NUMERICAL SOLUTION OF THE GENERALIZED KORTEWEG–DE VRIES–BURGERS EQUATION WITH ADAPTIVE GALERKIN METHODS*

J. L. BONA[1], V. A. DOUGALIS[2], O. A. KARAKASHIAN[3] and W. R. McKINNEY[4].

1. Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA.
2. Department of Mathematics, National Technical University, Zographou, 15773 Athens, Greece and IACM, FORTH, Iraklion, Greece.
3. Department of Mathematics, University of Tennessee, Knoxville, TN 37996, USA.
4. Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA.

ABSTRACT

The generalized Korteweg–de Vries equation with dissipation is solved numerically by a highly accurate and stable scheme based on a Galerkin (finite element) method in space and implicit Runge–Kutta time–stepping. The scheme is coupled with adaptive grid refinement in the spatial and temporal meshes and is used to investigate the blow–up instability of solitary–wave solutions of the nondissipative equation as well as the effect of dissipation on the singularities.

1. INTRODUCTION

We shall consider a model equation that describes the propagation of strongly nonlinear, dispersive waves in the presence of dissipation, the so–called Generalized Korteweg–de Vries–Burgers equation given by

$$u_t + u^p u_x + \varepsilon\, u_{xxx} - \delta\, u_{xx} = 0\,, \tag{1.1a}$$

where p is a nonnegative integer, $\varepsilon > 0$ and $\delta \geq 0$ are given constants and the unknown function (the amplitude of the wave) $u = u(x,t)$, $t \geq 0$, $x \in$

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[0, 1] satisfies periodic boundary conditions at the end points of the spatial interval [0, 1]. (1.1a) is supplemented by a 1-periodic initial condition

$$u(x,0) = u^0(x)\,. \tag{1.1b}$$

If $\delta = 0$, it is well-known that for $p < 4$ the initial-and periodic-value problem (1.1) possesses a unique smooth solution for all $t \geq 0$ provided u^0 is smooth. If $p \geq 4$, a solution is guaranteed to exist only locally in t. A recent theory[1] has shown that its special, travelling wave solutions known as solitary-waves are unstable if $p \geq 4$. The numerical calculations that will be summarized here and appear in detail in Bona *et al*[3,4] indicate that this instability leads to the formation of point blow-up singularities in finite time. If now the dissipation parameter δ is positive and sufficiently large, smooth solutions exist for all $t \geq 0$. If, however, δ is suitable small, then point blow-up singularities will occur again in finite time.

2. THE NUMERICAL SCHEME.

We shall describe one of the numerical methods considered in Bona *et al*[2,3,4] for the numerical solution of (1.1). Let $h = 1/N$, N positive integer, and denote by S_h the space of 1-periodic smooth cubic splines defined on the uniform partition of [0,1] given by $x_i = ih$, $i = 0, 1, \ldots, N$. The Galerkin semidiscrete approximation $v_h(\cdot, t) \in S_h$ to the solution $u(\cdot, t)$ of (1.1) is then the unique solution of the system of O.D.E.'s

$$\begin{cases} v_{ht} = F(v_h)\,, \qquad t \geq 0\,, \\ v_h(0) = \pi_h\, u^0\,, \end{cases} \tag{1.2}$$

where $(F(v_h), \chi) = (-v_h^p v_{hx} - \varepsilon v_{hxxx} + \delta v_{hxx}, \chi)$ for $\chi \in S_h$, $(f,g) = \int_0^1 fg\,dx$, and $\pi_h u^0$ is any conveniently chosen element of S_h approximating u^0 (e.g. interpolant) so that $\|u^0 - \pi_h u^0\| \leq ch^4$ for some constant c independent of h. (Here $\|\cdot\|$ denotes the L^2 norm on [0,1].)

