorem 7.2 are exactly those of the Implicit-Function Theorem applied to T_1 at the point (0, 0), a result that is known to apply on account of Proposition 7.3. □

8. Determination of $T_\gamma(0, 0)$

This somewhat technical section is concerned with evaluating the vitally important linear operator $T_\gamma(0, 0)$ in the function-space setting being used throughout. The principal result is Proposition 8.2, which paves the way to the analysis of $I - T_\gamma(0, 0)$ in Section 9. The computations that lead to the formulas (8.4) and (8.5) are straightforward but tedious to derive. We content ourselves with a few remarks about their derivation designed to help the interested reader arrive efficiently at our conclusions.

One thing that simplifies the calculations is the following elementary observation. Consider an expression of the form $P(\gamma, \ldots, \gamma'''')\, \partial_{\tilde{x}}^i \partial_{\tilde{y}}^j$ where P is some analytic function. Its derivative with respect to γ at $(g, \gamma) = (0, 0)$ vanishes whenever P contains only terms of second degree or higher in γ and its derivatives. For instance, an expression like $\gamma\gamma'\partial_{\tilde{y}}$ has a derivative with respect to γ in the direction h given by $(\gamma h' + h\gamma')\,\partial_{\tilde{y}}$, which equals zero when $\gamma \equiv 0$. This elementary point, Lemma 4.4 and Remarks 4.1 and 4.3 are the main tools that come to the fore in the detailed verification of (8.4) and (8.5).

With this observation in mind, we begin the calculations. First consider the mapping introduced in Section 4, and the associated mappings $\mathscr{A}$ and $\mathscr{A}_g^\gamma$ (see (4.8) *et seq.*). For $g \equiv 0$, write

$$\mathscr{A}_0^\gamma \hat{\psi} = -\mathscr{A}_0^\gamma \hat{\Phi} + (0, 0, \tfrac{1}{3} C_1 y_0^3, 0, 0).$$

Differentiating with respect to γ in the direction h and evaluating at $\gamma \equiv 0$ yields

$$(D_\gamma \mathscr{A}_0^\gamma(0, 0)\, h)\, \hat{\psi}_0 + \mathscr{A} D_\gamma \hat{\psi}(0, 0)\, h = -(D_\gamma A_0^\gamma(0, 0)\, h)\, \hat{\Phi}, \qquad (8.1)$$

where $\hat{\psi}_0$ is the solution of (4.2) with $\gamma = g \equiv 0$; thus by our choice of $\hat{\Phi}$, it follows that $\hat{\psi}_0 \equiv 0$. If one defines $w(h) := D_\gamma \hat{\psi}(0, 0)\, h$, then w is the solution of the equation

$$\mathscr{A} w(h) = -(D_\gamma A_0^\gamma(0, 0)\, h)\, \hat{\Phi}. \qquad (8.2)$$

Making use of the observation mentioned above and the explicit form $C_1(\frac{1}{2} y_0 \hat{y}^2 - \frac{1}{6}\hat{y}^3)$ of $\hat{\Phi}$ then leads to the relation

$$D_\gamma A_0^\gamma \hat{\Phi}(0, 0)\, h$$
$$= \left(-C_1 \frac{\hat{y}}{y_0}(y_0 \hat{y} - \tfrac{1}{2}\hat{y}^2)\, h'''' + 2\left(C_1 \frac{\hat{y}}{y_0} - \frac{2C_1(y_0 - y)}{y_0}\right) h'',\ 0, 0, 0, \tfrac{1}{4} C_1 y_0^2 h''\right). \qquad (8.3)$$

Combining (8.2) and (8.3) and the relation exposed in Section 4 between $\mathscr{A}_g^\gamma$ and the mapping S that associates to (g, γ) a velocity field in Ω_g^γ leads directly to the following lemma.

