THE EFFECT OF DISSIPATION ON SOLUTIONS OF THE GENERALIZED KORTEWEG-DE VRIES EQUATION

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Abstract. It was indicated in recent numerical simulations that the initial-value problem for the generalized Korteweg-de Vries equation is not globally well posed when the nonlinearity is strong enough. Indeed, even C^∞ -initial data that is spatially periodic is observed to form singularities in finite time. The generalized Korteweg-de Vries equations are Hamiltonian systems that feature a competition between nonlinear and dispersive effects. A natural question that comes to the fore in consequence of the observed singularity formation is whether or not the addition of a term modelling the effect of dissipation will eliminate singularities and so result in an initial-value problem that is globally well posed. It is the purpose of the present paper to study this question both analytically and numerically. Our concern will be mainly with the addition of a Burgers-type dissipative term because of its frequent appearance in practical modelling problems. Some commentary is also provided about the situations that obtain when other dissipative mechanisms are introduced. It seems that singularity formation persists in the presence of small amounts of dissipation, but ceases at a certain critical level whose general form is studied both numerically and analytically.

1. INTRODUCTION

The present paper is inspired by an earlier one of the same authors and aims to add considerably to the conclusions drawn therein. In the previous study (Bona *et al.* 1994), attention was given to the development and use of numerical approximations of solutions to the initial- and periodic-boundary-value problem for the generalized Korteweg-de Vries equation (GKdV equation henceforth)

$$u_t + u^p u_x + \varepsilon u_{xxx} = 0. \tag{1.1a}$$

Here, the dependent variable $u = u(x,t)$ is a 1-periodic, real-valued function of the spatial variable $x \in [0,1]$ and the temporal variable $t \geq 0$ which is prescribed at $t = 0$ by

$$u(x,0) = u_0(x) \tag{1.1b}$$

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for $0 \leq x \leq 1$. In (1.1a), ε is a fixed, positive number that is related to a generalization of the classical Stokes number of surface water-wave theory (see Albert & Bona 1991, Bona & Sciálom 1993) and p is a positive integer. Special consideration was given in our previous work to understanding the instability of the travelling-wave solutions of (1.1a) called solitary waves, and it transpired that this instability manifests itself in blow-up in finite time. More precisely, if $p \geq 4$ in (1.1a), then perturbations of solitary waves form a similarity structure under the evolution (1.1a) and this structure in turn blows up, leading to the inference that there is a point (x^*, t^*) such that $u(x,t) \longrightarrow +\infty$ as $(x,t) \longrightarrow (x^*, t^*)$. These earlier numerical simulations showed also that this special blow-up phenomenon has more scope than might be expected. A broad class of initial data u_0 has the property that, under the evolution (1.1a), the resulting solutions rapidly decompose into a finite number of pulses resembling solitary waves, the first and largest of which then becomes unstable, forms a similarity structure and blows up in finite time. As nearly as could be discerned from the numerical simulations, the process of singularity formation for general initial data u_0 was the same as that appearing in the instability of the solitary wave.

It is worth noting that the blow-up phenomenon just described subsists on both nonlinear and dispersive effects, and it can only occur if the nonlinearity overpowers the dispersion. Three facts support this conclusion. First, if $\varepsilon = 0$, so that dispersion is absent, there are large classes of initial data whose corresponding solutions form singularities in finite time, but it is the derivative that becomes unbounded, not the solution itself. Second, if $p < 4$, then smooth initial data u_0 lead to global solutions u of the initial-value problem (1.1), regardless of the size of the data (see Kato 1983, Albert *et al.* 1988). Finally, even for $p \geq 4$, if the initial data u_0 is reasonably smooth and small enough so that one expects nonlinear effects to be relatively insignificant, then the initial-value problem (1.1) still has global solutions (cf. Strauss 1974, Schechter 1978, Kato 1983).

The GKdV equations arise in modelling the propagation of small-amplitude, long waves in nonlinear dispersive media (see Benjamin *et al.* 1972, Benjamin 1974). The case $p = 1$

is the classical Korteweg-de Vries (KdV) equation about which much has been written in the last three decades, and which arises in a number of interesting physical situations (cf. Benjamin 1974, Jeffrey & Kakutani 1972, Scott *et al.* 1973). In real physical situations, dissipative effects are often as important as nonlinear and dispersive effects (see the experimental study of Bona *et al.* 1981) and this fact has given currency to the study of the Korteweg-de Vries-Burgers equation

$$u_t + uu_x - \delta u_{xx} + \varepsilon u_{xxx} = 0 \tag{1.2}$$

as a model that incorporates all three effects (see Grad & Hu 1967, Johnson 1970, Bona & Smith 1975, Bona *et al.* 1981, Bona & Schonbek 1985, Pego 1985, Amick *et al.* 1989, Bona *et al.* 1992). In (1.2), the parameter ε is as before and $\delta > 0$ is another parameter expressing the relative strength of dissipative to nonlinear effects.

A natural question arises as to whether dissipative effects in the form of a Burgers-type term, say, overcome the nonlinear-dispersive interaction that leads to blow-up. It is to this and related questions that the present work is directed. For the most part, attention is given to the initial-value problem

$$\begin{aligned} u_t + u^p u_x - \delta u_{xx} + \varepsilon u_{xxx} &= 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{1.3}$$

where $p \geq 4$, u_0 is a reasonably smooth, 1-periodic, real-valued function on the real line $\mathbb{R}$, and ε and δ are positive constants as indicated previously. It is straightforward to show that the initial-value problem (1.3) has unique solutions corresponding to reasonably smooth initial data, at least locally in time, by semigroup methods (Kato 1975, 1983) or by regularization techniques (Bona & Smith 1975).

It is also easy to ascertain that a solution of (1.3) defined locally in time has a global extension if it remains bounded in a suitable norm on bounded time intervals (see Kato 1983, Albert *et al.* 1988 or Bona *et al.* 1987). However, standard energy techniques seem unable to establish the *a priori* deduced bounds needed to guarantee global existence.

It will be shown below that for fixed $\varepsilon > 0$ and for any given initial datum u_0, there is a $\delta_c > 0$ such that if $\delta > \delta_c$, then the local solution of (1.3) emanating from u_0 has a global continuation as a smooth solution of the differential equation. This global result was motivated by the outcome of a series of numerical experiments simulating solutions of (1.3) which were designed to cast light on the effect of the dissipative term. Other interesting points are indicated by these numerical results which are described in the detailed outline of the paper to be presented now.

