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Model equations for waves in stratified fluids

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This paper is dedicated to T. Brooke Benjamin,
friend, colleague and mentor to all three of us.
His encouragement and ideas pervade our report. — footnote. please leave as it is.

Model equations for gravity waves in horizontally stratified fluids are considered. The theories to be addressed focus on stratifications featuring either a single pycnocline or neighbouring pairs of pycnoclines. Particular models investigated include the general version of the intermediate long-wave equation derived by Kubota, Ko and Dobbs to simulate waves in a model system consisting of two homogeneous layers separated by a narrow region of variable density, and the related system of equations derived by Liu, Ko and Pereira for the transfer of energy between waves running along neighbouring pycnoclines. Issues given rigorous mathematical treatment herein include the well-posedness of the initial value problem for these models, the question of existence of solitary wave solutions, and theoretical results about the stability of these solitary waves.

1. Introduction

Considered here are physical systems that serve as models for waves in laboratory studies and in certain regimes in oceans and lakes. While these systems are idealized, they nevertheless present some of the more important aspects of naturally occurring configurations, and, consequently, they are taken to be worthy of sustained investigation as a guide to practical issues.

In natural environments, various effects conspire to produce water basins having density variations with regard to depth. Often these variations consist of rather thin regions of substantial variation concatenated with larger regions of essentially homogeneous fluid. In this situation, a region of sharp variation is termed a pycnocline (see figures 1 and 2). Because of the density variation around a pycnocline, they may act as conduits of gravity-wave motion, just as does the density variation at a water–air interface. Such internal wave motions have been found to be a common feature of ocean and lake environments (cf. Apel *et al.* 1975; Farmer & Smith 1978; Fu & Holt 1982; Haury *et al.* 1978; Hunkins & Fliegel 1973; Osborne & Burch 1980; Sandstrom & Elliot 1984).

The model systems that are of concern here are composed of a fluid confined between two horizontal planes that possesses a stable uniform stratification ρ depending

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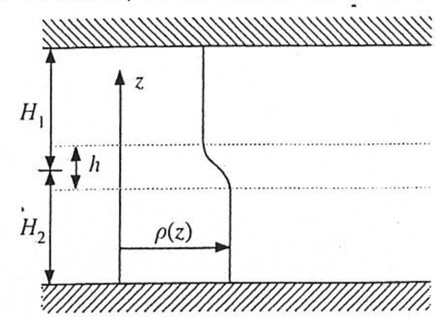
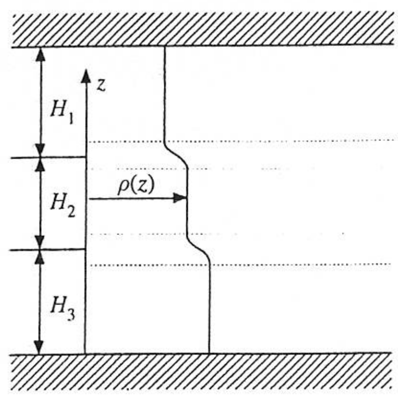


Figure 1.

Two-fluid system with a single pycnocline



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Two-fluid system with a pair of pycnoclines

Figure 2.

only upon the vertical coordinate z, say. Wave motion is considered which is uniform in the y-coordinate of a standard Cartesian frame oriented so that the x–y-plane is coplanar with the confining surfaces and z is oriented positively in the direction opposite to that in which gravity acts. In consequence, a two-dimensional analysis is appropriate in which the wavetrains propagate in the direction of the x-axis with a uniform structure in the y variable. While naturally occurring basins do not have rigid upper surfaces, constant depths, or uniform stratifications, these special aspects are sometimes a reasonable idealization. Moreover, conclusions drawn under the aegis of these idealizations appear to correspond to events occurring in the laboratory and in nature. For example, Davis & Acrivos (1967) found that the amplitude–wavespeed relationship predicted by a model equation of the type considered here showed good agreement with their laboratory observations; while the model system of Liu *et al.* (1980, see (1.4) below) successfully predicted the 'leapfrogging' phenomenon for internal waves which was observed later in the experiments of Weidman & Johnson (1982). Hence these model systems are taken to be an interesting and informative object for study.