Lemma 8.1. *For $h \in B_c^{4,\lambda}(\mathbb{R})$, let $w = w(h)$ denote the derivative $D_\gamma S(0, 0)\, h$. Then $w(h)$ is the solution of the boundary-value problem*

$$\begin{aligned}
\Delta^2 w &= C_1 \frac{\hat{y}}{y_0}(y_0 \hat{y} - \tfrac{1}{2}\hat{y}^2)\, h''''(\hat{x}) - \frac{2C_1}{y_0}(3\hat{y} - 2y_0)\, h''(\hat{x}), \quad (\hat{x}, \hat{y}) \in \Sigma,\\
w(\hat{x}, 0) &= w(\hat{x}, y_0) = 0, \quad \hat{x} \in \mathbb{R},\\
\frac{\partial w}{\partial \hat{y}}(\hat{x}, 0) &= 0, \quad \hat{x} \in \mathbb{R},\\
\frac{\partial^2 w}{\partial \hat{y}^2}(\hat{x}, y_0) &= -\tfrac{1}{2} C_1 y_0^2 h''(\hat{x}), \quad \hat{x} \in \mathbb{R}.
\end{aligned} \qquad (8.4)$$

To complete the analysis of a determining problem for $T_\gamma(0, 0)$, one needs to find the partial derivative of the mapping $(g, \gamma) \mapsto D_x((1 + \gamma'^2)^{3/2} \times (-q + \chi))$ where q and χ are as in (5.2) and (6.2), respectively. This proceeds much as the just-indicated calculation of $D_\gamma S(0, 0)$. Putting the resulting expression together with Lemma 8.1 and formula (6.2) leads to the following proposition.

Proposition 8.2. *Let h lie in $B_c^{4,\lambda}(\mathbb{R})$. The mapping $T_y(0, 0)$ is the continuous linear operator that associates to h the solution $v = v(h)$ of the equation*

$$\mathscr{L}v = v''' - \tau C_1 y_0 v'' + \tau C_2 v' + 3\tau \frac{C_1}{y_0} v = \tau\left(-2 \frac{\partial^3 w}{\partial x^2 \partial y}(\cdot, y_0) - \frac{\partial^3 w}{\partial y^3}(\cdot, y_0)\right) \tag{8.5}$$

where w is the solution of (8.6) *associated with h. (Here, and henceforth, the circumflexes have been dropped.)*

9. Invertibility of $I - T_y(0, 0)$

In this section a proof of Proposition 7.3 is provided. This lemma was crucial for justifying the application of the Implicit-Function Theorem. Let f be given in $B_c^{4,\lambda}(\mathbb{R})$ and consider solving for h the equation

$$(I - T_y(0, 0))\, h = f,$$

where I denotes the identity mapping. If we consider the difference $h - f$, then this quantity is a solution of (8.5), which is to say that

$$\mathscr{L}(h - f) = -\tau\left(2 \frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3}\right)(\cdot, y_0)$$

where w is the solution of (8.4) and $\mathscr{L}$ is the linear ordinary differential operator defined in (6.1), or what is the same,

$$\mathscr{L}h = -\tau\left(2 \frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3}\right)(\cdot, y_0) + \mathscr{L}f. \tag{9.1}$$

The proof of existence of a unique solution h of (9.1) rests upon the following technical lemmas.

Lemma 9.1. *Let $f \in B_c^{1,\lambda}(\mathbb{R})$ and consider the differential equation*

$$v''' - a_1 \varepsilon v'' - a_2 \sqrt{1 - \varepsilon^2}\, v' + a_3 \varepsilon v = f \tag{9.2}$$

for $x \in \mathbb{R}$, where the a_i are positive constants, $1 \leqq i \leqq 3$. There exist an $\varepsilon_1 > 0$ and a constant K_1 depending only on a_1, a_2, a_3 such that whenever $0 < \varepsilon < \varepsilon_1$, then
(i) *The polynomial $P_\varepsilon(z) = z^3 - a_1 \varepsilon z^2 - a_2 \sqrt{1 - \varepsilon^2}\, z + a_3 \varepsilon$ has three distinct real roots $r_1(\varepsilon), r_2(\varepsilon), r_3(\varepsilon)$ with $r_1(\varepsilon) < -r_2(\varepsilon) < 0 < r_2(\varepsilon) < r_3(\varepsilon)$.*
(ii) *For every c with $0 < c < r_2(\varepsilon)$, and every f in $B_c^{1,\lambda}(\mathbb{R})$,* (9.2) *has a unique solution v in $B_c^{4,\lambda}(\mathbb{R})$ and, moreover, v satisfies the estimate*

$$\|v\|_{c,4,\lambda} \leqq K_1 \left(1 + \frac{1}{r_2(\varepsilon) - c}\right) \|f\|_{c,1,\lambda}.$$