The plan of the paper is straightforward. Section 2 contains theoretical results appertaining to (1.3) and one of its near relatives wherein the dissipative term $-\delta u_{xx}$ is replaced by σu. We are able to establish in both cases that if the parameter δ or σ is sufficiently large relative to certain norms of the initial data u_0, then the solution u emanating from u_0 exists and is uniformly bounded over the entire temporal half-axis $[0, \infty)$. Moreover, the proofs lead to explicit formulas for an upper bound on the critical values δ_c and σ_c at least when u_0 is a perturbed solitary wave that would blow up in finite time in the absence of dissipation. When a solution is global in time, results about its decay to a quiescent state are also derived.

Section 3 contains a description of the numerical method used to integrate (1.3). As in our earlier study, the scheme is based on a Galerkin spatial discretization with periodic, cubic splines coupled with a time-stepping procedure which combines a two-stage Gauss-Legendre implicit Runge-Kutta method with a version of Newton's method for solving the system of nonlinear equations that arise at each time step. This basic scheme is augmented by adaptive mechanisms that adjust the temporal and local spatial grids in an effort to retain accuracy in the face of large values of the dependent variable.

Section 4 reports on numerical experiments carried out using a computer code derived from the numerical scheme described in Section 3. In Subsection 4a, computations show that solutions of (1.3) still blow up in finite time for $p = 5, 6$ or 7 provided the dissipative coefficient δ is small enough. The computed rates of blow-up and the structure of solutions as they become singular are virtually identical to those observed for the non-dissipative problem in which $\delta = 0$. An analysis of the data presented shows the interesting conclusion that the theoretically derived forms for the critical values δ_c and σ_c coincide with those obtained in practice.

Subsection 4b records some data connected with the decay of solutions when the parameter δ is large enough to prevent blow-up. It is shown in Section 2 that in this circumstance solutions approach exponentially a constant equal to the mean value of u_0. The lapse rate in the exponential decay depends linearly on δ and tends to zero as δ tends to zero. The numerically obtained evidence supports the contention that for any p, the long-term behavior of global solutions is determined by the linearized form of the GKdV equation, a conclusion that agrees with Biler's sharp decay results for the case $p = 1$ (Biler 1984). Commentary is also offered about the transitory, oscillatory break-up of initial data that occurs at early stages in the evolution prior to the long-term, exponential asymptotics becoming dominant.

The paper concludes with a summary section that also features remarks on potentially interesting avenues for further research.

2. Theoretical Results

After a review of notational conventions, the principal theorem in our theoretical development is stated and proved. Several useful corollaries are then derived which act as a foil for some of the numerical simulations presented in Section 4.

Notation. In the sequel L_q, $1 \leq q < \infty$ will denote the collection of L-periodic functions which are q^{th}-power integrable over $[0, L]$ endowed with the norm

$$|f|_q = \left(\int_0^L |f(x)|^q \, dx \right)^{1/q},$$

with the usual modification if $q = \infty$. For $s \geq 0$, the space $H^s = H^s(0, L)$ is the Sobolev class of L-periodic functions which, along with their first s derivatives belong to L_2. The usual norm on H^s is denoted by $\|\cdot\|_s$. The norm of $L_2 = H^0$ appears frequently and will be denoted $|\cdot|_2$ rather than $\|\cdot\|_0$; the associated inner product is the only Hilbert-space structure to intervene in the analysis and it is written simply as $(\cdot,\cdot)$. In Sections 3 and 4, we shall restrict attention to the case where $L = 1$. For the periodic problem, this simply amounts to a rescaling of the spatial variable x, and no loss of generality results from this presumption. However, in the present section, it will be convenient to leave L arbitrary for reasons that will become apparent shortly.

It deserves remark that while it is convenient to present analytical results first, early numerical results helped motivate the theory which in turn provided significant insights and guidance into later numerical experiments.

As an example, which sets the stage for Theorem 2.1, the reader may consult Figures 1a and 2 in Section 4 that depict the outcome of two numerical simulations of (1.2), both with $p = 5$, ε fixed and the same initial data. The difference between the two simulations lies with the value of the dissipative parameter δ; in Figure 1a, δ is rather small while in Figure 2 it is five times larger. As documented in detail in Subsection 4a, the smaller value of δ seems to allow the associated solution to form a singularity in finite time, whereas the larger value of δ appears to prevent the single-point blow-up observed in Figure 1a.

Armed with these, and other like results, for different values of p, the following theorem was conjectured and proved.

THEOREM 2.1. *Let u_0 be given initial data that is periodic of period $L > 0$ and suppose u_0 to lie in $H^s(0, L)$ for some $s \geq 2$. Let $\varepsilon, \delta > 0$ be given.*

(1) If $p < 4$, then there is a unique global solution u of (1.3) corresponding to the above specification of data and parameters, that lies in $C(0, T; H^s(0, L))$ for every $T > 0$. Moreover, $\|u(\cdot, t)\|_1$ is uniformly bounded in t.

(2) If $p \geq 4$, there is a $T_0 > 0$ depending on $\|u_0\|_1$ and a unique solution $u \in C(0, T_0; H^s(0, L))$ of (1.3) with initial data u_0. If $\|u_0\|_1$ is sufficiently small with respect to

δ, then T can be taken arbitrarily large, the solution is global, and $\|u(\cdot,t)\|_1$ is uniformly bounded for $t \in [0,+\infty)$.

In all the above cases, the solution u depends continuously on the initial data u_0 in that the mapping $u_0 \mapsto u$ is continuous from H^s to $C(0,T;H^s)$.

Remark. Part (1) and the local existence theory in Part (2) may be found more or less as stated in the literature (cf. Bona & Smith 1975, Kato 1975, 1983, Albert *et al.* 1988 Albert & Bona 1991). It deserves remark that the correspondence $u_0 \mapsto u$ has recently been investigated in more detail by Bourgain (1993) and Zhang (1993), with the outcome that values of s smaller than 2 can be accomodated and the correspondence, much more than being continuous, is analytic. Moreover, since $\delta > 0$, coarse data becomes smooth for $t > 0$. None of these subtle aspects are important in our analysis, however, so we pass over them in favor of the simpler description in Theorem 2.1.

Proof. Attention will be given only to the case $p \geq 4$. As mentioned above, a theory, local in time, of existence, uniqueness and continuous dependence for (1.3) may be concluded using standard semigroup theory, and the details are therefore omitted. The focus of attention here will be to provide *a priori* deduced bounds that allow the local theory to be continued indefinitely. Finer results from the local theory can be deduced, but for our purposes it will suffice to note that if the initial data u_0 lies in H^s for some $s \geq 2$, and if the solution u is bounded, at least on any interval of the form $[0,T]$ for finite $T > 0$, then the solution is global in time and lies in $C(0,T;H^s)$ for all finite T. This state of affairs can be ascertained from Kato's theory (Kato 1975, 1983, Albert *et al.* 1988) or from the estimates to be derived now.

