The Initial - Value Problem for the Forced Korteweg-de Vries Equation *

Jerry L. Bona
Department of Mathematics and Computational & Applied Mathematics Program
University of Texas
Austin, TX 78712

Bing-Yu Zhang
Department of Mathematical Sciences
University of Cincinnati
Cincinnati, Ohio 45221-0025

Abstract

The initial-value problem for the Korteweg-de Vries equation with a forcing term has recently gained prominence as a model for a number of interesting physical situations. At the same time, the modern theory for the initial-value problem for the unforced Korteweg-de Vries equation has taken great strides forward. The mathematical theory pertaining to the forced equation is currently set in narrow function classes and has not kept up with recent advances for the homogeneous equation. This aspect is rectified here with the development of a theory for the initial-value problem for the forced Korteweg-de Vries equation that entails weak assumptions on both the initial wave configuration and the forcing. The results obtained include analytic dependence of solutions on the auxiliary data and allows the external forcing to lie in function classes sufficiently large that a Dirac δ-function or its derivative is included. Analyticity is proved by an infinite-dimensional analog of Picard iteration. A consequence is that solutions may be approximated arbitrarily well on any bounded time interval by solving a finite number of linear initial-value problems.

*This work was supported in part by the National Science Foundation and the W. M. Keck Foundation. The paper was completed while the first author held a CNRS position at the Centre de Mathématiques et de Leurs Applications, Ecole Normale Superieure de Cachan and the second author held a post-doctoral position in industrial mathematics at the Institute for Mathematics and its Applications at the University of Minnesota. The authors thank G. Ponce for advice and encouragement at the outset of the present study.

1 Introduction

Considered herein is the initial-value problem (IVP) for the forced Korteweg-de Vries (KdV) equation

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x^3 u = f(x,t), \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

for $x,\ t \in R$. Here, the dependent variable $u = u(x,t)$ is a real-valued function of the independent variables x and t that in most situations where the equation appears as a model, are related to distance measured in the direction of the waves propagation and elapsed time, respectively. The initial value u_0 and external forcing f will be suitably restricted presently.

The problem (1.1) arises naturally in situations close to those that lead to the Korteweg-de Vries equation as an approximate model, but which feature suitably small non-homogeneities. For example, the IVP (1.1) has arisen in studying wave motion over a flat, horizontal bottom that has a localized perturbation (see Cole [9] or Grimshaw and Smyth [14]) and in attempting to determine the response in a channel to a surface disturbance moving into undisturbed fluid (cf. Akylas [1], Lee [23] and Wu [28]). The forcing term f may also be thought of as providing a rough accounting of terms that are neglected in arriving at the tidy KdV equation (1.2) below.

Interest in (1.1) lagged behind that associated with the IVP

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x^3 u = 0, & x,\ t \in R, \\ u(x,0) = u_0(x), & x \in R, \end{cases} \tag{1.2}$$

for the KdV equation. The intensive investigation of (1.2) by an army of scientists was sparked in large measure by the inverse-scattering theory pertaining thereto. So far, no effective means has been found to bring this fruitful theory to bear upon the forced KdV equation (1.1). Indeed, this has remained a major open problem since the 1970's (see Miura [24]).

The well-posedness of the IVP (1.2) in the classical, L_2-based Sobolev spaces $H^s(R)$ for $s > 3/2$ was well established in the mid-1970's (cf. Bona and Smith [3], Bona and Scott [4], Kato [15], [16], Saut and Temam [26] and the references contained therein). In the early 1980's Kato [17] discovered a subtle and rather general smoothing effect for the IVP (1.2). While this effect had been apparent in the early work of Cohen [8] (see also Sachs [25]), the simplicity and power of Kato's observations inspired new consideration of the IVP (1.2) by Constantin and Saut [10], Ginibre and Velo [12] and Kenig, Ponce and Vega [18], [19], for example. This theory shows the IVP (1.2) to be locally (resp. globally) well-posed in H^s provided that $s > 3/4$ (resp. $s \geq 1$) [19], [20]. Recently, Bourgain [7] demonstrated (1.2) to be globally well posed in $H^0(R) = L^2(R)$ using a contraction-mapping argument in a very cleverly chosen space. Combining Bourgain's theory with their estimates in [20], Kenig, Ponce and Vega [21] showed shortly thereafter that (1.2) is locally well-posed in the space $H^s(R)$ provided only that $s > -5/8$.

By contrast, the theory pertaining to the IVP (1.1) for the forced KdV equation has remained less developed. The following result was given by Bona and Smith [3] in the early 1970's.

Theorem 1.1 *For given $T > 0$ and $s \geq 3$, if (i) $u_0 \in H^s(R)$, (ii) $f \in C(-T,T;H^s(R))$, and (iii) $f_t \in C(-T,T;L^2(R))$, then the IVP (1.1) has a unique solution $u \in C(-T,T;H^s(R)) \cap C^1(-T,T;L^2(R))$. In addition, the solution u depends continuously in $C(-T,T;H^s(R))$ on u_0 in $H^s(R)$ and f in $C(-T,T;H^s(R)) \cap C^1(-T,T;L^2(R))$.*

This result was strengthened recently by Zhang in [29] where he showed that the conclusion of Theorem 1.1 holds without assumption (iii).

It is our purpose here to bring the theory for the IVP (1.1) into the general range of what is known for the IVP (1.2). Four aspects of the IVP (1.1) will occupy us in the body of the paper. Use will be made throughout of the recent developments for the unforced problem (1.2) (cf. [6], [7], [12], [13], [18], [19], [20], [21], [30], [31]). While in most aspects it is only required to adapt the tools available in previous works, the theory that emerges is very much more satisfactory than the earlier results quoted above.

Before going into a little more detail, it is convenient to discuss briefly our notational conventions.

Notation

In general, if X is a Banach space of functions of one or two variables, its norm will be denoted by $\|\cdot\|_X$ except for the abbreviations listed now. The norm for $L^2(R)$ will be written without decoration as simply $\|\cdot\|$ and the standard norm

$$\|g\|_s^2 = \int_{-T}^{T} (1+|\xi|^2)^s |\hat{g}(\xi)|^2 d\xi \tag{1.3}$$

for the L^2-based Sobolev space $H^s(R)$, s any fixed real number, will be written as indicated in (1.3). Here and elsewhere, a circumflex adorning a function of one or two real variables denotes that function's Fourier transform. In one instance, it will be useful to consider the Fourier transform of a function $g = g(x,t)$ in only the first variable x, and this will be indicated by the non-standard notation $\tilde{g} = \tilde{g}(\xi, t)$. As this partial transform only appears briefly, its notation should not cause confusion.

If X is a Banach space, $C(a,b;X)$ is the set of functions $u: [a,b] \to X$ which are continuous. This is a Banach space with the norm

$$\sup_{a\le t\le b} \|u(t)\|_X.$$

In case $a=-\infty$ or $b=\infty$, we will append a subscript b to connote that the mappings u are bounded. Thus $C_b(R;X)$ is the space of bounded, continuous mappings of R into X equipped with the norm just displayed. The collection $L^p(a,b;X)$ is defined similarly, as are the Sobolev classes $W^{k,p}(a,b;X)$ of functions whose first k derivatives lie in $L^p(a,b;X)$.

I. *Well-posedness of the IVP (1.1) in the space $H^s(R)$*

Our goal is to update the known results for the IVP (1.1) to the general level of Kenig, Ponce and Vega's recent work [21] on the IVP (1.2). Indeed, armed with the new tools introduced by Kenig, Ponce and Vega [18], [21] and Bourgain [7], we are able to show that

> *for given $T>0$ and $s>-5/8$, the IVP (1.1) is locally well-posed for initial data u_0 in the space $H^s(R)$ and forcing $f\in L^2(-T,T;H^s(R))$ ($f\in L^1(-T,T;H^s(R))$ if $s>3/4$).*

As a consequence of the above well-posedness result, the IVP (1.1) establishes a nonlinear map K_I from the space $H^s(R)\times L^2(-T,T;H^s(R))$ to the space $C(-T,T;H^s(R))$ by the specification $K_I(u_0,f):=u$, where u is the solution of the IVP (1.1) corresponding to the initial data u_0 and the forcing function f.

II. *Regularity of the map K_I.*

For the homogeneous KdV equation, the IVP (1.2) also defines a nonlinear map K_H from the space $H^s(R)$ to $C(-T,T;H^s(R))$. Bona and Smith [3], and Kato [15], [16] showed that K_H is continuous from the space $H^s(R)$ to the space $C(-T,T;H^s(R))$. Then, Saut and Temam [26] proved that K_H is Hölder continuous with exponent $\frac{1}{2}$ from the space $H^{s+1/2}(R)$ to the space $L^\infty(-T,T;H^s(R))$. These early results did not use smoothing properties of the equation. Much stronger regularity can be established by taking advantage of the various smoothing properties possessed by the KdV equation. Simply as a by-product of their contraction-principle approach to the IVP (1.2), Kenig, Ponce and Vega [20] showed that the map K_H is Lipschitz continuous from the space $H^s(R)$ to the space $C(-T,T;H^s(R))$. Zhang [30] then proved that the map K_H is infinitely many times Fréchet differentiable from the space $H^s(R)$ to the space $C(-T,T;H^s(R))$ and that for $\delta>0$ sufficiently small, the formal Taylor series expansion

$$K_H(\phi+h)=\sum_{n=0}^{\infty}\frac{K_H^{(n)}(\phi)[h^n]}{n!} \tag{1.4}$$

converges in $C(-T,T;H^s(R))$ uniformly for $\|h\|_s \leq \delta$, which is the same as saying that the map K_H is analytic from the space $H^s(R)$ to the space $C(-T,T;H^s(R))$. Here, $K_H^{(n)}(\phi)$ is the n-th derivative of K_H at ϕ, an n-multilinear map from the n-fold product of $H^s(R)$ to $C(-T,T;H^s(R))$.

We show in this paper that

> *the map K_I corresponding to the IVP (1.1) is analytic from the space $H^s(R) \times L^2(-T,T;H^s(R))$ to the space $C(-T,T;H^s(R))$.*

As a result, the solution u of the IVP (1.1) can be expanded as a Taylor series with respect to its initial data u_0 and the forcing function f. Since each term in the Taylor series may be obtained by solving a linearized KdV equation, any solution of the nonlinear IVP (1.1) can be written as a series of solutions of associated linear problems.

III. *Smoothing properties of the IVP (1.1).*

It is a standard issue arising in the study of inhomogeneous partial differential equations to determine whether solutions have higher regularity than the forcing term. For the IVP (1.1), the regularity of solutions $u(x,t)$ in the spatial variable x is usually the same as that of the forcing term $f(x,t)$. However, for the associated linear problem,

$$\partial_t u + \partial_x^3 u = f, \quad u(x,0) = 0,$$

standard semigroup theory shows that the solution u lies in $C(-T,T;H^{s+3}(R))$ if $f \in W^{1,1}(-T,T;H^s(R))$. The price paid for the extra spatial regularity is that f is required to have stronger regularity in the temporal variable t. We present here similar results for the nonlinear problem (1.1). Indeed, we shall be able to prove the following sort of theorem, stated here with zero initial data for simplicity.

