matemática
contemporânea

Matemática Contemporânea, Vol 8, 21-37

ON THE COMPARISON OF SOLUTIONS OF MODEL EQUATIONS FOR LONG WAVES*

J. L. Bona **M. Scialom**

Abstract

The purpose of this note is to understand the dependence of solutions of nonlinear, dispersive equations on the nonlinearity and on the dispersion relation. It focuses on the relatively specific, but practically important context of Korteweg-de Vries-type equations. The general thrust of the results are that small perturbations of a given dispersion relation or nonlinearity make only a small difference in the solution over a relatively long time scale.

1. Introduction

We are interested in equations of the form

$$u_t + u_x + f(u)_x - Mu_x = 0 \ , \tag{1.1}$$

where $f : I\!R \to I\!R$ and M is a Fourier multiplier operator defined via its Fourier transform as

$$\widehat{Mh}\ (k) = m(k)\hat{h}(k); \tag{1.2}$$

where in is the symbol of the operator.

The equations in (1.1) arise in a wide range of physical context as models for the propagation of waves (see [1] and [2]). Typically both nonlinear effects, modelled by f and dispersive effects, modelled by M, are only approximations to a more complete accounting of these aspects of wave propagation. In consequence it becomes interesting to understand to what extent the detailed structure of the nonlinearity or the dispersion is reflected in solutions of

*Work partially supported by the National Science Foundation and the Keck Foundation, USA; FAPESP and by CNPq, Brasil

the equations. Attention will be given to this issue in the context of the pure initial-value problem for (1.1), in which the dependent variable u is specified for all x at some fixed time t, say $t = 0$, so that

$$u(x,0) = \varphi(x)$$

for all $x \in \mathbb{R}$. In the present work we shall consider variations of both the nonlinearity and the dispersion relation. The question that will be posed is, for a fixed initial datum φ, if the dispersion relation m is perturbed, or if the nonlinearity f is changed, what can be said about the resulting variation of the solution u? The basic conclusion of the study is that, on a long time scale T naturally related to the underlying physical situation, the equations predict the same outcome to within their implied order of accuracy.

The most famous example in the form (1.1) is the equation proposed by Korteweg and de Vries in [3],

$$\eta_t + \eta_x + \frac{3}{2}\,\eta\eta_x + \frac{1}{6}\,\eta_{xxx} = 0. \tag{1.3}$$

This model is often used to describe the unidirectional propagation of irrotational, weakly nonlinear, dispersive waves on the surface of an ideal liquid in a uniform channel. In this equation, $\eta = \eta(x,t)$ represents the vertical displacement of the surface of the liquid from its equilibrium position, t is the time and x is the horizontal coordinate (which increases in the direction of propagation of the waves). Equation (1.3) is written in dimensionless form, with the length scale taken to be the undisturbed depth h of the liquid and the time scale to be $(\frac{h}{g})^{1/2}$ g, where is the gravity constant. It is assumed in the derivation of (1.3) that the maximum amplitude ε of the waves is small and that the waves can be characterized by a wane δ^{-1}, which is large. In particular, it is crucial that the amplitude scale and the wavelength of the waves are such that $\varepsilon\delta^{-2}$ is of order one so that the nonlinear and dispersive corrections to the primary wave equation $\eta_t + \eta_x = 0$ are of comparable importance (cf, [4]).

Turning back to equation (1.1), let ε be a representative value of the amplitude of the motions in question and λ a typical value of the wavelength, where it is presumed that both there quantities have been non-dimensionalized with respect to underlying length scale present in the problem. In these circumstances the initial wave profile is naturally scaled as

$$\varphi(x) = \varepsilon\psi(\lambda^{-1}x), \tag{1.4}$$

where ψ and its derivatives are of order one. If we suppose the nonlinearity f and the dispersion m to be homogeneous so that $f(u)_x = u^p u_x$ and $m(k) = |k|^\alpha$, then the conditions that nonlinear and dispersive effects are small and balanced are the requirement that $\varepsilon^p\lambda^\alpha$ is of order one, while ε and λ^{-1} are both small. The quantity $S = \varepsilon^p\lambda^\alpha$ is a natural generalization of the classical Stokes or Ursell number of shallow-water theory (see Stokes [5], Ursell [6], Whitham [7] and Bona, Pritchard & Scott [8]). If the small parameter δ is defined to be ε^p, then λ has order $\delta^{-1/\alpha}$ and the relation (1.4) can be expressed in terms of the single parameter δ as

