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Higher-order asymptotics of decaying solutions of some nonlinear, dispersive, dissipative wave equations

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Abstract. Considered herein are the generalized Korteweg–de Vries equations with a homogeneous dissipative term appended. Solutions of these equations that start with finite energy decay to zero as time tends toward infinity. We present an asymptotic form which renders explicit the relative strengths of the dissipative, dispersive, and nonlinear effects in this decay.

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1. Introduction

The present work contributes to the discussion of how nonlinearity, dispersion and dissipation interact in wave propagation. Our commentary will be based on the class of one-dimensional model equations

$$u_t + u_x + g(u)_x - Lu_x + Mu = 0 \tag{1.1}$$

where $u = u(x,t)$ represents the displacement of the medium of propagation from its equilibrium position, x is proportional to distance in the direction of propagation and t is proportional to time. Here, subscripts denote partial differentiation, the function $g : \mathbb{R} \to \mathbb{R}$ is usually smooth, often a polynomial, and L and M are Fourier multiplier operators given by

$$\widehat{Lv}(\xi) = \alpha(\xi)\widehat{v}(\xi) \qquad \text{and} \qquad \widehat{Mv}(\xi) = \beta(\xi)\widehat{v}(\xi).$$

The symbols α and β of L and M are typically real-valued, even, non-negative, functions that increase at $\pm\infty$, and consequently L and M represent dispersive and dissipative effects, respectively. This class of models has been discussed in several recent works (see Biler 1984, Dix 1992).

Model equations like those appearing in (1.1) arise when the weak effects of nonlinearity, dispersion and dissipation are appended to a basic model $u_t + u_x = 0$ for uni-directional wave propagation. Such effects often make their appearance at second order in some rational scheme of approximation (see Albert and Bona 1991, Bona and Scialom 1995).

Perhaps the best-known example in the class (1.1) is the Korteweg–de Vries equation

$$u_t + u_x + uu_x + u_{xxx} = 0.$$

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Originally derived as a model for small-amplitude, long-wavelength, surface water waves (Korteweg and de Vries 1895), this equation and its natural generalization

$$u_t + u_x + u^p u_x + u_{xxx} = 0 \tag{1.2}$$

where p is a positive integer, have found application in a wide variety of physically interesting situations. In terms of the class depicted in (1.1), (1.2) has $g(z) = z^{p+1}/p+1$, $\alpha(\xi) = \xi^2$ and $\beta(\xi) \equiv 0$. The equations in (1.2) also arise as one of a number of interesting and approachable classes to study in attempting to understand the way dispersion and nonlinearity can interact (see the recent papers of Amick *et al* (1989), Bona and Luo (1993), Dix (1992), and especially the monograph of Naumkin and Shishmarev (1994) that describes a great deal of the work carried out over the last decade or so).

Nonlinear, dispersive wave equations like those in (1.2) or those in (1.1) with $M = 0$ have come to the fore in the last few decades not only because of the range of their applications, but also because of their interesting and sometimes subtle mathematical properties. Especially intriguing are the travelling-wave solutions called solitary waves which often play a central role in the long-term evolution of initial data (see Albert *et al* 1987, Pego and Weinstein 1992). These aspects are a consequence of the fact, among others, that only nonlinear and dispersive effects are retained in the model. Because of this, equations (1.2), or (1.1) with $M = 0$ comprise Hamiltonian systems that conserve the functional

$$\int_{-\infty}^{\infty} u^2(x,t)\,\mathrm{d}x. \tag{1.3}$$

That is, if u is a smooth solution of one of the just-mentioned evolution equations which decays suitably as $x \to \pm\infty$, then the quantity displayed in (1.3) is independent of t.

In many practically important situations, dissipative mechanisms have the same general strength as those of nonlinearity and dispersion. When dissipation is included in the model, as when $M \neq 0$ in (1.1), most of the special properties just mentioned no longer hold exactly. For example, the quantity in (1.3) typically tends to zero as $t \to +\infty$, rather than being conserved by the evolution. In this circumstance, while the ghosts of solitary waves still play a substantial role in the short term (see Bona and Soyeur 1994), the long-time behavior may be dominated by the decay induced from the dissipation, seen clearly in the fact that $u(\cdot,t)$ tends to zero as $t \to +\infty$, at least in $L_2(\mathbb{R})$. It is our purpose here to explore in some detail the asymptotic structure that obtains for a class of model equations of the form (1.1).

In the remainder of this paper we will study models which are a specialization of those described in (1.1), namely equations having the form

$$u_t + u^p u_x + u_{xxx} + M_\beta u = 0 \tag{1.4a}$$

where a shift has been made to a moving frame of reference to eliminate the u_x term, p is a positive integer as before and M_β is the homogeneous dissipative operator with symbol $\widehat{M}_\beta(\xi) = |\xi|^{2\beta}$. This particular class in which the dispersive term is fixed as the Korteweg–de Vries dispersion $\alpha(\xi) = \xi^2$, the nonlinearity is a monomial and the dissipation is homogeneous, albeit non-local, provides perhaps the simplest class of models in which to study the three effects. The dispersion is local and the strength of the nonlinearity is determined by specifying the integer p. Because $\beta \geqslant 0$ is arbitrary, a wide range of relative strengths of nonlinearity, dispersion and dissipation are encompassed by the class (1.3), but with the advantages that only one non-local operator and a very simple nonlinearity intervene in the analysis. It is expected that the theory obtained for the equations (1.4) will guide us to the correct conclusions for the broader class displayed in (1.1).

In the present report, attention will be given to the initial-value problem in which (1.4*a*) is posed together with the starting configuration

$$u(\cdot, 0) = f(\cdot) \tag{1.4b}$$

and for the range $\frac{1}{2} < \beta \leqslant 1$ and $2\beta \leqslant p$. We intend to extend the previous studies of the long-term behavior of (1.4) (see Dix 1992, Naumkin and Shishmarev 1994, Bona, Promislow and Wayne 1994) to obtain higher-order asymptotics of the decay of solutions. The earlier developments concluded that if the initial data f is smooth and small enough, then the long-time behavior of u may be described via the associated function f^* defined to be

$$f^*(x) = \int_{-\infty}^{\infty} e^{i\xi x} e^{-|\xi|^{2\beta}} d\xi. \tag{1.5}$$

Indeed, it was shown that for any $\epsilon > 0$, there exist positive constants c_2 and c_∞ depending on f such that for all $t \geqslant 1$,

$$\begin{aligned} \left| u(\cdot, t) - \frac{A_0}{t^{1/2\beta}} f^*(\cdot/t^{1/2\beta}) \right|_2 &\leqslant \frac{c_2}{t^{(3/4\beta-\epsilon)}} \\ \left| u(\cdot, t) - \frac{A_0}{t^{1/2\beta}} f^*(\cdot/t^{1/2\beta}) \right|_\infty &\leqslant \frac{c_\infty}{t^{(1/\beta-\epsilon)}} \end{aligned} \tag{1.6}$$

where the norms are those of $L_2(\mathbb{R})$ and $L_\infty(\mathbb{R})$, respectively, and $A_0 = \int_{-\infty}^{\infty} f(x)\, dx$ is the total mass of the initial data (a conserved quantity even when dissipation is present). These results may be interpreted in the following way. Let v be the solution of the linear initial-value problem

$$\begin{aligned} v_t + M_\beta v &= 0 \\ v(\cdot, 0) &= h(\cdot) \end{aligned} \tag{1.7}$$

and suppose h has the same total mass A_0 as f. Then v and the solution u of (1.4) with initial data f have the same asymptotic form as $t \to \infty$. Thus at lowest order, the asymptotic state of solutions of the initial-value problem (1.4) for $p \geqslant 2\beta$ does not depend upon the nonlinearity, the dispersion, nor indeed on the initial data save through its mass.

It will be seen presently that a more refined asymptotic analysis is needed to discern the long-term effects of nonlinearity and dispersion. Consider the two-parameter family $\Gamma = \Gamma_{A,B}$ of functions of (x, t) defined via their Fourier transform with respect to x to be

$$\begin{aligned} \widehat{\Gamma}(k, t) = \widehat{\Gamma}_{A,B}(k, t) &= A e^{-|k|^{2\beta}t} + [-iAtk^3 + iBk] e^{-|k|^{2\beta}t} \\ &\quad + \frac{ik}{p+1} \int_1^t e^{-|k|^{2\beta}(t-s)} \mathcal{F}\left(\left[\frac{A}{s^{1/2\beta}} f^*\left(\cdot/s^{1/2\beta}\right) \right]^{p+1} \right)(k)\, ds \end{aligned} \tag{1.8}$$

where f^* is as in (1.5) and $\mathcal{F}(h)$ denotes the Fourier transform of h. Then for given initial data f, there is a choice of the constants A and B so that for any $\epsilon > 0$, there exist constants c_2' and c_∞' depending on f for which

$$\begin{aligned} \left| u(\cdot, t) - \Gamma_{A,B}(\cdot, t) \right|_2 &\leqslant c_2' t^{-(\frac{5}{4\beta}-\epsilon)} \\ \left| u(\cdot, t) - \Gamma_{A,B}(\cdot, t) \right|_\infty &\leqslant c_\infty' t^{-(\frac{3}{2\beta}-\epsilon)} \end{aligned} \tag{1.9}$$

and, if $w(x) = 1 + |x|$, then

$$\begin{aligned} \left|w(\cdot)[u(\cdot,t) - \Gamma_{A,B}(\cdot,t)]\right|_2 &\leqslant c_2' t^{-(\frac{3}{4\beta}-\epsilon)} \\ \left|w(\cdot)[u(\cdot,t) - \Gamma_{A,B}(\cdot,t)]\right|_\infty &\leqslant c_\infty' t^{-(\frac{1}{\beta}-\epsilon)} \end{aligned} \tag{1.10}$$

for $t \geqslant 0$, where u is the solution of (1.4) corresponding to f and β is the parameter defining the dissipative operator M_β. The dependence of the constant B on f will be made precise in due course, but one can see immediately from comparison with the first order asymptotics in (1.6) that $A = A_0 = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$.

The estimates (1.9)–(1.10) showing Γ to be a more accurate approximation to the asymptotics of solutions of (1.4) comprise our principal goal. The paper is laid out as follows. Section 2 provides the notation and mathematical structure used later. In particular, we define a renormalization mapping that was introduced in another context by Bricmont *et al* (1994), and which plays a central role in our analysis. The introductory material is then followed by a technical section composed of preparatory estimates. This in turn is followed by the statement and proof of our main results in section 4, namely, precise versions of the inequalities (1.9) and (1.10). These are obtained via a contraction–mapping argument in Fourier-transformed variables using the inequalities derived in section 3. The paper concludes with a brief summary and a discussion of directions potentially worth further inquiry.

2. Notation

Here, function classes are introduced, notational conventions set forth, and the renormalization operators defined.

The norm of a function f in the standard class $L_p = L_p(\mathbb{R})$ is denoted $|f|_p$, for $1 \leqslant p \leqslant \infty$. The classes $C^k = C^k(\mathbb{R})$, $k = 0, 1, 2, \cdots$, comprise the functions which, along with their first k derivatives, are continuous. Less standard are the Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ defined as follows. A function f lies in $\mathcal{B}_1$ if its Fourier transform $\widehat{f}$ lies in C^1 and the norm

$$\|f\|_{\mathcal{B}_1} = \sup_k \left\{(1+|k|^3)|\widehat{f}(k)| + (1+|k|^2)\left|\frac{\mathrm{d}}{\mathrm{d}k}\widehat{f}(k)\right|\right\} \tag{2.1}$$

is finite. Similarly, f lies in $\mathcal{B}_2$ if $\widehat{f}$ lies in C^2 and its $\mathcal{B}_2$-norm

$$\|f\|_{\mathcal{B}_2} = \sup_k \left\{(1+|k|^3)|\widehat{f}(k)| + (1+|k|^2)\left|\frac{\mathrm{d}}{\mathrm{d}k}\widehat{f}(k)\right| + (1+|k|)\left|\frac{\mathrm{d}^2}{\mathrm{d}k^2}\widehat{f}(k)\right|\right\} \tag{2.2}$$

is finite.

The definition of the renormalization operator requires some preliminary notions. Let $u = u(x,t)$ be a real-valued function of $(x,t) \in \mathbb{R}^2$. For $n = 0, 1, 2, \cdots$, $\beta > 0$ and $L > 1$, define a sequence $\{u_n\}_{n=0}^{\infty}$ of rescaled functions inductively by $u_0 = u$ and, for $n \geqslant 1$,

$$u_n(x,t) = L u_{n-1}(Lx, L^{2\beta}t) = L^n u(L^n x, L^{2\beta n}t). \tag{2.3}$$

Later, L will be taken to be large so that $1/L$ becomes a small parameter. If the original function u happens to satisfy the initial-value problem (1.4) and the β in (2.3) is identified with the parameter β in the symbol of the dissipative operator $M = M_\beta$, then u_n satisfies the initial-value problem

$$\begin{aligned}&\partial_t u_n + M u_n + \alpha^n \partial_x^3 u_n + \frac{\gamma^n}{p+1}\partial_x\left(u_n^{p+1}\right) = 0\\ &f_n(x) = u_n(x,1) = L^n u(L^n x, L^{2\beta n})\end{aligned} \tag{2.4$_n$}$$

for $n = 1, 2, \cdots$, where $\alpha = L^{2\beta-3}$ and $\gamma = L^{2\beta-(p+1)}$. For technical reasons (see the comments following proposition 2) the renormalized initial-value problems are posed at $t = 1$, in particular $f_0(\cdot) \equiv u(\cdot, 1)$ where u solves (1.4a, b). Note that both α and γ are small if L is large; indeed, $\alpha, \gamma \leqslant L^{-1}$ since $\beta \leqslant 1$ and $2\beta \leqslant p$.