(iii) *If in addition* $f = \tilde{f}'$ *for some* $\tilde{f}$ *in* $B_c^{2,\lambda}(\mathbb{R})$, *then* v *satisfies the further estimate*

$$\|v\|_{c,4,\lambda} \leqq K_1 \left(1 + \frac{r_2(\varepsilon)}{r_2(\varepsilon) - c}\right) \|\tilde{f}\|_{c,2,\lambda}.$$

Proof. As $\varepsilon \downarrow 0$, $r_1(\varepsilon)$ and $r_3(\varepsilon)$ tend to $-\sqrt{a_2}$ and $\sqrt{a_2}$, respectively, while $r_2(\varepsilon)$ tends to zero. It follows more or less immediately that for small ε, P_ε has three real roots as asserted in part (i). Furthermore, one easily verifies that $r_2(\varepsilon)/\varepsilon$ and $\varepsilon/r_2(\varepsilon)$ are bounded in any bounded neighborhood of 0.

To establish (ii), proceed as follows. First form the Green's function for the constant-coefficient equation (9.2) with zero boundary conditions at $\pm\infty$ in the standard way. Since the Green's function is a linear combination of exponentials, it happens that the solution v of (9.2) corresponding to an f given in $B_c^{1,\lambda}(\mathbb{R})$ satisfies the estimates

$$\sup_{x\in\mathbb{R}} |v^{(k)}(x)|\, e^{c|x|} \leqq \|f\|_{c,1,\lambda} \left\{ \sum_{i=1}^{3} 2|\lambda_i|\, |r_i|^k \left(\frac{1}{|c - r_i|} + \frac{1}{|c + r_i|} \right) \right\} \tag{9.3}$$

for $0 \leqq k \leqq 3$. As for the norm of v'''', we simply differentiate once the equation satisfied by v and use the estimates in (9.3) to obtain that

$$\sup_{x\in\mathbb{R}} e^{c|x|}\, |v''''(x)| \leqq \Big\{ \|f\|_{c,1,\lambda} + a_3\varepsilon \sup_{x\in\mathbb{R}} e^{c|x|}\, |v'(x)|$$
$$+ a_2\sqrt{1-\varepsilon^2} \sup_{x\in\mathbb{R}} e^{c|x|}\, |v''(x)| + a_1\varepsilon \sup_{x\in\mathbb{R}} e^{c|x|}\, |v'''(x)| \Big\},$$

and that

$$\sup_{\substack{(x,x')\in\mathbb{R}^2 \\ x\neq x'}} \frac{|v''''(x) - v''''(x')|}{|x - x'|^\lambda} \leqq 2 \sup_{x\in\mathbb{R}} |v''''(x)| + \sup_{\substack{|x-x'|<1 \\ x\neq x'}} \frac{|v''''(x) - v''''(x')|}{|x - x'|^\lambda}$$
$$\leqq 2 \sup_{x\in\mathbb{R}} |v''''(x)| + \sup_{x\neq x'} \frac{|f'(x) - f'(x')|}{|x - x'|^\lambda}$$
$$+ a_3\varepsilon \sup_{x\in\mathbb{R}} |v''(x)| + a_2\sqrt{1-\varepsilon^2} \sup_{x\in\mathbb{R}} |v'''(x)|$$
$$+ a_1\varepsilon \sup_{x\in\mathbb{R}} |v''''(x)|,$$

where we used the equation a second time in the last step. Whenever $0 < \varepsilon < \varepsilon_1$ and $0 < c < r_2(\varepsilon)$, the quantities $\dfrac{r_i^k}{|r_i \pm c|}$ for $i = 1, 3$, are uniformly bounded. Appropriately grouping the terms appearing above, we obtain (ii) with some constant K_1 that may be chosen independently of ε in the range $0 < \varepsilon < \varepsilon_1$.

The proof of (iii) also follows by representing v in terms of the Green's function integrated against $f = \tilde{f}'$, integrating this expression by parts and then estimating as in the proof of (ii). □

Corollary 9.2. *Under the assumptions of Lemma 9.1,*

$$\|v\|_{\frac{1}{2}r_2(\varepsilon),4,\lambda} \leqq K_2 \|\tilde{f}\|_{\frac{1}{2}r_2(\varepsilon),2,\lambda}$$

whenever v is the solution of (9.2) in which the right-hand side is $\tilde{f}'$, and K_2 depends only on a_1, a_2 and a_3.