$$\varphi(x) = \delta^{1/p}\psi(\delta^{1/\alpha}x). \tag{1.5}$$

The central question that will attract attention here is the following. With initial data as in (1.4), suppose two different dispersion relations m_1 and m_2 or two different nonlinearities f_1 and f_2 to be given, and let u_1 and u_2 be the corresponding solutions of (1.1) emanating from the initial-value φ. For relatively small values δ, it is expected that both u_1 and u_2 will be small, but, depending on the difference $m_1 - m_2$ and $f_1 - f_2$, it may be that $u_1 - u_2$ is smaller still, at least over certain time intervals.

A result of this sort may be interpreted as saying that the difference between using m_2 and f_2 rather than m_1 and f_1 is relatively negligible, at least over certain time intervals. As will appear below, this time interval is often large, proportional to an inverse power of δ, and, under reasonable hypothesis, coincides with the time scale over which interesting nonlinear and dispersive effects appear at the leading order (see [8] and [9]).

The notation employed througout this paper will be that which is currently standard in the theory of partial differential equations. Thus $L_p = L_p(\mathbb{R})$, for $1 \le p \le \infty$ is the usual Banach space of p^{th}-power integrable functions (essentially bounded functions if $p = \infty$) whose norm will be denoted by $|\cdot|_p$. A circumflex adorning a function connotes that function's Fourier transform. The solutions of (1.1) or (1.3) which will be discussed are, for each instant of time, members of Sobolev spaces H^s for some $s \geq 0$. If $f \in H^s$, then

$$||f||_s = \left[\int_{-\infty}^{\infty} (1+\xi^2)^s |\hat{f}(\xi)|^2 d\xi\right]^{1/2} .$$

Notice that the L_2-norm has two different notations, between which systematic preference will be given to $|\cdot|_2$. The spaces H^s are Hilbert spaces, but the only inner product needed here is that of L_2, which will be denoted simply as $(\cdot,\cdot)$. If X is any Banach space, the space $C(a,b;X)$ is the collection of continuous maps $u : [a,b] \to X$ with the norm

$$||u||_{C(a,b;X)} = \sup_{a \leq t \leq b} ||u(t)||_X ,$$

where $|| \ ||_X$ dennotes the norm on X.

The plan of the paper is as follows. In section 2, a general Theorem of comparison for equations of the form (1.1) is formulated and proved. In Theorem 2, it is assumed that the initial-value problem for (1.1) has global solutions corresponding to reasonable smooth initial data. A similar result, without this assumption is shown in [9]. Section 3 contains examples of particular comparisons made with the use of Theorem 2 together with interpretation in terms of the physical problems being modelled.

2. The comparison theorem

The main result for equations of type (1.1) is formulated and proved below as Theorem 2.

In this section we assume that in equation (1.1), the nonlineatity f is quadratic and the dispersion m is homogeneous, so that $f(u)_x = uu_x$ and $m(k) = |k|^\alpha$. The more general case wherein $f(u)_x = u^p u_x$, $p \geq 1$, will be discussed in [9].

Consider the initial value problems

$$\begin{aligned} &u_t + uu_x - Mu_x = 0 , \\ &v_t + vv_x + v^q v_x - Nv_x = 0 , \end{aligned} \tag{2.1}$$

where $q > 1$ is integer, the dispersion operators M and N are given and the initial data is scaled as

$$u(x,0) = v(x,0) = \varepsilon\varphi(\varepsilon^\alpha x) . \tag{2.2}$$

Rescale u and v by the relations

$$\begin{aligned} &u(x,t) = \varepsilon U(\varepsilon^\alpha x, \varepsilon^\beta t) , \\ &v(x,t) = \varepsilon V(\varepsilon^\alpha x, \varepsilon^\beta t) , \end{aligned} \tag{2.3}$$