By applying the Fourier transform with regard to the spatial variable to (2.4)$_n$ and solving the resulting ordinary differential equation by the variation of constants formula, a formal representation is discovered, namely

$$\begin{aligned}\widehat{u_n}(k,t) &= \mathrm{e}^{-(|k|^{2\beta}+\mathrm{i}\alpha^n k^3)(t-1)}\widehat{f_n}(k)\\ &\quad + \frac{ik\gamma^n}{p+1}\int_1^t \mathrm{e}^{-(|k|^{2\beta}+\mathrm{i}\alpha^n k^3)(t-s)}\widehat{u_n^{p+1}}(k,s)\,\mathrm{d}s.\end{aligned} \tag{2.5}$$

After taking the inverse Fourier transform of (2.5), we note that the first term on the right-hand side of (2.5) is just the linear semigroup applied to f_n, namely

$$S_n(t) f_n(x) = \int_{-\infty}^{\infty} \mathrm{e}^{-(|k|^{2\beta}+\mathrm{i}\alpha^n k^3)(t-1)-\mathrm{i}kx}\widehat{f_n}(k)\,\mathrm{d}k \tag{2.6}$$

while the nonlinear term is $N^n_{1,t}(u)$ where, for $b \geqslant a$,

$$\mathcal{F}\left(N^n_{a,b}(u)\right)(k) = \frac{ik\gamma^n}{p+1}\int_a^b \mathcal{F}\left(S_n(b-s)u_n^{p+1}\right)(k,s)\,\mathrm{d}s. \tag{2.7}$$

We will be interested in comparing (2.4)$_n$ with the linear, dispersionless equation (1.7) which results from the formal limit $n \to \infty$, and its associated semigroup

$$S_\infty(t) f(x) = \int_{-\infty}^{\infty} \mathrm{e}^{-|k|^{2\beta}(t-1)-\mathrm{i}kx}\widehat{f}(k)\,\mathrm{d}k. \tag{2.8}$$

The renormalization group operators R_n are now defined. For $n = 0, 1, 2, \cdots$, let v_n be the solution of (2.4)$_n$ corresponding to given initial data f. Define $R_n f$ to be

$$(R_n f)(x) = L\, v_n(Lx, L^{2\beta}). \tag{2.9}$$

Notice that (2.9) and the definition of f_n imply that

$$R_n f_n(x) = L\, u_n(Lx, L^{2\beta}) = L^{n+1} u\left(L^{n+1}x, L^{2\beta(n+1)}\right) = f_{n+1}(x).$$

The linear and the linear dispersionless renormalization group operators will also appear in the analysis in sections 3 and 4. They are denoted R_n^0 and R_∞, respectively, and the outcome of their application to a function f is

$$\begin{aligned}(R_n^0 f)(x) &= L(S_n(L^{2\beta})f)(Lx)\\ (R_\infty f)(x) &= L(S_\infty(L^{2\beta})f)(Lx).\end{aligned} \tag{2.10}$$

There are several observations that, taken together, indicate the renormalization group operators $\{R_n\}_{n=0}^{\infty}$, $\{R_n^0\}_{n=0}^{\infty}$, and R_∞ are objects worthy of study in attempting to understand the long-term behaviour of solutions of the initial-value problem (1.4). First, for $L > 1$, the long-time asymptotics of the solution u of (1.4) with given initial data, turns out to be related to fixed-point problems for these renormalization operators which can be set in the Banach space $\mathcal{B}_2$. Secondly, as n grows, it is expected that $R_n \to R_\infty$. Finally, as solutions decay, it is expected that the action of R_n and R_n^0 on them will be nearly the same.

In consequence of these observations, we now turn to a study of the various renormalization operators, their fixed points and the relations between them.

3. Technical estimates

The similarity function f^* defined in (1.5) is a fixed point of the linear dispersionless renormalization group R_∞, so that

$$R_\infty f^* = f^* \tag{3.1}$$

and it is our goal to show that successive applications of the nonlinear renormalization group R_n drive one towards f^*. That is, we intend to give a detailed analysis of the convergence

$$L^n u(L^n x, L^{2\beta n}) = R_{n-1} \circ \cdots \circ R_0 f_0 \underset{n\to\infty}{\longrightarrow} f^*. \tag{3.2}$$

The remainder of this section is devoted to technical estimates required in the demonstration of this convergence. From the formal representation (2.5), the nonlinear renormalization group R_n may be decomposed into linear and nonlinear parts, namely

$$R_n f_n = R_n^0 f_n + L\, N^n_{1,L^{2\beta}}(u_n)\,(Lx). \tag{3.3}$$

The analysis begins with lemmas 1 and 2 below, which show that the linear map R_n^0 is contractive and that f^* is close to being a fixed point of R_n^0 for n large. Lemma 3 demonstrates regularity properties of the linear semigroup while proposition 1 provides technical estimates on the nonlinear term.

Lemma 1. *Let the power β in the dissipative operator be larger than $\frac{1}{2}$. Then there exist positive constants $C_1 = C_1(\beta)$ and $C_2 = C_2(\beta)$ such that for all $g \in \mathcal{B}_2$ satisfying $\widehat{g}(0) = 0$, it follows that*

$$\|R_n^0 g\|_{\mathcal{B}_2} \leqslant C_1(\beta) L^{-1} \|g\|_{\mathcal{B}_2} \tag{3.4}$$

and if additionally $\widehat{g}'(0) = 0$, then

$$\|R_n^0 g\|_{\mathcal{B}_2} \leqslant C_2(\beta) L^{-2} \|g\|_{\mathcal{B}_2}. \tag{3.5}$$

Proof. From the definition (2.10) of R_n^0 and the formula (2.6) for the semigroup S_n, it follows that

$$\widehat{R_n^0 g}(k) = \mathrm{e}^{-(|k|^{2\beta} + \mathrm{i}\, L^{2\beta-3} \alpha^n k^3)(1 - L^{-2\beta})} \widehat{g}\left(\frac{k}{L}\right).$$

Since $\widehat{g}(0) = 0$ and $\widehat{g} \in C^1(\mathbb{R})$, the Mean-Value Theorem implies that for any k, there is a point $\xi = \xi_k$ with $|\xi_k| \leqslant |k|/L$ such that

$$\left|\widehat{g}\left(\frac{k}{L}\right)\right| \leqslant \frac{|k|}{|L|}\left|\widehat{g}'(\xi_k)\right|.$$

In consequence, it follows that

$$\begin{aligned}
\sup_k \left\{(1+|k|^3)|\widehat{R_n^0 g}(k)|\right\} &\leqslant \frac{1}{L}\sup_k\{(1+|k|^3)\mathrm{e}^{-|k|^{2\beta}(1-L^{-2\beta})}|k|\,\widehat{g}'(\xi_k)|\} \\
&\leqslant \frac{1}{L}\sup_k\{(1+|k|^3)\mathrm{e}^{-|k|^{2\beta}(1-L^{-2\beta})}|k|\}\sup_\xi\{(1+|\xi|^2)|\widehat{g}'(\xi)|\} \\
&\leqslant c(\beta)\frac{1}{L}\|g\|_{\mathcal{B}_2}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}k}\widehat{R_n^0 g}(k) = &\left[(2\beta|k|^{2\beta-1} + 3\mathrm{i}\,\alpha^{n+1}k^2)\widehat{g}\left(\frac{k}{L}\right) + \frac{1}{L}\widehat{g}'\left(\frac{k}{L}\right)\right] \\
&\times \mathrm{e}^{-(|k|^{2\beta}+\mathrm{i}\,\alpha^{n+1}k^3)(1-L^{-2\beta})}
\end{aligned}$$

and bounding $\widehat{g}\left(\frac{k}{L}\right)$ via the Mean-Value Theorem as above leads to the estimate

$$\begin{aligned}
&\sup_k \left\{(1+|k|^2)\left|\widehat{R_n^0 g}'(k)\right|\right\} \\
&\quad\leqslant \frac{1}{L}\sup_k\left\{(1+|k|^2)(2\beta|k|^{2\beta} + 3\alpha^{n+1}k^3 + 1)\mathrm{e}^{-|k|^{2\beta}(1-L^{-2\beta})}\right\}\sup_\xi|\widehat{g}'(\xi)| \\
&\quad\leqslant \frac{1}{L}c(\beta)\|g\|_{\mathcal{B}_2}.
\end{aligned}$$

In the same vein, one discovers that

$$\begin{aligned}
\left|\frac{\mathrm{d}^2}{\mathrm{d}k^2}\widehat{R_n^0 g}(k)\right| \leqslant c&\left\{\left(|k|^{2\beta-2} + \alpha^{n+1}|k| + (|k|^{2\beta-1} + \alpha^{n+1}k^2)^2\right)\left|\frac{k}{L}\right||\widehat{g}'(\xi_k)|\right. \\
&\left. + 2\left(|k|^{2\beta-1} + \alpha^{n+1}k^2\right)\frac{1}{L}\left|\widehat{g}'\left(\frac{k}{L}\right)\right| + \frac{1}{L^2}\left|\widehat{g}''\left(\frac{k}{L}\right)\right|\right\}\mathrm{e}^{-|k|^{2\beta}(1-L^{-2\beta})}
\end{aligned}$$

whence

$$\sup_k\left\{(1+|k|)\left|\widehat{R_n^0 g}''(k)\right|\right\} \leqslant \frac{c(\beta)}{L}\|g\|_{\mathcal{B}_2}$$

and the result (3.4) follows. For the second inequality (3.5), it is assumed that $\widehat{g}(0) = \widehat{g}'(0) = 0$, which implies via Taylor's Theorem and the definition of $\mathcal{B}_2$, that

$$\begin{aligned}
\left|\widehat{g}\left(\frac{k}{L}\right)\right| &\leqslant \frac{k^2}{L^2}|\widehat{g}''(\xi_{1,k})| \leqslant \frac{k^2}{L^2}\|g\|_{\mathcal{B}_2} \\
\left|\widehat{g}'\left(\frac{k}{L}\right)\right| &\leqslant \frac{k}{L}|\widehat{g}''(\xi_{2,k})| \leqslant \frac{k}{L}\|g\|_{\mathcal{B}_2}
\end{aligned}$$

where $|\xi_{i,k}| \leqslant \frac{|k|}{L}$ for $i = 1, 2$. Arguments similar to those provided above then yield the desired result. □

It will be useful presently to understand the action of R_n^0 on f^*, as well as its action on the function f_1^* defined by

$$f_1^*(x) = \int_{-\infty}^{\infty} \mathrm{i}k\, \mathrm{e}^{-\mathrm{i}kx} \mathrm{e}^{-|k|^{2\beta}}\, \mathrm{d}k. \tag{3.6}$$

Lemma 2. *Suppose that $\frac{1}{2} < \beta < 1$. Then there are constants $L_0 = L_0(\beta)$ and $c = c(\beta)$ such that for any $L \geqslant L_0$, one has*

$$\|R_n^0 f^* - f^*\|_{B_2} \leqslant c\, L^{-(n+1)} \tag{3.7}$$

and

$$\|R_n^0 f_1^* - \frac{1}{L} f_1^*\|_{B_2} \leqslant c\, L^{-(n+2)} \tag{3.8}$$

for $n = 1, 2, 3, \cdots$, where f^* is defined in (1.5) and f_1^* is as in (3.6).