Proof. This follows immediately from Lemma 9.1, part (iii), with $K_2 = 3K_1$. □

Lemma 9.3. *Let W be the solution of the boundary value problem*

$$\Delta^2 W(x, y) = \left(y^2 - \frac{1}{2}\frac{y^3}{y_0}\right) h''''(x) - \frac{2}{y_0}(3y - 2y_0)\, h''(x) \quad \textit{for } (x, y) \in \Sigma,$$

$$W(x, 0) = W(x, y_0) = \frac{\partial W}{\partial y}(x, 0) = 0 \quad \textit{for } x \in \mathbb{R}, \tag{9.4}$$

$$\frac{\partial^2 W}{\partial y^2}(x, y_0) = -\tfrac{1}{2} y_0^2 h''(x) \quad \textit{for } x \in \mathbb{R},$$

where h is a given function in $B_c^{4,\lambda}(\mathbb{R})$. Define ρ to be

$$\rho(x) = C_0\tau \left\{2 \frac{\partial^3 W}{\partial x^2 \partial y}(x, y_0) + \frac{\partial^3 W}{\partial y^3}(x, y_0)\right\}.$$

Then there exists a $c_1 > 0$ such that whenever $0 \leqq c \leqq c_1$, the following conditions hold:

(i) *$\rho(x) \in B_c^{1,\lambda}(\mathbb{R})$ and $\rho(x) = \tilde{\rho}'(x)$ for some function $\tilde{\rho}$ in $B_c^{2,\lambda}(\mathbb{R})$.*

(ii) *The mapping $h \to \tilde{\rho}$ is linear and continuous from $B_c^{4,\lambda}(\mathbb{R})$ into $B_c^{2,\lambda}(\mathbb{R})$.*

(iii) *There exists a constant K_3 depending only on y_0 with the property that $\|\tilde{\rho}\|_{c,2,\lambda} \leqq \tau C_0 K_3 \|h\|_{c,4,\lambda}$.*

Proof. Consider the solution Z of

$$\Delta^2 Z(x, y) = \left(y^2 - \frac{1}{2}\frac{y^3}{y_0}\right) h'''(x) - \frac{2}{y_0}(3y - 2y_0)\, h'(x) \quad \text{for } (x, y) \in \Sigma,$$

$$Z(x, 0) = Z(x, y_0) = \frac{\partial Z}{\partial y}(x, 0) = 0 \quad \text{for } x \in \mathbb{R}, \tag{9.4$'$}$$

$$\frac{\partial^2 Z}{\partial y^2}(x, y_0) = -\tfrac{1}{2} y_0^2 h'(x) \quad \text{for } x \in \mathbb{R}.$$

By Corollary 4.3, we deduce that $W \in B_c^{5,\lambda}(\bar{\Sigma})$ for $0 < c < \tilde{c}$, where $\tilde{c}$ is as in Proposition 4.2. Plainly we have that

$$W(x, y) = \frac{\partial Z}{\partial x}(x, y) \quad \text{for } (x, y) \in \bar{\Sigma}.$$

As a matter of fact, W and $\frac{\partial Z}{\partial x}$ are solutions of the same boundary-value problem, and this solution is unique by Proposition 4.2. This conclusion implies that $\tilde{\rho}(x) := C_0\tau\left\{2\frac{\partial^3 Z}{\partial x^2 y}(x, y_0) + \frac{\partial^3 Z}{\partial y^3}(x, y_0)\right\}$ is such that

$$\rho(x) = \tilde{\rho}'(x),$$

and (i) is proved. Property (ii) is obvious, and (iii) is proved as follows. Using Proposition 4.2 (and see also Corollary 4.3), we fix a value c_1 with $0 < c_1 < \tilde{c}$ and obtain a constant K_3 depending only on y_0 such that

$$\|Z\|_{c,5,\lambda} \leqq K_3\|h\|_{c,4,\lambda}$$

whenever $0 \leqq c \leqq c_1$. Thus (iii) is established. □

Here is the main result of this section.

Lemma 9.4. *Consider the equation*

$$h''' - a_1\varepsilon h'' - a_2\sqrt{1-\varepsilon^2}\,h' + a_3\varepsilon h + \varepsilon\rho = (f''' - a_1\varepsilon f'' - a_2\sqrt{1-\varepsilon^2}\,f' + a_3\varepsilon f) \tag{9.5}$$

where $f \in B_c^{4,\lambda}(\mathbb{R})$, and ρ is defined in Lemma 9.3. There exists an $\varepsilon_0 > 0$ such that (9.5) has a unique solution h in $B_{c(\varepsilon)}^{4,\lambda}(\mathbb{R})$ whenever $0 < \varepsilon < \varepsilon_0$, where $c(\varepsilon) = \frac{1}{2}r_2(\varepsilon)$ and $r_2(\varepsilon)$ is as introduced in Lemma 9.1, and f is a given function in $B_{c(\varepsilon)}^{4,\lambda}(\mathbb{R})$.