> *For $u_0 = 0$ and $s > -5/8$, if the forcing term $f \in W^{1,2}(-T,T;H^s(R))$, then the solution $u \in C(-T,T;H^{s+3}(R))$ and if $s > -17/8$, then $f \in W^{\frac{1}{2},2}(-T,T;H^s(R))$ implies that $u \in C(-T,T;H^{s'}(R))$ for any $s' < s + 3/2$.*

As a particular example, the theory allows one to take a Dirac δ-function (or even the derivative of a δ-function) as a forcing function acting on the right-hand side of the KdV equation and conclude the corresponding solution u lies in $C(-T,T;H^s(R))$ for any $s < 5/2$ (for any $s < 0$).

Smoothing properties of the IVP (1.1) with respect to its initial data are also considered. The IVP (1.1) is shown to have the same smoothing properties as those proved by Kenig, Ponce and Vega [19] for the IVP (1.2).

IV. *Global existence of solutions in the space $H^s(R)$.*

It is familiar in nonlinear analysis that a global existence result for an initial-value problem usually follows from a local existence result and appropriate global *a priori* estimates. For the IVP (1.1), the needed global estimates when $s = 0$ or $s \geq 1$ can be established with the aid of forced versions of the conservation laws appertaining to the unforced KdV equation (1.2). Consequently, we are able to show that for any $u_0 \in H^s(R)$ and $f \in L^2(R; H^s(R))$, the corresponding solution of the IVP (1.1) exists globally in the space $H^s(R)$. (For $0 < s < 1$, bounds can be obtained by interpolation between $s = 0$ and $s = 1$, for example, but we leave this point aside in the present development.) However, when $-5/8 < s < 0$, the needed estimates in $H^s(R)$ are not available. The question arises naturally as to whether a solution of the IVP (1.1) exists for all time or blows up in a finite time in the space $H^s(R)$ when $-5/8 < s < 0$. This is an open question, even for the homogeneous IVP (1.2) (cf. [21]).

An interesting point that casts some light on this last mentioned issue follows from the analyticity of the map K_I. For any $s > -5/8$ and $T > 0$, let $\mathcal{D}_s^T$ be the collection of all $(u_0, f) \in H^s(R) \times L^2(R, H^s(R))$ for which the corresponding solution u of the IVP (1.1) exists in the whole interval $(-T, T)$.

> *For $-5/8 < s < 0$ and any $T > 0$, $\mathcal{D}_s^T$ is a dense open subset of $H^s(R) \times L^2(R, H^s(R))$.*

The paper is organized as follows. In section 2, useful linear estimates from [21] are briefly reviewed. Then consideration is given to the associated linear IVP

$$\begin{cases} \partial_t u + \partial_x(vu) + \partial_x^3 u = f, \\ u(x,0) = u_0(x), \end{cases} \tag{1.5}$$

for $x, t \in R$ where $v = v(x,t)$ is a given function. The well-posedness of the IVP (1.5) in the space $H^s(R)$ is established and estimates of the solution in terms of u_0 and f are provided. This theory is the basis for the demonstration of analyticity of the map K_I. In section 3, the well-posedness of the IVP (1.1) in the space $H^s(R)$ ($s > -5/8$) and the analyticity of the map K_I are established. Instead of dealing directly with the nonlinear system (1.1) itself, we first consider the infinite system of linear equations

$$\begin{cases} \partial_t y_1 + \partial_x(u y_1) + \partial_x^3 y_1 = h_f, \\ y_1(x,0) = h_{u_0}, \end{cases} \tag{1.6}$$

and, for $n \geq 2$,

$$\begin{cases} \partial_t y_n + \partial_x(u y_n) + \partial_x^3 y_n = -\frac{1}{2}\sum_{k=0}^{n-1} \begin{pmatrix} n \\ k \end{pmatrix} \partial_x(y_k y_{n-k}), \\ y_n(x,0) = 0, \end{cases} \tag{1.7}$$

where u is assumed to be a solution of the IVP (1.1) corresponding to the initial data u_0 and the forcing term f. According to the theory in section 2, the linear system (1.6)-(1.7) is solvable. It is then shown that

$$v = u + \sum_{k=1}^{\infty} \frac{y_k}{k!}$$

is a solution of the IVP (1.1) corresponding to the initial data $u_0 + h_{u_0}$ and the forcing function $f + h_f$ provided the size of h_{u_0} and h_f is small in a particular sense. The well-posedness of the IVP (1.1) and the analyticity of the map K_I follow as corollaries. In section 4, we discuss smoothing properties of the system (1.1), while section 5 provides global existence of solutions of the IVP (1.1) in $H^s(R)$-spaces. In section 6, theory is developed for the periodic IVP for the forced KdV equation, namely (1.1) where u_0 is chosen from appropriate classes of periodic functions. Results similar to those established for the IVP (1.1) posed on the entire real line R are obtained.

2 Linear Estimates

To begin, we introduce a special Sobolev-type space used by Kenig, Ponce and Vega in [21], which is a modified version of the space first introduced by Bourgain in [7].

For any $s, b \in R$, let $Y_{s,b}$ be the completion of the space $S(R^2)$ of tempered test functions with respect to the norm

$$\|f\|^2_{Y_{s,b}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 + |\tau - \xi^3|)^{2b} (1 + |\xi|)^{2s} |\hat{f}(\xi, \tau)|^2 d\xi d\tau,$$

where $\hat{f}(\xi, \tau)$ denotes the Fourier transform of $f(x,t)$. As shown in [21], if $u \in Y_{s,b}$ with $s > -1$ and $b > 1/2$, one has

$$u \in C^{\alpha}_{loc}(R; L^2_t(R))$$

for any $\alpha \in (0, 1+s)$, and consequently

$$u \in L^p_{x,loc}(R; L^2_t(R)),$$

for $1 \leq p \leq \infty$.

Let $\{W(t)\}^{+\infty}_{-\infty}$ denote the unitary group generated by the operator

$$Af = -f'''$$

in the space $L^2(R)$. Suppose that $\psi \in C_0^\infty(R)$ with $supp\,\psi \subset (-1,1)$ and $\psi(x) = 1$ for every $x \in [-1/2, 1/2]$.

The following lemmas are results or simple corrolaries of results in [21].

Lemma 2.1 *Let there be given $s > -5/8$ and $\sigma \in C_0^\infty(R)$ with its support in the interval $(-1,1)$. Then there exists a $\beta_0 \in (1/2, 1)$ such that for any $b \in (1/2, \beta_0)$, there is a $\theta_0 > (2b-1)/2$ for which*

$$\|\sigma^2(\delta^{-1}t)\partial_x(uv)\|_{Y_{s,b-1}} \le c\delta^{\theta_0}\|u\|_{Y_{s,b}}\|v\|_{Y_{s,b}} \tag{2.1}$$

for any u, $v \in Y_{s,b}$ and $\delta \in (0,1)$.

Lemma 2.2 *Let s, σ and b be as in Lemma 2.1. Then, for any u, $v \in Y_{s,b}$ and $T > 0$, there is a $c_1 = c_1(T)$ such that*

$$\|\sigma(T^{-1}t)\partial_x(uv)\|_{Y_{s,b-1}} \le c_1\|u\|_{Y_{s,b}}\|v\|_{Y_{s,b}}. \tag{2.2}$$

Lemma 2.3 *Let $b > 1/2$ and $s \in R$ be given. Then $Y_{s,b} \subset C_b(R; H^s(R))$ and there is a constant $c > 0$ such that for any $f \in Y_{s,b}$,*

$$\sup_{t\in R}\|f(\cdot,t)\|_s \le c\|f\|_{Y_{s,b}}.$$

Lemma 2.4 *For given $s \in R$ and $b \in (1/2, 1]$, there is a constant c such that*

$$\|\psi(\delta^{-1}t)W(t)u_0\|_{Y_{s,b}} \le c\delta^{(1-2b)/2}\|u_0\|_s \tag{2.3}$$

and

$$\|\psi(\delta^{-1}t)\int_0^t W(t-\tau)f(\tau)d\tau\|_{Y_{s,b}} \le c\delta^{(1-2b)/2}\|f\|_{Y_{s,b-1}} \tag{2.4}$$

for any $\delta \in (0,1)$.

Remark 2.5 Combining (2.1) and (2.4), one has

$$\|\psi(\delta^{-1}t)\int_0^t W(t-\tau)\sigma^2(\tau/\delta)(\partial_x(uv))(\cdot,\tau)d\tau\|_{Y_{s,b}} \le c\delta^{\theta_0}\|u\|_{Y_{s,b}}\|v\|_{Y_{s,b}}. \tag{2.5}$$

This comprises a global smoothing property of the linear KdV equation.

Attention is now turned to the linear problem

$$\begin{cases} \partial_t u + \partial_x(vu) + \partial_x^3 u = f, \\ u(x,0) = u_0(x), \end{cases} \tag{2.6}$$

for $x, t \in R$, where $v = v(x,t)$ is a given function.

Theorem 2.6 *Suppose to be given $s > -5/8$, $T > 0$, b as in Lemma 2.1 with $1/2 < b \le 1$ and $v \in Y_{s,b}$. Then for any $u_0 \in H^s(R)$ and $f \in Y_{s,b-1}$, the IVP (2.6) has a unique solution $u \in C(-T,T; H^s(R))$ which is the restriction to $(-T,T)$ of a function $\bar{u} \in Y_{s,b}$ that satisfies the estimate*

$$\|\bar{u}\|_{Y_{s,b}} \le c_1\left(\|u_0\|_s + \|f\|_{Y_{s,b-1}}\right), \tag{2.7}$$

where $c_1 = c_1(T, \|v\|_{Y_{s,b}})$.

Proof: We attend first to the local existence of a solution using Kenig, Ponce and Vega's contraction-mapping argument [20]. First rewrite (2.6) in its equivalent integral-equation form

$$u(\cdot,t) = W(t)u_0(\cdot) + \int_0^t W(t-\tau)f(\cdot,\tau)d\tau - \int_0^t W(t-\tau)\partial_x(uv)(\cdot,\tau)d\tau.$$

For the given auxiliary data (u_0, f) and $\delta \in (0,1)$, define a map Γ of the space $Y_{s,b}$ as follows:

$$\begin{aligned}\Gamma(w)(\cdot) &= \psi(\delta^{-1}t)W(t)u_0(\cdot) + \psi(\delta^{-1}t)\int_0^t W(t-\tau)f(\cdot,\tau)d\tau - \\ &\quad \psi(\delta^{-1}t)\int_0^t W(t-\tau)\sigma^2(\delta^{-1}\tau)\partial_x(vw)(\cdot,\tau)d\tau,\end{aligned}$$

for any $w \in Y_{s,b}$, where $\sigma \in C_0^\infty(R)$ with $\sigma = 1$ on the support of ψ, and the support of $\sigma \subset (-1,1)$. Using the preceding lemmas, one sees as in [21] that if δ is chosen such that

$$2c\delta^{\theta_0+\frac{1-2b}{2}}\|v\|_{Y_{s,b}} \le 1, \tag{2.8}$$

then the map Γ is a contraction in the ball

$$H_M = \{w \in Y_{s,b};\ \|w\|_{Y_{s,b}} \le M\},$$

where

$$M = 2c\delta^{(1-2b)/2}(\|u_0\|_s + \|f\|_{Y_{s,b-1}}). \tag{2.9}$$

As a consequence, there exists a $u \in H_M$ such that

$$\begin{aligned}u(\cdot,t) &= \psi(\delta^{-1}t)W(t)u_0(\cdot) + \psi(\delta^{-1}t)\int_0^t W(t-\tau)f(\cdot,\tau)d\tau \\ &\quad -\psi(\delta^{-1}t)\int_0^t W(t-\tau)\sigma^2(\delta^{-1}\tau)\partial_x(uv)(\cdot,\tau)d\tau.\end{aligned}$$