Proof. We begin with the demonstration of (3.8). Observe that from (2.10) and (2.6),

$$\begin{aligned}\widehat{R_n^0 f_1^*}(k) &= \mathrm{e}^{-(|k|^{2\beta} + \mathrm{i}\alpha^{n+1}k^3)(1-L^{-2\beta})} \widehat{f_1^*}\left(\frac{k}{L}\right)\\ &= \frac{\mathrm{i}k}{L} \mathrm{e}^{-|k|^{2\beta}} \mathrm{e}^{-\mathrm{i}\alpha^{n+1}k^3(1-L^{-2\beta})}.\end{aligned}$$

Consequently, one sees immediately that

$$\widehat{R_n^0 f_1^*}(k) - \frac{1}{L}\widehat{f_1^*}(k) = \frac{\mathrm{i}k}{L} \mathrm{e}^{-|k|^{2\beta}} \left(\mathrm{e}^{-\mathrm{i}\alpha^{n+1}k^3(1-L^{-2\beta})} - 1\right).$$

We now estimate $\sup(1 + |k|^3) \left|\widehat{R_n^0 f_1^*} - \frac{1}{L}\widehat{f_1^*}\right|$ which is accomplished in two steps. First suppose that k, L and n are such that $|k|^3 < \alpha^{-n\mu}$ for some fixed positive $\mu < 1$, where $\alpha = L^{2\beta-3} < 1$. Then one sees at once that

$$\left|\mathrm{e}^{-\mathrm{i}\alpha^{n+1}k^3(1-L^{-2\beta})} - 1\right| \leqslant c\, \alpha^{(n+1)(1-\mu)}$$

and

$$\left|(1 + |k|^3)\frac{|k|}{L} \mathrm{e}^{-|k|^{2\beta}}\right| \leqslant \frac{c(\beta)}{L}$$

which together imply

$$\sup_{|k|^3 < \alpha^{-n\mu}} \left\{(1 + |k|^3)\left|\left(\widehat{R_n^0 f_1^*} - \frac{1}{L}\widehat{R_n^0 f_1^*}\right)\right|\right\} \leqslant \frac{c(\beta)}{L} \alpha^{(n+1)(1-\mu)}.$$

If, on the other hand, $|k|^3 \geqslant \alpha^{-n\mu}$ then

$$\left|\mathrm{e}^{-\mathrm{i}\alpha^{n+1}k^3(1-L^{-2\beta})} - 1\right| \leqslant 2$$

and so

$$\sup_{|k|^3 \geqslant \alpha^{-n\mu}} \left\{(1+|k|^3)\frac{|k|}{L}e^{-|k|^{2\beta}}\right\} \leqslant \frac{1}{L}e^{-\frac{1}{2}\alpha^{-\left(\frac{2\beta n\mu}{3}\right)}} \sup_{|k|^3 \geqslant \alpha^{-n\mu}} \left\{(1+|k|^4)e^{-\frac{1}{2}|k|^{2\beta}}\right\}$$
$$\leqslant c(\beta)\frac{1}{L}e^{-\frac{1}{2}L^{\left(\frac{(3-2\beta)2\beta n\mu}{3}\right)}}$$
$$\leqslant c(\beta)\left(\frac{1}{L}\right)^{n+2}$$

if $L \geqslant L_0$ where L_0 depends upon β and μ but not on n. If μ is restricted to lie in the interval $\left(0, \frac{2-2\beta}{3-2\beta}\right]$, then

$$\alpha^{(n+1)(1-\mu)} = L^{(2\beta-3)(1-\mu)(n+1)} \leqslant \left(\frac{1}{L}\right)^{n+1}.$$

As a consequence of the above inequalities, it is adduced that there exist constants C and L_0 depending only on β such that for $L \geqslant L_0$,

$$\sup_k \left\{(1+|k|^3)\left|\mathcal{F}\left(R_n^0 f_1^* - \frac{1}{L}f_1^*\right)(k)\right|\right\} \leqslant C\left(\frac{1}{L}\right)^{n+2}.$$

Similar sets of inequalities show that

$$\sup_k \left\{(1+|k|^{3-i})\left|\frac{d^i}{dk^i}\mathcal{F}\left(R_n^0 f_1^* - \frac{1}{L}f_1^*\right)(k)\right|\right\} \leqslant c_i\left(\frac{1}{L}\right)^{n+2}$$

for $i = 1, 2$ and L large, where the c_i depend only on β. The inequality (3.8) follows.

The derivation of (3.7) follows in the same vein except for the term

$$\sup_{|k|^3 < \alpha^{-n\mu}} \left\{(1+|k|)\left|\frac{d^2}{dk^2}\left(\widehat{R_n^0 f^*} - \widehat{f^*}\right)(k)\right|\right\}$$
$$\leqslant \sup_{|k|^3 < \alpha^{-n\mu}} \left\{(1+|k|)\left|\frac{d^2}{dk^2}e^{-|k|^{2\beta}}\left(e^{-i\alpha^{n+1}k^3(1-L^{-2\beta})} - 1\right)\right|\right\}$$
$$\leqslant \sup_{|k|^3 < \alpha^{-n\mu}} \left\{(1+|k|)\left|\frac{d^2}{dk^2}(k\,e^{-|k|^{2\beta}})\left(\frac{e^{-i\alpha^{n+1}k^3} - 1}{k}\right)\right|\right\}.$$

Now $k\,e^{-|k|^{2\beta}}$ is uniformly bounded in C^2 provided $\beta > \frac{1}{2}$, and therefore

$$\sup_{|k|^3 < \alpha^{-n\mu}} \left\{(1+|k|)\left|\frac{d^2}{dk^2}\left(\widehat{R_n^0 f^*} - \widehat{f^*}\right)(k)\right)\right\} \leqslant c(\beta)\alpha^{(n+1)(1-\mu)}.$$

The remaining estimates follow lines already indicated; the lemma is thereby established. □

Lemma 3. *Let $T > 1$ and $\frac{1}{2} < \beta \leqslant 1$ be given. Then there exists a constant $C_T > 0$ such that for $n = 0, 1, 2, \cdots$, the linear semigroup S_n in (2.6) satisfies the estimates*

$$\|S_n(t)g\|_{B_2} \leqslant c_T\|g\|_{B_2} \tag{3.9}$$

for $1 \leqslant t \leqslant T$, *for all* $g \in B_2$ *with total mass equal to zero.*

Proof. By its definition, $S_n(t)g$ is given in Fourier-transformed variables as

$$\widehat{S_n(t)g}(k) = \mathrm{e}^{-(|k|^{2\beta}+\mathrm{i}\alpha^n k^3)(t-1)}\widehat{g}(k). \tag{3.10}$$

To estimate $\|S_n(t)g\|_{B_2}$, consider first the term

$$\begin{aligned}\sup_k\left\{(1+|k|)\left|\frac{\mathrm{d}^2}{\mathrm{d}k^2}\widehat{S_n(t)g}(k)\right|\right\} &\leqslant \sup_k\left\{(1+|k|)\left|\mathrm{e}^{-\Theta(k)(t-1)}\widehat{g}''(k)\right|\right\}\\ &+\sup_k\left\{(1+|k|)\left|2\Theta'(k)(t-1)\mathrm{e}^{-\Theta(k)(t-1)}\widehat{g}'(k)\right|\right\}\\ &+\sup_k\left\{(1+|k|)\left|\left[\Theta''(k)(t-1)+(\Theta'(k)(t-1))^2\right]\right.\right.\\ &\left.\left.\times\ \mathrm{e}^{-\Theta(k)(t-1)}\widehat{g}(k)\right|\right\}\end{aligned} \tag{3.11}$$

where $\Theta(k) = |k|^{2\beta} + \mathrm{i}\alpha^n k^3$. Denoting the three terms on the right-hand side of (3.1) by Σ_1, Σ_2, and Σ_3, respectively, we see that

$$\Sigma_1 \leqslant \sup_k\{(1+|k|)|\widehat{g}''(k)|\} \leqslant \|g\|_{B_2} \tag{3.12}$$

and

$$\begin{aligned}\Sigma_2 &\leqslant \sup_k\left\{c(1+|k|)k^2(t-1)\mathrm{e}^{-|k|^{2\beta}(t-1)}\frac{1}{1+|k|^2}\right\}\|g\|_{B_2}\\ &\leqslant c(t-1)\sup\{(1+|k|)\mathrm{e}^{-|k|^{2\beta}(t-1)}\}\|g\|_{B_2} \leqslant c((t-1)+(t-1)^{1-\frac{1}{2\beta}})\|g\|_{B_2}.\end{aligned} \tag{3.13}$$

To estimate Σ_3, observe that $\Theta''(k)$ has a singularity at $k = 0$, so it is natural to write $\Sigma_3 \leqslant \Sigma_{3,1} + \Sigma_{3,2} + \Sigma_{3,3}$ where

$$\begin{aligned}\Sigma_{3,1} &= \sup_{|k|\leqslant 1}\left\{(1+|k|)\left|\Theta''(k)(t-1)\mathrm{e}^{-\Theta(k)(t-1)}\widehat{g}(k)\right|\right\}\\ \Sigma_{3,2} &= \sup_{|k|\geqslant 1}\left\{(1+|k|)\left|\Theta''(k)(t-1)\mathrm{e}^{-\Theta(k)(t-1)}\widehat{g}(k)\right|\right\}\\ \Sigma_{3,3} &= \sup_k\left\{(1+|k|)\left|(\Theta'(k)(t-1))^2\mathrm{e}^{-\Theta(k)(t-1)}\widehat{g}(k)\right|\right\}.\end{aligned}$$

Since $\widehat{g}(0) = \int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = 0$, we have $|\widehat{g}(k)| \leqslant |k||\widehat{g}'|_\infty \leqslant |k|\ \|g\|_{B_2}$, and therefore

$$\Sigma_{3,1} \leqslant c\sup_{|k|\leqslant 1}\{|k|^{2\beta-2}|k|\}(t-1)\|g\|_{B_2} \leqslant c(t-1)\|g\|_{B_2}. \tag{3.14}$$

On the other hand,

$$\Sigma_{3,2} \leqslant c\sup_{|k|\geqslant 1}\{(1+|k|)|k|(t-1)|\widehat{g}(k)|\} \leqslant c(t-1)\|g\|_{B_2} \tag{3.15}$$

and, finally,

$$\begin{aligned}\Sigma_{3,3} &\leqslant c\sup_k\left\{(1+|k|)k^4(t-1)^2\mathrm{e}^{-|k|^{2\beta}(t-1)}\frac{1}{1+|k|^3}\right\}\|g\|_{B_2}\\ &\leqslant c((t-1)+(t-1)^{2-\frac{1}{\beta}})\|g\|_{B_2}.\end{aligned} \tag{3.16}$$

Combining the estimates (3.12)–(3.16) and recalling that β lies in the interval $(\frac{1}{2}, 1]$, it is readily deduced that

$$\sup_k \left\{ (1+|k|) \left| \frac{d^2}{dk^2} \widehat{S_n(t)g}(k) \right| \right\} \leqslant c_T \|g\|_{B_2}.$$

Suitable estimates on $\sup_k\{(1+|k|^2)|\frac{d}{dk}\widehat{S_n(t)g}(k)|\}$ and $\sup_k\{(1+|k|^3)|\widehat{S_n(t)g}(k)\}$ are similarly derived. The result (3.9) follows. □

The remainder of this section is dedicated to development of estimates on the Fourier transform of the nonlinear map $N^n_{a,b}$ defined in (2.7). For this purpose, we will make use of the space-time norms

$$\begin{aligned} \|u\|_{L_\infty(a,b;B_1)} &= \sup_{a\leqslant t\leqslant b} \|u(t)\|_{B_1}, \\ \|u\|_{L_\infty(a,b;B_2)} &= \sup_{a\leqslant t\leqslant b} \|u(t)\|_{B_2}. \end{aligned} \tag{3.17}$$

Define also the Fourier multiplier operator Q with symbol q by

$$\widehat{Qf}(k) = q(k)\widehat{f}(k)$$

where $q \in C^2(\mathbb{R})$ and

$$q(k) = \begin{cases} 1 & k \geqslant 2 \\ k & |k| \leqslant 1 \\ -1 & k \leqslant 2 \end{cases} \tag{3.18}$$

and an associated quotient $\widetilde{Q}$ with symbol $\widetilde{q}$ given by

$$\widetilde{q}(k) = \begin{cases} k/q(k) & k \neq 0 \\ 1 & k = 0. \end{cases} \tag{3.19}$$

Note that $q(k)\widetilde{q}(k) = k$ and $\widetilde{q} \in C^2(\mathbb{R})$.

The following inequalities about Fourier transforms of products are used in the following.

Lemma 4. *Let $p \geqslant 1$ be an integer. Then there is a constant C depending only on p such that for any $f \in B_1$, we have*

$$|\widehat{f^{p+1}}(k)| \leqslant \frac{c}{1+|k|^3} \|f\|^{p+1}_{B_1} \tag{3.20}$$

and

$$\left| \frac{d}{dk} \widehat{f^{p+1}}(k) \right| \leqslant \frac{c}{1+|k|^2} \|f\|^{p+1}_{B_1}. \tag{3.21}$$

If moreover $h \in B_1$ and $Qh \in B_2$, where Q is the operator defined in (3.18), then

$$\left| \frac{d^2}{dk^2} \widehat{kh^{p+1}}(k) \right| \leqslant c(\|h\|_{B_1} + \|Qh\|_{B_2}) \|h\|^p_{B_1}. \tag{3.22}$$

Proof. The estimates (3.20) and (3.21) follow in a straightforward manner upon writing

$$\widehat{f^{p+1}}(k) = \widehat{f} * \cdots * \widehat{f}(k) = \int_{\mathbb{R}^p} \widehat{f}(k - (k_1 + \cdots + k_p))\widehat{f}(k_1) \cdots \widehat{f}(k_p)\, dk_1 \cdots dk_p$$

and

$$\frac{d}{dk}\widehat{f^{p+1}}(k) = \int_{\mathbb{R}^p} \widehat{f}'(k - (k_1 + \cdots + k_p))\widehat{f}(k_1) \cdots \widehat{f}(k_p)\, dk_1 \cdots dk_p.$$

For (3.22), proceed by writing

$$\left|\frac{d^2}{dk^2} k\widehat{h^{p+1}}(k)\right| = (p+1)\left|\frac{d^2}{dk^2}\int_{\mathbb{R}^p}\left(k - \sum_{i=1}^{p} k_i\right)\widehat{h}\left(k - \sum_{i=1}^{p} k_i\right) \times \widehat{h}(k_1)\ldots\widehat{h}(k_p)\, dk_1 \cdots dk_p\right|. \tag{3.23}$$

Since $h \in \mathcal{B}_1$ and $Qh \in \mathcal{B}_2$ we use the relation

$$\begin{aligned}(\xi\widehat{h}(\xi))'' &= (q(\xi)\widetilde{q}(\xi)\widehat{h}'(\xi))' + \widehat{h}'(\xi)\\ &= \widetilde{q}'(\xi)(q(\xi)\widehat{h}'(\xi)) + \widetilde{q}(\xi)(q(\xi)\widehat{h}'(\xi))' + \widehat{h}'(\xi)\end{aligned}$$

where q and $\widetilde{q}$ are defined in (3.18) and (3.19), and observe that $|\widetilde{q}(\xi)| \leqslant c(1 + |\xi|)$ and $|\widetilde{q}'(\xi)| \leqslant c$ to achieve the following bound on the second derivative:

$$|(\xi\widehat{h}(\xi))''| \leqslant c(\|h\|_{\mathcal{B}_1} + \|Qh\|_{\mathcal{B}_2}).$$

Returning to (3.23), bring the derivatives inside the integral, and use the estimate above with $\xi = k - \sum_{i=1}^{p} k_i$ to bound the L_∞-norm of the first term, thereby yielding

$$\begin{aligned}\left|\frac{d^2}{dk^2} k\widehat{h^{p+1}}(k)\right| &\leqslant c(\|h\|_{\mathcal{B}_1} + \|Qh\|_{\mathcal{B}_2})\int_{R^3}\frac{1}{1 + |k_1|^3}\cdots\frac{1}{1 + |k_p|^3}\|h\|_{\mathcal{B}_1}^p\, dk_1 \ldots dk_p\\ &\leqslant c(\|h\|_{\mathcal{B}_1} + \|Qh\|_{\mathcal{B}_2})\|h\|_{\mathcal{B}_1}^p.\end{aligned}$$

Thus (3.22) is proved and the lemma is complete. □

In the estimation of $N^n_{a,b}(u)$, a bound on the kernel of the linear propagator is needed. This is provided in the next lemma.