Proof. Denoting by $P_\varepsilon(D)$ the operator $D_x^3 - a_1\varepsilon D_x^2 - a_2\sqrt{1-\varepsilon^2}\,D_x + a_3\varepsilon$, we may write (9.5) in the form

$$h + \varepsilon(P_\varepsilon(D)^{-1}\rho) = f. \tag{9.5)'$$

From Lemma 9.3 and Corollary 9.2, one deduces that if $0 < \varepsilon < \varepsilon_1$ and $0 < \frac{1}{2}r_2(\varepsilon) \leqq c_1$, then

$$\|P_\varepsilon(D)^{-1}\rho\|_{c(\varepsilon),4,\lambda} \leqq \tau C_0 K_4\|h\|_{c(\varepsilon),4,\lambda},$$

where $K_4 = K_3K_2$ depends only on y_0. Therefore, the norm of the linear mapping $h \mapsto \varepsilon P_\varepsilon(D)^{-1}\rho$ is bounded from above by $\varepsilon C_0\tau K_4$.

From Lemma 9.1 part (i), we know it is possible to choose ε_0 meeting all of the following requirements:

$$0 < \varepsilon_0 < \varepsilon_1,\ 0 < \sup_{0<\varepsilon<\varepsilon_0}\{\tfrac{1}{2}r_2(\varepsilon)\} < c_0, \quad \varepsilon_0 C_0\tau K_4 < 1.$$

Then for any positive $\varepsilon < \varepsilon_0$, the operator $I + \varepsilon P_\varepsilon(D)^{-1}\rho$, when considered as a self-mapping of $B_{c(\varepsilon)}^{4,\lambda}(\mathbb{R})$, is boundedly invertible. Thus the lemma is established. □

Proof of Proposition 7.3. To conclude the proof of Proposition 7.3, apply Lemma 9.4 with $a_1 = \tau C_0 y_0$, $a_2 = \tau C_0$, $a_3 = 3\tau C_0/y_0$ and $\varepsilon = \sin(\alpha)$. □

10. Proof of Proposition 4.2 and Corollary 4.3

The final piece of the argument is completed here, namely a proof of Proposition 4.2 and its Corollary concerning the solvability of a certain boundary-value problem in the spaces appropriate to our theory.

First, consider the homogeneous problem

$$\begin{gathered}\Delta^2 v = f_1 \quad \text{in } \Sigma, \\ v(\cdot, 0) = v(\cdot, y_0) = \frac{\partial v}{\partial y}(\cdot, 0) = \frac{\partial^2 v}{\partial y^2}(\cdot, y_0) = 0 \quad \text{in } \mathbb{R},\end{gathered} \tag{10.1}$$

and introduce the Green's function $\mathscr{G}$ associated to (10.1). Because of the structure of the problem, $\mathscr{G}(x, y; x', y')$ can be written as $G(x - x', y, y')$ where G enjoys the following properties: there is a constant $C > 0$ such that

$$\begin{aligned}|D^4 G(x - x', y, y')| &\leqq C(|x - x'| + |y - y'|)^{-2}, \\ |D^3 G(x - x', y, y')| &\leqq C(|x - x'| + |y - y'|)^{-1}, \\ |D^2 G(x - x', y, y')| &\leqq C \log(|x - x'| + |y - y'|), \\ |DG(x - x', y, y')| &\leqq C, \\ |G(x - x', y, y')| &\leqq C,\end{aligned} \tag{10.2}$$

whenever $|x - x'| \leqq 2$ and D^k is any differential operator in the variables x and y of order k, and there is a positive constant $\bar{c}$ such that

$$|D^k G(x - x', y, y')| \leqq C e^{-\bar{c}|x - x'|} \tag{10.3}$$

for $|x - x'| \geqq 2$, and $0 \leqq k \leqq 4$. These results are essentially in AMICK (1977; 1978, Theorem 4.1), modulo minor modifications imposed by the boundary conditions. Under the assumption that $f_1 \in B_c^{0,\lambda}(\bar{\Sigma})$ for some c with $0 < c < \bar{c}$, we now prove that $v \in B_c^{4,\lambda}(\bar{\Sigma})$. First of all, there exists a weak solution