Let u be a solution in $C(0,T;H^s)$ corresponding to initial data u_0 of the initial-value problem (1.3), where $s \geq 2$. Without loss of generality, we may suppose that s is large enough that the formal calculations to follow are straightforwardly justified. Because of the continuous dependence of the solution on the initial data, one simply regularizes the initial data, makes the calculations with the associated smoother solutions, and after deriving the desired inequalities, then passes to the limit as the regularization disappears. So long as the inequalities in question do not involve derivatives higher than those appearing in the initial data, this procedure leads securely to the desired results.

We begin by multiplying the equation (1.2) by the solution u, integrating the result with respect to x over the period $[0, L]$ and with respect to t over the interval $[0, t_0]$, where $t_0 \leq T$. After suitable integrations by parts, and using periodicity to see that the boundary terms cancel, there appears the simple relation

$$|u(\cdot, t_0)|_2^2 + 2\delta \int_0^{t_0} |u_x(\cdot, t)|_2^2 \, dt = |u_0|_2^2. \tag{2.1}$$

(Throughout this proof, all the norms are computed with respect to the spatial region $[0, L]$.) It is deduced from this that $|u(\cdot, t)|_2$ is a decreasing function of time and that

$$\int_0^t |u_x(\cdot, t)|_2^2 \, dt$$

is bounded independently of t.

The next stage of the estimates is more complicated. As shown in Kato (1983) and Albert *et al.* (1988), all that is required in order to infer the boundedness of u in H^s on bounded time intervals, and thereby to deduce the conclusion of the theorem is to demonstrate that the L_∞-norm of the solution is bounded on bounded time intervals. For this, it suffices to show the H^1-norm of the solution is bounded on bounded time intervals. In fact, we shall show that the H^1-norm of solutions corresponding to initial data suitably small with respect to δ is bounded independently of t.

To this end, let $\overline{u}_0 = \frac{1}{L}\int_0^L u_0(x)dx$ be the average mass of the initial data. By integrating (1.2) with respect to x, one readily deduces from the spatial periodicity that for any $t > 0$ for which the solution exists on $[0, t]$, one has $\overline{u}(t) = \frac{1}{L}\int_0^L u(x,t)dx = \overline{u}_0$. (This is a reflection of conservation of mass in some applications of this class of equations to practical situations.)

Define a new dependent variable v by $v(x,t) = u(x,t) - \overline{u}_0$. Then the variable v has total mass zero and satisfies the initial-value problem

$$\begin{gathered} v_t + \frac{1}{p+1}\partial_x(v+\overline{u}_0)^{p+1} - \delta v_{xx} + \varepsilon v_{xxx} = 0, \\ v(x,0) = v_0(x) = u_0(x) - \overline{u}_0. \end{gathered} \tag{2.2}$$

Multiply the differential equation in (2.2) by v_{xx} and integrate over $[0,L]$ to obtain the following differential inequality:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L v_x^2 dx + \delta \int_0^L v_{xx}^2 dx &= \frac{1}{p+1}\int_0^L \partial_x(v+\overline{u}_0)^{p+1} v_{xx} dx \\ &= \frac{1}{p+1}\int_0^L \partial_x\left(\sum_{j=0}^{p+1}\binom{p+1}{j} v^{p+1-j}\,\overline{u}_0^j\right) v_{xx} dx \\ &= \frac{1}{p+1}\int_0^L \sum_{j=0}^{p-1}\binom{p+1}{j}(p+1-j)v^{p-j}v_x\overline{u}_0^j\, v_{xx} dx \\ &\le \frac{1}{p+1}\sum_{j=0}^{p-1}\binom{p+1}{j}(p+1-j)\,|\overline{u}_0|^j|v|_\infty^{p-j}\,|v_x|_2\,|v_{xx}|_2. \end{aligned} \tag{2.3}$$

It is elementary that if w is periodic with mean value equal to zero, then

$$|w|_\infty^2 \le |w|_2\,|w_x|_2, \qquad |w_x|_2^2 \le |w|_2\,|w_{xx}|_2,$$

and (2.4)

$$|w|_2 \le \frac{L}{2\pi}\,|w_x|_2.$$

If we use the first two of these relations systematically in (2.3), we ascertain that the right-hand member of (2.3) may be bounded above as follows:

$$\begin{aligned} &\frac{1}{p+1}\sum_{j=0}^{p-1}\binom{p+1}{j}(p+1-j)|\overline{u}_0|^j|v|_2^{\frac{p-j}{2}}\,|v_x|_2^{\frac{p+2-j}{2}}\,|v_{xx}|_2 \\ &\le \frac{1}{p+1}\left[\sum_{j=0}^{p-2}\binom{p+1}{j}(p+1-j)\,|\overline{u}_0|^j|v|_2^{\frac{p+2-j}{2}}\,|v_x|_2^{\frac{p-2-j}{2}}\,|v_{xx}|_2^{2}\right. \\ &\qquad \left. +2\binom{p+1}{p-1}\,|\overline{u}_0|^{p-1}|v|_2^{1/2}\,|v_x|_2^{\frac{3}{2}}\,|v_{xx}|_2\right]. \end{aligned} \tag{2.5}$$

The last summand on the right-hand side of (2.5) requires special treatment; using the second relation in (2.4), one sees that

$$
\begin{aligned}
|v|_2^{1/2} |v_x|_2^{3/2} |v_{xx}|_2 &\leq (L/2\pi)^{1/2} |v_x|_2^2 |v_{xx}|_2 \\
&\leq (L/2\pi)^{1/2} |v|_2 |v_{xx}|_2^2 .
\end{aligned}
$$

Putting this together with (2.5) and (2.3) leads to the differential inequality

$$
\frac{1}{2} \frac{d}{dt} |v_x(\cdot,t)|_2^2 + \delta\, |v_{xx}(\cdot,t)|_2^2 \leq \theta |v_{xx}(\cdot,t)|_2^2, \tag{2.6}
$$

where

$$
\begin{aligned}
\theta = \theta(|v|_2, |v_x|_2, |\overline{u}_0|) = \\
\frac{1}{p+1} \Bigg[\sum_{j=0}^{p-2} \binom{p+1}{j} (p+1-j)\, |\overline{u}_0|^j |v|_2^{\frac{p+2-j}{2}} |v_x|_2^{\frac{p-2-j}{2}} \\
+ 2 \binom{p+1}{p-1} (L/2\pi)^{1/2}\, |\overline{u}_0|^{p-1} |v|_2 \Bigg] .
\end{aligned}
$$

Note that since v has mean value zero, then

$$
|u(\cdot,t)|_2^2 = L\,\overline{u}_0^2 + |v(\cdot,t)|_2^2
$$

while (2.7)

$$
|u_x(\cdot,t)|_2^2 = |v_x(\cdot,t)|_2^2
$$

for all t for which the solution exists. In consequence of these inequalities, it suffices to show that $\|v(\cdot,t)\|_1$ is bounded, independently of t, in order that $\|u(\cdot,t)\|_1$ is bounded, independently of t.