where $\beta = 1 + \alpha$. Then U and V satisfy the initial-value problems

$$\begin{aligned} &U_t + UU_x - M_\varepsilon U_x = 0 , \\ &V_t + VV_x + \varepsilon^{q-1} V^q V_x - N_\varepsilon V_x = 0 , \end{aligned} \tag{2.4}$$

with

$$U(x,0) = V(x,0) = \varphi(x) .$$

The rescaled operator M_ϵ is defined such that if h is a function of the spatial variable x, then the Fourier transform of $M_\epsilon h$ is given as

$$\widehat{M_\varepsilon h}(\xi) = \varepsilon^{-1} m(\varepsilon^\alpha \xi)\hat{h}(\xi) = m_\varepsilon(\xi)\hat{h}(\xi) ,$$

and similarly for N_ϵ.

The next lemma refers to the pair of rescaled initial-value problems displayed in (2.4).

Lemma 1. *Suppose α is such that for all ξ and sufficiently small ε*

$$|m_\varepsilon(\xi) - n_\varepsilon(\xi)| \leq \varepsilon |P_{r-1}(\xi)|, \tag{2.5}$$

where $r \geq 2$ is an integer and P_{r-1} is a polynomial of degree $r-1$. Let $\varphi \in H^{k+r}$ where $k \geq 0$. Suppose that the initial-value problems (2.4) are globally well posed *in* H^{k+r} *and that the* H^{k+r}*-norms of* U *and* V *are bounded with a bound that depends only on the norm of* φ *in* H^{k+r}*, and not on* ε*, at least for* ε *small. Then there exists an* $\varepsilon_0 > 0$ *and constants* B_j *such that, for* $0 \le t \le 1$ *and* $0 < \varepsilon \le \varepsilon_0$*,*

$$|\partial_x^j(\cdot,t) - V(\cdot,t))|_2 \le \varepsilon B_j\, t \tag{2.6}$$

for $0 \le j \le k$*. The constants* B_j *depend only on the norms of the solutions* U *and* V *in* $C(0,T;H^{k+r})$*.*

Proof. The method employed here is to define w as the difference $U - V$ and then apply energy-type arguments for its estimation. Toward this end, note first that w satisfies the initial-value problem

$$\begin{aligned} &w_t + \tfrac{1}{2}(w^2)_x + (Vw)_x - M_\varepsilon w_x = (M_\varepsilon - N_\varepsilon)V_x - \varepsilon^{q-1}V^qV_x \\ &w(x,0) \equiv 0 \end{aligned} \tag{2.7}$$

at least in the sense of tempered distributions. The inequalities in (2.6) will be established by induction on j. In what follows, calculations will be made as if the solutions U and V are C^∞-functions, all of whose derivatives lie in L_2. The formulas that result therefrom will only involve spatial derivatives of order less than or equal to $k+r$. Consequently, these formulas may be justified by taking a sequence $\{\varphi_n\}_{n=1}^\infty$ of smooth functions with compact support that approach φ in H^{k+r}, making the calculations for the associated solutions U_n and V_n of (2.4), and then passing to the limit as n tends to infinity.

To begin, differentiate the equation in (2.7) j times with respect to x and multiply the result by $\partial_x^j w = w_{(j)}$, where a new notation has been introduced for partial derivatives with respect to the spatial variable x. Upon integrating the equation that arises from the just-described operations with respect to x over the entire real line and with respect to t over the interval $[0,t]$ and after suitable integrations by parts, using the fact that $w(\cdot,0) \equiv 0$, there appears the relationships

$$|w_{(j)}(\cdot,t)|_2^2 = -\int_0^t ([(w^2)_x]_{(j)}, w_{(j)})ds$$

$$\begin{aligned} &-2\int_0^t ([(Vw)_x]_{(j)}, w_{(j)})ds + 2\int_0^t ((M_\varepsilon - N_\varepsilon)V_{(j+1)}, w_{(j)})ds \\ &-2\varepsilon^{q-1}\int_0^t ((V^qV_x)_{(j)}, w_{(j)})ds \end{aligned} \tag{2.8}$$

for $0 \le j \le k$.