Lemma 5. *Let $\frac{1}{2} < \beta \leqslant 1$, and $0 < a < b$ be given. Then there are positive constants C_0, C_1 and C_2 such that for all η with $0 < \eta < 1$ and all k,*

$$\int_a^b \left|e^{-(|k|^{2\beta} + i\eta k^3)(b-s)}\right| ds \leqslant C_0\frac{(b-a)}{1 + |k|^{2\beta}} \tag{3.24a}$$

$$\int_a^b \left|k\frac{d}{dk}e^{-(|k|^{2\beta} + i\eta k^3)(b-s)}\right| ds \leqslant C_1(b-a)^2(1 + |k|^{3-4\beta}) \tag{3.24b}$$

$$\int_a^b \left|\frac{d^2}{dk^2}\left(ke^{-(|k|^{2\beta} + i\eta k^3)(b-s)}\right)\right| ds \leqslant C_2(1 + (b-a)^2)(1 + |k|)^{5-6\beta}. \tag{3.24c}$$

Proof. These inequalities follow by computing the integrals in question. □

This section is concluded by demonstrating that the nonlinear map $N^n_{a,b}$ takes $L_\infty(a,b;\mathcal{B}_2)$ into $\mathcal{B}_2$ and, in this setting, is bounded and Lipschitz on bounded sets.

Proposition 1. *Let $\frac{1}{2} < \beta \leqslant 1$, let $u_1, u_2 \in L_\infty(a,b;\mathcal{B}_1)$ be given, and suppose $Qu_1, Qu_2 \in L_\infty(a,b;\mathcal{B}_2)$ where Q is the Fourier multiplier introduced in (3.18). If $0 \leqslant a \leqslant b$, the following inequalities hold:*

$$\|N^n_{a,b}(u_1)\|_{\mathcal{B}_2} \leqslant c\,\gamma^n(1+(b-a)^2)\|u_1\|^p_{L_\infty(a,b;\mathcal{B}_1)}(\|Qu_1\|_{L_\infty(a,b;\mathcal{B}_2)} + \|u_1\|_{L_\infty(a,b;\mathcal{B}_1)}) \tag{3.25a}$$

$$\begin{aligned}\|N^n_{a,b}(u_1) - N^n_{a,b}(u_2)\|_{\mathcal{B}_2} &\leqslant c\,\gamma^n(1+(b-a)^2)\\ &\quad\times(\|u_1\|_{L_\infty(a,b;\mathcal{B}_1)} + \|u_2\|_{L_\infty(a,b;\mathcal{B}_1)})^p\|Q(u_1-u_2)\|_{L_\infty(a,b;\mathcal{B}_2)}\end{aligned} \tag{3.25b}$$

where $\gamma = L^{2\beta-(p+1)}$ and c is a constant.

Remark. Since $\|Qf\|_{\mathcal{B}_2} \leqslant c\|f\|_{\mathcal{B}_2}$, the results above imply bounds on the nonlinearity $N^n_{a,b}$ when considered as a map from $L_\infty(a,b;\mathcal{B}_2)$ into $\mathcal{B}_2$.

Proof. From (2.2), we have

$$\begin{aligned}\|N^n_{a,b}(u)\|_{\mathcal{B}_2} = \sup_k \Big\{&(1+|k|^3)|\widehat{N^n_{a,b}(u)}(k)|\\ &+(1+|k|^2)\left|\frac{\mathrm{d}}{\mathrm{d}k}\widehat{N^n_{a,b}(u)}(k)\right| + (1+|k|)\left|\frac{\mathrm{d}^2}{\mathrm{d}k^2}\widehat{N^n_{a,b}(u)}(k)\right|\Big\}\end{aligned} \tag{3.26}$$

where $N^n_{a,b}(u)$ is given by

$$\widehat{N^n_{a,b}(u)}(k) = \frac{\mathrm{i}\,k\,\gamma^n}{p+1}\int_a^b \mathrm{e}^{-(b-s)\Theta(k)}\widehat{u^{p+1}}(k,s)\,\mathrm{d}s$$

with $\Theta(k) = |k|^{2\beta} + \mathrm{i}\alpha^n k^3$ as in (2.7). Employing the relation $(kf(k)g(k))'' = (kf(k))''g(k) + 2kf'(k)g'(k) + (kg(k))''f(k)$ with f denoting the kernel and g the expression on which f acts by convolution in the expression above, it transpires that

$$\begin{aligned}\sup_k &\left\{(1+|k|)\left|\frac{\mathrm{d}^2}{\mathrm{d}k^2}\widehat{N^n_{a,b}(u)}\right|\right\}\\ &\leqslant c\gamma^n \sup_k\Big\{(1+|k|)\Big[\int_a^b\left|\frac{\mathrm{d}^2}{\mathrm{d}k^2}\left(k\mathrm{e}^{-(b-s)\Theta(k)}\right)\right|\mathrm{d}s\left|\widehat{u^{p+1}}(k,\cdot)\right|_{L_\infty(a,b)}\\ &\quad+2\int_a^b\left|k\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{e}^{-(b-s)\Theta(k)}\right|\mathrm{d}s\left|\widehat{u^{p+1}}'(k,\cdot)\right|_{L_\infty(a,b)}\\ &\quad+\int_a^b\left|\mathrm{e}^{-(b-s)\Theta(k)}\right|\mathrm{d}s\left|\frac{\mathrm{d}^2}{\mathrm{d}k^2}k\widehat{u^{p+1}}(k,\cdot)\right|_{L_\infty(a,b)}\Big]\Big\}.\end{aligned}$$

With the estimates afforded by lemmas 4 and 5, one deduces readily that

$$\begin{aligned}\sup_k &\left\{(1+|k|)\left|\frac{\mathrm{d}^2}{\mathrm{d}k^2}\widehat{N^n_{a,b}(u)}\right|\right\}\\ &\leqslant c\,\gamma^n\sup_k\Big\{(1+|k|)\Big[c_2(1+(b-a)^2)\frac{(1+|k|)^{5-6\beta}}{1+|k|^3}\|u\|^{p+1}_{L_\infty(a,b;\mathcal{B}_1)}\\ &\quad+c_1(b-a)\frac{1+|k|^{3-4\beta}}{1+|k|^2}\|u\|^{p+1}_{L_\infty(a,b;\mathcal{B}_1)}\\ &\quad+c_0\frac{(b-a)}{1+|k|^{2\beta}}\left(\|u\|_{L_\infty(a,b;\mathcal{B}_1)} + \|Qu\|_{L_\infty(a,b;\mathcal{B}_2)}\right)\|u\|^p_{L_\infty(a,b;\mathcal{B}_1)}\Big]\Big\}.\end{aligned}$$

Since $2\beta \geqslant 1$, it follows that

$$\sup_k \left\{ (1+|k|) \left| \frac{d^2}{dk^2} \mathcal{F}(N^n_{a,b}(u)(k) \right| \right\} \leqslant c\gamma^n (1+(b-a)^2) \|u\|^p_{L_\infty(a,b;\mathcal{B}_1)} (\|u\|_{L_\infty(a,b;\mathcal{B}_1)} + \|Qu\|_{L_\infty(a,b;\mathcal{B}_2)}).$$

The other two terms on the right-hand-side of (3.26) are estimated similarly, and (3.25*a*) results. The Lipschitz estimate (3.25*b*) follows from identical arguments applied to $N^n_{a,b}(u_1) - N^n_{a,b}(u_2)$ after the relation

$$\left| u_1^{p+1} - u_2^{p+1} \right| = \left| \sum_{i=0}^{p} (u_1 - u_2) u_1^{p-i} u_2^i \right| \leqslant c|u_1 - u_2|(|u_1|^p + |u_2|^p)$$

has been employed. □

4. The renormalization group maps

With the technical tools developed in section 3, we turn to the task of determining the asymptotic behaviour of solutions of (1.4*a*, *b*). This is accomplished in two steps. First, the well-posedness of the equations $(2.4)_n$, $n = 0, 1, 2, \cdots$, is established in the $\mathcal{B}_2$-norm via contraction-mapping arguments based upon the Lipschitz properties of the nonlinear term. Second, an inductive argument relying on the contractive properties of the linear renormalization group R^0_n is used to show the convergence of u to the asymptotic form Γ introduced in (1.8).

4.1. Well-posedness in $\mathcal{B}_2$

The leading term f^* of the asymptotic form Γ is not an element of the Banach space $\mathcal{B}_2$ as defined in (2.2). However, the difference

$$v(x,t) = u(x,t) - \frac{A}{t^{1/2\beta}} f^*(x/t^{1/2\beta})$$

is an element of $\mathcal{B}_2$, has zero total mass, and depends continuously on the initial data.

Proposition 2. *Let $\frac{1}{2} < \beta \leqslant 1$ and $p \geqslant 1$ be given. For $L > 0$, let $T = L^{2\beta}$ and let u_n denote the solution of the initial-value problem $(2.4)_n$, $n = 0, 1, 2, \cdots$. Then there exist positive constants C_T and ϵ_0 which are independent of n such that for all initial data f_n of $(2.4)_n$ of the form*

$$f_n = Af^* + g_n \tag{4.1}$$

where f^* is given by (1.5), $g_n \in \mathcal{B}_2$ has zero total mass, $A \in \mathbb{R}$, and

$$A + \|g_n\|_{\mathcal{B}_2} < \epsilon_0$$

the corresponding solution u_n is of the form

$$u_n(x,t) = \frac{A}{t^{1/2\beta}} f^*(x/t^{1/2\beta}) + v_n(x,t) \tag{4.2}$$

where $v_n \in L_\infty(1, T; \mathcal{B}_2)$, $\widehat{v_n}(0, t) = 0$ *and*

$$\|v_n\|_{L_\infty(1,T;\mathcal{B}_2)} \leqslant c_T(L^{-n}|A| + \|g_n\|_{\mathcal{B}_2}). \tag{4.3}$$

Proof. For each $n = 0, 1, 2, \cdots$, construct the map $T_n : L_\infty(1, T; \mathcal{B}_2) \to L_\infty(1, T; \mathcal{B}_2)$ defined by

$$\begin{aligned}(T_n v)(\cdot, t) &= S_n(t) f_n(\cdot) - A\psi(\cdot, t) + N^n_{1,t}(A\psi + v)(\cdot, t)\\ &\equiv v^0_n(\cdot, t) + N^n_{1,t}(A\psi + v)(\cdot, t)\end{aligned} \tag{4.4}$$

where S_n is the linear semigroup in (2.6), $N^n_{a,b}$ is given by (2.7) and $\psi(x, t) = t^{-1/2\beta} f^*\left(\frac{x}{t^{1/2\beta}}\right)$. In Fourier transformed variables, $v^0_n(t)$ has the form

$$\widehat{v^0_n}(k, t) = \mathrm{e}^{-(|k|^{2\beta}+\mathrm{i}\alpha^n k^3)(t-1)} \widehat{f_n}(k) - A\,\mathrm{e}^{-|k|^{2\beta} t}$$

and using the formula (4.1) for f_n, this simplifies to

$$\widehat{v^0_n}(k, t) = A\,\mathrm{e}^{-|k|^{2\beta} t}(\mathrm{e}^{\mathrm{i}\alpha^n k^3(t-1)} - 1) + \widehat{S_n(t) g_n}(k). \tag{4.5}$$

Clearly $\widehat{v^0_n}(0, t) = 0$, and an elementary calculation shows that for $1 \leqslant t \leqslant T$,

$$\|\mathcal{F}^{-1}(\mathrm{e}^{-|k|^{-2\beta} t}(\mathrm{e}^{-\mathrm{i}\alpha^n k^3(t-1)} - 1))\|_{\mathcal{B}_2} \leqslant c\,\alpha^n \leqslant c\,L^{-n} \tag{4.6}$$

where c is independent of n and T. The term $S_n(t) g_n$ is estimated via lemma 3, and combining these results, there obtains the inequality

$$\|v^0_n\|_{L_\infty(1,T;\mathcal{B}_2)} \leqslant C_T(|A| L^{-n} + \|g_n\|_{\mathcal{B}_2}) \tag{4.7}$$

where C_T depends on T, but not on n.