Consider the case of a single pycnocline pictured in figure 1 in which the density variation is confined to a thin layer of height h whose center is located at a distance H_1 below the upper surface and a distance H_2 above the lower surface. The total distance H between the two bounding surfaces is thus $H = H_1 + H_2$. Kubota *et al.* (1978) have shown that if attention is restricted to small-amplitude long-wavelength waves, and viscous and diffusive effects are ignored, then the integro-differential equation

$$u_t + c_0 u_x + \alpha u u_x - c_0 \partial_x^2 \int_{-\infty}^{\infty} G(x-\xi) u(\xi, t)\, \mathrm{d}\xi = 0, \tag{1.1}$$

may be derived as an approximation to the full Euler equations. Here x is, as above, proportional to distance in the direction of propagation, t is proportional to elapsed time, and subscripted variables denote partial derivatives. The dependent variable $u(x,t)$ is related to the stream function $\psi(x,z,t)$ by $\psi(x,z,t) = u(x,t)\eta(z)$, where $\eta(z)$ determines the vertical structure of the wave within the pycnocline and is a solution of the Sturm–Liouville problem

$$\begin{cases} (\rho_0(z)\eta'(z))' - \frac{1}{c_0^2}\rho_0'(z)\eta(z) = 0, & \text{for } 0 < z < 1, \\ \eta'(0) = \eta'(1) = 0. \end{cases}$$

The function $\rho_0(z)$ describes the undisturbed density profile within the pycnocline, and the variable z has been rescaled so that $z = 0$ and $z = 1$ correspond to the bottom and top of the pycnocline, respectively. The constant α and the kernel function G in (1.1) are given by

$$\alpha = \frac{3}{2I} \int_0^1 \rho_0(z)(\eta'(z))^3\, \mathrm{d}z$$

and

$$\begin{aligned} G(x) = {} & \frac{\beta_1}{2H_1}\left\{\coth\left(\frac{\pi x}{2H_1}\right) - \operatorname{sgn}(x)\right\} \\ & + \frac{\beta_2}{2H_2}\left\{\coth\left(\frac{\pi x}{2H_2}\right) - \operatorname{sgn}(x)\right\}, \end{aligned} \tag{1.2}$$

where

$$\beta_1 = \frac{\rho_0(1)\eta(1)^2}{2I}, \quad \beta_2 = \frac{\rho_0(0)\eta(0)^2}{2I}, \quad I = \int_0^1 \rho_0(z)(\eta'(z))^2\, \mathrm{d}z$$

and H_1 and H_2 are now dimensionless parameters (scaled by wavelength).

Equation (1.1) with the general kernel given in (1.2) has received relatively little attention despite its obvious importance. Rather more interest has been associated with cases of (1.1) which correspond to special geometries, namely the formal limit wherein one of the depths H_1 or H_2 tends to zero or the case where $H_1 = H_2$. In both these situations, (1.1) reduces to the well-studied intermediate long-wave equation (ILW equation henceforth) with kernel

$$G(x) = \frac{\beta}{2H}\left\{\coth\left(\frac{\pi x}{2H}\right) - \operatorname{sgn}(x)\right\}. \tag{1.3}$$

Although the ILW equation (1.1)–(1.3) is less widely applicable as a model equation than the more general version (1.1)–(1.2), it has attracted more effort because it falls within the class of equations which are solvable by an inverse-scattering transform (Kodama *et al.* 1982), and because it possesses explicit solitary-wave and multisoliton solutions (Joseph 1977; Joseph & Egri 1978). Moreover, the ILW equation is related in an interesting way to the Korteweg–de Vries and Benjamin–Ono equations, to which it reduces in appropriate limiting situations. The initial-value problem for the ILW equation along with the Benjamin–Ono and Korteweg–de Vries limits has been rigorously analyzed in the article of Abdelouhab *et al.* (1989), whilst the orbital stability of the solitary-wave solutions was established by Albert & Bona (1991).

In the present article, various aspects of (1.1) with the more general kernel (1.2) are examined. A global theory for the initial-value problem is established following the lines laid down in Abdelouhab *et al.* (1989), an existence theory for solitary-wave solutions is put forth making use of concentration-compactness arguments as in Weinstein (1987) and a certain range of these waves is shown to be stable.

A more complex situation is envisioned in figure 2 in which the underlying stratification features two pycnoclines. In case these pycnoclines are close together relative to the underlying wavelengths, a pair of coupled Korteweg–de Vries-type equations derived by Gear & Grimshaw (1984) gives an approximate description of the system. In case the pycnoclines are relatively far apart, but not so distant that motion on one is decoupled from the other, Liu *et al.* (1980) have derived a model consisting of a coupled pair of ILW-type equations, namely

$$\left.\begin{aligned} u_t + \alpha_1 u u_x - \gamma_1 \partial_x^2 \mathcal{H}_1(u) - \gamma_2(\partial_x^2 \mathcal{H}_2(u) - \partial_x^2 \mathcal{J}(v)) = 0, \\ v_t - \Delta c v_x + \alpha_2 v v_x - \gamma_3 \partial_x^2 \mathcal{H}_3(v) - \gamma_4(\partial_x^2 \mathcal{H}_2(v) - \partial_x^2 \mathcal{J}(u)) = 0, \end{aligned}\right\} \tag{1.4}$$