Formula (2.8) will be used inductively to derive the bounds advertised in (2.6). Consider first the case $j = 0$ for which (2.8) may be written in the form

$$\begin{aligned} |w(\cdot,t)|_2^2 = &-\int_0^t\int_{-\infty}^\infty ((w^2)_x w\,dx\,ds - 2\int_0^t\int_{-\infty}^\infty (Vw)_x w\,dx\,ds \\ &+2\int_0^t\int_{-\infty}^\infty (M_\varepsilon - N_\varepsilon)V_x w\,dx\,ds - 2\varepsilon^{q-1}\int_0^t\int_{-\infty}^\infty V^qV_x w\,dx\,ds\,. \end{aligned} \tag{2.9}$$

Estimating the first and second terms on the right-hand side of (2.9) in a standard way and applying Plancherel's theorem, the Cauchy-Schwarz inequality and (2.5) to the third and fourth terms lead to the inequality

$$\begin{aligned} |w(\cdot,t)|_2^2 \le &\int_0^t |V_x|_\infty |w|_2^2 ds \\ &+\varepsilon c_0\int_0^t ||V||_r |w|_2 ds + \frac{2\varepsilon^{q-1}}{q+1}\int_0^t ||V^{q+1}||_1 |w|_2 ds\,, \end{aligned}$$

where

$$c_0 = \sup_{\xi\in\mathbb{R}} \frac{2|P_{r-1}(\xi)|}{(1+\xi^2)^{r/2}} \tag{2.10}$$

If we define

$$A_0 = \max_{0\le t\le 1} |V_x(\cdot,t)|_\infty \le ||V||_{c(0,\infty,H^2)}$$

and

$$C_0 = \max_{0\le t\le m1} (c_0||(V(\cdot,t)||_r + \frac{2\varepsilon^{q-2}}{q+1}||V^{q+1}(\cdot,t)||_1),$$

where c_0 is as in (2.10), then Gronwall's lemma implies that

$$|w(\cdot,t)|_2 \le \varepsilon C_0 \frac{e^{\frac{1}{2}At} - 1}{A_0} \le \varepsilon B_0\, t. \tag{2.11}$$

Notice that B_0 depends only on the norm of the solutions U and V.

We turn now to the case $j = 1$. In this case the relation (2.8) can be put as

$$\begin{aligned} |w_x(\cdot,t)|_2^2 = &-3\int_0^t\int_{-\infty}^\infty V_x^2\,dx\,ds - \int_0^t\int_{-\infty}^\infty w_x^3\,dx\,ds - 2\int_0^t\int_{-\infty}^\infty V_{xx} w w_x\,dx\,ds \\ &+2\int_0^t\int_{-\infty}^\infty (M_\varepsilon - N_\varepsilon)V_{xx} w_x\,dx\,ds + \frac{2\varepsilon^{q-1}}{q+1}\int_0^t\int_{-\infty}^\infty (V^{q+1})_{xx} w_x\,dx\,ds. \end{aligned}$$

The second term on the right-hand side of the last relation is estimated using the embedding of $H^{1/6}$ into L_3 and interpolation. The other terms are estimated in obvious ways. The upshot is

$$|w_x(\cdot,t)|_2^2 \le -3\int_0^t |V_x|_\infty |w|_2^2 ds + k_0\int_0^t ||w_x||_{1/2}|w_x|_2^2 ds + 2\int_0^t |V_{xx}|_\infty |w|_2|w_x|_2 ds$$

$$+\varepsilon c_0\int_0^t ||V_x||_r|w_x|_2 ds + \frac{2\varepsilon^{q-1}}{q+1}\int_0^t |(V^{q+1})_{xx}|_2|w_x|_2 dx ds \,, \tag{2.12}$$

where k_0 is an embedding constant and c_0 is defined in (2.10). The third term on the right-hand side of (2.12) is further bounded above using (2.11) as follows:

$$\int_0^t |V_{xx}|_\infty |w|_2|w_x|_2 ds \le \varepsilon B_0\int_0^t |V_{xx}|_\infty |w_x|_2 ds \,. \tag{2.13}$$

Much as before, if we let

$$A_1 = \max_{0\le t\le 1}(3|V_x(\cdot,t)|_\infty + k_0||w_x(\cdot,t)||_{1/2})$$

and

$$C_1 = \max_{0\le t\le 1}(B_0|V_{xx}(\cdot,t)|_\infty + c_0||V_x(\cdot,t)||_r + \frac{2\varepsilon^{q-2}}{q+1}||V^{q+1}(\cdot,t)||_2) \,,$$

then Gronwall's inequality applied to (2.12) and (2.13) yields

$$|w_x(\cdot,t)|_2 \le \varepsilon C_1\frac{e^{\frac{1}{2}A_1 t}-1}{A_1} \le \varepsilon B_1\, t.$$

Again, B_1 depends only on norms of the solutions U and V.