The differential inequality (2.6) is employed to deduce a global bound on $\|v(\cdot,t)\|_1$. Notice that θ is monotone increasing as a function of its three arguments. Moreover, because of (2.1), $|u(\cdot,t)|_2 \le |u_0|_2$ whence $|v(\cdot,t)|_2 \le |u(\cdot,t)|_2 \le |u_0|_2$ in view of (2.7). In consequence of these two inequalities, we see that

$$
\begin{aligned}
\theta &\le \frac{1}{p+1}\sum_{j=0}^{p-2}\binom{p+1}{j}(p+1-j)\left(\frac{|u_0|_2}{L^{1/2}}\right)^j |u_0|_2^{\frac{p+2-j}{2}}|v_x|_2^{\frac{p-2-j}{2}} \\
&\quad+\frac{2}{p+1}\binom{p+1}{p-1}(L/2\pi)^{1/2}\left(\frac{|u_0|_2}{L^{1/2}}\right)^{p-1}|u_0|_2 \\
&\le |u_0|_2^{\frac{p+2}{2}}\left\{\sum_{j=0}^{p-3}\binom{p+1}{j}\frac{p+1-j}{p+1}\left(\frac{|u_0|_2}{L}\right)^{j/2}|v_x|_2^{\frac{p-2-j}{2}}\right. \\
&\quad\left.+\left[\frac{3}{p+1}\binom{p+1}{p-2}+\frac{(2\pi^{-1})^{1/2}}{p+1}\binom{p+1}{p-1}\right]\left(\frac{|u_0|_2}{L}\right)^{\frac{p-2}{2}}\right\} \\
&\le \lambda_p|u_0|_2^{\frac{p+2}{2}}\sum_{j=0}^{p-2}\binom{p-2}{j}\left(\frac{|u_0|_2}{L}\right)^{j/2}|v_x|^{\frac{p-2-j}{2}} \\
&\le \lambda_p|u_0|_2^{\frac{p+2}{2}}\left(\frac{|u_0|_2^{1/2}}{L^{1/2}}+|v_x|_2^{1/2}\right)^{p-2} = \overline{\theta}(|u_0|_2,|v_x|_2),
\end{aligned}
\tag{2.8}
$$

for some constant λ_p depending only on p. If one rewrites (2.6) as

$$
\frac{1}{2}\frac{d}{dt}|v_x(\cdot,t)|_2^2+(\delta-\overline{\theta})|v_{xx}(\cdot,t)|_2^2 \le 0,
$$

then it becomes clear that as soon as $\overline{\theta} \le \delta$, $|v_x(\cdot,t)|_2$ becomes monotone decreasing. In this range, $\overline{\theta}$ is also decreasing, and consequently if for some t_0 we find that $\overline{\theta} \le \delta$, then for all $t \ge t_0$ the same inequality holds. In particular, if at $t = 0$ it is the case that $\overline{\theta}(|u_0|_2,|v_{0x}|_2) = \overline{\theta}(|u_0|_2,|u_{0x}|_2) \le \delta$, then for all $t \ge 0$, $|v_x(\cdot,t)|_2 \le |u_{0x}|_2$, and so $\|v\|_1$ is seen to be bounded, independently of t.

Referring back to (2.8), if

$$
\lambda_p|u_0|_2^{\frac{p+2}{2}}\left(\frac{1}{L^{1/2}}|u_0|_2^{1/2}+|u_{0x}|_2^{1/2}\right)^{p-2} \le \delta, \tag{2.9}
$$

then $\|v(\cdot,t)\|_1$ is bounded, independently of t.

The theorem is thus established. □

Several interesting consequences can be drawn from this theorem and its proof. First, by letting the period L tend to $+\infty$, a result pertaining to the pure initial-value problem emerges. (The integrals in the norms mentioned in the following Corollary refer to the entire real axis $\mathbb{R}$.)

COROLLARY 2.2. *Let $u_0 \in H^s(\mathbb{R})$ for some $s \geq 2$ and let u be the corresponding solution of equation (1.2) for $x \in \mathbb{R}$ and $t > 0$ with parameters $\varepsilon, \delta > 0$ and $p \geq 4$. There is a constant $\mu = \mu_p$ depending only on p such that if*

$$|u_0|_2^{\frac{p+2}{2}} \, |u_{0x}|_2^{\frac{p-2}{2}} \leq \mu_p \delta, \tag{2.10}$$

then the solution lies in $C(0,T;H^s)$ for all $T > 0$ and $\|u(\cdot,t)\|_1 \leq \|u_0\|_1$ for all $t \geq 0$.

An interesting point arises relative to the inequality (2.10). It appears that this inequality is sharp in a certain way to be explained now. Consider the situation in which the initial data $u_0(x) = A\psi(Kx)$ for $x \in \mathbb{R}$, where A and K are positive constants. Then $|u_0|_2 = A|\psi|_2/K^{1/2}$ and $|u_{0x}|_2 = AK^{1/2}|\psi_x|_2$. Viewing ψ as fixed, but A, K as variable, we observe that inequality (2.10) becomes

$$\frac{A^p}{K} \leq \mu\delta \tag{2.11}$$

for a constant μ depending on p and norms of ψ. Of especial interest are the traveling-wave solutions of (1.1a) called solitary waves. These have the explicit form

$$u_s(x,t) = \mathrm{A}\mathrm{sech}^{2/p}[K(x - x_0) - \omega t] \tag{2.12a}$$

for any x_0, where the parameter K governing the spread of the solution and the speed of propagation ω are defined in terms of the amplitude A by

$$K = \left(\frac{p^2 A^p}{2\varepsilon(p+1)(p+2)}\right)^{1/2}, \quad \omega = \frac{2KA^p}{(p+1)(p+2)}. \tag{2.12b}$$

If the initial data u_0 lies close to a solitary-wave solution $u_s(\cdot, 0)$ as defined above, then (2.11) becomes

$$\mu\delta \geq \frac{A^p}{\left(\frac{p^2 A^p}{2\varepsilon(p+1)(p+2)}\right)^{1/2}} = d_p A^{p/2} \varepsilon^{1/2},$$

or what is the same,

$$\Delta = \frac{\delta^2}{\varepsilon A^p} \geq C_p, \tag{2.13}$$

where C_p is a constant depending on p and on the L_2-norms of $\text{sech}^{2/p}(x)$ and its first derivative, and so in fact depends only on p.