Employing (3.24*a*) of proposition 1 on the nonlinear term in (4.4) yields

$$\begin{aligned}&\|N^n_{1,t}(A\psi + v)\|_{\mathcal{B}_2}\\ &\quad\leqslant c\,\gamma^n t^2 \|A\psi + v\|^p_{L_\infty(1,T;\mathcal{B}_1)} (\|Q(A\psi + v)\|_{L_\infty(1,T;\mathcal{B}_2)} + \|A\psi + v\|_{L_\infty(1,T;\mathcal{B}_1)})\\ &\quad\leqslant c\,\gamma^n t^2 (A + \|v\|_{L_\infty(1,T;\mathcal{B}_1)})^p (\|\psi\|_{L_\infty(1,T;\mathcal{B}_1)} + \|Q\psi\|_{L_\infty(1,T;\mathcal{B}_2)} + \|v\|_{L_\infty(1,T;\mathcal{B}_2)}).\end{aligned}$$

But $\|\psi\|_{L_\infty(1,T;\mathcal{B}_1)} + \|Q\psi\|_{L_\infty(1,T;\mathcal{B}_2)} \leqslant c$ independent of $T \geqslant 1$, hence

$$\begin{aligned}\|N^n_{1,t}(A\psi + v)\|_{L_\infty(1,T;\mathcal{B}_1)} &\leqslant C_T \gamma^n (|A| + \|v\|_{L_\infty(1,T;\mathcal{B}_1)})^p (|A| + \|v\|_{L_\infty(1,T;\mathcal{B}_2)})\\ &\leqslant C_T (|A| + \|v\|_{L_\infty(1,T;\mathcal{B}_1)})^p (L^{-n}|A| + \|v\|_{L_\infty(1,T;\mathcal{B}_2)}).\end{aligned} \tag{4.8}$$

Define the set

$$\begin{aligned}B_n = \Big\{&v \in L_\infty(1, T; \mathcal{B}_2) :\\ &\|v - v^0_n\|_{L_\infty(1,T;\mathcal{B}_2)} \leqslant L^{-n}|A| + \|g_n\|_{\mathcal{B}_2},\ \widehat{v}(0, t) = 0 \text{ for } 0 \leqslant t \leqslant 1\Big\}\end{aligned} \tag{4.9}$$

and presume that $|A| + \|g_n\|_{\mathcal{B}_2} \leqslant \epsilon_0$, a constant whose value will be determined presently. Then from (4.8) it follows that for $v \in B_n$,

$$\begin{aligned}\|T_n v - v^0_n\|_{L_\infty(1,T;\mathcal{B}_2)} &\leqslant \|N^n_{1,T}(A\psi + v)\|_{L_\infty(1,T;\mathcal{B}_2)}\\ &\leqslant C_T(|A| + \|v\|_{L_\infty(1,T;\mathcal{B}_1)})^p (L^{-n}|A| + \|v\|_{L_\infty(1,T;\mathcal{B}_2)}).\end{aligned}$$

The fact that $v \in B_n$ together with (4.7) implies

$$\begin{aligned}\|v\|_{L_\infty(1,T;B_2)} &\leqslant \|v_n^0\|_{L_\infty(1,T;B_2)} + L^{-n}|A| + \|g_n\|_{B_2} \\ &\leqslant (1+C_T)(L^{-n}|A| + \|g_n\|_{B_2}).\end{aligned}$$

Thus we see that

$$\begin{aligned}\|T_n v - v_n^0\|_{L_\infty(1,T;B_2)} &\leqslant C_T(|A| + \|g_n\|_{B_2})^p(L^{-n}|A| + \|g_n\|_{B_2}) \\ &\leqslant C_T\epsilon_0^p(L^{-n}|A| + \|g_n\|_{B_2}).\end{aligned}$$

Choosing ϵ_0 small enough, it follows that $T_n : B_n \to B_n$. Moreover if v_1 and v_2 lie in B_n, then

$$\begin{aligned}&\|T_n v_1 - T_n v_2\|_{L_\infty(1,T;B_2)} \\ &\quad\leqslant \|N_{1,t}^n(A\psi + v_1) - N_{1,t}^n(A\psi + v_2)\|_{L_\infty(1,T;B_2)} \\ &\quad\leqslant C_T\,\gamma^2(\|A\psi + v_1\|_{L_\infty(1,T;B_1)} + \|A\psi + v_2\|_{L_\infty(1,T;B_1)})^p\|v_1 - v_2\|_{L_\infty(1,T;B_2)} \\ &\quad\leqslant C_T(|A| + \|g_n\|_{B_2})^p\|v_1 - v_2\|_{L_\infty(1,T;B_2)}\end{aligned}$$

and thus independently of n, ϵ_0 may be chosen small enough so that T_n is seen to be a strict contraction on B_n. With such a choice of ϵ_0, the contraction-mapping theorem implies that T_n has a unique fixed point $v_n \in B_n$. It follows that

$$u_n = A\psi + v_n$$

where u_n solves $(2.4)_n$. Moreover, since $v_n \in B_n$, we have the bound

$$\|v_n\|_{L_\infty(1,T;B_2)} \leqslant \|v_n^0\|_{L_\infty(1,T;B_2)} + L^{-n}|A| + \|g_n\|_{B_2} \leqslant C_T(L^{-n}|A| + \|g_n\|_{B_2})$$

and the proof of proposition 2 is completed. □

Since the assumption $f_0 = Af^* + g_0$ on the initial data made in proposition 2 is not generic, we pose (1.4*a*) with initial data $f \in B_2$ at time $t = 0$, show that $u(\cdot, t = 1)$ is of the from (4.1), and then take $f_0 \equiv u(\cdot, 1)$ as the initial data for $(2.4)_0$ at time $t = 1$. The solution u of (1.4*a*) is given formally by

$$u(\cdot, t) = S_0(t)f(\cdot) + N_{0,t}^0(S_0(t)f(\cdot) + (u(\cdot, t) - S_0(t)f(\cdot))).$$

Introduce the map $T : L_\infty(0, 1; B_2) \to L_\infty(0, 1; B_2)$ defined by

$$(Tv)(\cdot, t) = N_{0,T}^0(v(\cdot, t) + S_0(t)f(\cdot)).$$

It is clear that a fixed point v_0 of T satisfies

$$u(\cdot, t) = S_0(t)f(\cdot) + v_0(\cdot, t). \tag{4.10}$$

Much as in the proof of proposition 2, define the set

$$B = \{v \in L_\infty(0, 1; B_2) \;:\; \|v\|_{L_\infty(0,1;B_2)} \leqslant \|f\|_{B_2},\ \widehat{v}(0, t) = 0 \text{ for } 0 \leqslant t \leqslant 1\}.$$

Proposition 1 implies that

$$\|Tv\|_{L_\infty(0,1;B_2)} \leqslant c\|v+S_0(t)f\|^p_{L_\infty(0,1;B_1)}(\|v+S(t)f\|_{L_\infty(0,1;B_1)}+\|Q(v+S_0(t)f)\|_{L_\infty(0,1;B_2)}) \leqslant (\|v\|_{L_\infty(0,1;B_1)}+\|f\|_{B_1})^p(\|v\|_{L_\infty(0,1;B_2)}+\|S_0(t)Qf\|_{L_\infty(0,1;B_2)}).$$

Now $\widehat{Qf}(k)=q(k)\widehat{f}(k)$ and $\widehat{Qf}(0)=0$, so lemma 3 implies

$$\|S_0(t)Qf\|_{L_\infty(0,1;B_2)} \leqslant c\|Qf\|_{B_2} \leqslant c\|f\|_{B_2}.$$

Together these estimates yield that if $v \in B$, the there is a constant c independent of such v for which

$$\|Tv\|_{L_\infty(0,1;B_2)} \leqslant c\|f\|^{p+1}_{B_2}$$

and hence if $\|f\|_{B_2} \leqslant R$ with R small enough, then

$$\|Tv\|_{L_\infty(0,1;B_2)} \leqslant \|f\|_{B_2}.$$

For such values of R, T maps B into B. Similarly, if R is small enough, T is a strict contraction on B, and thus T has a unique fixed point $v_0 \in B$. Rewriting (4.10) then yields

$$u(x,t) = Af^* + (S_0(t)f - Af^*) + v_0$$

where $A=\widehat{f}(0)$. Defining $f_0(x)=u(x,1)$, it is seen that

$$f_0 = Af^* + g_0 \tag{4.11}$$

where $\widehat{g}_0(k)=e^{-|k|^{2\beta}}(e^{-ik^3}\widehat{f}(k)-\widehat{f}(0))+\widehat{v}_0(k,1)$. Hence $\widehat{g}_0(0)=0$ and $\|g_0\|_{B_2}\leqslant c\|f\|_{B_2}$, as required to satisfy the conditions of proposition 2.

4.2. *The main result*

The asymptotic behaviour of u, the solution of (1.4a, b), as $t\to\infty$, is linked to the limit, as $n\to\infty$, of the sequence $\{f_n\}_{n=0}^\infty$ of initial data in $(2.4)_n$, via the relation

$$L^n u(L^n x, L^{2\beta n}) = f_n(x). \tag{4.12}$$

Assuming that the conditions of proposition 2 hold uniformly in $n=0,1,2,\cdots$, then the results of section 4.1 imply that if the initial data $f \in B_2$, then $f_0 \equiv u(\cdot,1)$ and the f_n's defined above have the form

$$f_n = Af^* + g_n \qquad n=0,1,2\cdots \tag{4.13}$$

where A is the total mass of each f_n and the g_n lie in B_2 and satisfy $\widehat{g}_n(0)=0$. Moreover, the f_n satisfy the recursion relations

$$f_{n+1} = R_n f_n = R_n^0 f_n + LN^n_{1,T}(u_n)(\cdot L) \qquad n=0,1,2\cdots \tag{4.14}$$

where R_n and R_n^0 are the renormalization maps (2.9), (2.10), $N^n_{u,b}$ is the nonlinear term defined in (2.7), and $T=L^{2\beta}$. The relations (4.14) can be thought of as determining g_{n+1}

in terms of g_n and A via (4.13). Before we make these relations explicit it is useful for notational purposes to introduce the sequence of functions $\{\varphi_n\}_{n=0}^{\infty} \subset \mathcal{B}_1$, defined by

$$\begin{aligned} \varphi_0 &= Af^* \\ \varphi_{n+1} &= R_n^0 \varphi_n + N_{T^{-1},1}^{n+1}(A\psi) \qquad \text{for } n \geqslant 0 \end{aligned} \tag{4.15}$$

where, as before, $\psi(x,t) = \frac{1}{t^{\frac{1}{2\beta}}} f^*(x/t^{\frac{1}{2\beta}})$ or, equivalently, $\widehat{\psi}(k,t) = \mathrm{e}^{-|k|^{2\beta}t}$. This definition is motivated in part by the fact made apparent in (1.6) that $A\psi$ is a good approximation to u and hence, in view of proposition 1, $N_{T^{-1},1}^{n+1}(A\psi)$ is a good approximation to $N_{T^{-1},1}^{n+1}(u_n)$.

At this point, a technical result about ψ is needed.

Lemma 6. *The function ψ defined above satisfies*

$$(a)\ L^{p/2\beta}\widehat{\psi^{p+1}}(k,tL) = \widehat{\psi^{p+1}}(kL^{1/2\beta},t) \tag{4.16a}$$

and

$$(b)\ LN_{1,T}^n(A\psi)(\cdot L) = N_{T^{-1},1}^{n+1}(A\psi)(\cdot) \tag{4.16b}$$

Proof. (a) From the relation

$$\widehat{\psi^{p+1}}(kL^{1/2\beta},t) = \int_{\mathbb{R}^p} \widehat{\psi}(kL^{1/2\beta} - k_1 \cdots k_p, s)\widehat{\psi}(k_1,s)\cdots\widehat{\psi}(k_p,s)\,\mathrm{d}k_1\cdots\mathrm{d}k_p$$

and $\widehat{\psi}(L^{1/2\beta}k,s) = \mathrm{e}^{-|L^{1/2\beta}k|^{2\beta}s} = \widehat{\psi}(k,Ls)$, it transpires that

$$\begin{aligned} \widehat{\psi^{p+1}}(kL^{1/2\beta},t) &= \int_{\mathbb{R}^p} \widehat{\psi}(k - L^{-1/2\beta}(k_1 + \ldots + k_p), Ls)\widehat{\psi}(L^{-1/2\beta}k_1, Ls) \\ &\quad \ldots \widehat{\psi}(L^{-1/2\beta}k_p, Ls)\,\mathrm{d}k_1\ldots\mathrm{d}k_p \\ &= L^{p/2\beta}\widehat{\psi^{p+1}}(k,Ls). \end{aligned}$$