The proof is finished by an inductive argument wherein the desired result (2.6) is assumed to be valid for $j<m$ where $m<k$, and then, on that basis, the result is established for $j=m$. To this end, consider the right-hand side of the relation (2.8) with $j=m$. The first and second terms can be bounded above in a similar way as is now shown:

$$\int_{-\infty}^{\infty}[(w^2)_x]_{(m)}w_{(m)}d(x) = (2m-1)\int_{-\infty}^{\infty} w_x w_{(m)}^2 dx +$$

$$+\,2\sum_{k=2}^{m}\binom{m}{k}\int_{-\infty}^{\infty} w_{(k)}w_{(m-k+1)}w_m dx \,,$$

$$\int_{-\infty}^{\infty}[(Vw)_x]_{(m)}w_{(m)}d(x) = \int_{-\infty}^{\infty}[V_x w + V w_x]_{(m)}w_{(m)}dx.$$

Expanding the derivatives of the product in the right-hand side of the last equality, leads to an expression of the form

$$\int_{-\infty}^{\infty} F w_{(m)}^2 dx + \int_{-\infty}^{\infty} G w_{(m)} dx \,,$$

where F is a polynomial in V, V_x, w and w_x and G is a polynomial in $V, V_x, \dots, V_{(m)}, w, w_x, \dots, w_{(m-1)}$. Estimating the L_∞-norm of F and the L_2-norm of G, and using the induction hypothesis leads to an inequality of the form

$$\Big|\int_{-\infty}^{\infty}(Vw_x)_{(m)}(\cdot,t)w_{(m)}(\cdot,t)dx\Big| \le a|w_{(m)}(\cdot,t)|_2^2 + \varepsilon b|w_{(m)}(\cdot,t)|_2.$$

Making similar estimates of terms in the other sums on the right-hand side, and combining these with the bounds

$$\Big|\int_{-\infty}^{\infty}(M_\varepsilon - N_\varepsilon)V_{(m+1)}w_{(m)}dx\Big| \le \varepsilon c_0||V||_{r+m}|w_{(m)}|_2 \,,$$

and

$$\varepsilon^{q-1}\Big|\int_{-\infty}^{\infty}(V^{q+1})_{(m+1)}w_{(m)}dx\Big| \le \varepsilon^{q-1}||V^{q+1}||_{(m+1)}|w_{(m)}|_2 \,,$$

there appears the inequality

$$|w_{(m)}(\cdot,t)|_2^2 \le A_m\int_0^t |w_{(m)}(\cdot,s)|_2^2 ds + \varepsilon C_m\int_0^t |w_m(\cdot,s)|_2 ds$$

from which it follows that

$$|w_m(\cdot,t)|_2 \le \varepsilon C_m\frac{e^{\frac{1}{2}A_m t}-1}{A_m} \le \varepsilon B_m t$$

for $0\le t\le 1$. The constant B_m depends as before on norms of U and V and on the previous constants $B_0,\dots,B_{m-1}$. The inductive step being established, it is concluded that (2.6) holds for all $j\le k$ and the proof is completed. □

As an immediate corollary of this lemma and the transformations (2.3), the principal technical result to be used in the next section is obtained.

Theorem 2. *Suppose that condition (2.5) is valid for the scaled operators M_ε and N_ε for some value of r and a fixed value of α with $\beta = 1+\alpha$. Suppose in addition that the initial-value problems (2.1) are globally well posed in H^{k+r} for some $k \geq 0$, where $k+r \geq 2$. Let $\varphi \in H^{k+r}$ be given and let u_ε and v_ε be the respective solutions of (2.1) with initial data as in (2.2). Then there are constants B_j, $0 \leq j \leq k$ which depend only on the norm of φ in H^{k+r} such that for $0 \leq j \leq k$,*