The system (1.2) is further discretized in time by an implicit Runge-Kutta method[5]. For example, one may choose the *two-stage Gauss-Legendre* method defined by the constants $\alpha_{11} = \alpha_{22} = \frac{1}{4}$, $\alpha_{12} = \frac{1}{4} - \frac{1}{2\sqrt{3}}$, $\alpha_{21} = \frac{1}{4} + \frac{1}{2\sqrt{3}}$, $\beta_1 = \beta_2 = \frac{1}{2}$. If $k > 0$ is the (constant) time-step and $t^n = nk$, $n = 0, 1, 2, \ldots, J$, where $nJ = T > 0$, these constants generate fully discrete approximations $u_h^n \in S_h$ to $u(x, t^n)$ as follows:

$$\begin{cases} u_h^0 = \pi_h u^0 \\ \text{For } \ n = 0, 1, 2, \ldots, J-1: \\ u_h^{n,j} = u_h^n + k \sum_{j=1}^2 \alpha_{ij}\, F(u_h^{n,j}) \qquad j = 1, 2, \\ u_h^{n+1} = u_h^n + k \sum_{j=1}^2 \beta_j\, F(u_h^{n,j})\,. \end{cases} \tag{1.3}$$

It is shown in Bona *et al*[3,4] that the solution u_h^n of (1.3) exists uniquely (if kh^{-1} is sufficiently small) and satisfies the optimal–order in space and time L^2–error estimate

$$\max_{0\leq n\leq J} \|u(t^n) - u_h^n\| \leq c\,(k^4 + h^4),$$

for some constant $c = c(u, T)$ independent of k and h, provided $u(x,t)$ is sufficiently smooth for $0 \leq t \leq T$. In practice, the nonlinear system defining the intermediate stages $u_h^{n,i}$, $i = 1, 2$, at each tine step is solved by a suitable modification of Newton's method in which the equations decoumple. The efficiency (achieved accuracy vs computational cost) of the resulting fully discrete scheme has been studied in detail in Bona *et al*[2,3], where it is shown that it is indeed a fast and highly accurate method for the integration of (1.1). We shall use an ***adaptive version*** of (1.3) to integrate numerically solutions of (1.1) that appear to develop singularities in finite time.

3. GRID REFINEMENT.

It is evident that it is not possible to approximate in a satisfactory way solutions $u(x,t)$ of (1.1) that blow up in L^∞ as (x,t) tends to some point (x^*, t^*), $t^* < \infty$, with a method that uses fixed h and k. To overcome this difficulty an automatic grid refinement was implemented to supplement the base scheme described in §2. The adaptive mechanism in our code consists of three main parts:

(i) local refinement of the spatial grid,
(ii) selection of a temporal step size k,
(iii) spatial translations of the solution.

In what follows we shall be interested in simulating the evolution of solitary–wave initial profiles for (1.1). If $\delta = 0$, it is well–known that (1.1a) possesses ***solitary wave*** solutions, which are travelling waves of the form

$$u(x,t) = A\ \text{sech}^{2/p}\left[B(x - x_0 - Ct)\right], \tag{3.1}$$

where

$$C = \frac{2A^p}{(p+1)(p+2)}, \qquad B = \frac{p}{2}\sqrt{\frac{C}{\varepsilon}},$$

where A represents the amplitude of the waves. Although (3.1) is an exact solution of the pure initial–value problem for (1.1a), if A is large and ε is taken sufficiently small, then it should also be an approximate solution in the periodic case since the tails of the solitary wave decay exponentially. We shall usually take $x_0 = 1/2$. These solutions are known[1,2] to be unstable if $p > 4$. To investigate the nature of this instability (and hasten its onset) we use as initial data in (1.1b) the slightly pertubed solitary wave profile

$$u^0(x) = 1.01\, A\, \text{sech}^{2/p}\left[B(x - x_0)\right]. \tag{3.2}$$

Early numerical experiments (cf. Bona *et al*[2]) with fixed grids and small grid sizes revealed that an initial profile like (3.2) soon evolves into a thin peak that proceeds to blow up in L^∞ at some (x^*, t^*). Therefore, the adaptive mechanism of the code is geared towards approximating well solutions with a single peak that blows up at a point. Consequently, our spatial refinement will consist of adding new nodes, distributed evenly about the mid point $x = .5$, but in successively smaller neighborhoods of the midpoint. This local refinement is combined with (iii), which takes advantage of the fact that the solution is a travelling–wave, to keep the peak near the midpoint $x = .5$ in the region of highest density of nodes, and away from a region of coarse mesh. This accomplished by occasionally translating the solution and centering the peak at $x = .5$. (In effect, we translate the solution to conform to the grids we construct rather than have a moving grid which conforms to the solution.)