(b) From (a) and (2.7), it is found that

$$\begin{aligned} &\mathcal{F}(LN_{1,T}^n(A\psi)(\cdot L))(k) \\ &\quad = \mathcal{F}(N_{1,T}^n(A\psi))(k/L) \\ &\quad = \frac{\mathrm{i}\gamma^n}{p+1}\left(\frac{k}{L}\right)\int_1^T \mathrm{e}^{-\left(|\frac{k}{L}|^{2\beta}+\mathrm{i}\alpha^n|\frac{k}{L}|^3\right)(T-s)}\widehat{(A\psi)^{p+1}}\left(\frac{k}{L},s\right)\mathrm{d}s \\ &\quad = \frac{\mathrm{i}\gamma^{n+1}k}{p+1}\int_{T^{-1}}^1 \mathrm{e}^{-(|k|^{2\beta}+\mathrm{i}\alpha^{n+1}k^3)(1-\widetilde{s})}\widehat{(A\psi)^{p+1}}(k,\widetilde{s})\,\mathrm{d}\widetilde{s} \end{aligned}$$

where $\widetilde{s} = s/L^{2\beta} = s/T$. □

The results of lemma A of the appendix show that $f_n - \varphi_n \in \mathcal{B}_2$ and satisfies $\widehat{f}_n(0) - \widehat{\varphi}_n(0) = 0$. Thus there are constants $B_n \in \mathbb{R}$ such that the relations (4.13) may be rewritten as

$$f_n = \varphi_n + B_n f_1^* + h_n \qquad n = 0, 1, 2\ldots \tag{4.17$_n$}$$

where $f_1^* \in \mathcal{B}_2$ is given by (3.6) and satisfies $\widehat{f_1^*}(0) = 0$, $\widehat{f_1^*}'(0) = i$, and $h_n \in \mathcal{B}_2$ satisfies $\widehat{h}_n(0) = \widehat{h}'_n(0) = 0$. Now the relations (4.14) determine h_{n+1} and B_{n+1} in terms of h_n, B_n, and A via $(4.17)_n$, (4.16*b*) and the relation

$$\begin{aligned} f_{n+1} &= R_n^0(\varphi_n + B_n f_1^* + h_n) + LN_{1,T}^n(u_n)(\cdot L) \\ &= \varphi_{n+1} + \left(\frac{B_n}{L} + C_n\right) f_1^* + \left\{ B_n \left(R_n^0 f_1^* - \frac{1}{L} f_1^* \right) + LN_{1,T}^n(u_n)(\cdot L) \right. \\ &\quad \left. - LN_{1,T}^n(A\psi)(\cdot L) - C_n f_1^* + R_n^0 h_n \right\} \end{aligned} \tag{4.18}$$

where C_n is defined by

$$C_n = \frac{1}{L} \frac{\mathrm{d}}{\mathrm{d}k} \mathcal{F}(N_{1,T}^n(u_n) - N_{1,T}^n(A\psi))(0). \tag{4.19}$$

Thus we have the recurrence relations

$$B_{n+1} = \frac{B_n}{L} + C_n \tag{4.20}$$

and

$$h_{n+1} = B_n \left(R_n^0 f_1^* - \frac{1}{L} f_1^* \right) + LN_{1,T}^n(u_n)(\cdot L) - LN_{1,T}^n(A\psi)(\cdot L) - C_n f_1^* + R_n^0 h_n. \tag{4.21}$$

From proposition 1, it follows that $h_{n+1} \in \mathcal{B}_2$, while an inspection of (4.21) shows that $\widehat{h}_n(0) = 0$ and $\widehat{h}_n{}'(0) = 0$ since C_n satisfies (4.19).

It will be useful to estimate the $\mathcal{B}_2$-norm of h_n. First, from (4.19), we have

$$|C_n| \leqslant \frac{1}{L} \| N_{1,T}^n(u_n) - N_{1,T}^n(A\psi) \|_{\mathcal{B}_1}$$

the right-hand side of which may be estimated via proposition 1 to give

$$|C_n| \leqslant \frac{c}{L} T \gamma^n (\|u_n\|_{L_\infty(1,T;\mathcal{B}_1)} + \|A\psi\|_{L_\infty(1,T;\mathcal{B}_1)})^p \|u_n - A\psi\|_{L_\infty(1,T;\mathcal{B}_1)}.$$

It was shown in Bona *et al* (1994) that for smooth and sufficiently small initial data f_0, the following inequality holds:

$$\|u_n - A\psi\|_{L_\infty(1,T;\mathcal{B}_1)} \leqslant c(n+1)L^{-n} \|f_0\|_{\mathcal{B}_1}.$$

Hence one derives

$$\begin{aligned} |C_n| &\leqslant \frac{c}{L} T \gamma^n \|f_0\|_{\mathcal{B}_1}^{p+1} (n+1) L^{-n} \\ &\leqslant C_L (n+1) L^{-2(n+1)} \|f\|_{\mathcal{B}_2} (\|f\|_{\mathcal{B}_2}^p L^{2\beta+1}). \end{aligned}$$

Thus, if $C_L \|f\|_{\mathcal{B}_2}^p L^{2\beta+1} \leqslant 1$, C_n may be estimated thusly:

$$|C_n| \leqslant c(n+1)L^{-2(n+1)} \|f\|_{\mathcal{B}_2}. \tag{4.22}$$

Consequently, the constant B_n satisfies

$$|B_n| \leqslant \frac{|B_{n-1}|}{L} + |C_{n-1}|$$
$$\leqslant \frac{|B_0|}{L^n} + \sum_{k=1}^{n} \frac{C_{k-1}}{L^{n-k}} \leqslant \frac{1}{L^n}\left(|B_0| + C\|f\|_{\mathcal{B}_2} \sum_{k=1}^{n} (k+1)L^{-k}\right)$$

and since $B_0 = \widehat{u}'(0, 1)$, it follows that $|B_0| \leqslant c\|f\|_{\mathcal{B}_2}$, so one obtains

$$|B_n| \leqslant c\, L^{-n} \|f\|_{\mathcal{B}_2}. \tag{4.23}$$

Taking the $\mathcal{B}_2$-norm of equation (4.21), and applying the triangle inequality to the right-hand side leads to the inequality

$$\begin{aligned}\|h_{n+1}\|_{\mathcal{B}_2} &\leqslant \|R_n^0 h_n\|_{\mathcal{B}_2} + |B_n|\, \|R_n^0 f_1^* - \frac{1}{L} f_1^*\|_{\mathcal{B}_2} \\ &\quad + \|L(N_{1,T}^n(u_n)(\cdot L) - N_{1,T}^n(A\psi)(\cdot L))\|_{\mathcal{B}_2} \\ &\quad + |C_n|\, \|f_1^*\|_{\mathcal{B}_2}\end{aligned} \tag{4.24}$$

valid for all $n \geqslant 0$. Making use of lemma 1 to estimate the first term on the right-hand side of (4.24) yields

$$\|R_n^0 h_n\|_{\mathcal{B}_2} \leqslant cL^{-2}\|h_n\|_{\mathcal{B}_2} \tag{4.25}$$

while lemma 2 and (4.23) applied to the second term implies

$$|B_n|\, \|R_n^0 f_1^* - \frac{1}{L} f_1^*\|_{\mathcal{B}_2} \leqslant cL^{-n}\|f\|_{\mathcal{B}_2} cL^{-(n+2)} \leqslant cL^{-2(n+1)}\|f\|_{\mathcal{B}_2}. \tag{4.26}$$

From (4.22), it follows readily that

$$|C_n|\, \|f_1^*\|_{\mathcal{B}_2} \leqslant c(n+1)L^{-2(n+1)}\|f\|_{\mathcal{B}_2} \tag{4.27}$$

provided that $\|f\|_{\mathcal{B}_2}$ is small enough with respect to L. It remains to bound the nonlinear term in (4.24). Using proposition 2, we may write $u_n = A\psi + v_n$, where $\|v_n\|_{L_\infty(1,T;\mathcal{B}_2)} \leqslant c_T(L^{-n}|A| + \|g_n\|_{\mathcal{B}_2})$ and

$$g_n = f_n - Af^* = \varphi_n - Af^* + B_n f_1^* + h_n.$$

Rescaling the x-variable, applying proposition 1, and substituting $u_n = A\psi + v_n$, leads to the inequality

$$\begin{aligned}&\|L(N_{1,T}^n(u_n)(\cdot L) - N_{1,T}^n(A\psi)(\cdot L))\|_{\mathcal{B}_2} \\ &\quad\leqslant L^3\|N_{1,T}^n(u_n) - N_{1,T}^n(A\psi)\|_{\mathcal{B}_2} \\ &\quad\leqslant cL^3T^2\gamma^n(\|v_n\|_{L_\infty(1,T;\mathcal{B}_1)} + \|A\psi\|_{L_\infty(1,T;\mathcal{B}_1)})^p\|v_n\|_{L_\infty(1,T;\mathcal{B}_2)}.\end{aligned} \tag{4.28}$$

As just noted, $\|v_n\|_{L_\infty(1,T;\mathcal{B}_2)} \leqslant C_T(L^{-n}|A| + \|g_n\|_{\mathcal{B}_2})$ and the formula defining g_n leads immediately to the inequality

$$\|g_n\|_{\mathcal{B}_2} \leqslant \|\varphi_n - Af^*\|_{\mathcal{B}_2} + |B_n|\|f_1^*\|_{\mathcal{B}_2} + \|h_n\|_{\mathcal{B}_2}. \tag{4.29}$$

Formula (A.3) of lemma A of the appendix implies, for $|A| \leqslant 1$, that

$$\|\varphi_n - Af^*\|_{B_2} \leqslant c|A|L^{-n}.$$

Additionally, $|B_n| \leqslant cL^{-n}\|f\|_{B_2}$ and $|A| \leqslant \|f\|_{B_2}$, so the preceding estimates and (4.29) imply

$$\|g_n\|_{B_2} \leqslant c\, L^{-n}\|f\|_{B_2} + \|h_n\|_{B_2}$$

whence

$$\|v_n\|_{L_\infty(1,T;B_2)} \leqslant C_T(L^{-n}\|f\|_{B_2} + \|h_n\|_{B_2}).$$

Combining the bound above with (4.28) gives the useful result

$$\begin{aligned}\|L(N^n_{1,T}(u)(\cdot L) - N^n_{1,T}(A\psi)(\cdot L))\|_{B_2} \\ \leqslant C_L\, L^{-n}(\|f\|_{B_2} + \|h_n\|_{B_2})^p(L^{-n}\|f\|_{B_2} + \|h_n\|_{B_2}).\end{aligned} \tag{4.30}$$

Together, the inequalities (4.25), (4.26), (4.27), and (4.30), when used in conjunction with (4.24), yield the recurrence relation

$$\begin{aligned}\|h_{n+1}\|_{B_2} \leqslant C(n+1)L^{-2(n+1)}\|f\|_{B_2} + c\, L^{-2}\|h_n\|_{B_2} \\ + C_L\, L^{-n}(\|f\|_{B_2} + \|h_n\|_{B_2})^p(L^{-n}\|f\|_{B_2} + \|h_n\|_{B_2})\end{aligned} \tag{4.31}$$

valid for all $n \geqslant 0$.

Fixing $\epsilon > 0$ in accordance with the earlier condition guaranteeing the existence of various fixed points, we consider the following induction hypothesis: for $n = 0, 1, 2, \cdots$,

(i)$_n$ $L^2C_L(\|f\|_{B_2} + \|h_n\|_{B_2})^p < 1$

(ii)$_n$ $\|h_n\|_{B_2} \leqslant \widetilde{C}\, L^{-2n(1-\epsilon)}\|f\|_{B_2}$

where C_L is the constant in (4.31) depending upon L and $\widetilde{C}$ will be specified below. For $n = 0$, $h_0 = g_0 - B_0 f_1^*$ where $B_0 = -i\widehat{g}_0'(0)$, and $\|g_0\|_{B_2} \leqslant C\|f\|_{B_2}$. Thus it is adduced that $\|h_0\|_{B_2} \leqslant C\|f\|_{B_2}$ and we see that (i)$_0$ is verified if $\|f\|_{B_2} < R$ for any $R > 0$ satisfying

$$L^2C_L((1+C)R)^p \leqslant 1.$$

The condition (ii)$_0$ is trivially satisfied at $n = 0$. Assume (i)$_n$ and (ii)$_n$ hold for $n = 0, 1, \cdots, m$. We show they hold for $n = m+1$. Applying (4.31) with $n = m$, we have

$$\begin{aligned}\|h_{m+1}\|_{B_2} \leqslant C(m+1)L^{-2(m+1)}\|f\|_{B_2} + c\, L^{-2}\|h_m\|_{B_2} \\ + C_L L^{-m}(\|f\|_{B_2} + \|h_m\|_{B_2})^p(L^{-m}\|f\|_{B_2} + \|h_m\|_{B_2})\end{aligned}$$

and the induction hypothesis (ii)$_m$ yields

$$\begin{aligned}\|h_{m+1}\|_{B_2} &\leqslant (C(m+1)+1)L^{-2(m+1)}\|f\|_{B_2} + (C\, L^{-2} + L^{-m-2})\|h_m\|_{B_2} \cdot \\ &\leqslant (C(m+1)+1)L^{-2(m+1)}\|f\|_{B_2} + 2C\, L^{-2}\|h_m\|_{B_2} \\ &\leqslant \left[(C(m+1)+1)L^{-2\epsilon(m+1)} + 2C\, L^{-2\epsilon}\widetilde{C}\right] L^{-2m(1-\epsilon)}\|f\|_{B_2} \\ &\leqslant \widetilde{C}L^{-2m(1-\epsilon)}\|f\|_{B_2}\end{aligned}$$

where, if $L \geqslant L_0(\epsilon, \widetilde{C})$ and L_0 is chosen large enough that

$$\widetilde{C} \geqslant \sup_{m \geqslant 0} \{(C(m+1)+1)L^{-2\epsilon(m+1)} + 2C\,L^{-2\epsilon}\widetilde{C}\}$$

then $(\mathrm{ii})_{m+1}$ follows. The relation $(\mathrm{i})_{m+1}$ is a consequence of $(\mathrm{ii})_{m+1}$ if $\|f\|_{B_2} \leqslant R$ where R is small enough to satisfy $L^2 C_L (1+\widetilde{C})^p R^p \leqslant 1$. Thus the induction is complete. These results are summarized in the following proposition.