In particular, one has

$$u(\cdot,t) = W(t)u_0(\cdot) + \int_0^t W(t-\tau)f(\cdot,\tau)d\tau - \int_0^t W(t-\tau)\partial_x(uv)(\cdot,\tau)d\tau$$

for $-\delta/2 \le t \le \delta/2$. Hence $u(x,t)$ is a solution of (2.6) for $-\delta/2 \le t \le \delta/2$ and in this range of t, satisfies

$$\|u(\cdot,t)\|_{Y_{s,b}} \le 2c\delta^{(1-2b)/2}(\|u_0\|_s + \|f\|_{Y_{s,b-1}}). \tag{2.10}$$

This local solution is easily extended to the entire interval $(-T,T)$. Indeed, because the time of existence δ depends only on v and the solution u possesses the bound (2.10) in $Y_{s,b}$, a straightforward iteration of the contraction-mapping argument starting with $u(\cdot,t')$ as initial data at successively larger values of $|t'|$ allows one to conclude in a finite number of steps existence and uniqueness on $(-T,T)$. The function u obtained by pasting these local solutions together plainly lies in $C(-T,T;H^s(R))$. That it is the restriction to $(-T,T)$ of a function in $Y_{s,b}$ follows by writing it as a finite sum of elements of $Y_{s,b}$ using a partition of unity of $[-T,T]$ whose supports are based on the local patches of solution obtained via the contraction-mapping principle. The bound in (2.7) is obtained by applying (2.10) to each summand in the partition-of-unity representation of u. The proof is complete. □

For any given $T > 0$, $s \in R$ and $b > 1/2$, define

$$w^* = \{v \in Y_{s,b} : \; v(\cdot,t) = w(\cdot,t), \; \forall t \in (-T,T)\}$$

and

$$Y_{s,b}^T = \{w^*, \; w \in Y_{s,b}\}.$$

The space $Y_{s,b}^T$ is a Banach space equipped with the quotient norm

$$\|w^*\|_{Y_{s,b}^T} := \inf_{v \in w^*} \|v\|_{Y_{s,b}}.$$

According to the definition, $v \in Y_{s,b}^T$ is a family of elements in the space $Y_{s,b}$. It will occasionally be convenient to ignore the distinction between the equivalence class $v \in Y_{s,b}^T$ and a representative of this class. As long as values of $t \in (-T,T)$ are in question, this abuse causes no difficulty.

Remark 2.7 In the above notation, and keeping in mind the convention concerning equivalence classes and their representatives, Theorem 2.6 may be restated as follows.

Let $s > -5/8$, $T > 0$ and $v \in Y_{s,b}$. Let b in the range $(1/2, 1]$ be chosen as in Lemma 2.1. Then for any $u_0 \in H^s(R)$ and $f \in Y_{s,b-1}$, there is a unique $u \in Y_{s,b}^T$ which is a solution of (2.6) in the time interval $(-T,T)$. Moreover, one has

$$\|u\|_{Y_{s,b}^T} \le c_1 \left(\|u_0\|_s + \|f\|_{Y_{s,b-1}} \right), \tag{2.11}$$

where $c_1 = c_1(T, \|v\|_{Y_{s,b}})$ may tend to $+\infty$ if $T \to +\infty$ or $\|v\|_{Y_{s,b}} \to +\infty$.

3 Well-posedness and Analyticity

Throughout this section it is assumed that $s > -5/8$ and $b > 1/2$ are as in Lemma 2.1. Define the product space $X_{s,b}$ to be

$$X_{s,b} := H^s(R) \times Y_{s,b-1}.$$

It follows from Theorem 2.6 that for given $T > 0$ and $(u_0, f) \in X_{s,b}$, there corresponds at most one $u \in Y_{s,b}^T$ which is a solution of the nonlinear IVP (1.1) in the time interval

$(-T,T)$. Thus solving the IVP (1.1) defines a map K_I from $X_{s,b}$ to $Y^T_{s,b}$ given by $u = K_I(\phi)$, where $\phi = (u_0, f) \in X_{s,b}$ and u is the corresponding solution of the IVP (1.1) if it exists.

Let $\mathcal{D}^T_s = \mathcal{D}^T_s(K_I)$ denote the domain of the map K_I in the space $X_{s,b}$. Obviously, $\mathcal{D}^T_s$ is not empty since $0 \in \mathcal{D}^T_s$. We show that $\mathcal{D}^T_s$ is an open set in the space $X_{s,b}$ and that the nonlinear map K_I is analytic from $\mathcal{D}^T_s$ to $Y^T_{s,b}$.

Formally, if K_I is an analytic mapping from $\mathcal{D}^T_s$ to $Y^T_{s,b}$, then, for $n = 0, 1, 2, \cdots$, its n-th order Fréchet derivative $K_I^{(n)}(\phi)$ at $\phi \in \mathcal{D}^T_s$ exists and is the symmetric, n-linear map from $X_{s,b}$ to $Y^T_{s,b}$ given as

$$K_I^{(n)}(\phi)[h_1, ..., h_n] = \left\{ \frac{\partial^n}{\partial \xi_1 ... \partial \xi_n} K_I(\phi + \sum_{k=1}^n \xi_k h_k) \right\}_{0,...,0}$$

for any $h_1, h_2, ..., h_n \in X_{s,b}$. The homogeneous polynomial $K_I^{(n)}(\phi)[h^n]$ of degree n induced by $K_I^{(n)}(\phi)$, where $h^n = (h, h, ..., h)$ (n-components), is

$$K_I^{(n)}(\phi)[h^n] = \left\{ \frac{d^n}{d\xi^n} K_I(\phi + \xi h) \right\}_{\xi=0}$$

for $h = (h_{u_0}, h_f) \in X_{s,b}$. If we define y_n by

$$y_n = K_I^{(n)}(\phi)[h^n],$$

then it is formally ascertained that for $-T < t < T$,

$$\begin{cases} \partial_t y_1 + \partial_x(u y_1) + \partial_x^3 y_1 = h_f, \\ y_1(x, 0) = h_{u_0}, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \partial_t y_n + \partial_x(u y_n) + \partial_x^3 y_n = -\frac{1}{2} \sum_{k=0}^{n-1} \binom{n}{k} \partial_x(y_k y_{n-k}), \\ y_n(x, 0) = 0, \end{cases} \tag{3.2}$$

for $n \geq 2$, where $u = K_I(\phi)$ and $h = (h_{u_0}, h_f) \in X_{s,b}$.

On the other hand, for any $\phi = (u_0, f) \in \mathcal{D}^T_s$, let $u = K_I(\phi)$ and consider solving the linear systems

$$\begin{cases} \partial_t y_1 + \partial_x(u y_1) + \partial_x^3 y_1 = h_f, \\ y_1(x, 0) = h_{u_0}, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \partial_t y_n + \partial_x(u y_n) + \partial_x^3 y_n = -\frac{1}{2} \psi(\frac{t}{2T}) \sum_{k=0}^{n-1} \binom{n}{k} \partial_x(y_k y_{n-k}), \\ y_n(x, 0) = 0, \end{cases} \tag{3.4}$$

for $n \geq 2$, where $h = (h_{u_0}, h_f) \in X_{s,b}$ and ψ is as in the previous section. It follows from Theorem 2.6 that (3.3)-(3.4) defines a homogeneous polynomial of degree n from $X_{s,b}$ to $Y_{s,b}^T$ as described by the following proposition.

Proposition 3.1 *Let $T > 0$ and $\phi \in \mathcal{D}_s^T = \mathcal{D}_s^T(K_I)$ be given and let $u = K_I(\phi)$. Then (3.3)-(3.4) defines a homogeneous polynomial $K_I^{(n)}(\phi)[h^n]$ of degree n from $X_{s,b}$ to $Y_{s,b}^T$. Moreover, there exists a constant c_3 such that*

$$\|y_n\|_{Y_{s,b}^T} \leq c_3^n n! \|h\|_{X_{s,b}}^n \tag{3.5}$$

for any $n \geq 2$, where $c_3 = c_3(T, \|u\|_{Y_{s,b}^T})$, and it may be that $c_3 \to +\infty$ as $T \to +\infty$ or $\|u\|_{Y_{s,b}^T} \to +\infty$, but in any case $c_3 \to 0$ if $T \to 0$.

Proof: The proof is a straightforward consequence of Lemma 2.1 – Lemma 2.4 (cf. [30, Prop. 3.3]. □

Define a Taylor polynomial $P_n(h)$ of degree n, for $h \in X_{s,b}$, by

$$P_n(h) = \sum_{k=0}^{n} \frac{K_I^{(k)}(\phi)[h^k]}{k!} = K_I(\phi) + \sum_{k=1}^{n} \frac{y_k}{k!}, \tag{3.6}$$

and a Taylor series by

$$P(h) = \sum_{k=0}^{\infty} \frac{K_I^{(k)}(\phi)[h^k]}{k!}. \tag{3.7}$$

Proposition 3.2 *For any $\phi = (u_0, f) \in \mathcal{D}_s^T = \mathcal{D}_s^T(K_I)$, there exists an $\eta > 0$ depending only on $\|K_I(\phi)\|_{Y_{s,b}^T}$ such that the formal Taylor series (3.7) is uniformly convergent in the space $Y_{s,b}^T$ with respect to $h \in X_{s,b}$ with $\|h\|_{X_{s,b}} \leq \eta$. Moreover, if $v = P(h)$, then $v \in Y_{s,b}^T$ solves the IVP*

$$\begin{cases} \partial_t v + v\partial_x v + \partial_x^3 v = f + h_f, \\ v(x,0) = u_0 + h_{u_0} \end{cases} \tag{3.8}$$

for $-T \leq t \leq T$.

Proof: It is readily seen that the sequence $\{P_n(h)\}_{n=0}^{\infty}$ of Taylor polynomials is Cauchy in $Y_{s,b}^T$ uniformly for h in the ball of radius η in $X_{s,b}$ for suitable η. Indeed, because of Proposition 3.1, it transpires that, for $m \geq n \geq 0$,

$$\begin{aligned} \|P_n(h) - P_m(h)\|_{Y_{s,b}^T} &= \|\sum_{k=n}^{m} \frac{y_k}{k!}\|_{Y_{s,b}^T} \\ &\leq \sum_{k=n}^{m} \frac{\|y_k\|_{Y_{s,b}^T}}{k!} \\ &\leq \sum_{k=n}^{m} c_3^k \|h\|_{X_{s,b}}^k. \end{aligned}$$

If η is chosen so that

$$\eta \leq 1/(2c_3), \tag{3.9}$$

then for $h \in X_{s,b}$ with $\|h\|_{X_{s,b}} \leq \eta$, one has

$$\|P_n(h) - P_m(h)\|_{Y^T_{s,b}} \leq \sum_{k=n}^{m} \frac{1}{2^k}$$

which goes to zero uniformly as $n,\, m \to \infty$.