Spatial Grid Refinement. Let NSPLIT stand for the number of times nodes are to be added and let Ω^* represent the neighborhood of the midpoint $x = .5$ which is the region with the finest grid, with grid size h^*. Each time nodes are added, NSPLIT is increased by 1, and both Ω^* and h^* are cut in half. As an example, with NSPLIT$=1$, we might have $\Omega^* = [0.4, 0.6]$ and $h^* = 1/100$. Then as NSPLIT increases to 2, $\Omega^* = [0.45, 0.55]$, $h^* = 1/200$; when NSPLIT=3, $\Omega^* = [0.475, 0.525]$, $h^* = 1/400$, and so on. The effect of the occasional spatial translations is to insure that the peak of the solution remains in the region Ω^*.

From the $L^\infty - L^2$ inverse inequality satisfied by elements of S_h and the discrete conservation law $\|u_h^n\| = \|\pi_h u^0\|$ satisfied by (1.3) for $\delta = 0$, we have

$$\max_x |u_h^n(x)| \le ch^{-1/2}\|u_h^n\| = ch^{-1/2}\|\pi_h u^0\| \quad \text{for } n = 0, \ldots, J. \tag{3.3}$$

Similarly, for a non–uniform grid the minimal grid size h^* must become arbitrary small if the numerical solution is to develop an arbitrary large peak. Our particular choice of spatial refinement is based on a local $L^\infty - L^2$ inverse property on Ω^*. Specifically, we use the following test to determine if NSPLIT needs to be increased. At each step, compute

$$Z_\infty = \max_x |u_h^n(x)| \quad \text{and} \quad Z_2 = \left(\int_{\Omega^*} (u_h^n)^2\, dx \right)^{1/2}.$$

We then increase NSPLIT by 1, add new nodes, and modify Ω^* and h^* if

$$\frac{Z_\infty \sqrt{h^*}}{Z_2} > \mathrm{TOL}_1 \,. \tag{3.4}$$

Here TOL_1 must be chosen small enough in accordance with (3.3) to allow for new nodes to be added. In our program, we have used values of TOL_1 equal to 0.1 or 0.2, typically. The same procedure is used if $\delta > 0$, since the only change is then that $\|u_h^n\| \le \|\pi_h u^0\|$.

Temporal Step Size Reduction. The temporal step size is adjusted in an attempt to preserve the third invariant of the GKdV equation. This invariant is defined by

$$I_3(v) = \int_0^1 \left[v^{p+2} - \frac{(p+1)(p+2)\varepsilon}{2} (v_x)^2 \right] dx \,.$$

For exact solutions $u(x,t)$ of (1.1) for $\delta = 0$, $I_3(u(\cdot,t)) = I_3(u^0)$, independent of the values of t.

Given u_h^n, a possible u_h^{n+1} is computed using the current step size k. It is accepted if

$$\frac{|I_3(u_h^{n+1}) - I_3(u_h^n)|}{\int_0^1 [(u_h^{n+1})_x]^2 dx} < \mathrm{TOL}_2 \,, \tag{3.5}$$

where TOL_2 is a small parameter. In the results to be presented, a typical value of TOL_2 is 10^{-5}. If (3.5) is not satisfied, then k is cut in half and the process is repeated. The denominator in (3.5) is a convienient normalization factor. Although I_3 is not an invariant of the dissipative equation (i.e. when $\delta > 0$ in (1.1a)), nevertheless, the criterion (3.5) proved quite effective in cutting appropriately the temporal step for $\delta > 0$ as well.