$$|\partial_x^j(u_\varepsilon - v_\varepsilon)|_2 \leq \varepsilon^{2+\alpha(j-\frac{1}{2})} B_j \varepsilon^\beta t \tag{2.14}$$

provided $0 \leq t \leq \varepsilon^{-\beta}$. By interpolation, therefore, it follows that

$$|\partial_x^j(u_\varepsilon - v_\varepsilon)|_\infty \leq \varepsilon^{2+\alpha j} C_j \varepsilon^\beta t \tag{2.15}$$

for $0 \leq j < k$ and $0 \leq t \leq \varepsilon^{-\beta}$, where $C_j = (B_j B_{j+1})^{1/2}$.

□

3. Applications

In this section, a number of examples will be set forth which show the efficacy of the theory developed in Section 2. A wide range of others examples applying an extension of the present theory will be presented in [9].

3a. Perturbations of the Symbol

The situation envisioned here is perhaps the most straightforward application of Theorem 2 as it involues only the dispersion relation. The idea is that the two symbols m and n of the operators M and N are the same except that one has a higher-order correction to the modelling of dispersion. A paradigm for this situation is provided by the surface water-wave problem in which the full, linearized dispersion relation is

$$c(k) = \left(\frac{\tanh(k)}{k}\right)^{1/2} \tag{3.1}$$

in suitably normalized variables. The Korteweg-de Vries equation is obtained when c is replaced by the first two terms in its Taylor expansion about the origin, namely

$$c_{KdV}(k) = 1 - \frac{1}{6}k^2, \tag{3.2}$$

which will be a good approximation provided only small values of k (long waves) are in question. It may happen that one needs to model dispersive effects more accurately, however, while still staying in the realm of long-wave models, and in this situation it is natural to take an additional term in the Taylor expansion, namely

$$\tilde{c}(k) = 1 - \frac{1}{6}k^2 + \frac{19}{360}k^4 \tag{3.3}$$

(cf. Abdelouhab *et al.* [10]). The partial differential equations corresponding to (3.2) and (3.3) are

$$u_t + u_x + \frac{3}{2}uu_x + \frac{1}{6}u_{xxx} = 0\ ,$$

and (3.4.a)

$$v_t + v_x + \frac{3}{2}vv_x + \frac{1}{6}v_{xxx} + \frac{19}{360}v_{xxxxx} = 0,$$

respectively (The factor $\frac{3}{2}$ in the nonlinear term comes out naturally in the usual non-dimensionalization of variables, as in Benjamin *et al.* [11]). By moving to a traveling frame of reference, the linear translational term u_x can be eliminated, and one is then left with the two equations

$$u_t + \frac{3}{2}uu_x + \frac{1}{6}u_{xxx} = 0\ ,$$

and (3.4.b)

$$v_t + \frac{3}{2}vv_x + \frac{1}{6}v_{xxx} + \frac{19}{360}v_{xxxxx} = 0\ .$$

The natural scaling for small-amplitude long waves on the surface of shallow water is that of (2.3) with $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$ (cf. Bona & Smith [12]). Letting M connote the operator $-\frac{1}{6}\partial_x^2$ and N the operator $-\frac{1}{6}\partial_x^2 - \frac{19}{360}\partial_x^4$, we find ourselves in the situation envisioned in (2.1) with $q = 2$. Moreover, both equations in (3.4) are globally well posed in H^k for any $k \geq 2$ (see Kato [13], Bona & Smith [12]). Hence to apply Theorem 2 in this situation, it is only necessary to check condition (2.5) of Lemma 1. A straightforward calculation reveals that the associated operators M_ε and N_ε satisfy the relations

$$|m_\varepsilon(\xi) - n_\varepsilon(\xi)| \leq \frac{19}{360}\varepsilon\xi^4 .$$