Since $\{P_n(h)\}_{n=0}^{\infty}$ is a Cauchy sequence in the space $Y^T_{s,b}$, it makes sense to define $v = P(h)$ as its limit as $n \to \infty$. Then $v \in Y^T_{s,b}$ and v solves the IVP (3.8). To see this, note first that

$$\begin{aligned} v(x,0) &= \sum_{k=0}^{\infty} \frac{y_k(x,0)}{k!} \\ &= u(x,0) + y_1(x,0) \\ &= u_0(x) + h_{u_0}(x). \end{aligned}$$

Moreover, since the series $P(h)$ is absolutely convergent in the space $Y^T_{s,b} \subset C(-T,T;H^s(R))$, it follows that

$$\begin{aligned} v^2 &= \left(u + \sum_{k=1}^{\infty} \frac{y_k}{k!}\right)^2 \\ &= u^2 + 2\sum_{k=1}^{\infty} \frac{u y_k}{k!} + \left(\sum_{k=1}^{\infty} \frac{y_k}{k!}\right)^2 \\ &= 2\left(\frac{1}{2}u^2 + \sum_{k=1}^{\infty} \frac{u y_k}{k!} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=0}^{k-1} \binom{k}{n} y_n y_{n-k}\right). \end{aligned}$$

In consequence, we have

$$\begin{aligned} \partial_t v + \frac{1}{2}\partial_x(v^2) + \partial_x^3 v &= \partial_t u + \sum_{k=1}^{\infty} \frac{\partial_t y_k}{k!} + \partial_x^3 u + \\ &+ \sum_{k=1}^{\infty} \frac{\partial_x^3 y_k}{k!} + \frac{1}{2}\partial_x(u^2) + \sum_{k=1}^{\infty} \left\{ \frac{\partial_x(u y_k)}{k!} + \frac{1}{2k!} \sum_{n=0}^{k-1} \binom{k}{n} \partial_x(y_n y_{n-k}) \right\} \\ &= \left(\partial_t u + \frac{1}{2}\partial_x(u^2) + \partial_x^3 u\right) + \left(\partial_t y_1 + \partial_x(u y_1) + \partial_x^3 y_1\right) + \\ &+ \sum_{k=2}^{\infty} \frac{1}{k!} \left\{ \partial_t y_k + \partial_x(u y_k) + \frac{1}{2} \sum_{n=0}^{k-1} \binom{k}{n} \partial_x(y_n y_{n-k}) + \partial_x^3 y_k \right\} \\ &= f + h_f. \end{aligned}$$

The proof is complete. □

The following theorem is now readily adduced.

Theorem 3.3 (Analyticity) *Let $s > -5/8$ be given and let $b > 1/2$ be as in Lemma 2.1. For any $T > 0$, the IVP (1.1) establishes a map K_I from the space $X_{s,b}$ to the space $Y^T_{s,b}$ having as its domain $\mathcal{D}^T_s = \mathcal{D}^T_s(K_I)$ a nonempty open subset of $X_{s,b}$. The map K_I is analytic from $\mathcal{D}^T_s$ to $Y^T_{s,b}$ in the sense that for any $\phi \in \mathcal{D}^T_s$, there exists an $\eta > 0$ such that for any $h \in X_{s,b}$ with $\|h\|_{X_{s,b}} \leq \eta$, the Taylor series expansion*

$$K_I(\phi + h) = \sum_{n=0}^{\infty} \frac{K_I^{(n)}(\phi)[h^n]}{n!}$$

converges in the space $Y^T_{s,b}$. Moreover, the convergence is uniform with regard to h in the aforementioned ball in $X_{s,b}$.

In particular, since $0 \in \mathcal{D}^T_s$, there exists an $\eta > 0$ depending on T, such that for any $\phi = (u_0, f) \in X_{s,b}$ with $\|\phi\|_{X_{s,b}} \leq \eta$, the IVP (1.1) has a unique solution $u \in Y^T_{s,b}$ defined at least in the time interval $(-T, T)$. Moreover, according to (3.9) and Proposition 3.1, $\eta \to \infty$ as $T \to 0$. The local well-posedness of the IVP (1.1) thus follows as a corollary to Theorem 3.3.

Theorem 3.4 (**Local well-posedness**) *For any $u_0 \in H^s(R)$ and $f \in Y_{s,b-1}$, there exists a $T = T(\|u_0\|_s, \|f\|_{Y_{s,b-1}})$ and a unique $u \in Y^T_{s,b}$ which is a solution of the IVP (1.1) on the time interval $(-T, T)$ and which satisfies*

$$\|u\|_{Y^T_{s,b}} \leq c(\|u_0\|_s + \|f\|_{Y_{s,b-1}})$$

for some constant $c = c(\|u_0\|_s, \|f\|_{Y_{s,b-1}}) > 0$. Moreover, for any $T' < T$, there exists a neighborhood U of (u_0, f) in the space $X_{s,b}$ such that the map K_I is analytic from U to $Y^{T'}_{s,b}$.

Remark 3.5 The approach to the IVP (1.1) taken here is to demonstrate the analyticity of the map K_I by establishing the solvability of the n-linear system (3.1)-(3-2). The well-posedness of the IVP (1.1) follows as a corollary. An interesting aspect of this approach is that it shows the solution of the nonlinear IVP (1.1) can be obtained by solving a series of linear problems. Another interesting point emerging from the theory is the fact that for given T and $s > -5/8$, the domain $\mathcal{D}^T_s$ of the map K_I is a non-empty open subset of the space $X_{s,b}$. This is potentially useful information in the study of the global well-posedness of the IVP (1.1) as will be seen in section 5.

The well-posedness of the IVP (1.1) can also be established using the same contraction-mapping argument that Kenig, Ponce and Vega developed in [21]. Moreover, as pointed out to us by G. Ponce, [1] the analyticity of the map K_I can be

[1] Private communication

established by using the implicit-function theorem. In outline, the argument would proceed as follows.

Let u be the solution in $Y^T_{s,b}$ corresponding to $(u_0, f) \in H^s(R) \times Y^T_{s,b-1}$. Consider neighborhoods U_u, V_{u_0}, D_f of u, u_0, f in $Y^T_{s,b}$, $H^s(R)$, $Y^T_{s,b-1}$, respectively. Define a map F from $U_u \times V_{u_0} \times D_f$ to $Y^T_{s,b}$ by

$$F(v, v_0, f) = v - W(t)v_0 - \int_0^t W(t-\tau)(vv_x - f)d\tau.$$

It is clear that $F(u, u_0, f) = 0$ and that

$$\frac{\partial F}{\partial u}(u, u_0, f)v = v - \int_0^t W(t-\tau)(uv_x + vu_x)d\tau.$$

Thus $\frac{\partial F}{\partial u}(u, u_0, f)$ is a map from $Y^T_{s,b}$ to $Y^T_{s,b}$ which may be written as

$$\frac{\partial F}{\partial u}(u, u_0, f) = I - G$$

where I is the identity map and G is the linear map from $Y^T_{s,b}$ to $Y^T_{s,b}$ defined by

$$Gv = \int_0^t W(t-\tau)(uv)_x d\tau.$$

If T is chosen appropriately small so that the operator norm of G is strictly less than one, then it follows from elementary considerations that the map $\frac{\partial F}{\partial u}(u, u_0, f)$ is invertible. There thus exists $g: \tilde{V}_{u_0} \times \tilde{D}_f \subset V_{u_0} \times D_f \to Y^T_{s,b}$ such that

$$F(g(\bar{u}_0, \bar{f}), \bar{u}_0, \bar{f}) \equiv 0, \quad \text{for all } (\bar{u}, \bar{f}) \in \tilde{V}_{u_0} \times \tilde{D}_f. \tag{3.10}$$

Since the map F depends analytically on its variables, the map g also depends analytically on the variables $\bar{u}_0$ and $\bar{f}$.

4 Smoothing Properties

Consideration is first given to smoothing properties of the IVP (1.1) with regard to the forcing term f.

Lemma 4.1 *For any $s, b \in R$, the embedding*

$$W^{1-b,2}(R, H^{s-3(1-b)}(R)) \subset Y_{s,b-1}$$

is continuous.

Proof: If $v = \partial_t u$, then v is a solution of the IVP

$$\begin{cases} \partial_t v + \partial_x(uv) + \partial_x^3 v = f_t, \\ v(x,0) = f(x,0) - u_0'''(x) - u_0(x)u_0'(x). \end{cases}$$

Because of the regularity of the initial data and the forcing term, it is deduced from the preceding theory that u and $\partial_t u$ lie in $Y_{s,b}^{T_1}$ for some $T_1 > 0$. In Fourier transformed variables, this means

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|\xi|)^{2s}(1+|\tau|)^2(1+|\tau-\xi^3|)^{2b}|\hat{u}(\xi,\tau)|^2 d\xi d\tau \ < \infty.$$

Since $b < 1$, there is a constant c such that

$$(1+|\xi|)^{6b} \le c(1+|\tau-\xi^3|)^{2b}(1+|\tau|)^2.$$

Hence it follows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|\xi|)^{2(s+3b)}(1+|\tau|)^{1-b}|\hat{u}(\xi,\tau)|^2 d\xi d\tau$$

$$\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|\xi|)^{2s}(1+|\tau|)^2(1+|\tau-\xi^3|)^{2b}|\hat{u}(\xi,\tau)|^2 d\xi d\tau.$$

The right-hand side of the last inqeuality is bounded, so

$$u \in W^{1-b,2}(R; H^{s+3b}(R)),$$

which in turn implies

$$\partial_x u \in L^2(-T_1, T_1; H^{s+3b-1}(R))$$

and

$$u\partial_x u \in L^2(-T_1, T_1; H^s(R)).$$

Finally, it is seen by writing the equation in the form

$$u_{xxx} = f - uu_x - u_t$$

that

$$u \in C(-T_1, T_1; H^{s+3}(R)).$$

The proof is complete. □

Remark 4.5 If it is only assumed f, $f_t \in Y_{s,b-1}$ rather than $f \in W^{1,2}(-T,T;H^s(R))$, we still have

$$u \in W^{1-b,2}(R : H^{s+3b}(R)).$$

This follows immediately from the sort of calculation appearing in the proof of the last theorem.

Proof: Since

$$\begin{aligned}1+|\xi| &= 1+|\xi^3|^{1/3}\\ &\leq 1+(|\xi^3-\tau|+|\tau|)^{1/3}\\ &\leq c(1+|\xi^3-\tau|)^{1/3}(1+|\tau|)^{1/3}\end{aligned}$$

for some positive numerical constant c, it follows that for any $u \in W^{1-b,2}(R, H^{s-3(b-1)}(R))$,

$$\|u\|^2_{W^{1-b,2}(R,H^{s-3(1-b)}(R))} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|\xi|)^{2s}(1+|\tau-\xi^3|)^{2(b-1)}|u(\xi,\tau)|^2 d\xi d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|\xi|)^{2(s-3(1-b))}\frac{(1+|\tau|)^{2(1-b)}(1+|\xi|)^{6(1-b)}}{(1+|\tau|)^{2(1-b)}(1+|\tau-\xi^3|)^{2(1-b)}}|u(\xi,\tau)|^2 d\xi d\tau$$

$$\leq c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|\xi|)^{2(s-3(1-b))}(1+|\tau|)^{2(1-b)}|u(\xi,\tau)|^2 d\xi d\tau$$

$$= c\|u\|^2_{Y_{s,b-1}}.$$

The proof is complete. □

The next theorem now follows directly from Theorem 3.4.

Theorem 4.2 *Let $s > -5/8$ be given and $b > 1/2$ be as in Lemma 2.1. Then for any $u_0 \in H^s(R)$ and $f \in W^{1-b,2}(R, H^{s-3(1-b)}(R))$, there is a $T_1 > 0$ and a unique $u \in Y_{s,b}^{T_1}$ which defines a solution of the IVP (1.1) corresponding to the initial data u_0 and the forcing f.*

Note that the constant b in the above theorem can be chosen as close to $\frac{1}{2}$ as one likes. As a consequence, we have the following result.