Proposition 3. *Let $\frac{1}{2} < \beta \leqslant 1$ and $p \geqslant 2\beta$. Then for any $\epsilon > 0$ and $C > 0$ there exists L_0 such that if $L \geqslant L_0$ and $\|f\|_{B_2}$ is small enough, then the functions f_n given by (4.12) satisfy*

$$\left\| f_n - (\varphi_n + B_n f_1^*) \right\|_{B_2} \leqslant C\,L^{-2n(1-\epsilon)} \|f\|_{B_2} \tag{4.32}$$

when φ_n is given in (A.1) and B_n in (4.20).

Proof. From $(4.13)_n$ we have

$$f_n = \varphi_n + B_n f_1^* + h_n$$

and from the induction argument above, h_n satisfies $\|h_n\|_{B_2} \leqslant C\,L^{-2n(1-\epsilon)}\|f\|_{B_2}$. Moreover, this bound on h_n and our previous estimate

$$\|g_n\|_{B_2} \leqslant cL^{-n}\|f\|_{B_2} + \|h_n\|_{B_2},$$

together show that the g_n are arbitrarily small, uniformly in n, if $\|f\|_{B_2}$ is chosen small enough. This satisfies the conditions of proposition 2 and justifies the decomposition (4.13).

The case $\beta = 1$ requires a slight additional argument, since lemma 2 requires $\frac{1}{2} < \beta < 1$. In fact (3.7) and (3.8) hold when $\beta = 1$ provided the exponent of L is multiplied by an additional factor of $1 - \delta$ for any small $\delta > 0$. This δ may then be absorbed in the ϵ of (4.32). □

To obtain the main result, it is useful to simplify the expression for φ_n and derive a limiting expression for B_n.

Lemma 7. *Let $\widetilde{\varphi}_n$ be given by*

$$\widehat{\widetilde{\varphi}}_n(k) = A\,\mathrm{e}^{-|k|^{2\beta}}(1 - \mathrm{i}\alpha^n k^3) + \mathrm{i}\frac{k}{p+1}\gamma^n \int_{T^{-n}}^{1} \mathrm{e}^{-|k|^{2\beta}(1-s)} \widehat{(A\psi)^{p+1}}(k,s)\,\mathrm{d}s$$

where $\psi(x,t) = \frac{1}{t^{1/2\beta}} f^*(x/t^{1/2\beta})$. Then it follows that

$$\|\varphi_n - \widetilde{\varphi}_n\|_{B_2} \leqslant C\,L^{-2n}|A|. \tag{4.33}$$

Proof. From formula (A.1),

$$\widehat{\varphi}_n(k) = A\,\mathrm{e}^{-|k|^{2\beta}} \mathrm{e}^{-\mathrm{i}\alpha^n k^3(1-T^n)} + \widehat{N^n_{T^{-n},1}(A\psi)}(k)$$

so

$$\begin{aligned}(\widehat{\varphi}_n - \widehat{\widetilde{\varphi}}_n)(k) &= A\,\mathrm{e}^{-|k|^{2\beta}}(\mathrm{e}^{-\mathrm{i}\alpha^n k^3(1-T^{-n})} - (1 - \mathrm{i}\alpha^n k^3)) \\ &\quad + \mathrm{i}\gamma^n \frac{k}{p+1} \int_{T^{-n}}^{1} \mathrm{e}^{-|k|^{2\beta}(1-s)} \widehat{(A\psi)^{p+1}}(k,s)(\mathrm{e}^{-\mathrm{i}\alpha^n k^3(1-s)} - 1)\,\mathrm{d}s.\end{aligned} \tag{4.34}$$

We also have, for $j = 0, 1, 2$ and $0 \leqslant s \leqslant 1$,

$$\left|\frac{\mathrm{d}^j}{\mathrm{d}k^j}(\mathrm{e}^{-\mathrm{i}\alpha^n k^3(1-T^{-n})} - (1 - \mathrm{i}\alpha^n k^3))\right| \leqslant C\alpha^{2n}(1 + |k|)^6$$

and

$$\left|\frac{\mathrm{d}^j}{\mathrm{d}k^j}(\mathrm{e}^{-\mathrm{i}\alpha^n k^3(1-s)} - 1)\right| \leqslant C\alpha^n(1 + |k|)^3.$$

These estimates, plus an argument similar to that used in the estimation of the nonlinear term in (A.2), when applied to (4.34) yield

$$\|\varphi_n - \widetilde{\varphi}_n\|_{B_2} \leqslant C|A|\alpha^{2n} + C|A|^p L^{-n}\alpha^n \leqslant C|A|L^{-2n}.$$

□

Lemma 8. *Let the sequence* $\{B_n\}$ *be given by (4.20). Then there exists a constant* $B \in \mathbb{R}$ *given by (4.36) below such that* $B_n L^n \underset{n\to\infty}{\longrightarrow} B$, *and for any* $\epsilon > 0$ *there is a constant* $c > 0$ *such that*

$$|L^n B_n - B| \leqslant c\, L^{-n(1-\epsilon)}\|f\|_{B_2}$$

Proof. Equation (4.20) implies that

$$B_n = \frac{B_0}{L^n} + \sum_{k=0}^{n-1} L^{-(n-1-k)} C_k \tag{4.35}$$

where $B_0 = \mathrm{i}\widehat{u}'(0, 1)$ and C_n is given by (4.17). The relation (2.7) for $N^n_{a,b}$, in conjunction with (4.17), upon evaluating the derivative at $k = 0$, yields

$$\begin{aligned} C_n &= \frac{1}{L}\frac{\gamma^n \mathrm{i}}{p+1}\int_1^T \left(\widehat{u_n^{p+1}}(0, s) - \widehat{(A\psi)^{p+1}}(0, s)\right) \mathrm{d}s \\ &= \frac{1}{L}\frac{\gamma^n \mathrm{i}}{p+1}\int_1^T \int_{-\infty}^{\infty} (u_n^{p+1}(x, s) - (A\psi)^{p+1}(x, s))\, \mathrm{d}x\, \mathrm{d}s \end{aligned}$$

where $T = L^{2\beta}$. The substitution $u_n(x, s) = L^n u(L^n x, T^n s)$, a change of variables, and (4.18*a*) obtains the formula

$$C_n = \frac{\mathrm{i}}{p+1} L^{-(n+1)} \int_{T^n}^{T^{n+1}} \int_{-\infty}^{\infty} (u^{p+1}(x, s) - (A\psi)^{p+1}(x, s))\, \mathrm{d}x\, \mathrm{d}s$$

which, in view of (4.34), produces the relations

$$B_n = L^{-n}\left(B_0 + \frac{\mathrm{i}}{p+1}\int_1^{T^n} \int_{-\infty}^{\infty} \left(u^{p+1}(x, s) - (A\psi)^{p+1}(x, s)\right)\right) \mathrm{d}x\, \mathrm{d}s$$

for $n = 0, 1, 2, \cdots$. If we define B by

$$B = B_0 + \frac{\mathrm{i}}{p+1}\int_1^{\infty}\int_{-\infty}^{\infty} \left(u^{p+1}(x, s) - (A\psi)^{p+1}(x, s)\right) \mathrm{d}x\, \mathrm{d}s \tag{4.36}$$

then

$$|L^n B_n - B| = \frac{1}{p+1}\int_{T^n}^{\infty}\int_{-\infty}^{\infty}(u^{p+1}(x,s) - (A\psi)^{p+1}(x,s))\,\mathrm{d}x\,\mathrm{d}s$$
$$\leqslant c\int_{T_n}^{\infty}\sum_{k=1}^{p}\int_{-\infty}^{\infty}|u - A\psi|\,|u|^{p-k}|A\psi|^k\,\mathrm{d}x\,\mathrm{d}s. \qquad (4.37)$$

But $\psi(x,t) = \frac{1}{t^{1/2\beta}} f^*(x/t^{1/2\beta})$, so

$$\|\psi(\cdot,t)\|_{L^2} \leqslant c\,t^{-\frac{1}{4\beta}}$$

and

$$\|\psi(\cdot,t)\|_{L^\infty} \leqslant c\,t^{-\frac{1}{2\beta}}$$

while (1.6) implies that for any $\tilde{\epsilon} > 0$, there is a C such that for $\|f\|_{B_2}$ small enough,

$$\|u(\cdot,t) - A\psi(\cdot,t)\|_{L^2} \leqslant c\frac{\|f\|_{B_2}}{t^{3/4\beta-\tilde{\epsilon}}}$$

and

$$\|u(\cdot,t) - A\psi(\cdot,t)\|_{L^\infty} \leqslant c\frac{\|f\|_{B_2}}{t^{1/\beta-\tilde{\epsilon}}}.$$

These bounds and (4.36) together produce the estimate

$$|L^n B_n - B| \leqslant c\int_{T^n}^{\infty}|u - A\psi|_{L^2}|u|_{L^2}|u|_{L^\infty}^{p-k-1}|A\psi|_{L^\infty}\,\mathrm{d}s$$
$$\leqslant c\|f\|_{B_2}^{p+1}\int_{T^n}^{\infty}s^{-\frac{3}{4\beta}+\tilde{\epsilon}}s^{-\frac{1}{4\beta}}s^{-\frac{p-1}{2\beta}}\,\mathrm{d}s$$
$$\leqslant c\|f\|_{B_2}^{p+1}(T^n)^{-\frac{p+1}{2\beta}+1+\tilde{\epsilon}} \leqslant c\|f\|_{B_2}^{p+1}L^{-(p+1-2\beta)n+\tilde{\epsilon}2\beta n}.$$

But, $p+1-2\beta \geqslant 1$ since $2\beta \leqslant p$, so taking $\epsilon = 2\beta\tilde{\epsilon}$, and $\|f\|_{B_2} \leqslant 1$, we have

$$|L^n B_n - B| \leqslant c\,L^{-n(1-\epsilon)}\|f\|_{B_2}(L^{n(\tilde{\epsilon}2\beta-\epsilon)}) = c\,L^{-n(1-\epsilon)}\|f\|_{B_2}.$$

□

We are now in a position to state and prove our main result.

Theorem 1. *Let $1/2 < \beta \leqslant 1$ and $p \geqslant 2\beta$ be given. If $\|f\|_{B_2}$ is small enough, there exist constants A and B depending upon f such that for any $\epsilon > 0$ the solution u of (1.4a,b) satisfies*

$$\|u(\cdot t^{1/2\beta},t) - \Gamma_{A,B}(\cdot t^{1/2\beta},t)\|_{B_2} \leqslant c\,t^{-3/2\beta+\epsilon}\|f\|_{B_2} \qquad (4.38)$$

where

$$\widehat{\Gamma_{A,B}}(k,t) = A\,\mathrm{e}^{-|k|^{2\beta}t}(1 - \mathrm{i}\,t\,k^3) + \frac{\mathrm{i}\,k}{p+1}\int_1^t \mathrm{e}^{-|k|^{2\beta}(t-s)}\widehat{(A\psi)^{p+1}}(k,s)\,\mathrm{d}s$$
$$+\,\mathrm{i}\,B\,k\,\mathrm{e}^{-|k|^{2\beta}t}. \qquad (4.39)$$

Proof. The preliminary results in proposition 3, and lemmas 7 and 8 imply

$$\begin{aligned}\left\| f_n - \left(\widetilde{\varphi}_n + \frac{B}{L^2} f_1^*\right)\right\|_{\mathcal{B}_2} \\ &\leqslant \|f_n - (\varphi_n + B_n f_1^*)\|_{\mathcal{B}_2} + \|\varphi_n - \widetilde{\varphi}_n\|_{\mathcal{B}_2} + \left|\frac{B}{L^n} - B_n\right| \|f_1^*\|_{\mathcal{B}_2} \\ &\leqslant c\, L^{-2n(1-\widetilde{\epsilon})} \|f\|_{\mathcal{B}_2}.\end{aligned}$$

But, $f_n = L^n u(L^n x, L^{2\beta n})$, and setting $t = L^{2\beta n}$, we find

$$\|t^{\frac{1}{2\beta}}(u(\cdot t^{1/2\beta}, t) - \Gamma_{A,B}(\cdot t^{1/2\beta}, t))\|_{\mathcal{B}_2} \leqslant c\, t^{-\frac{1}{\beta}(1-\widetilde{\epsilon})} \|f\|_{\mathcal{B}_2} \tag{4.40}$$

where

$$\Gamma_{A,B}(x,t) = t^{-1/2\beta}\widetilde{\varphi}_n(x/t^{1/2\beta}) + B f_1^*(x/t^{1/2\beta}). \tag{4.41}$$

Dividing by $t^{1/2\beta}$ and setting $\epsilon = \widetilde{\epsilon}/\beta$, (4.38) results. The equivalence of (4.39) and (4.41) follows from the definitions of $\widetilde{\varphi}_n$ and f_1^* given by lemma 7 and (3.6). □

5. L_2- and L_∞-bounds

The result (4.38) of theorem 1 can be reinterpreted in terms of the L_2- and L_∞-spatial norms. From the definition of the $\mathcal{B}_2$-norm, (4.38) implies

$$\begin{aligned}\sup_k \Big\{(1 + |k|^3)|\widehat{u}(k t^{-\frac{1}{2\beta}}, t) - \widehat{\Gamma}_{A,B}(k t^{-\frac{1}{2\beta}}, t)| \\ + (1 + |k|^2) t^{-\frac{1}{2\beta}} |\widehat{u}'(k t^{-\frac{1}{2\beta}}, t) - \widehat{\Gamma}'_{A,B}(k t^{-\frac{1}{2\beta}}, t)|\Big\} \leqslant c\, t^{-\frac{1}{\beta}+\epsilon} \|f\|_{\mathcal{B}_2}.\end{aligned} \tag{5.1}$$