where the dispersion operator $\mathcal{H}_i$ ($i = 1$, 2 or 3) is defined to be convolution with the kernel G_i given in (1.3) with $\beta = 1$ and H replaced by H_i and $\mathcal{J}$ is convolution with a kernel J given by

$$J(x) = \frac{1}{2H_2} \tanh\left(\frac{\pi x}{2\,H_2}\right). \tag{1.5}$$

The system (1.4) is written in a frame of reference moving with the speed c_1 of infinitesimal waves of extreme length on the upper pycnocline. The dependent values $u(x,t)$ represent the deviation of the centre of the upper pycnocline from its equilibrium position at the point x at time t and $v(x,t)$ represents, similarly, the deviation of the lower pycnocline. The parameters α_1, α_2, γ_1, γ_2, γ_3 and γ_4 correspond to the wave environment, with α_1, γ_1 and γ_2 relating to the upper pycnocline and α_2, γ_3 and γ_4 to the lower pycnocline, while Δc is the difference $c_1 - c_2$ between the linear long-wave velocities on the two pycnoclines. (The choice of signs in the individual terms of (1.4) will be convenient in § 4 below.) The model of Gear & Grimshaw is analysed in a companion paper (Bona *et al.* 1992). Here, attention is given to the coupled system (1.4) as its properties parallel those of the single equation (1.1). In particular, a global well-posedness result is proved for (1.4), and the existence of travelling-wave solutions is demonstrated.

The plan of the paper is as follows. Section 2 deals with the initial-value problems for both (1.1) and (1.4). Issues concerning travelling-wave solutions receive attention in §§ 3 and 4. The final section features a summary of the earlier accomplishments together with some discussion of possible related lines of inquiry.

2. The Cauchy problems

Consideration is given to the pure initial-value problems for (1.1) and for (1.4). The theory for (1.1) follows readily from results in Abdelouhab *et al.* (1989). Interestingly, considerations similar to those that come to the fore for (1.1) suffice to conclude a theory for the system (1.4).

The notation used throughout is that which is currently standard in the theory of partial differential equations. Thus for $1 \leqslant p < \infty$, $L_p(\Omega)$ stands for the pth-power integrable, real-valued functions defined on a subset Ω of $\mathbb{R}$, the real line, with the usual modification if $p = \infty$. In most instances here where $L_p(\Omega)$ arises, the subset Ω will be the whole of $\mathbb{R}$, and in this case we will usually write L_p for $L_p(\mathbb{R})$. The standard norm on L_p will be denoted by $|\cdot|_p$. For $s \geqslant 0$, the L_2-based Sobolev spaces $H^s = W_2^s$ of L_2-functions whose derivatives up to order s lie in L_2 will also intervene substantially. The space H^s is a Hilbert space with inner product given by

$(1+k^2)^s$

$$\langle f, g\rangle_s = \int_{-\infty}^{\infty} (1+k^2)^{s\cancel{/2}}\, \widehat{f(k)}\overline{\widehat{g(k)}}\, \mathrm{d}k$$

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and norm defined to be

$$\|f\|_s = \langle f, f\rangle_s^{1/2}.$$

(Here and throughout the paper, a circumflex over a function f will denote that function's Fourier transform with respect to the spatial variable x, defined for $k \in \mathbb{R}$ by $\hat{f}(k) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x)\,\mathrm{d}x$. The overbar signifies complex conjugation.) Finally, if X and Y are any Banach spaces, then $B(X, Y)$ will denote the space of bounded linear maps from X to Y with the operator norm, while for $T > 0$, $C(0, T; X)$ is the collection of continuous maps $g : [0, T] \to X$ with the maximum norm.

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Consider first equation (1.1). After moving to a travelling frame of reference and rescaling, the initial-value problem for the general ILW equation (1.1) takes the form

$$u_t + uu_x - \beta_1 M_1 u_x - \beta_2 M_2 u_x = 0, \quad u|_{t=0} = u_0, \tag{2.1}$$

where M_i is the Fourier multiplier operator defined by

$$\widehat{M_i w}(k) = m_i(k)\widehat{w}(k) \tag{2.2}$$

and the symbol m_i of M_i is

$$m_i(k) = k\coth(kH_i) - \frac{1}{H_i} \tag{2.3}$$

for $i = 1, 2$.