Thus taking $r = 5$ and supposing the initial data φ lies in H^{k+5} for some non-negative value of k, it is deduced at once that

$$|\partial_x^j(u - v)|_\infty \leq C_j\varepsilon^{2+j/2+3/2}t \tag{3.5}$$

for $0 \leq j < k$, provided $0 \leq t \leq \varepsilon^{-3/2}$, with similar estimates for the L_2-norm of the difference. One is thus led to the conclusion that the inclusion of higher-order dispersive effects is without consequence at the level of modelling inherent in either equation in (3.4). In the Korteweg-de Vries equation written in the form (3.4) and with small-amplitude, long-wavelength initial data $\varepsilon\varphi(\varepsilon^{1/2}x)$, we know that nonlinear and dispersive effects accumulate to make an order-one relative contribution to the wave profile at time t of order $\varepsilon^{-3/2}$ (see Bona *et al.* [8]). Equally, at time t of order $\varepsilon^{-5/2}$, the error terms inherent in the Korteweg-de Vries model could in principle make an order-one relative contribution to the wave profile, thus rendering the Korteweg-de Vries approximation invalid. The same remarks apply to the extended model because higher-order nonlinear effects have not been included. Now, refering to (3.5), we see that while u and v are both order ε, their difference is of order ε^2 at $t = \varepsilon^{-3/2}$. As ε^2 is the order that would be contributed by the neglected terms in either model, it is inferred that the effect of the higher-order dispersion relation in the extended model is of no consequence on time scales wherein the neglected effects remain relatively negligible.

3b. Perturbations of the Nonlinearity

Here we consider the effect of including a higher-order nonlinear term in the model equation. A case that arises often in pratice is the inclusion of a cubic nonlinearity in the Korteweg-de Vries equation. The general nature of u^2u_x as the next term in the approximation of nonlinear effects is explained in Benjamin *et al.* [11], where it is argued heuristically that this form will generically appear at the next order of approximation.

Attention is thus given to the two equations

$$\begin{aligned} u_t + uu_x + u_{xxx} &= 0 \\ \text{and} \qquad v_t + vv_x + v^2v_x + v_{xxx} &= 0 . \end{aligned} \tag{3.6}$$

Both equations in (3.6) are globally well posed in H^k for any $k \geq 2$ (cf. Kato [14]). Taking the scaling appropriate to the Korteweg-de Vries equation, namely $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$ in (2.3) and applying Theorem 2 leads directly to the conclusion that for $\varphi \in H^k$ and $0 \leq j < k$,

$$|\partial_x^j(u - v)|_\infty \leq C_j\varepsilon^{2+j/2+3/2}t , \tag{3.7}$$

at least for $0 \leq t \leq \varepsilon^{-3/2}$, just as in (3.5). Refering to the discussion in Section 3a, it is concluded that during the time period over which significant alteration of the initial profile takes place due to nonlinear and dispersive effects, the cubic nonlinearity remains relatively negligible for data that satisfies the basic Korteweg-de Vries-type scaling.

3c. Comparison between the Korteweg-de Vries equation and Smith's equation

This comparison is a little more subtle than the simple pertubation featured in Section 3a. The evolution equation

$$u_t + uu_x - Mu_x = 0,$$

where the symbol m of M is (3.8)

$$m(\xi) = (\sqrt{1+\xi^2} - 1),$$

was derived by Smith [15] as a model for continental shelf waves (the form of the symbol in (3.8) corrects a minor oversight in Smith's paper). Because m is smooth and has the approximate form $\frac{1}{2}\xi^2$ near $\xi = 0$, it is natural to ask whether or not an appropriate version of the Korteweg-de Vries equation might be just as good as a model for the phenomena in question. This depends upon the scaling assumption that applies to the initial data. If the waves to which the model is to be applied are adequately represented by the scaling $\varepsilon g(\varepsilon^{1/2}x)$, then we will show now that one might as well use the Korteweg-de Vries equation as a model.

Turning to a detailed analysis of the last assertion, we attempt to apply Theorem 2 to equation (3.8) and the Korteweg-de Vries equation in the form

$$v_t + vv_x + \frac{1}{2}v_{xxx} = 0. \tag{3.9}$$

Again we choose $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$ which is consistent with the use of the Korteweg-de Vries equation. The crux of the matter is to establish condition (2.5) in Lemma 1. As the symbol m is not homogeneous, this is slightly more complicated than for the perturbation in Section 3a. The operator M_ε is given by

$$\widehat{M_\varepsilon f}(\xi) = \left(\frac{\sqrt{1+\varepsilon\xi^2} - 1}{\varepsilon}\right)\hat{f}(\xi)$$