Corollary 4.3 *Let $s > -17/8$ be given. For any $f \in W^{\frac{1}{2},2}(R, H^s(R))$ and $u_0 \in H^{s+\frac{3}{2}}(R)$, there exists a $T_1 > 0$ such that the corresponding solution u of the IVP (1.1) lies in the space $C(-T_1, T_1, H^{s'}(R))$ for any $s' < s + 3/2$.*

As Corollary 4.3 indicates, the price paid for a gain in spatial regularity of the solution of the IVP (1.1) is the assumption of more temporal regularity in the forcing function f. If one is willing to assume further temporal regularity, then the conclusion of the last result can be strengthened.

Theorem 4.4 *Let $T > 0$ and $s > -5/8$ be given and let $b > 1/2$ be chosen as in Lemma 2.1. Then for any $u_0 \in H^{s+3}(R)$ and $f \in W^{1,2}(-T, T; H^s(R))$, there exists a $T_1 > 0$ and a unique $u \in Y_{s,b}^{T_1}$ which is a solution of the IVP (1.1) over the time interval $(-T_1, T_1)$ where $T_1 \leq T$ depends on $\|u_0\|_{s+3}$ and $\|f\|_{Y_{s,b-1}}$. Moreover,*

$$\partial_t u \in Y_{s,b}^T \quad \textit{and} \quad u \in C(-T_1, T_1; H^{s+3}(R)). \tag{4.1}$$

Attention is now given to smoothing properties of the IVP (1.1) with regard to its initial data, which is the property that solutions may be more regular in their spatial variable than the initial data. Let $s > 0$ and $T > 0$ be given. For a function $w: R \times [-T,T] \to R$, define the quantities

$$\lambda_1(T,w) = \sup_{[-T,T]} \|w(\cdot,t)\|_s,$$

$$\lambda_2(T,w) = \left(\sup_x \int_{-T}^{T} |D^s \partial_x w(x,t)|^2 dt\right)^{1/2},$$

$$\lambda_3(T,w;l) = \left(\int_{-T}^{T} \|J^l \partial_x w(\cdot,t)\|_\infty^4 dt\right)^{1/4}$$

with $l \in [0, s-3/4]$ where $J^s = (1-\partial_x^2)^{s/2}$,

$$\lambda_4(T,w;\,r) = (1+T)^{-\rho}\left(\int_R \sup_{[-T,T]} |J^r w(x,t)|^2 dx\right)^{1/2}$$

with $r \in [0, s-3/4]$ and $\rho > 3/4$ a fixed constant, and

$$\Lambda_{l,r}^s(T;\,w) = \max\{\lambda_1(T,w),\ \lambda_2(T,w),\ \lambda_3(T,w;\,l),\ \lambda_4(T,w;\,r)\}.$$

Define also the function class $X_{r,l}^{T,s}$ by

$$X_{r,l}^{T,s} = \{w \in C(-T,T;H^s(R):\ \Lambda_{r,l}^s(T,w) < \infty\}$$

for $(r,l) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$. This linear space is a Banach space when equipped with the norm

$$\|w\|_{X_{r,l}^{T,s}} := \Lambda_{r,l}^s(T,w)$$

introduced by Kenig, Ponce and Vega [20]. Clearly, $X_{r,l}^{T,s}$ is a subspace of $C(-T,T;H^s(R))$ with a stronger topology. It has the following properties established in [20].

Proposition 4.6 *Let $T > 0$, $s > 3/4$ and $(l,r) \in [0,3/4] \times [0,3/4)$ be given. Then there is a constant c independent of u_0, f, u and v such that*

$$\|W(t)u_0\|_{X_{r,l}^{T,s}} \le c\|u_0\|_s, \tag{4.2}$$

$$\|\int_0^t W(t-\tau)f(\cdot,\tau)d\tau\|_{X_{r,l}^{T,s}} \le c(1+T)^\rho \int_{-T}^{T} \|f(\cdot,\tau)\|_s d\tau, \tag{4.3}$$

and

$$\int_{-T}^{T} \|\partial_x(uv)\|_s dt \le cT^{1/2}(1+T)^\rho \|u\|_{X_{0,0}^{T,s}} \|v\|_{X_{0,0}^{T,s}}, \tag{4.4}$$

where $\rho > 3/4$ is another constant that is independent of u and v.

The answer is affirmative if $s > 3/2$ and the proof is an easy exercise. The issue remains open when $s \leq 3/2$. There are many discussions in the literature about the uniqueness problem for the IVP (1.2) for the KdV equation (cf. [13] and [22]). But the results thus far extant require that either the solution u is in a stronger space than $C(-T,T;H^s(R))$ or the initial data decays at a certain rate as $x \to \pm\infty$. The uniqueness results appearing in Theorem 3.4 and Theorem 4.8 also require solutions in spaces stronger than $C(-T,T;H^s(R))$, namely $Y^T_{s,b}$ and $X^{T,s}_{r,l}$, respectively. On the other hand, these uniqueness results do imply uniqueness of the so-called strong solutions of the IVP (1.1). These are defined as follows.

Definition: *A function $u \in C(-T,T;H^s(R))$ is called a strong solution of the IVP (1.1) if there exists a sequence $\{u_m\}_{m=1}^{\infty}$ lying in $C^\infty(-T,T;H^\infty(R))$ such that*

$$\partial_t u_m + u_m \partial_x u_m + \partial_x^3 u_m = f_m$$

for $x \in R$, $t \in (-T,T)$,

$$\lim_{m\to\infty} \sup_{-T<t<T} \|u_m(\cdot,t) - u(\cdot,t)\|_s = 0,$$

and for which

$$\lim_{m\to\infty} \|f_m - f\|_{L^2(-T,T;H^s(R))} = 0.$$

Proposition 4.9 *Let $s > -5/8$ and $T > 0$ be given. Then the IVP (1.1) has at most one strong solution $u \in C(-T,T;H^s(R))$ for any given $u_0 \in H^s(R)$ and $f \in L^2(-T,T;H^s(R))$.*

Proof: It suffices to show that if $u \in C(-T,T;H^s(R))$ is a strong solution of the IVP (1.1), then $u \in Y^T_{s,b}$ since the IVP (1.1) has at most one solution in this space.

Let $\{u_m\}_{m=1}^{\infty}$ be a sequence of functions corresponding to the assumption that u is a strong solution of the IVP (1.1). Our theory shows that for all m, $u_m \in Y^T_{s,b}$. Note that f_m tends to f in the space $L^2(-T,T;H^s(R))$ and that $u_m(\cdot,0)$ tends to $u_0(\cdot)$ in the space $H^s(R)$. In addition, $\sup_{-T<t<T} \|u_m(\cdot,t)\|_s$ is bounded independently of m. It follows from the local well-posedness result in Theorem 4.8 that the sequence $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in the space $Y^T_{s,b}$ whose limit v is a solution of the IVP (1.1) corresponding to the given u_0 and f. Since u is a limit of $\{u_m\}$ in the space $C(-T,T;H^s(R))$ and the space $Y^T_{s,b}$ is a stronger subspace of $C(-T,T;H^s(R))$, we must have $u \equiv v$. □

Corollary 4.10 *Let $s > 3/4$ and $T > 0$ be given. Let u_1 be the unique solution of the IVP (1.1) in the space $X^{T,s}_{r,l}$ and u_2 the unique solution of the IVP (1.1) in the space $Y^T_{s,b}$. Then $u_1 \equiv u_2$ for $-T \leq t \leq T$.*

Remark 4.7 The estimate (4.2) reveals a stronger version of both the Kato smoothing effect and the Strichartz' global smoothing effect for the unitary group $W(t)$ (see [20]). The estimates (4.3) and (4.4) show a global smoothing property which is similar to (2.5).

In [20], Kenig, Ponce and Vega showed that for any $u_0 \in H^s(R)$ with $s > 3/4$, the IVP (1.2) for the homogeneous KdV equation has a unique solution $u \in X_{r,l}^{T,s}$. A similar result holds for the IVP (1.1) of the forced KdV equation. Indeed, the same arguments as those used in section 3 provide the following result.

Theorem 4.8 *Let $s > 3/4$, $T > 0$ and $(l, r) \in [0, s - 3/4] \times [0, s - 3/4)$ be given. Then for any $(u_0, f) \in H^s(R) \times L^1(-T, T; H^s(R))$, there are positive constants*

$$c = c(\|u_0\|_s, \|f\|_{L^1(-T,T;H^s(R))})$$

and

$$T_1 = T_1(\|u_0\|_s, \|f\|_{L^1(-T,T;H^s(R))}) \leq T,$$

and a unique solution $u \in X_{r,l}^{T_1,s}$ to the IVP (1.1) satisfying

$$\|u\|_{X_{r,l}^{T_1,s}} \leq c(\|u_0\|_s + \|f\|_{L^1(-T,T;H^s(R))}).$$

Moreover, for any $T' < T_1$, there exists a neighborhood U of (u_0, f) in the space $H^s(R) \times L^1(-T, T; H^s(R))$ such that the map K_I is analytic from U to the space $X_{l,r}^{T',s}$.

The space $L^2(-T, T; H^s(R))$ is a subspace of the space $L^1(-T, T; H^s(R))$ and the space $Y_{s,b-1}$. But the space $Y_{s,b}^T$ and the space $X_{r,l}^{T,s}$ are not related by one being included in the other. Neither are the spaces $L^1(-T, T; H^s(R))$ and $Y_{s,b-1}$ so related. This raises an interesting question. Suppose $u_0 \in H^s(R)$ and $f \in L^2(-T, T; H^s(R))$ with $s > 3/4$. Then Theorem 3.4 provides a solution $u_1 \in Y_{s,b}^T$ for the IVP (1.1) and Theorem 4.8 provides us another solution $u_2 \in X_{r,l}^{T,s}$. Are these two solutions the same? This question is related to the following uniqueness problem for the IVP (1.1).

Uniqueness Problem: *Suppose u_1, $u_2 \in C(-T, T; H^s(R))$ are both solutions of the equation*

$$\partial_t u + u\partial_x u + \partial_x^3 u = f$$

for some $f \in L^2(-T, T, H^s(R))$. Does the fact $u_1(\cdot, 0) = u_2(\cdot, 0)$ in the space $H^s(R)$ imply $u_1(\cdot, t) \equiv u_2(\cdot, t)$ in $H^s(R)$ for $t \in [-T, T]$?

Proof: It follows from Proposition 4.9 and the fact that both u_1 and u_2 are strong solutions of the IVP (1.1). □

In consequence of these developments, we have the following result.

Theorem 4.11 *Let $s > 3/4$ and $T > 0$ be given. For any $u_0 \in H^s(R)$ and $f \in L^2(-T,T;H^s(R))$, there exists a $T_1 = T_1(\|u_0\|_s, \|f\|_{L^2(-T,T;H^s(R))}) \leq T$ such that the IVP (1.1) has a unique strong solution $u \in C(-T_1,T_1;H^s(R))$. Moreover, the solution lies in $Y_{s,b}^{T_1}$ and $X_{l,r}^{T_1,s}$, where $b > 1/2$ is as in Lemma 2.1 and (l,r) is as in Proposition 4.6.*

5 Global Existence Results in the Space $H^s(R)$

The local well-posedness theory developed in the preceding sections leads naturally to consideration of whether or not the solutions can generally be continued in time, so becoming global solutions of the IVP. Of course, if the auxiliary data (u_0, f) is sufficiently regular, global solutions are known to obtain therefrom via the earlier theory [3], [29]. However, for weaker specification, it is possible that solutions might blow up in finite time, so ceasing to exist at a certain point. This issue arises pointedly when one is concerned with the IVP (1.1) as an approximate description of wave phenomena.