As mentioned earlier, if $p \geq 4$, the solitary wave solutions of (1.2) with $\delta = 0$ are unstable (Bona *et al.* 1987, Pego & Weinstein 1992), and small perturbations were seen to lead to blow up in finite time (Bona *et al.* 1986, 1994). Consider now a situation where $\varepsilon > 0$ and $p \geq 5$ are fixed and initial data is specified to be $u_0(x) = \lambda u_s(x, 0)$ where u_s is the solitary-wave solution in (2.12a) and λ is slightly greater than one, (e.g. $\lambda = 1.01$ as in §5 of Bona *et al.* 1994). With $\delta = 0$ and this initial data, the numerical approximation of the resulting solution of (1.2) indicates that it forms a singularity in finite time. Theorem 2.1 shows that for δ sufficiently large, the solution of (1.2) emanating from this type of initial data is uniformly bounded in t. One therefore expects a critical value δ_c of the dissipative parameter δ which defines the boundary between blow-up and global existence. From the condition in (2.13), it is known that $\delta_c^2 < C_p \varepsilon A^p$.

In Subsection 4a, an approximation of $\delta_c = \delta_c(A, \varepsilon)$ will be determined by making sequences of runs where u_0 is fixed as above and δ is varied systematically. It transpires that the combination denoted Δ in (2.13) is central to determining whether or not one has blow-up, at least for these perturbed solitary waves. It is a little unusual that the relatively crude energy estimates leading to the conclusion enunciated in (2.10) has this sharp aspect.

A final point that presents itself as a consequence of Theorem 2.1 is the decay of solutions to a quiescent state.

COROLLARY 2.3. *If u is a solution of the initial- and periodic-boundary-value problem (1.3) with $\delta > 0$ corresponding to initial data $u_0 \in H^s(0,L)$, then $|u(\cdot,t) - \overline{u}_0|_2 \leq e^{-\delta(\frac{2\pi}{L})^2 t}|u_0 - \overline{u}_0|_2$ for all t for which the solution exists. If u_0 satisfies condition (2.9) relative to δ, then $|u_x|_2$ also decays exponentially, as does $|\partial_x^j u|_2$ for all $j \leq s$.*

Proof. First, consider again the differential equation (2.2) satisfied by $v = u - \overline{u}_0$ and write it in the form

$$v_t + (v + \overline{u}_0)^p v_x + \varepsilon v_{xxx} - \delta v_{xx} = 0.$$

Multiply this by v and integrate the result over the period $[0, L]$ to reach the differential relation

$$\frac{1}{2}\frac{d}{dt}\int_0^L v^2 dx + \delta \int_0^L v_x^2 dx = 0.$$

Making use of (2.4) then implies that

$$\frac{d}{dt}\int_0^L v^2 dx + 2\delta \left(\frac{2\pi}{L}\right)^2 \int_0^L v^2 dx \leq 0,$$

whence $|v(\cdot,t)|_2 \leq |v(\cdot,0)|_2 e^{-\delta(\frac{2\pi}{L})^2 t}$ or what is the same, $|u(\cdot,t) - \overline{u}_0|_2 = O(e^{-\delta(\frac{2\pi}{L})^2 t})$ as $t \to +\infty$. Notice that this result is independent of p and the size of the data.

Now suppose the initial data u_0 satisfies the condition in (2.9). Since the L_2-norm of v is strictly decreasing and the H^1-semi-norm $|v_x|_2$ is non-increasing, it follows that for $t > 0$, $\theta = \theta(|v|_2, |v_x|_2, \overline{u}_0) < \delta$. Upon applying (2.4) to $|v_{xx}(\cdot,t)|_2$ in (2.6) we obtain that

$$\frac{1}{2}\frac{d}{dt}|v_x(\cdot,t)|_2^2 + (\delta - \theta)\left(\frac{2\pi}{L}\right)^2 |v_x(\cdot,t)|_2^2 \leq 0,$$

from which it is deduced that

$$|v_x(\cdot,t)|_2 \leq |v_x(\cdot,0)|_2 \exp\left[\left(\frac{2\pi}{L}\right)^2 \int_0^t (\theta - \delta) ds\right].$$

Thus $|v_x(\cdot,t)|_2$ is seen to be exponentially decreasing to zero, and since by (2.8) $\theta(t) \leq \overline{\theta} \leq \delta$, the asymptotic form of this decay is $O(e^{(\overline{\theta}-\delta)(\frac{2\pi}{L})^2 t})$ as $t \to +\infty$.

Similar considerations apply to higher-order semi-norms. We pass over the details. □

Remarks.

(i) Biler (1984) has obtained detailed decay estimates for periodic solutions in the case $p < 2$. See subsection 4b below for more commentary on his work.

(ii) This result is in marked contrast to the behavior of global solutions of the *pure* initial-value problem for (1.2) corresponding to initial data $u_0 \in H^s(\mathbb{R})$, $s \geq 2$. Such solutions are expected to decay to zero as $t \to +\infty$. However, the rate of decay is algebraic in t as witnessed by the results of Amick *et al.* (1989), where for $p = 1$ it was shown that

$$\int_{-\infty}^{\infty} u^2(x,t)dx = O\left(t^{-1/2}\right)$$

and that this rate was sharp in general. (See also the results of Bona & Luo 1993, Bona, Promislow & Wayne 1994, Dix 1992, and Zhang 1994 for $p \geq 2$.)

Dissipative mechanisms other than the Burgers-type appearing in (1.2) arise in practice. One particularly appealing dissipative mechanism is a simple, zero$^{\text{th}}$-order term corresponding to the initial-value problem

$$u_t + u^p u_x + \varepsilon u_{xxx} + \sigma u = 0, \tag{2.14}$$
$$u(x,0) = u_0(x),$$

where $\sigma > 0$. A theory entirely similar to that worked out for the initial-value problem (1.3) applies to (2.14). As there are some interesting mathematical points that arise, and because it ties in with some of the numerical simulations in Section 4, a sketch of the theory for (2.14) is provided.

THEOREM 2.4. *Let u_0 be given initial data that is periodic of period L and suppose u_0 to lie in $H^s(0,L)$ for some $s \geq 2$. Let $\varepsilon > 0$ and $\sigma \geq 0$ be given.*

(1) If $p < 4$, then there is a unique global solution u of (2.14) corresponding to the above specification of data and parameters which is periodic in x of period L that lies in $C(0,T;H^s)$ for every $T > 0$. Moreover, $\|u(\cdot,t)\|_1$ is uniformly bounded for $t \in [0,\infty)$.

(2) If $p \geq 4$, there is a $T_0 > 0$ depending on $\|u_0\|_1$ and a unique periodic solution $u \in C(0,T_0;H^s)$ of (2.14) with initial data u_0. If σ is sufficiently large with respect to $\|u_0\|_2$, then T_0 can be taken arbitrarily large, the solution u is global, and $\|u(\cdot,t)\|_1$ is uniformly bounded for $t \in [0,\infty)$.