$$v \in \tilde{H}^4(\Sigma) := \left\{ v \in H^4(\Sigma) : v(\cdot, 0) = v(\cdot, y_0) = \frac{\partial v}{\partial y}(\cdot, y_0) = \frac{\partial^2 v}{\partial y^2}(\cdot, y_0) = 0 \right\}$$

because Δ^2 is a coercive, self-adjoint operator from $\tilde{H}^4(\Sigma)$ into $L_2(\Sigma)$ and f_1 obviously belongs to $L_2(\Sigma)$. Applying standard arguments, we obtain that v is a classical solution in $C^{4,\lambda}(\bar{\Sigma})$. Concerning the weighted norm, use the Green's function to write

$$v(x, y) = \int_{\Sigma} G(x - x', y, y') \, f_1(x', y') \, dx' \, dy' \tag{10.4}$$

so that

$$\begin{aligned}&|v(x, y)| \, e^{c|x|} \\ &\leqq e^{c|x|} \left(C \int_{|x - x'| \geqq 2} \int_0^{y_0} e^{-\bar{c}|x - x'|} f_1(x', y') \, dy' \, dx' + C \int_{|x - x'| \leqq 2} \int_0^{y_0} f_1(x', y') \, dy' \, dx' \right).\end{aligned}$$

Let $M = \sup_{(x,y)\in\Sigma} e^{c|x|}\,|f_1(x, y)|$ and continue the estimate in the last display as follows:

$$|v(x, y)|\, e^{c|x|} \leqq e^{c|x|}\, CM\Bigg(\int_{-\infty}^{x-2} y_0 e^{-\bar{c}(x-x')}\, e^{-c|x'|}\, dx' \tag{10.5}$$
$$+ \int_{x+2}^{+\infty} y_0 e^{-\bar{c}(x'-x)}\, e^{-c|x'|}\, dx' + \int_{x-2}^{x+2} y_0\, e^{-c|x'|}\, dx'\Bigg).$$

Attention is first drawn to the case wherein $x \to -\infty$. Under the assumption that $x + 2 \leqq 0$, the inequality (10.5) yields

$$|v(x, y)|\, e^{c|x|} \leqq CMy_0 e^{c|x|}\Bigg(\int_{-\infty}^{x-2} e^{-\bar{c}(x-x')}\, e^{cx'}\, dx' + \int_{x+2}^{0} e^{-\bar{c}(x'-x)}\, e^{cx'}\, dx'$$
$$+ \int_{0}^{+\infty} e^{-\bar{c}(x'-x)}\, e^{-cx'}\, dx' + 4e^{-c|x-2|}\Bigg)$$
$$\leqq CMy_0 e^{c|x|}\Bigg(e^{-\bar{c}x}\left[\frac{e^{(c+\bar{c})x'}}{c+\bar{c}}\right]_{-\infty}^{x-2} + e^{\bar{c}x}\left[\frac{e^{(c-\bar{c})x'}}{c-\bar{c}}\right]_{x+2}^{2}$$
$$+ e^{\bar{c}x}\left[\frac{-e^{-(c+\bar{c})x'}}{c+\bar{c}}\right]_{0}^{+\infty} + 4e^{-2c}\, e^{cx}\Bigg)$$
$$\leqq CMy_0 e^{-cx}\left(\frac{e^{cx}e^{-2(c+\bar{c})}}{c+\bar{c}} + \frac{e^{cx}e^{2(c-\bar{c})} - e^{\bar{c}x}}{\bar{c}-c} + \frac{e^{\bar{c}x}}{c+\bar{c}} + 4e^{-2c}e^{cx}\right),$$

and therefore

$$|v(x, y)|\, e^{c|x|} \leqq CMy_0\left(\frac{e^{-2(c+\bar{c})}}{c+\bar{c}} + \frac{e^{2(c-\bar{c})} - e^{(\bar{c}-c)x}}{\bar{c}-c} + \frac{e^{(\bar{c}-c)x}}{c+\bar{c}} + 4e^{-2c}\right).$$