The following two criteria for whether or not a solution of the IVP (1.1) ceases to exist in finite time follow from Theorem 3.4 and 4.8, respectively, by a standard argument. Throughout this section, we continue to suppose that $b > 1/2$ is fixed in accordance with the requirement that Lemma 2.1 is valid.

Proposition 5.1 *Let $s > -5/8$ and $T > 0$ be given. Then for any $u_0 \in H^s(R)$ and $f \in Y_{s,b-1}$, there exists a maximal value T_1 with $0 < T_1 \leq T$ such that the IVP (1.1) has a unique solution $u \in Y_{s,b}^{T_2}$ for any $T_2 < T_1$. The maximal value $T_1 < T$ if and only if*

$$\lim_{t \to T_1^-} \|u(\cdot,t)\|_s = +\infty.$$

Proposition 5.2 *Let $s > 3/4$, $(l,r) \in [0, s-3/4] \times [0, s-3/4)$ and $T > 0$ be given. Then for any $(u_0, f) \in H^s(R) \times L^1(-T,T;H^s(R))$, there is a maximal value T_1 with $0 < T_1 \leq T$ such that for any $T_2 < T_1$, the IVP (1.1) has a unique solution $u \in X_{l,r}^{T_2,s}$ and $T_1 < T$ if and only if*

$$\lim_{t \to T_1^-} \|u(\cdot,t)\|_s = +\infty.$$

In consequence of these results, global *a priori* estimates for solutions of the IVP (1.1) in the space $H^s(R)$ suffice to infer the global well-posedness of the IVP (1.1). It is to the provision of such bounds that attention is now turned.

Lemma 5.3 *Let there be given $T > 0$ and $u \in C(-T,T;H^\infty(R))$ a solution of the IVP (1.1). Then the following inequalities appertain to u:*

$$\sup_{[-T,T]} \|u(\cdot,t)\| \leq c\left(\|u_0\| + \int_{-T}^{T} \|f(\cdot,t)\|dt\right), \tag{5.1}$$

$$\sup_{[-T,T]} \|u(\cdot,t)\|_1 \leq c\left(\|u_0\|^2 + \|u_0\|_1 + \int_{-T}^{T} \|f(\cdot,t)\|_1 dt\right), \tag{5.2}$$

and for any $s > 0$,

$$\sup_{[-T,T]} \|u(\cdot,t)\|_s \leq c_s exp\left(\int_{-T}^{T} \|\partial_x u(\cdot,\tau)\|_\infty d\tau\right)\left(\|u_0\|_s + \int_{-T}^{T} \|f(\cdot,t)\|_s dt\right), \tag{5.3}$$

where the numerical constants c and c_s are independent of u_0 and f.

Proof: The proof of the estimates (5.1) and (5.2) is standard. The estimate (5.3) follows from the argument used in the proof of Lemma 3.2 in [19]. □

Theorem 5.4 (Global well-posedness) *Let $s \geq 1$, $T > 0$ and $(l,r) \in [0,s-3/4] \times [0,s-3/4)$. Then, for any $(u_0,f) \in H^s(R) \times L^1(-T,T;H^s(R))$, the IVP (1.1) has a unique solution $u \in X_{r,l}^{T,s}$. Moreover, the corresponding map K_I is analytic from $H^s(R) \times L^1(-T.T;H^s(R))$ to the space $X_{r,l}^{T,s}$.*

Proof: This follows from the global *a priori* estimates (5.2), (5.3), Proposition 5.2 and Theorem 4.8. □

Theorem 5.5 (Global well-posedness) *Let $T > 0$ be given.*

(i) For any $(u_0,f) \in L^2(R) \times L^2(-T,T;L^2(R))$, the IVP (1.1) has a unique solution $u \in Y_{0,b}^T$ and the associated correspondence K_I is an analytic mapping between these spaces.

(ii) For any $s \geq 1$ and $(u_0,f) \in H^s(R) \times L^2(-T,T;H^s(R))$, the IVP (1.1) has a unique solution $u \in Y_{s,b}^T \cap X_{r,l}^{T,s}$ and in this case also K_I is an analytic mapping between these function classes.

Proof: In the case $s = 0$, the result follows directly from the *a priori* estimates (5.1) and Proposition 5.1 by a standard argument.

If $s \geq 1$, the IVP (1.1) has a solution $u \in X_{r,l}^{T,s}$ according to Theorem 5.4. In particular,

$$\sup_{[-T,T]} \|u(\cdot,t)\|_s < +\infty. \tag{5.4}$$

On the other hand, from Corollary 4.10,

$$u(\cdot,t) = v(\cdot,t) \quad \text{in the space } H^s(R), \text{ for } t \in (-T_1,T_1), \tag{5.5}$$

where v is the unique solution of the IVP (1.1) in the space $Y^{T_1}_{s,b}$ corresponding to (u_0, f) for some $T_1 > 0$. Because of the estimate (5.4) and Theorem 3.4, the solution v can be extended to the whole interval $(-T, T)$ in such a way that $v \in Y^T_{s,b}$ and (5.5) holds for $t \in (-T, T)$. The proof is complete. □,

The case wherein $-5/8 < s < 0$ is interesting since the needed global *a priori* estimates are not available, although the IVP (1.1) has been shown to be locally well-posed in the relevant $H^s(R)$-space. This raises the question mentioned before of whether the corresponding solutions blow up in finite time or exist globally in the space $H^s(R)$.

As an application of the analyticity of the map K_I, a partial answer to this problem can be provided. For $(u_0, f) \in X_{s,b}$, the corresponding solution u of the IVP (1.1) exists globally in the space $H^s(R)$ if and only if

$$(u_0, f) \in \mathcal{D}^T_s = \mathcal{D}^T_s(K_I), \quad \text{for any } T > 0.$$

Now we know that if $-5/8 < s \le 0$, then

$$L^2(R) \times L^2(R; L^2(R)) \subset \mathcal{D}^T_s$$

for any $T > 0$. In addition, $\mathcal{D}^T_s$ is a non-empty open subset of $X_{s,b}$ according to Theorem 3.1. The following theorem is then obvious.

Theorem 5.6 *Let $s > -5/8$ be given. Then, for any $T > 0$, $\mathcal{D}^T_s = \mathcal{D}^T_s(K_I)$ is a dense open subset of the space $X_{s,b}$.*

We close this section by giving still another global existence result.

Theorem 5.7 *Let $T > 0$ and $s \ge 0$ be given. For any*

$$u_0 \in H^{s+3}(R) \qquad \text{and} \qquad f \in W^{1,2}(-T, T; H^s(R)),$$

there exists a unique $u \in Y^T_{s,b}$ which is a solution of the IVP (1.1) in the time interval $(-T, T)$. Moreover,

$$\partial_t u \in Y^T_{s,b} \quad \text{and} \quad u \in C(-T, T; H^{s+3}(R)).$$

Proof: It follows from Theorem 5.5 ($s = 0$) and Theorem 2.6 that $u, u_t \in Y^T_{0,b}$. Then using the argument presented in the proof of Theorem 4.4, one has $u \in L^2(-T, T; H^{s+3}(R))$. In particular,

$$\int_{-T}^{T} \|\partial_x u(\cdot, t)\|_{L^\infty(R)} dt < \infty.$$

It then follows from (5.3) that

$$\sup_{[-T,T]} \|u(\cdot,t)\|_s < \infty$$

for any $s \geq 0$. As a result, $u \in Y^T_{s,b}$. Then Theorem 4.11 implies

$$\partial_t u \in Y^T_{s,b}, \quad u \in C(-T,T; H^{s+3}(R))$$

for any $T > 0$. The proof is complete. □

6 The Periodic Forced KdV Equation

In this final section of the paper, we analyze the periodic initial-value problem for the forced KdV equation. The situation in view assumes that both the initial data and the forcing function as regards its spatial variation are periodic of the same period, and focuses on solutions having the same periodicity in space. This problem is usually somewhat artificial as regards application to physically interesting situations. However, it frequently arises in numerical simulation where unbounded domains are hard to model and the relative simplicity of imposing periodic boundary conditions is very attractive. As a model of physical reality, the idea is usually that the initial disturbance and the forcing often take place far from boundaries, and hence the imposition of periodicity should not affect the solution significantly provided the period is large enough. According to the theory in [5], this approach is justified over certain time scales provided the initial data is suitably evanescent at $\pm\infty$.

Here, we treat the periodic IVP with a finite period, and leave aside the question of the relationship between the periodic IVP and the pure IVP. Because the period λ is finite and the equation is written in a frame of reference having no linear transport term $\partial_x u$, a simple change of variables allows us to assume $\lambda = 1$, and this convenient normalization will be adopted hereafter.

Let $H^s(S)$ denote the real Sobolev space of order s ($s \geq 0$) on the unit-length circle S in the plane. $H^s(S)$ may be characterized as the space of real 1-periodic functions v whose Fourier series

$$v(x) \sim \sum_{-\infty}^{\infty} v_k exp(2i\pi kx) \tag{6.1}$$

is such that

$$\|v\|_s = \left\{ \sum_{-\infty}^{\infty} (1+|k|)^{2s}|v_k|^2 \right\}^{\frac{1}{2}} < +\infty. \tag{6.2}$$

The left-hand side of (6.2) defines a Hilbert-space norm on the linear space $H^s(S)$. Let D^s represent the fractional derivative of order s, so if v has the Fourier series in

(6.1), then

$$D^s v \sim \sum_{-\infty}^{\infty} v_k |k|^s exp(2i\pi kx).$$

We consider the IVP for the forced KdV equation, namely

$$\partial_t u + u\partial_x u + \partial_x^3 u = f, \quad u(x,0) = u_0(x), \tag{6.3}$$

for $x \in S$, $t \in R$.

The IVP (6.3) is first normalized by subtracting the mean-value of a putative solution. For any integrable, real-valued function g defined on S, its mean value is denoted by $[g]$ and is given by

$$[g] = \int_S g(x)\, dx.$$

Let u be a solution of the IVP (6.3) and let $v = u - [u_0]$. Then v is a solution of the IVP

$$\begin{cases} \partial_t v + [u_0]\partial_x v + v\partial_x v + \partial_x^3 v = f, \\ v(x,0) = u_0(x) - [u_0]. \end{cases} \tag{6.4}$$

It is straightforward to see that $[v(\cdot,t)] \equiv 0$ provided that $[f(\cdot,t)] \equiv 0$.

Let $s \geq 0$ and $\beta \in R$ be given. For $w: S \times R \to R$, define $\Lambda_j^\beta(w)$, $j = 1,2,3$, to be

$$\Lambda_1^\beta(w) = \left(\sum_{n=-\infty}^{\infty} (1+|n|)^{2s} \int_{-\infty}^{\infty} (1 + |\tau - n^3 + \beta n|) |\hat{w}(n,\tau)|^2 d\tau \right)^{1/2},$$

$$\Lambda_2^\beta(w) = \left(\sum_{n=-\infty}^{\infty} (1+|n|)^{2s} \int_{-\infty}^{\infty} \frac{|\hat{w}(n,\tau)|^2}{1 + |\tau - n^3 + \beta n|} d\tau \right)^{1/2},$$

$$\Lambda_3^\beta(w) = \left(\sum_{n=-\infty}^{\infty} (1+|n|)^{2s} \left[\int_{-\infty}^{\infty} \frac{|\hat{w}(n,\tau)|}{1 + |\tau - n^3 + \beta n|} d\tau \right]^2 \right)^{1/2}.$$

Define the Hilbert space Z_β^s by

$$Z_\beta^s = \left\{ w \in L^2(S \times R) : \ \Lambda_1^\beta(w) < \infty \right\}$$

equipped with the norm

$$\|w\|_{Z_\beta^s} = \Lambda_1^\beta(w).$$

In addition, let

$$F_\beta^s = \{ f(x,t) : \ x \in S, \ t \in R; \ \Lambda_2^\beta(f) + \Lambda_3^\beta(f) < \infty \}$$

with the norm

$$\|f\|_{F^s_\beta} := \Lambda^\beta_2(f) + \Lambda^\beta_3(f).$$

Obviously, we have

$$L^2(R, H^s(S)) \subset F^s_\beta.$$

The space Z^s_β was first introduced by Bourgain [7] to deal with the periodic IVP for the homogeneous KdV equation. It has the following properties.