In all the above cases, the mapping $u_0 \mapsto u$ is continuous from H^s to $C(0,T;H^s)$.

Proof. As in Theorem 2.1, attention is concentrated on the case $p \geq 4$. The case $p < 4$ is essentially contained in the existing literature, and the local well-posedness theory for $p \geq 4$ is a straightforward application of nonlinear semigroup theory. As before, the crux of the matter is an *a priori* deduced, L_∞-bound on u, and it suffices for this to show the H^2-norm is bounded, at least on bounded time intervals.

The analog of (2.1) is

$$|u(\cdot,t)|_2^2 + 2\sigma \int_0^t |u(\cdot,s)|_2^2 ds = |u_0|_2^2, \tag{2.15}$$

from which one deduces immediately that

$$|u(\cdot,t)|_2 = e^{-\sigma t}|u_0|_2. \tag{2.16}$$

The next step, if one were following the line of argument in Theorem 2.1, would be to subtract the mean value, multiply the resulting equation by v_{xx}, where $v = u - \overline{u}_0$, and integrate over $[0,L]$. This does not appear to be effective in the present case. Indeed, solutions of (2.14) have exponentially decaying rather than constant mean values. However, multiplying equation (2.14) by u_{xxxx} and integrating over $[0,L]$ is useful; after suitable integrations by parts, this procedure leads to

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L u_{xx}^2 dx + \sigma\int_0^L u_{xx}^2\, dx &= \int_0^L (u^p u_{xx} + pu^{p-1}u_x^2)u_{xxx}dx \\ &= -\frac{5p}{2}\int_0^L u^{p-1}u_x u_{xx}^2 dx - p(p-1)\int_0^L u^{p-2}u_x^3 u_{xx}\, dx \\ &\leq \frac{5p}{2}|u|_\infty^{p-1}|u_x|_\infty|u_{xx}|_2^2 + p(p-1)|u|_\infty^{p-2}|u_x|_\infty^2|u_x|_2|u_{xx}|_2. \end{aligned} \tag{2.17}$$

Making systematic use of (2.4) for the zero-mean, periodic function u_x, together with the elementary inequality

$$|u|_\infty^2 \leq \frac{1}{L}|u|_2^2 + 2|u|_2|u_x|_2, \tag{2.18}$$

it is deduced from (2.17) that

$$\frac{1}{2}\frac{d}{dt}|u_{xx}(\cdot,t)|_2^2 + (\sigma - \Omega)|u_{xx}(\cdot,t)|_2^2 \leq 0, \tag{2.19}$$

where

$$\begin{aligned} \Omega = \Omega(|u|_2, |u_{xx}|_2) = \frac{5p}{2}\left(\frac{1}{L}|u|_2^2 + 2|u|_2^{3/2}|u_{xx}|_2^{1/2}\right)^{\frac{p-1}{2}} |u|_2^{1/4}|u_{xx}|_2^{3/4} \\ +p(p-1)\left(\frac{1}{L}|u|_2^2 + 2|u|_2^{3/2}|u_{xx}|_2^{1/2}\right)^{\frac{p-2}{2}} |u|_2|u_{xx}|_2. \end{aligned}$$

The function Ω is an increasing function of both its arguments. According to (2.16), $|u(\cdot,t)|_2$ is a decreasing function of $t \geq 0$. Hence if $\Omega|_{t=0} \leq \sigma$, then $|u_{xx}(\cdot,t)|_2$ is non-increasing for $t \geq 0$. In particular, if

$$\Omega\left(|u_0|_2, |u_{0_{xx}}|_2\right) \leq \sigma, \tag{2.20}$$

then the H^2-semi-norm $|u_{xx}(\cdot,t)|_2$ is bounded by its value at $t = 0$.

This concludes the proof of the theorem. □

COROLLARY 2.5. *Let $u_0 \in H^s(\mathbb{R})$ for some $s \geq 2$ and let u be the corresponding solution of the initial-value problem (2.14) with parameters $\varepsilon, \sigma > 0$ and $p \geq 4$. There is a constant $\nu = \nu_p$ depending only on p such that if*

$$|u_0|_2^{\frac{3p-2}{4}} |u_{0_{xx}}|_2^{\frac{p+2}{4}} \leq \nu_p\sigma, \tag{2.21}$$

then $\|u(\cdot,t)\|_2$ *is uniformly bounded for all* $t \geq 0$.

Proof. Simply take the limit as $L \to +\infty$ in (2.20). □

Consider again the situation where $u_0(x) = A\psi(Kx)$ for some positive constants A and K. In this case, (2.21) amounts to the inequality

$$A^p K \leq \nu\sigma, \tag{2.22}$$

where ν is a constant depending on p, $|\psi|_2$, and $|\psi_{xx}|_2$. In particular, if $u_0 = \lambda u_s(x,0)$ is the perturbed solitary wave discussed earlier, then our theory implies that the solution emanating therefrom will exist globally in time provided

$$\Sigma = \frac{\sigma^{2/3}\varepsilon^{1/3}}{A^p} \geq C_p', \tag{2.23}$$

where C_p' is a constant depending only on p.

Just as for the initial-value problem (1.3) with $\delta > 0$, solutions of (2.14) that satisfy the initial restriction (2.20), decay to zero exponentially in t. This is already established for the L_2-norm, and for data respecting (2.20), the differential inequality (2.19) implies it for the H^2-semi-norm. Other semi-norms also decay exponentially. Note that in this case, the exponential decay rates are still valid in the limit as $L \to +\infty$ applicable to initial data in $H^s(\mathbb{R})$.

The theory propounded in this section will provide a framework for the numerical simulations of (1.3) and (2.14) reported in Section 4. In the next section, a careful description and associated benchmarks of our numerical schemes are provided.

3. The Numerical Method

After a brief review of preliminaries about splines and Runge-Kutta methods, the numerical technique used to approximate solutions of (1.3) and (2.14) is presented. Throughout this section and the next, the spatial period L will be normalized to the value 1.

The numerical scheme is a straightforward adaptation of one of the fully discrete Galerkin methods for (1.1) that was described in detail and analyzed in Bona *et al.* (1994). This scheme will be briefly reviewed below. We shall study its application to the initial- and periodic-boundary-value problem (1.3). Entirely analogous considerations apply when the scheme is used to solve the problem (2.14).