4. INSTABILITY OF SOLITARY WAVES FOR $\delta = 0$.

With the adaptive mechanism just described in place, we took (3.2) as initial value with $A = 2$, $p = 5$, $\varepsilon = 5 \times 10^{-4}$, $x_0 = 1/2$ and integrated (1.1) numerically using $h = 1/192$ and $k = 1/1000$ as initial mesh parameters. The graphs in Figure 1 illustrate the growth in the peak of the solution. The plots are given at NSPLIT=2, 4, 6, 9. Note that as the peak

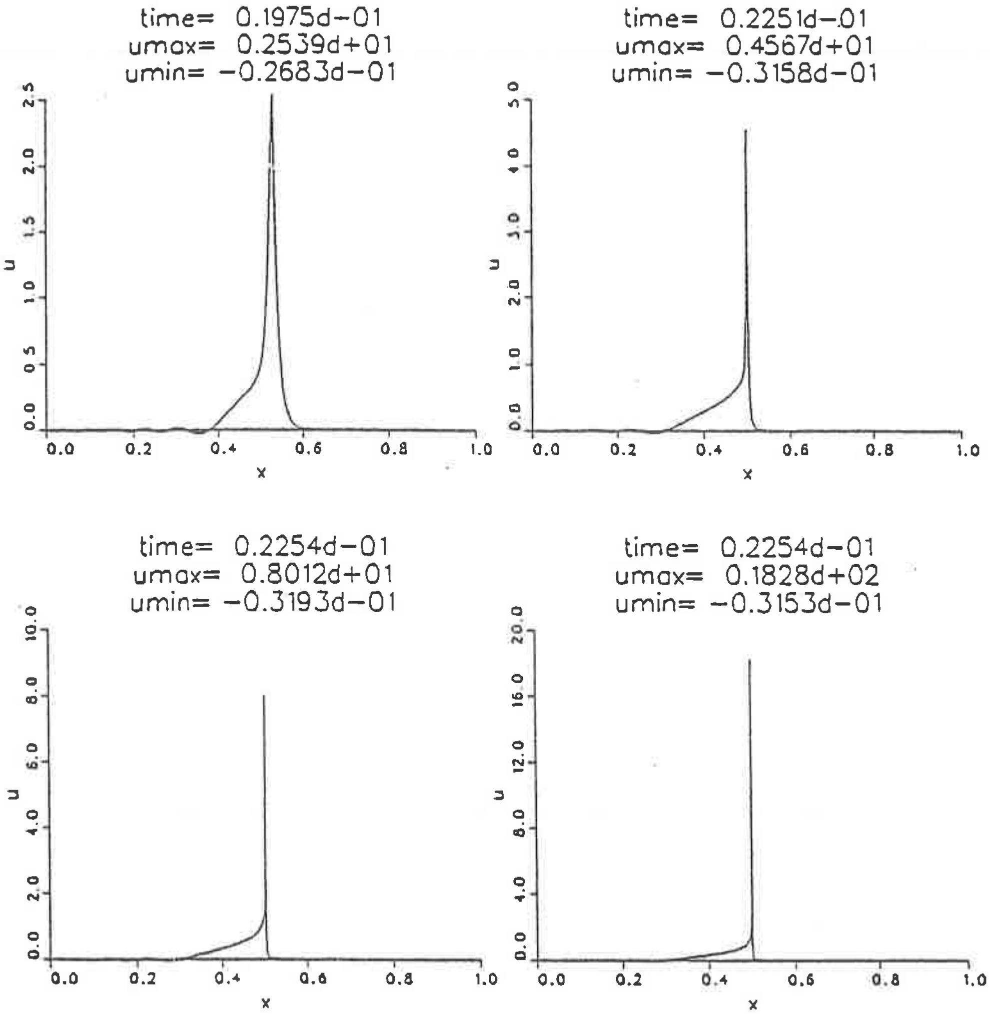

Figure 1. Numerical simulation of a solitary–wave solution with grid refinement, u–axis scaled.

(*umax* in the plots) significantly increases, we are forced to rescale the vertical axis in order to plot the entire solution. This experimenet shows that the solitary–wave type initial condition evolves into a solution that blows up in L^∞ as $(x,t) \to (x^*,t^*)$ where $x^* = .61333$, $t^* = 0.022549$.