Lemma 6.1 *Let $s \geq 0$, $\beta \in R$ and let u, $v \in Z^s_\beta$ satisfy*

$$[u](t) \equiv 0, \qquad [v](t) \equiv 0. \tag{6.5}$$

Then it follows that

$$\|\psi(\delta^{-1}t)\partial_x(uv)\|_{F^s_\beta} \leq c\delta^{\frac{1}{12}}\|u\|_{Z^s_\beta}\|v\|_{Z^s_\beta}, \tag{6.6}$$

where ψ is as in Lemma 2.4, an element of $C^\infty_0(R)$, supp$\psi \subset (-1,1)$ and $\psi \equiv 1$ on $[-1/2, 1/2]$.

Remark 6.2 This is very minor modification of Lemma 7.41 and 7.42 in Bourgain [7]. A nice feature of the spaces Z^s_β is that their structure absorbs the dispersion relation of the KdV equation. The estimate (6.6) reveals a very subtle smoothing effect in this context.

The following two technical lemmas may be found worked out in Bourgain [7] and Zhang [31].

Lemma 6.3 *For any $h \in Z^s_\beta$,*

$$\|\psi(\delta^{-1}t)h\|_{Z^s_\beta} \leq c(b)\delta^{(1-2b)/2}\|h\|_{Z^s_\beta}, \tag{6.7}$$

for any b with $1/2 < b \leq 1$, where ψ is as above. The constant $c(b)$ may tend to $+\infty$ as $b \to 1/2$.

Let $\{W_\beta(t)\}^\infty_{-\infty}$ denote the unitary group generated in $L^2(S)$ by the operator

$$A_\beta f = -f''' - \beta f',$$

which is defined for any $f \in H^3(S)$.

Lemma 6.4 *Let ψ be as above, let $\beta \in R$ be fixed and let $u_0 \in H^s(S)$ and $f \in F^s_\beta$, where $s \geq 0$. Then we have*

$$\|\psi(\delta^{-1}t)W_\beta(t)u_0\|_{Z^s_\beta} \leq c\|u_0\|_s, \tag{6.8}$$

$$\|\psi(\delta^{-1}t)\int_0^t W_\beta(t-\tau)f(\tau)d\tau\|_{Z^s_\beta} \le c\delta^{\frac{1-2b}{2}}\|f\|_{F^s_\beta}, \tag{6.9}$$

for any $b > 1/2$, where c may depend on b. For any T with $0 < T \le \infty$ and $f \in F^s_\beta$,

$$\sup_{[-T,T]}\left\|\psi(\delta^{-1}t)\int_0^t W_\beta(t-\tau)f(\cdot,\tau)d\tau\right\|_s \le c\Lambda^\beta_3(f).$$

Define another space X^s_β by

$$X^s_\beta = \{(u_0, f) \in H^s(S)\times F^s_\beta \ : \ [u_0] = \beta, \quad [f(\cdot,t)] \equiv 0\}$$

and for any $T > 0$ and $s \ge 0$,

$$\mathcal{Z}^s_{\beta,T} := Z^s_\beta \cap C(-T,T;H^s(R)).$$

As in the non-periodic case, the IVP (6.3) defines a nonlinear map K_P from the space X^s_β to the space $\mathcal{Z}^s_{\beta,T}$ via the correspondence $K_P((u_0,f)) := u$, where $u \in \mathcal{Z}^s_{\beta,T}$ is the solution of the IVP (6.3) on the time interval $(-T,T)$ corresponding to the initial data u_0 and the forcing term f.

Let $\mathcal{D}^T_s(K_P)$ be the domain of the map K_P. Then an argument entirely similar to that used in the non-periodic case gives the following.

Theorem 6.5 *Let $s \ge 0$, $T > 0$ and $\beta \in R$ be given. Then the following results hold.*

1. *(Analyticity) $\mathcal{D}^T_s(K_P)$ is a non-empty, dense, open set in the space X^s_β and the map K_P is analytic from its domain to the space $\mathcal{Z}^s_{\beta,T}$ (cf. [31]).*

2. *(local Well-posedness) For any $(u_0,f) \in X^s_\beta$, there exists a $T_1 = T_1(\|u\|_s, \|f\|_{F^s_\beta})$ with $0 < T_1 \le T$ such that the IVP (6.3) has a unique solution $u \in \mathcal{Z}^s_{\beta,T_1}$. Moreover, for any $T' < T_1$, there is a neighborhood U of (u_0,f) in the space X^s_β such that the map K_P is analytic from U to the space $\mathcal{Z}^s_{\beta,T'}$.*

3. *(Global well-posedness) If $s = 0$, or $s = 1$, or $s \ge 2$, then*

$$H^s(S)\times L^2(-T,T;H^s(S)) \subset \mathcal{D}^T_s(K_P).$$

The following is a version of Theorem 5.7 for the periodic IVP.

Theorem 6.6 *Let $s \ge 0$ and $T > 0$ be given. Then, for any $u_0 \in H^{s+3}(S)$ with $[u_0] = \beta$ and $f \in W^{1,2}(-T,T;H^s(S))$ with $[f(\cdot,t)] \equiv 0$, the IVP (6.3) has a unique solution $u \in \mathcal{Z}^s_{\beta,T}$. Moreover,*

$$\partial_t u \in \mathcal{Z}^s_{\beta,T} \quad \textit{and} \quad u \in C(-T,T;H^{s+3}(S)).$$

A version of Lemma 4.1 for periodic function classes is the following.

Lemma 6.7 *Let $s, \beta \in R$ be given. Then*

$$W^{1/2,2}(R, H^s(S)) \subset F_\beta^{s'}$$

for any $s' < s + 3/2$.

As a consequence, one can show by methods that are now familiar the following theorem.

Theorem 6.8 *Let $s \geq -3/2$ be given. For any $u_0 \in H^{s+\frac{3}{2}}(S)$ with $[u_0] = \beta$ and $f \in W^{1/2,2}(R, H^s(S))$ with $[f(\cdot, t)] \equiv 0$, the corresponding solution u of the IVP (6.3) lies in the space $\mathcal{Z}_{\beta,T}^{s'}$ for any $s' < s + 3/2$, where $T > 0$ may depend on $\|u_0\|_{s+3/2}$ and $\|f\|_{F_\beta^{s+3/2}}$.*

There are two serious restrictions in the above results. First they require $[f(\cdot, t)] \equiv 0$ to get an existence result. Secondly, the map K_P is only shown to be continuous from the space X_β^s to the space $\mathcal{Z}_{\beta,T}^s$ rather than from $H^s(S) \times L^2(R, H^s(S))$ to $C(-T, T; H^s(S))$. For instance, if $f = 0$, $v_n = u_0/n$ with u_0 a non-zero element of $H^s(S)$, the above result does not imply that the corresponding solution u_n of associated IVP (6.3) goes to zero as $n \to \infty$! These restrictions result from the assumptions entailed in Lemma 6.1. It is not clear whether they can be removed.

On the other hand, it has been known for many years that the map K corresponding to the IVP of the homogeneous KdV equation is continuous from the space $H^s(S)$ to the space $C(-T, T; H^s(S))$ when $s > 3/2$ (cf. [17] and [26]). In the following we show that a similar result holds for the forced KdV equation using energy estimates.

Theorem 6.9 *For $s \geq 2$ and $T > 0$, if $u_0 \in H^s(S)$ and $f \in L^1(-T, T; H^s(S))$, then the IVP (6.3) has a unique solution $u \in C(-T, T; H^s(S))$. Moreover, the solution u depends continuously on u_0 in $H^s(S)$ and f in $L^1(-T, T; H^s(S))$.*

Remark 6.10 If $3/2 < s < 2$, a local well-posedness result can be obtained using the same approach.

Proof of Theorem 6.9: It suffices to show the existence and continuous dependence. The proof of the uniqueness is a simple exercise for this range of s.

First choose a family $\{f_\epsilon\} \subset C^1(0, T; H^\infty(S))$ and a family $\{\phi_\epsilon\} \subset H^\infty(S)$ such that, for any $r \geq 0$ and $\epsilon > 0$,

$$\|\phi_\epsilon\|_{s+r} = O(\epsilon^{-\frac{r}{6}}), \quad \|\phi_\epsilon - u_0\|_{s-r} = o(\epsilon^{-\frac{r}{6}}), \tag{6.10}$$

$$\|f_\epsilon\|_{L^1(-T,T;H^s(S))} = O(\epsilon^{-\frac{r}{6}}) \quad \text{and} \quad \|f_\epsilon - f\|_{L^1(-T,T;H^{s-r}(S))} = o(\epsilon^{\frac{r}{6}}), \tag{6.11}$$

as $\epsilon \downarrow 0$. The construction of such approximating families is straightforward and may be found in [3], for example.

For $\epsilon > 0$, consider the regularized IVP

$$\partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + \partial_x^3 u^\epsilon = f_\epsilon, \quad u^\epsilon(x,0) = \phi_\epsilon(x). \tag{6.12}$$

It has a unique solution $u^\epsilon \in C(-T,T;H^\infty(S))$ (cf. [3] and [26]).