Let $r \geqq 3$ be an integer and $S_h = S_h^r$ be the N-dimensional vector space of 1-periodic smooth splines of order r (piecewise polynomials of degree $r-1$) on $[0,1]$ with uniform mesh length $h = 1/N$, where N is a positive integer. As usual, the standard semi-discretization of (1.2) in the space S_h is then defined to be the differentiable map $u_h : [0,T] \to S_h$ satisfying

$$(u_{ht} + u_h^p u_{hx}, \chi) - \varepsilon(u_{hxx}, \chi_x) + \delta(u_{hx}, \chi_x) = 0 \qquad (3.1a)$$

for all $\chi \in S_h$ and $0 \leq t \leq T$, which is such that

$$u_h(0) = \Pi_h u_0. \qquad (3.1b)$$

Here $\Pi_h\, u_0$ is any of several approximations of u_0 in S_h (for example, L_2-projection, interpolant, etc.) that satisfy an estimate of the form

$$|\Pi_h\, u_0 - u_0|_2 \leq ch^r\, \|u_0\|_r \qquad (3.2)$$

for $u_0 \in H^r$, and where c is a constant independent of u_0 and h. (Constants independent of the discretization parameters will frequently occur in the sequel and will be denoted by c, C, etc.) For smooth, periodic initial data u_0 for which (3.2) holds, and assuming that the associated solution $u(x,t)$ of (1.3) is sufficiently smooth on $[0,T]$, it may be proved, following the analysis in Baker *et al.* (1983) that there is a constant $c = c(u)$ depending on the solution u, but not on the discretization parameter h, for which

$$\max_{0 \leq t \leq T} |u_h - u|_2 \leq c(u)\, h^r. \qquad (3.3)$$

Upon choosing a basis for S_h and representing u_h in terms of this basis, the problem (3.1) is seen to be an initial-value problem for a system of ordinary differential equations which may be written compactly in the form

$$\begin{aligned} u_{ht} &= F(u_h)\,,\ 0 \le t \le T, \\ u_h(0) &= \Pi_h\, u_0, \end{aligned} \tag{3.4}$$

where $F : S_h \longrightarrow S_h$ is defined by

$$(F(v), \chi) = -(v^p\, v_x, \chi) + (\varepsilon v_{xx} - \delta v_x, \chi_x) \tag{3.5}$$

for all $\chi \in S_h$. Having recognized (3.1) as the initial-value problem (3.4), an appropriate numerical method for the approximation of systems of ordinary differential equations leads to a fully discrete approximation to (1.3). In our companion paper on the non-dissipative case, use was made of the family of implicit Runge-Kutta methods of Gauss-Legendre type. These were found to possess favourable accuracy and stability properties when applied to (1.1). They can be extended in a straightforward way to the dissipative case at hand. In particular, the fact that these methods don't generate artificial damping is very helpful when small values of ν or σ are in question. For simplicity, consideration is given here only to the two-stage member of the Gauss-Legendre family. Let k be the time step (considered constant for the moment) and let $t_n = nk$, $n = 0, 1, 2, \ldots, J$, where J is some positive integer such that $Jk = T$. For each integer $n \in [0, J]$, we seek a function $U^n \in S_h$, with

$$U^0 = \Pi_h\, u_0 \tag{3.6}$$

and which approximates $u_n = u(\cdot, t_n)$, where $u(x,t)$ is the solution of (1.3). For $n = 0, 1, 2, \ldots, J-1$ the approximation U^{n+1} is constructed from U^n through two intermediate stages $U^{n,1}$ and $U^{n,2}$ in S_h that are solutions of the system of nonlinear equations

$$U^{n,i} = U^n + k \sum_{j=1}^{2} a_{ij}\, F(U^{n,j}), \qquad i = 1, 2\,, \tag{3.7a}$$

by the formula

$$U^{n+1} = U^n + k \sum_{j=1}^{2} b_j \, F(U^{n,j}), \tag{3.7b}$$

where the 2×2 matrix $A = (a_{ij})$ and the 2-vector $b = (b_1, b_2)^T$ that define the two-stage Gauss-Legendre method are given in the following tableau:

$$\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \hline b_1 & b_2 \end{array} = \begin{array}{cc} \frac{1}{4} & \frac{1}{4} - \frac{1}{2\sqrt{3}} \\ \frac{1}{4} + \frac{1}{2\sqrt{3}} & \frac{1}{4} \\ \hline \frac{1}{2} & \frac{1}{2} \end{array} .$$

In view of Lemma 3.2 of Bona *et al.* (1994), it is straightforward to generalize Proposition 3.1 of this reference to include the system under consideration here and thereby prove that for any given $U^n \in S_h$, there are elements $U^{n,1}$, $U^{n,2}$ of S_h that satisfy (3.7a). The scheme that then assigns to U^n the function U^{n+1} defined by (3.7b) is stable in L_2, which is to say that

$$|U^n|_2 \le |U^0|_2 \quad \text{for } 1 \le n \le J. \tag{3.8}$$

The latter property follows from the well known conservative nature of the Gauss-Legendre implicit Runge-Kutta schemes implied by the fact that $b_i a_{ij} + b_j a_{ji} - b_i b_j = 0$ for all the relevant i, j. The fact that the time-stepping scheme is conservative means that $|U^n|_2 = |U^0|_2$ for $n = 1, \ldots, J$ in the non-dissipative case $\delta = 0$ since then $(F(v), v) = 0$ for all $v \in S_h$, whereas the inequality (3.8) holds when $\delta > 0$ since then $(F(v), v) \le 0$ for all $v \in S_h$. It deserves remark that the use of a conservative time-stepping scheme to approximate solutions of a dissipative partial differential equation seemed to work very well in the numerical experiments to be reported later.

The following remarks are meant to summarize the convergence theory pertaining to the scheme just outlined. The theoretical analysis of this scheme for (1.3) with $\delta > 0$ follows in detail that already derived for the initial-value problem (1.1) with $\delta = 0$ in Bona *et al.* (1994). Adapting the convergence proof contained in the last-cited reference leads immediately to the conclusion that, provided the solution is smooth enough on the time interval $[0, T]$ and k/h is sufficiently small, there is a unique solution U^n of (3.7) and it satisfies the optimal-order L_2-error estimate

$$\max_{0 \le n \le J} |U^n - u(\cdot, t^n)|_2 \le c(k^4 + h^r). \tag{3.9}$$

Analogous estimates may be established for higher-order accurate Runge-Kutta time-stepping methods following the arguments in Karakashian & McKinney (1990). In the case of a q-stage Gauss-Legendre method with the same hypotheses about smoothness and k/h, (3.8) holds and (3.9) generalizes to the optimal-order L_2-estimate

$$\max_{0 \le n \le J} |U^n - u(\cdot, t^n)|_2 \le c(k^{2q} + h^r).$$

It is well known that the temporal rate $2q$ obtained for the Gauss-Legendre method is the best that can be achieved by a q-stage Runge-Kutta method.