in fact, and it follows easily that if $N = \frac{1}{2}\partial_x^2$, then

$$|m_\varepsilon(\xi) - n_\varepsilon(\xi)| \le \frac{1}{4}\varepsilon\xi^4.$$

Thus, supposing that $\varphi \in H^{k+5}$ for some $k > 0$, the initial value problem associated to (3.8) is globally well posed in H^{k+5} (see Abdelouhab *et al.* [10] and Iorio [16]), it may be inferred from Theorem 2 that

$$|\partial_x^j(u-v)|_\infty \le C_j\varepsilon^{2+j/2+3/2}t \tag{3.10}$$

for $0 \le t \le \varepsilon^{-3/2}$ and for $0 \le j \le k-1$.

One infers from (3.10) that for data that has an amplitude to wavelength relationship well approximated by the form $\varepsilon\varphi(\varepsilon^{1/2}x)$, it doesn't matter whether (3.8) or (3.9) is used to model the wave's evolution. Both give the same answer to within the inerent order of accuracy of either.

Remark. A host of other interesting modelling situations may be analysed using the theory developed in Section 2, or its generalization to appear in [9]. In addition to KdV-type equations, comparisons may be effected between regularized long-wave-type equations and Schödinger-type equations using a suitable variant of the theory developed here. These developments will be set out in [9].

References

[1] Ablowitz, M.J. and Clarkson, P.A., *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, London Mathematical Society Lecture Note Series **149**, (1991).

[2] Jeffrey, A. and Kakutani, T., *Weak nonlinear dispersive waves: a discussion centered around the Korteweg - de Vries equation*, SIAM Rev. **14** (1972), 582-643.

[3] Korteweg, D.J. & de Vries, G., *On the change of form of long waves advancing in a rectangular channel, and on a new type of long stationary wave*, Phil. Mag. **39** (1895), 422-443.

[4] Meyer, R.E., *Note on the long wave equations*, Bull. Calcutta Math. Soc. **71** (1979), 121-130.

[5] Stokes, G.G., *On the theory of oscillatory waves*, Trans. Camb. Phil. Soc. **8**, (1849), 441-455.

[6] Ursell, F., *The long-wave paradox in the theory of gravity waves*, Proc. Camb. Phil. Soc. **49** (1953), 685-694.

[7] Whitham, G.B., *Linear and nonlinear waves*, John Wiley & Sons, New York (1974).

[8] Bona, J.L., Pritchard, W.G. and Scott, L.R., *A comparison of solutions of two model equations for long waves*, Lectures in Applied Mathematics **20** (1983), 235-267.

[9] Bona, J.L. and Scialom, M., *The effect of change in the nonlinearity and the dispersion relation of model equations for long waves*, (1995). To appear in Canadian Applied Mathematics Quartely.

[10] Abdelouhab, L., Bona, J.L., Felland, M. and Saut, J.-C., *Non-local models for nonlinear, dispersive waves*, Physica D **40** (1989), 360-392.

[11] Benjamin, T.B., Bona, J.L. and Mahony, J.J., *Model equations for long waves in nonlinear dispersive systems*, Phil. Trans. Royal Soc. London A **272** (1972), 47-78.

[12] Bona, J.L. and Smith, R., *The initial-value problem for the Korteweg-de Vries equation*, Phil. Trans. R. Soc. London A **278** (1975), 555-604.

[13] Kato, T., *Quasi-linear equations of evolution with applications to partial differential equations*, Lectures Notes in Math **448** (1975), 25-70.

[14] Kato, T., *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Studies in Applied Mathematics, Advances in Mathematics Supplementary Studies **8** (1983), 93-128.

[15] Smith, R., *Nonlinear Kelvin and continental-shelf waves*, J. Fluid Mech. **57** (1972), 379-391.

[16] Iorio, R.J., *KdV, BO and friends in weighted sobolev spaces*, Lecture Notes in Math. **1450** (1990), 104-121

J. L. Bona
Department of Mathematics
Pennsylvania State University
University Park, PA 16802
USA

J. L. Bona
Applied Research Laboratory
Pennsylvania State University
University Park, PA 16802
USA

M. Scialom
Departamento de Matemática
Universidade Estadual de Campinas
13081-970, Campinas, SP
Brasil