Claim 1. Given $T > 0$, there are constants K_s independent of ϵ such that,

$$\sup_{[-T,T]} \|u^\epsilon(\cdot,t)\|_s \le K_s, \tag{6.13}$$

and for any $r \ge 0$,

$$\sup_{[-T,T]} \|u^\epsilon(\cdot,t)\|_{s+r} = O\left(\epsilon^{-\frac{r}{6}}\right) \tag{6.14}$$

as $\epsilon \downarrow 0$

It is not difficult to show that (6.13) is true when $s = 2$ by using the forced version of the conservation laws for the KdV equation. This bound implies $\|u^\epsilon\|_2$ is bounded by a constant K_2, independently of $1 \ge \epsilon > 0$ and $t \in [-T,T]$. In general, for $l \ge s$, we apply the operator D^l to both sides of the equation in (6.12) and take the L^2-inner product of the resulting equation with $D^l u^\epsilon$, coming thereby to the relation

$$\frac{1}{2}\frac{d}{dt}\|D^l u^\epsilon\|_2^2 + \left(D^l(u^\epsilon \partial_x u^\epsilon), D^l u^\epsilon\right) = \left(D^l f_\epsilon, D^l u^\epsilon\right).$$

Write the second term on the left-hand side of the last equation as

$$\left(u^\epsilon D^l \partial_x u^\epsilon, D^l u^\epsilon\right) + \left([D^l : u^\epsilon]\partial_x u^\epsilon, D^l u^\epsilon\right),$$

where the commutator $[D^l : u]v = D^l(uv) - uD^l v$. Applying Lemma 1.1 in [26] and using the fact that D^l and ∂_x commute shows that, on account of the just mentioned bound on $\|u^\epsilon\|_2$,

$$\begin{aligned}
\left|\left(D^l(u^\epsilon \partial_x u^\epsilon), D^l u^\epsilon\right)\right| &\le \|u^\epsilon\|_l \|u^\epsilon\|_2 \|D^l u^\epsilon\| + \left|\left(u^\epsilon D^l \partial_x u^\epsilon, D^l u^\epsilon\right)\right| \\
&\le K_2 \|D^l u^\epsilon\| \|u^\epsilon\|_l + \frac{1}{2}\left|\left(\partial_x u^\epsilon D^l u^\epsilon, D^l u^\epsilon\right)\right| \\
&\le \left(cK_2(\|u^\epsilon\| + \|D^l u^\epsilon\|) + \frac{1}{2}\|u^\epsilon\|_2 \|D^l u^\epsilon\|\right) \|D^l u^\epsilon\| \\
&\le c_1 \|D^l u^\epsilon\| + c_2 \|D^l u^\epsilon\|^2.
\end{aligned}$$

In consequence, there appears the differential inequality

$$\frac{1}{2}\frac{d}{dt}\|D^l u^\epsilon\|^2 \le c_2 \|D^l u^\epsilon\|^2 + \|D^l u^\epsilon\|(\|f_\epsilon\|_l + c_1).$$

This in turn implies that

$$\|D^l u^\epsilon\| \;\leq\; c\left(\|\phi_\epsilon\|_l + \int_{-T}^{T}\|f_\epsilon\|_l dt\right)$$

for a suitable constant c that depends on T and the bound K_2 on $\|u^\epsilon\|_2$. Reference to (6.10) and (6.11) shows that both (6.13) and (6.14) hold.

Next it is shown that $\{u^\epsilon\}_{\epsilon>0}$ is a Cauchy sequence in $C(-T,T;H^s(S))$. Its limit as $\epsilon \to 0$ is the desired solution of the problem. This approach has the advantage that it provides the continuous dependence of solutions on the auxiliary data almost automatically (see [3] and [19]).

Assume $0 < \epsilon' < \epsilon$ and let $w = u^\epsilon - u^{\epsilon'}$. Then w solves the IVP

$$\partial_t w + u^\epsilon \partial_x w + w\partial_x u^{\epsilon'} + \partial_x^3 w = \triangle f, \quad w(x,0) = \triangle\psi, \tag{6.15}$$

where

$$\triangle f = f_\epsilon - f_{\epsilon'}, \quad \triangle\psi = \phi_\epsilon - \psi_{\epsilon'}.$$

Claim 2.

$$\sup_{[-T,T]} \|w(\cdot,t)\|_1 = o\left(\epsilon^{\frac{s-1}{6}}\right) \tag{6.16}$$

as $\epsilon \downarrow 0$.

In fact, taking the L^2-scalar product of each member of the equation in (6.15) with w, there appears

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + (\partial_x u^\epsilon w, w) - \frac{1}{2}\left(\partial_x u^{\epsilon'}, w\right) = (\triangle f, w),$$

which implies that

$$\begin{aligned}\sup_{[-T,T]} \|w(\cdot,t)\| \;&\leq\; c\left(\|\triangle\psi\| + \int_{-T}^{T}\|\triangle f\| dt\right)\\ &=\; o\left(\epsilon^{\frac{s}{6}}\right)\end{aligned} \tag{6.17}$$

because of (6.13). Then, applying ∂_x to each member of the equation in (6.15) and taking the L^2-scalar product with $\partial_x w$, we see after integration by parts that

$$\begin{aligned}\frac{d}{dt}\|\partial_x w\|^2 \;&=\; -\left((\partial_x u^\epsilon + 2\partial_x u^{\epsilon'})\partial_x w, \partial_x w\right) - 2\left(\partial_x^2 u^{\epsilon'} w, \partial_x w\right) + 2\left(\partial_x \triangle f, \partial_x w\right)\\ &\leq\; \|u^\epsilon + 2u^{\epsilon'}\|_2 \|\partial_x w\|^2 + \|u^{\epsilon'}\|_2 \|w\|_\infty \|\partial_x w\| + \|\triangle f\| \|\partial_x w\|\\ &\leq\; c_1\|\partial_x w\|^2 + c_2\|w\|_\infty\|\partial_x w\| + \|\partial_x \triangle f\|\|\partial_x w\|\end{aligned}$$

by (6.13), which, when combined with (6.17), implies (6.16).

Now we prove that

$$\sup_{[-T,T]} \|w(\cdot,t)\|_s = o(1) \tag{6.18}$$

as $\epsilon \downarrow 0$. Applying D^s to each member of the equation in (6.15) and taking the L^2-scalar product with $D^s w$, there obtains the differential equation

$$\begin{aligned}\frac{d}{dt}\|D^s w\|^2 &= -2\left(D^s(u^\epsilon \partial_x w), D^s w\right) - 2\left(D^s(w\partial_x u^{\epsilon'}), D^s w\right) + \\ &+ 2\left(D^s \triangle f, D^s w\right).\end{aligned} \tag{6.19}$$

By again using Lemma 1.1 in [26] and (6.13) in (6.19), it is found as above that

$$\begin{aligned}|\left(D^s(u^\epsilon \partial_x w), D^s w\right)| &= |([D^s : u^\epsilon]\partial_x w, D^s w) + \left(u^\epsilon D^{s+1} w, D^s w\right)| \\ &\leq c\|D^s w\|^2\end{aligned}$$

and

$$\begin{aligned}\left|\left(D^s(w\partial_x u^{\epsilon'}), D^s w\right)\right| &= \left|\left([D^s : w]u^{\epsilon'}, D^s w\right) + \left(wD^{s+1}u^{\epsilon'}, D^s w\right)\right| \\ &\leq \|[D^s : w]u^{\epsilon'}\|\|D^s w\| + \|D^{s+1}u^{\epsilon'}\|\|w\|_1\|D^s w\| \\ &\leq c\|D^s w\|^2 + o(\epsilon^{\frac{s-1}{6}})O(\epsilon^{-\frac{1}{6}})\|D^s w\| \\ &\leq c\|D^s w\|^2 + o(1)\|D^s w\|.\end{aligned}$$

It thus follows from (6.19) that

$$\frac{d}{dt}\|D^s w\|^2 \leq c\|D^s w\|^2 + o(1)\|D^s w\| + \|D^s \triangle f\|\|D^s w\|,$$

which implies (6.18). The proof is complete. □

References

[1] T. R. Akylas, On the excitation of long nonlinear water waves by a moving pressure distribution, *J. Fluid Mech.* **141** (1984), 455 – 466.

[2] T. B. Benjamin, J. L. Bona and J. J. Mahony, Model equations for long waves in nonlinear, dispersive media, *Philos. Trans. Roy. Soc. London A* **272** (1972), 47 – 78.

[3] J. L. Bona and R. Smith, The initial-value problem for the Korteweg-de Vries equation, *Philos. Trans. Roy. Soc. London A* **278** (1975), 555 – 601.

[4] J. L. Bona and L. R. Scott, Solutions of the Korteweg-de Vries equations in fractional order Sobolev spaces, *Duke Math. J.* **43** (1976), 87 – 99.

[5] J. L. Bona, Convergence of periodic wavetrains in the limit of large wavelength, *Appl. Scientific Res.* **37** (1981), 21 – 30.

[6] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to non-linear evolution equations, part I: Schrödinger equations, *Geometric and Functional Analysis* **3** (1993), 107 – 156.

[7] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to non-linear evolution equations, part II: the KdV equation, *Geometric and Functional Analysis* **3** (1993), 209 – 262.

[8] A. Cohen, Solutions of the Korteweg-de Vries equation from irregular data, *Duke Math. J.* **45** (1978), 149 – 181.

[9] S. L. Cole, Transient waves produced by flow past a bump, *Wave Motion* **7** (1985), 579 – 587.

[10] P. Constantin and J.-C. Saut, Local smoothing properties of dispersive equations, *J. Amer. Math. Soc.* **1** (1988), 413 – 446.

[11] W. Craig, T. Kappeler and W. A. Strauss, Gain of regularity for equations of the Korteweg-de Vries type, *Ann. Inst. Henri Poincaré* **9** (1992), 147 – 186.

[12] J. Ginibre and G. Velo, Smoothing properties and retarded estimates for some dispersive evolution equations, preprint.

[13] J. Ginibre and Y. Tsutsumi, Uniqueness for the generalized Korteweg-de Vries equations, *SIAM J. Math. Anal.* **20** (1989), 1388 – 1425.

[14] R. H. J. Grimshaw and N. Smyth, Resonant flow of a stratified fluid over topography, *J. Fluid Mech.* **169** (1986), 429 – 464.

[15] T. Kato, Quasilinear equations of evolutions, with applications to partial differential equation, *Lecture Notes in Math.* **448** (1975), Springer-Verlag, 27 – 50.

[16] T. Kato, On the Korteweg-de Vries equation, *Manuscripta Math.* **29** (1979), 89 – 99.

[17] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equations, *Advances in Mathematics Supplementary Studies, Studies in Applied Math.* **8** (1983), 93 – 128.

[18] C. E. Kenig, G. Ponce and L. Vega, Oscillatory integrals and regularity of dispersive equations, *Indiana University Math. J.* **40** (1991), 33 – 69.

[19] C. E. Kenig, G. Ponce and L. Vega, Well-posedness of the initial value problem for the KdV equation, *J. Amer. Math. Soc.*, **4** (1991), 323 – 347.

[20] C. E. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equations via the contraction principle, *Comm. Pure Appl. Math.* **46** (1993), 527 – 620.

[21] C. E. Kenig, G. Ponce and L. Vega, The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices, *Duke Math. J.* **71** (1993), 1 – 21.

[22] S. N. Kruzhkov and A. V. Faminskii, Generalized solutions of the Cauchy problem for the Korteweg-de Vries equation, *Math. U.S.S.R. Sbornik* **48** (1984), 93 – 138.

[23] S.-J. Lee, Generation of long water waves by moving disturbances, *Ph. D. thesis*, 1985, California Institute of Technology.

[24] R. M. Miura, The Korteweg-de Vries equation: A survey of results, *SIAM review* **18** (1976), 412 – 459.

[25] R. L. Sachs, Classical solutions of the Korteweg-de Vries equation for non-smooth initial data via inverse scattering, *Comm. P.D.E.* **10** (1985), 29 – 89.

[26] J.-C. Saut and R. Temam, Remarks on the Korteweg-de Vries equation, *Israel J. Math.* **24** (1976), 78 – 87.

[27] Y. Tsutsumi, The Cauchy problem for the Korteweg-de Vries equation with measures as initial data, *SIAM J. Math. Anal.* **20** (1989), 582 – 588.

[28] T. Y. Wu, Generation of upstream advancing solitons by moving disturbances, *J. Fluid Mech.* **184** (1987), 75 – 99.

[29] B.-Y. Zhang, Some results for the nonlinear dispersive wave equations with applications to control, *Ph. D. thesis*, University of Wisconsin-Madison, 1990.

[30] B.-Y. Zhang, Taylor series expansion for solutions of the KdV equation with respect to their initial values, *the IMA preprint, Series # 1015* Aug. 1992 (to appear in *J. Funct. Anal.*).

[31] B.-Y. Zhang, A remark on the Cauchy problem for the Korteweg-de Vries equation on a periodic domain, *the IMA preprint, Series # 1155* July 1993.