A word is appropriate about the numerical linear algebra involved in the implementation of the scheme described above. At each time step the $2 \dim S_h \times 2 \dim S_h$ nonlinear system represented in (3.7a) is solved by a doubly iterative scheme based on Newton's method. The work is organized in such a way that each time step only requires the solution of a small number of sparse, $\dim S_h \times \dim S_h$ complex linear systems with the same coefficient matrix. The details of the construction and implementation of this solver are virtually the same as those used earlier in the non-dissipative case described in Section 4 of Bona *et al.* (1994). In the numerical experiments whose outcome is presented here, use was made of only the simplest iterative scheme considered in the last-cited reference, namely the one corresponding to one "outer" (Newton) and two "inner" iterations at each time step; this scheme requires solving only two sparse complex systems of size $\dim S_h \times \dim S_h$ per time step. We pass over the details since they are adequately covered in the previous work. For a theoretical analysis of the approximation of solutions of nonlinear systems such as (3.7a) by Newton's method, see Karakashian & McKinney (1994).

It transpires that some of the solutions whose approximation is of interest feature very rapid spatial and temporal changes. To keep track of the solution in such circumstances, it proved necessary to introduce adaptive mechanisms into the numerical procedure. These mechanisms took two distinct forms. First, a criterion was designed to refine the temporal step size as the solution began to evolve rapidly, and then a procedure was developed to cut the spatial meshlength in a neighborhood of points where large values of the dependent variable are detected. The experiments reported here were all performed using a computer code that featured both of these developments. Their implementation is presented and discussed in Section 5 of Bona *et al.* (1994). It is geared toward approximating solutions that are developing a single peak that apparently becomes infinitely high at a finite time t^* at a well-defined point x^*. The spatial refinement is controlled by making use of a local version of the inverse $L_\infty - L_2$ inequality satisfied by members of S_h, and the temporal step is defined by reference to the local, temporal variation of the quantity

$$I_3(v) = \int_0^1 \left[v^{p+2}(x) - \varepsilon \frac{(p+1)(p+2)}{2} \, v_x^2(x) \right] dx. \tag{3.10}$$

More precisely, the computer code looks at the variation of $I_3(U^n)$ with n and cuts the time step when a normalized version of this quantity's change exceeds a specified tolerance. The functional I_3 came to the fore in the earlier work on the GKdV equation (1.1) because I_3 is an exact invariant of this evolution. That is, if $u = u(x,t)$ is an H^1-solution of (1.1), then $I_3(u(\cdot,t))$ is time independent. Although I_3 is no longer an invariant of the evolution generated by (1.2) when $\delta > 0$, its variation was still found to generate an effective criterion for keeping errors under control by refining the temporal discretization.

4. Numerical Experiments

The scheme just presented was used by Bona *et al.* (1994) in a detailed study of the initial-value problem for equation (1.1). As mentioned already, considerable attention was paid to issues surrounding the solitary-wave solutions written in (2.12). Although the function $u_s(x,t)$ in (2.12a) is an exact solution of (1.1) when this equation is posed on the entire real line, one may use it to define a solution on $[0,1]$, say, with periodic boundary conditions imposed at the endpoints, as discussed in detail in Bona (1981). If A/ε is large, then u_s decays very rapidly away from its peak value. Hence if at $t=0$ the peak is centered at the midpoint of the spatial interval ($x_0 = 1/2$ in (2.12a)), then to machine accuracy it defines initial data supported in $[0,1]$. Consequently, it may be extended to define periodic initial data thusly,

$$\tilde{u}_0(x) = \sum_{j=-\infty}^{\infty} u_s(x+j,0) \tag{4.1a}$$

and this data used to determine a spatially periodic solution of (1.2), say. As shown in Bona (1981), the periodic solution of (1.1a) emanating from $\tilde{u}_0$ above is, to very good approximation over relatively long time scales, given by

$$u(x,t) = \sum_{j=-\infty}^{\infty} u_s(x+j,t). \tag{4.1b}$$

The same remarks are valid for any initial data that decays rapidly to zero outside a finite region of space, although the longer the spatial period $[0,L]$, say, the longer the time scale over which the solution of the periodic initial-value problem is well approximated by (4.1b).

It is known from earlier theory (see Bona *et al.* 1987) that the solitary-wave solutions u_s in (2.12) are orbitally stable for $p < 4$ and unstable if $p \geq 4$. Also, while (1.1) is always locally well posed in reasonable function classes, it is known to be globally well posed for initial data unrestricted in size only when $p < 4$, (see Kato 1979, 1983, or Schechter 1978). Two natural questions arise from these theoretical considerations. First, what happens when an unstable solitary wave is perturbed? Second, is (1.1) globally well posed if $p \geq 4$? The numerical simulations in Bona *et al.* (1986) and Bona *et al.* (1994) indicate the answers to these two questions are related. The conclusions to which reference was just made were based on the outcome of numerical experiments conducted with the fully discrete, adaptive scheme presented in the previous section. It appears that the instability of the solitary-wave solution manifests itself in a transformation to a similarity solution that goes on to develop a single-point blow-up in finite time. That is, there is a point $(x^*, t^*) \in [0,1] \times (0, \infty)$ such that $|u(x,t)| \to +\infty$ as $(x,t) \to (x^*, t^*)$, $t < t^*$. A detailed analysis of a considerable collection of numerical simulations support the more precise conjecture that the similarity solution corresponding to the blow-up has the form

$$u(x,t) = \frac{1}{(t^* - t)^{2/3p}} \chi \left(\frac{x^* - x}{(t^* - t)^{1/3}} \right) + \text{ bounded terms}, \tag{4.2}$$

where χ is a smooth, bounded function. These tentative conclusions in turn yield a negative answer to the question of whether or not the initial-value problem (1.3) is globally well posed for $p \geq 4$. Further experimentation showed that more general classes of initial data rapidly decomposed into profiles resembling a sequence of solitary waves, the largest of which loses stability and evolves into a similarity solution of the type indicated in (4.2) and thus proceeds to form a singularity in finite time.

4A. Blow-up for Small Dissipation

It is the aim in this section to understand how the results just reviewed are modified by the addition of a Burgers-type dissipative term as in equation (1.2) with small $\delta > 0$. As was seen in Section 2, for δ sufficiently large, the solution of the initial- and periodic-boundary-value problem (1.3) exists globally in time and decays to the mean of the initial data exponentially. However, if one sets as initial data a perturbed solitary wave that apparently blows up when $\delta = 0$, the numerical experiments indicate that for δ below a critical value δ_c, the resulting solution of (1.2) seems also to blow up. Moreover, the various diagnoses, to be introduced presently, pertaining to the putative blow-up give results that are virtually identical with those that obtain in the absence of dissipation, though the time t^* of blow-up is retarded by the dissipation. The value of δ_c depends upon the initial data of course. The computations show that for a slightly perturbed solitary wave of amplitude A, the critical value of δ_c has the form $\delta_c = \varepsilon A^p c_p$ where c_p is a constant depending only upon p. It is especially interesting to recall that the theoretical analysis of Section 2