Setting $\widetilde{k} = k t^{-1/2\beta}$ in (5.1) yields

$$\begin{aligned}\sup_{\widetilde{k}} \{(1 + t^{\frac{3}{2\beta}} |\widetilde{k}|^3)|\widehat{u}(\widetilde{k}, t) - \widehat{\Gamma}_{A,B}(\widetilde{k}, t)| \\ + (1 + t^{\frac{1}{\beta}} |\widetilde{k}|^2) t^{-\frac{1}{2\beta}} |\widehat{u}'(\widetilde{k}, t) - \widehat{\Gamma}'_{A,B}(\widetilde{k}, t)|\} \leqslant c\, t^{-\frac{1}{\beta}+\epsilon} \|f\|_{\mathcal{B}_2}.\end{aligned} \tag{5.2}$$

In particular

$$|\widehat{u}(\widetilde{k}, t) - \widehat{\Gamma}_{A,B}(\widetilde{k}, t)| \leqslant \frac{c\, t^{-\frac{1}{\beta}+\epsilon} \|f\|_{\mathcal{B}_2}}{1 + t^{3/2\beta}|\widetilde{k}|^3}$$

and

$$|\widehat{u}'(\widetilde{k}, t) - \widehat{\Gamma}'_{A,B}(\widetilde{k}, t)| \leqslant \frac{c\, t^{-\frac{1}{2\beta}+\epsilon} \|f\|_{\mathcal{B}_2}}{1 + t^{1/\beta}|\widetilde{k}|^2}$$

which yield L_2- and L_1-bounds

$$\begin{aligned}|\widehat{u}(\cdot, t) - \widehat{\Gamma}_{A,B}(\cdot, t)|_{L_2} &\leqslant c\, t^{-\frac{5}{4\beta}+\epsilon} \|f\|_{\mathcal{B}_2} \\ |\widehat{u}(\cdot, t) - \widehat{\Gamma}_{A,B}(\cdot, t)|_{L_1} &\leqslant c\, t^{-\frac{3}{2\beta}+\epsilon} \|f\|_{\mathcal{B}_2}\end{aligned} \tag{5.3}$$

and the L_2- and L_1-bounds

$$\begin{aligned}|\widehat{u}'(\cdot, t) - \widehat{\Gamma}'_{A,B}(\cdot, t)|_{L_2} &\leqslant c\, t^{-\frac{3}{4\beta}+\epsilon} \|f\|_{\mathcal{B}_2} \\ |\widehat{u}'(\cdot, t) - \widehat{\Gamma}'_{A,B}(\cdot, t)|_{L_1} &\leqslant c\, t^{-\frac{1}{\beta}+\epsilon} \|f\|_{\mathcal{B}_2}.\end{aligned} \tag{5.4}$$

By Plancherel's theorem $|f|_{L_2} = |\widehat{f}|_{L_2}$, and additionally $|f|_{L_\infty} \leqslant |\widehat{f}|_{L_1}$, while $i\widehat{f}' = \widehat{xf}$. Combining these elementary relations with (5.3) and (5.4) yields precisely the estimates claimed in (1.9) and (1.10). □

6. Discussion

In providing a detailed asymptotic form for the decay of solutions of the model equation (1.4), the foregoing theory makes clear the relative strengths of dissipative, dispersive, and nonlinear effects. While dissipation dominates the decay, dispersion and nonlinearity are evident in second-order terms. In fact, these results may be interpreted as providing an asymptotic form for the difference $u - v$ of the solution of the model equation and the solution of the linear, dissipative equation (1.7). A subtle dependence upon the initial data also obtains in capturing the leading order term in the asymptotics when the disturbance has zero total mass.

Going well beyond our initial study Bona *et al* (1994) using the renormalization group methods of Bricmont *et al* (1994), the present work shows more clearly the efficacy of these techniques. Interesting further lines of inquiry include determining the complete temporal asymptotics and the application of these techniques to more general equations. The former has been accomplished, for instance, by Wayne (1994), where the long-time asymptotics to arbitrary order are derived for a class of parabolic equations which include local dissipation and nonlinearity but not dispersion.

Appendix

Lemma A. *The φ_n's introduced in (4.15) admit the following explicit formulae in Fourier transformed variables:*

$$\widehat{\varphi}_n(k) = A\,e^{-|k|^{2\beta}}e^{-i\alpha^n k^3(1-T^{-n})} + \widehat{N^n_{T^{-n},1}(A\psi)}(k). \tag{A.1}$$

Moreover,

$$\|N^n_{T^{-n},1}(A\psi)\|_{B_2} \leqslant c\,L^{-n}|A|^p \tag{A.2}$$

and

$$\|\varphi_n - Af^*\|_{B_2} \leqslant c|A|\alpha^n + c\,L^{-n}|A|^p \tag{A.3}$$

where $\psi(x,t) = \frac{1}{t^{1/2\beta}} f^*(x/t^{1/2\beta})$.

Proof. The formula (A.1) follows from the relation (4.15), (4.16a, b) and induction. To bound the nonlinear term, we first establish that

$$|\widehat{\psi^{p+1}}(k,t)| \leqslant c\,t^{-\frac{p}{2\beta}}\widehat{\psi}(\rho k, t) \tag{A.4}$$

for some $\rho \in (0,1)$ depending only on p. Indeed, for $p = 1$,

$$\begin{aligned}\widehat{\psi^2}(k,t) &= \int_{-\infty}^{\infty}\widehat{\psi}(k-k_1,t)\widehat{\psi}(k_1,t)\,dk_1 \\ &= \int_{-\infty}^{k/2}\widehat{\psi}(k-k_1)\widehat{\psi}(k_1,t)\,dk_1 + \int_{k/2}^{\infty}\widehat{\psi}(k-k_1)\widehat{\psi}(k_1,t)\,dk_1.\end{aligned} \tag{A.5}$$

Since $\widehat{\psi}(k,t) = e^{-|k|^{2\beta}t}$, $\widehat{\psi}$ is even in k, hence so is $\widehat{\psi^{p+1}}$. Assume without loss of generality that $k \geqslant 0$. Then, it follows that

$$\max_{k_1\in(-\infty,k/2]}|\widehat{\psi}(k-k_1,t)| = \widehat{\psi}(k/2,t)$$

while

$$\max_{k_1\in[k/2,\infty)} |\widehat{\psi}(k_1,t)| = \widehat{\psi}(k/2,t)$$

and the two integrals in (4.23) may therefore be bounded above by

$$\left|\int_{-\infty}^{k/2} \widehat{\psi}(k-k_1,t)\widehat{\psi}(k_1,t)\,\mathrm{d}k_1\right| \leqslant \widehat{\psi}(k/2,t)|\widehat{\psi}(t)|_{L^1} \leqslant c\frac{1}{t^{1/2\beta}}\widehat{\psi}(k/2,t)$$
$$\left|\int_{k/2}^{-\infty} \widehat{\psi}(k-k_1,t)\widehat{\psi}(k_1,t)\,\mathrm{d}k_1\right| \leqslant \widehat{\psi}(k/2,t)|\widehat{\psi}(t)|_{L^1} \leqslant c\frac{1}{t^{1/2\beta}}\widehat{\psi}(k/2,t).$$

Thus $\left|\widehat{\psi^2}(k,t)\right| \leqslant ct^{-\frac{1}{2\beta}}(k/2,t)$. An inductive argument gives (A.4) for $p = 2, 3, \cdots$.

From the definition (2.7) of $N^n_{a,b}$ we have

$$\mathcal{F}(N^n_{T^{-n},1}(A\psi))(k) = \frac{\mathrm{i}k\gamma^n}{p+1}\int_{T^{-n}}^{1} \mathrm{e}^{-(|k|^{2\beta}+\mathrm{i}\alpha^n k^3)(1-s)}\widehat{\psi^{p+1}}(k,s)\,\mathrm{d}s$$

and hence

$$|\mathcal{F}(N^n_{T^{-n},1}(A\psi))(k)| \leqslant c|A|^p\gamma^n|k|\mathrm{e}^{-|k|^{2\beta}}\int_{T^{-n}}^{1} \mathrm{e}^{|k|^{2\beta}s}s^{-\frac{p}{2\beta}}\widehat{\psi}(\rho k,s)\,\mathrm{d}s$$
$$\leqslant c|A|^p\gamma^n|k|\mathrm{e}^{-|k|^{2\beta}}\int_{T^{-n}}^{1} \mathrm{e}^{(1-\rho^{2\beta})|k|^{2\beta}s}s^{-\frac{p}{2\beta}}\,\mathrm{d}s.$$

Bounding the exponential term in the integral by its L_∞-norm, and the polynomial term by its L_1-norm, there obtains

$$|\mathcal{F}(N^n_{T^{-n},1}(A\psi))(k)| \leqslant c|A|^p|k|\mathrm{e}^{-|\rho k|^{2\beta}}(\gamma T^{\frac{p}{2\beta}-1})^n \leqslant c|A|^p|k|\mathrm{e}^{-|\rho k|^{2\beta}}L^{-n}$$

where $\gamma = L^{2\beta-(p+1)}$ and $T = L^{2\beta}$, and hence $\gamma T^{\frac{p}{2\beta}-1} = L^{-1}$. In consequence, it transpires that

$$\sup_k\{(1+|k|^3)|\mathcal{F}(N^n_{T^{-n},1}(A\psi))|\} \leqslant c|A|^pL^{-n}\sup_k\{(1+|k|^4)\mathrm{e}^{-|\rho k|^{2\beta}}\} \leqslant c|A|^pL^{-n}.$$

Since $\frac{\mathrm{d}}{\mathrm{d}k}\widehat{\psi^{p+1}} = \widehat{\psi}' * \widehat{\psi} * \cdots * \widehat{\psi}$, and $\widehat{\psi}'(k,t) = 2\beta t|k|^{2\beta-1}\widehat{\psi}(k,t)$, the term

$$\sup_k\left\{(1+|k|^2)|\frac{\mathrm{d}}{\mathrm{d}k}(\mathcal{F}(N^n_{T^{-n},1}(A\psi))(k)| + (1+|k|)\left|\frac{\mathrm{d}^2}{\mathrm{d}k^2}(\mathcal{F}(N^n_{T^{-n},1}(A\psi))(k)\right|\right\}$$

is similarly bounded, and (A.2) follows. The upper bound (A.3) follows immediately from the triangle inequality, (A.1), and (A.2). □

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References

Albert J and Bona J L 1991 Comparisons between model equations for long waves *J. Nonlinear Sci.* **1** 345–74

Albert J, Bona J L and Henry D 1987 Sufficient conditions for stability of solitary-wave solutions of model equations for long waves *Physica* **24D** 343–66

Amick C, Bona J L and Schonbek M 1989 Decay of solutions of some nonlinear wave equations *J. Diff. Eq.* **81** 1–49

Biler P 1984 Asymptotic behaviour in time of solutions to some equations generalizing the Korteweg–de Vries equation *Bull. Polish Acad. Sci.* **32** 275–82

Bona J L, Dougalis V, Karakashin O and McKinney W 1992 Computations of blow-up and decay for periodic solutions of the generalized Korteweg-de Vries equation *Appl. Num. Math.* **10** 335–55

Bona J L and Luo L 1993 Decay of solutions to nonlinear, dispersive wave equations *Diff. Int. Eq.* **6** 961–80

Bona J L, Pritchard W G and Scott L R 1981 An evaluation of a model equation for water waves *Phil. Trans. R. Soc.* A **302** 457–510

Bona J L, Promislow K. and Wayne C E 1994 On the asymptotic behaviour of solutions to nonlinear, dispersive, dissipative wave equations *J. Math and Computers in Simulation* **37** 264–77

Bona J L and Scialom M. 1995 On the comparison of solutions of model equations for long waves. *Canadian Appl. Math. Quart.* in press

Bona J L and Soyeur A 1994 On the stability of solitary-wave solutions of model equations for long waves *J. Nonlinear Sci.* **4** 449–70

Bricmont J, Kupiainen A and Lin G 1994 Renormalization group and asymptotics of solutions of nonlinear parabolic equations *Commun. Pure Appl. Math.* **47** 893–922

Dix D 1992 The dissipation of nonlinear dispersive waves: the case of asymptotically weak nonlinearity *Commun. PDE* **17** 1665–93

Kakutani T and Matsuachi K 1975 Effect of viscosity on long gravity waves *J. Phys. Soc. Japan* **39** 237–46

Korteweg D and DeVries G 1895 On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves *Phil. Mag.* **39** 422–33

Naumkin P I and Shishmarev I 1994 *Nonlinear nonlocal equations in the theory of waves* (American Mathematical Society Series: Transl. of Math. Mono.) vol 133

Ott E and Sudan R 1970 Damping of solitary waves *Phys. Fluids* **13** 1432–4

Pego R and Weinstein M 1992 On asymptotic stability of solitary waves *Phys. Lett.* **162A** 263–8

—— 1992 Eigenvalues and instabilities of solitary waves *Phil. Trans. R. Soc.* A **340** 47–94

Schonbek M 1980 Decay of solutions to parabolic conservation laws *Commun. PDE* **7** 449–73

—— 1985 L^2 decay for weak solutions of the Navier–Stokes equation *Arch. Rat. Mech. Anal.* **88** 209–22

Wayne C E, Invariant manifolds for parabolic partial differential equations on unbounded domains *Preprint*