When $x \to +\infty$, similar considerations apply, and we are led to the conclusion that there exists a constant K_5 for which

$$\sup_{(x,y)\in\bar{\Sigma}} |v(x, y)|\, e^{c|x|} \leqq K_5\left(\frac{1}{c+\bar{c}} + \frac{1}{\bar{c}-c} + 1\right)\|f_1\|_{0,c,\lambda},$$

where K_5 depends only on y_0. Furthermore, it is clear that as a function of c, K_5 is bounded on any compact subinterval of $[0, \bar{c})$. We proceed in exactly the same fashion to obtain an estimate for the weighted norm of the derivatives of v of order up to three. The technique is identical to that just outlined because differentiation under the integral sign is legitimate since the second- and third-order derivatives of G have integrable singularities at $x = x'$, $y = y'$. Once again, we obtain constants that are bounded when c is bounded away from $\bar{c}$. Finally, to estimate the weighted norm of the fourth-order derivatives of v, consider the function $w(x)\, v(x, y)$ where $w \in C^\infty(\mathbb{R})$, $w = e^{c|x|}$ for $|x| \geqq 1$ and $w \geqq w_0 > 0$, and work out the boundary-value problem to which this function wv is the solution. The conclusion follows from the classical Hölder estimates for elliptic equations since all the lower-order derivatives of wv are already known to be bounded.

For the case of non-homogenous boundary conditions, we use a succession of lifting operators. We want to solve (4.6), namely,

$$\Delta^2 v = f_1 \quad \text{for } (x, y) \in \Sigma,$$
$$\frac{\partial v}{\partial y}(\cdot, 0) = f_3, \quad \frac{\partial^2 v}{\partial y^2}(\cdot, y_0) = f_5, \tag{10.6}$$
$$v(\cdot, 0) = f_2, \quad v(\cdot, y_0) = f_4 \quad \text{for } x \in \mathbb{R}.$$

Begin by subtracting from v the function $f_2(y_0 - y)/y_0 + f_4 y/y_0$ to obtain the function v_1, which is a solution of

$$\Delta^2 v_1 = f_1 - \Delta^2\left(\frac{y_0 - y}{y_0} f_2 + \frac{y}{y_0} f_4\right) := k_1 \qquad \text{in } \Sigma,$$
$$v_1(\cdot, 0) = v_1(\cdot, y_0) = 0 \qquad \text{in } \mathbb{R},$$
$$\frac{\partial v_1}{\partial y}(\cdot, y_0) = f_3 - \frac{\partial}{\partial y}\left(\frac{y_0 - y}{y_0} f_2 + \frac{y}{y_0} f_4\right) := k_2 \qquad \text{in } \mathbb{R},$$
$$\frac{\partial^2 v_1}{\partial y^2}(\cdot, y_0) = f_5 \qquad \text{in } \mathbb{R},$$

and $v = v_1 + \left(\frac{y_0 - y}{y_0} f_2 + \frac{y}{y_0} f_4\right)$. Next, write

$$v_1 = v_2 + \left(1 - \frac{y}{y_0}\right) \int_x^{x+y} k_2(s)\, ds,$$

where v_2 is the solution of

$$\Delta^2 v_2 = k_1 - \Delta^2\left[\left(1 - \frac{y}{y_3}\right) \int_x^{x+y} k_2(s)\, ds\right] := k_3 \qquad \text{in } \Sigma,$$
$$v_2(\cdot, 0) = v_2(\cdot, y_0) = 0, \qquad \text{in } \mathbb{R},$$
$$\frac{\partial v_2}{\partial y}(\cdot, 0) = 0, \quad \frac{\partial^2 v_2}{\partial y^2}(\cdot, y_0) = f_5 + \frac{1}{y_0} \int_x^{x+y_0} k_2(s)\, ds := k_4 \qquad \textit{in } \mathbb{R}.$$

Finally, we set

$$v_2 = v_3 + \left(-\frac{2y^3}{y_0^3} + 3\frac{y^2}{y_0^2}\right) \int_x^{x+y_0-y} \int_x^t k_4(s)\, ds\, dt;$$

v_3 is now the solution of the homogeneous problem in (10.1). It just remains to check that the modifications above take place in the proper function spaces, and this is straightforward. □

Proof of Corollary 4.3. First reduce the problem to a homogeneous one using the method in the last proof, and then use the representation formula (10.4) for v to prove that $\frac{\partial v}{\partial x}$ is in $B_c^{1,\lambda}(\bar{\Sigma})$. In due course, use is made of the equation to establish that $\frac{\partial^4 v}{\partial y^4}$ is in $B_c^{1,\lambda}(\bar{\Sigma})$. The conclusion follows. □

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