In order to examine the nature of the solution near the peak, as the peak continues to grow, one needs to rescale the horizontal axis. Recall that Ω^* is the region of the finest grids with center at $x = .5$ and define $\Omega^0 = \Omega^* - 0.5$. In Figure 2 the solution is translated by 0.5 so that the peak is near 0 and then graphed using Ω^0 as the x–axis. The plots given are with NSPLIT=10, 21, 29, 40. By the final plot, the local spatial and temporal mesh sizes have decreased to approximately $h \approx 10^{-14}$ and $k \approx 10^{-38}$ respectively. Our code is able to follow the peak until it reaches approximately a maximum of 500,000 while maintaining a smooth profile everywhere including Ω^*. The graphs in Figure 2 suggest that instabilities of solitary waves can lead to blow–up of solutions in L^∞ in finite time and that the blow–up seems to be *self-similar*. Indeed, one of the major

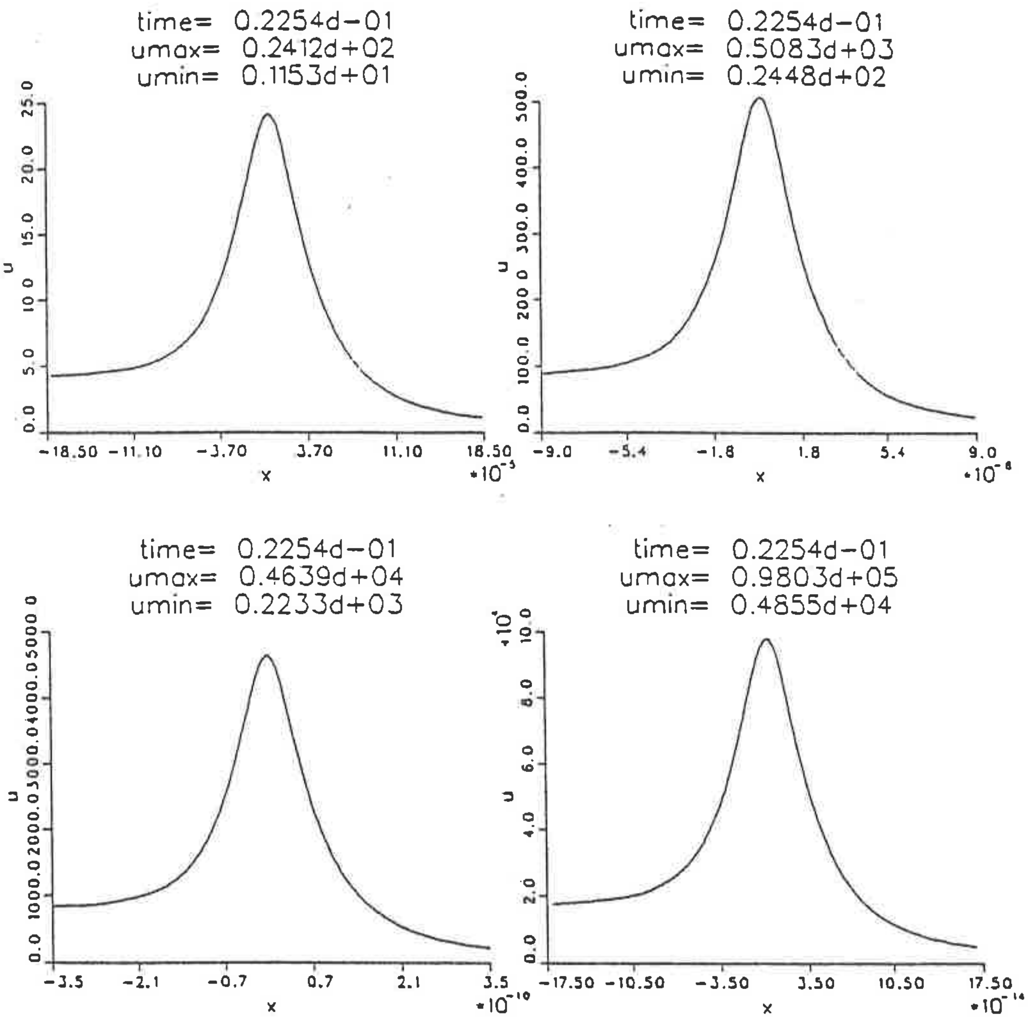

Figure 2. Numerical simulation of a solitary–wave solution near blow–up (both axes scaled).

points of Bona *et al*[3] is the accurate computation of *blow-up rates* of various norms of the solution $u(x,t)$. Such computations are consistent with the conjecture advanced in that reference that the blow–up is of similarity type, indeeed that, as $(x,t) \to (x^*,t^*)$

$$(3.6) \qquad u(x,t) \sim (t^* - t)^{-\frac{2}{3p}} \varphi\Big(\frac{x}{(t^* - t)^{1/3}}\Big) + \text{bounded terms} \ ,$$

where φ is a smooth, bounded function.

5. THE EFFECT OF DISSIPATION.

What happens to the solutions of (1.1) if $\delta > 0$, when we take as initial data e.g. solitary–wave–type initial profiles of the form (3.2)? Our computations and theory that appear in detail in Bona *et al*[4] show that if δ is sufficiently small, then the solution will blow up at a point (x^*, t^*), $t^* < \infty$, but the blow up will be delayed somewhat depending on δ. For example, if we take, A, p, ε and x_0 as in the numerical experiment of Figures 1–2 and add a dissipative term of $\delta = 2 \times 10^{-4}$, the solution blows up at $t^* = 0.038624$, $x^* = .65812$ with practically the same blow-up rates of the norms as in the nondissipative case. This leads us to believe that the similarity solution (3.6) is still valid provided δ is small enough. Moreover, our computations[4] show that in the case of solitary–wave initial profiles of the form (3.2) there exists a critical value c_* of the parameter $\frac{\delta^2}{\varepsilon A^p}$, ($c_*$ is a function only of p), below which blow–up occurs. If $\frac{\delta^2}{\varepsilon A^p}$ stays above c_* the solution will eventually *decay*. For example, Figure 3 shows the initial oscillatory break–up and decay of a solitary–wave initial profile with $\delta = 10^{-3}$ (other parameters as in previous experiments). As t grows, it is demonstrated in Bona *et al*[4] that the solution decays *exponentially* (more precisely, like $\exp[-(2\pi)^2 \delta t]$ due to the periodic boundary conditions) to the integral mean of $u^0(x)$ and that the asymptotic profile is a sinusoidal wave which travels as it decays. Such qualitative features of very long time decay in the presence of sufficiently large dissipation can be essentially determined by the linearized form of the equation (1.1a). We refer the reader to reference 4 for details.

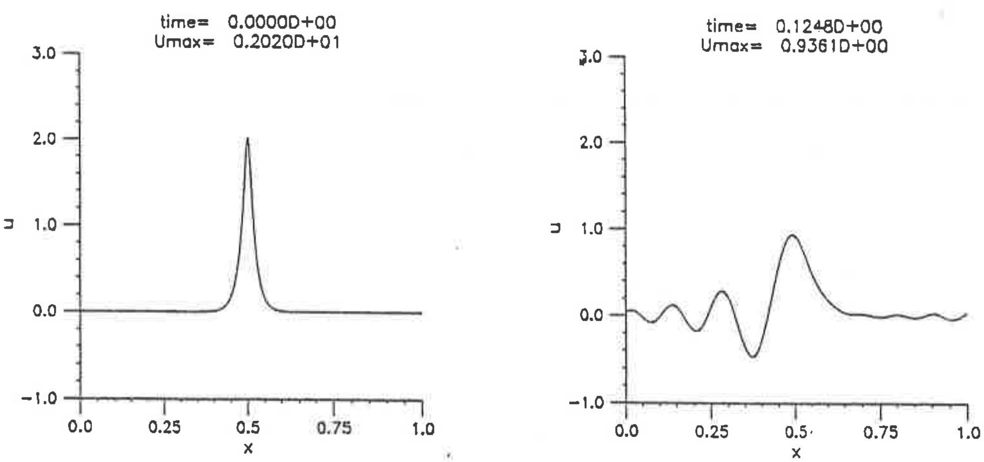

Figure 3. Oscillatory break–up and decay of a solitary–wave initial profile for sufficiently large dissipation.

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