Following the line of argument appearing in § 7 of Abdelouhab (1989), the operator M_i is decomposed as $M_i = -H\partial_x - K_i$, where H here denotes the Hilbert transform. A short calculation reveals that K_i is a Fourier multiplier operator with symbol a_i given as

$$a_i(k) = |k| - k\coth(kH_i) + \frac{1}{H_i} \tag{2.4}$$

for $i = 1, 2$, and that for all k,

$$-\frac{1}{H_i} \leqslant a_i(k) \leqslant 0. \tag{2.5}$$

In consequence of this decomposition, the equation in (2.1) is seen to be of the form

$$u_t + uu_x + (\beta_1 + \beta_2)Hu_{xx} + Ku_x = 0, \tag{2.6}$$

where $K = \beta_1 K_1 + \beta_2 K_2$ is a self-adjoint operator of order 0 and so bounded on all the L_2-based Sobolev spaces H^s. Because of these observations, the initial-value problem (2.1) is realized as an initial-value problem for the Benjamin–Ono equation perturbed by the term Ku_x. This allows us to take over intact some of the results in § 6 of Abdelouhab (1989) that apply to the ILW equation (1.1)–(1.3).

Theorem 2.1. *Let u_0 be initial data for the initial-value problem (2.1).*

(i) If $u_0 \in H^{j/2}$ for $j = 0$, 1, 2 or 3, then there exists a weak solution u of (2.1) with initial value u_0 which, for each $T > 0$, lies in $L_\infty(\mathbb{R}^+; H^{j/2}) \cap L_2(0, T; H^{(j+1)/2}_{\text{loc}})$.

(ii) If $u_0 \in H^s$ where $s > 3/2$, then this solution is unique and, for each $T > 0$, belongs to $C^k(0, T; H^{s-2k})$ for k such that $s - 2k \geqslant -1$. Moreover, for each $T > 0$, the correspondence that associates u to u_0 is continuous from H^s to $C^k(0, T; H^{s-2k})$ for all such k and continuous from H^s into $L_2(0, T; H^{s+1/2}_{\text{loc}})$.

Remarks. The existence of weak solutions goes back to the paper of Saut (1979). The fact that the equation preserves the spaces $H^{j/2}$, $1 \leqslant j \leqslant 3$, or H^s for $s > 3/2$ as it evolves from initial data is a consequence of the arguments developed by Abdelouhab *et al.* for the Smith equation in § 7 of Abdelouhab (1989). The local smoothing results stated in the theorem are like those proved for the Benjamin–Ono equation by Ginibre & Velo (1991), Ponce (1990) and Tom (1990). They are valid for equation (2.1) because it can be written in the form (2.6) and the smoothing results are stable to a perturbation of the Benjamin–Ono equation by a term of the form Ku_x, where K is a bounded self-adjoint operator on the spaces H^s, $s \geqslant 0$. The local smoothing allows one to strengthen the just-quoted results for weak solutions. Indeed, by following the arguments in the references above, but as applied to the perturbed Benjamin–Ono equation (2.6), one derives readily the next result.

et al.

Corollary 2.2. *Let u_0 be initial data for the problem (2.1). Then for $j = 0$, 1, 2, 3, if $u_0 \in H^{j/2}$, then there is a solution u of (2.1) with initial data u_0 which, for each $T > 0$, lies in $C(0, T; H^{j/2}) \cap L_2(0, T; H^{(j+1)/2}_{\text{loc}})$.*

We turn now to the Liu *et al.* system (1.4), which can be rewritten in the form

$$\left.\begin{aligned} u_t + \alpha_1 u u_x - \gamma_1 (M_1 u)_x - \gamma_2[(M_2 u)_x - (Nv)_x] = 0, \\ v_t - \Delta c v_x + \alpha_2 v v_x - \gamma_3 (M_3 v)_x - \gamma_4[(M_2 v)_x - (Nu)_x] = 0, \end{aligned}\right\} \tag{2.7}$$

where the operators M_i ($i = 1$, 2, 3) are defined by (2.2) and (2.3), and N is the Fourier multiplier operator defined by

$$\widehat{Nw}(k) = n(k)\widehat{w}(k), \tag{2.8}$$

whose symbol n is

$$n(k) = \frac{k}{\sinh kH_2}. \tag{2.9}$$

Note that, apart from the coupling terms involving $(Nv)_x$ and $(Nu)_x$, each of the equations in (2.7) is of the same form as the general ILW equation. Moreover, from (2.8) and (2.9) it is apparent that the operator $-\partial_x N$ is a smoothing operator, since its symbol $(ik^2/\sinh kH_2)$ vanishes rapidly as $|k| \to \infty$. (In fact, for any $s \in \mathbb{R}$, $-\partial_x N$ carries H^s to $H^\infty = \cap_{k\in\mathbb{Z}} H^k$.) System (2.7) has, therefore, the following structure:

$$\left.\begin{aligned} u_t + \alpha_1 u u_x + (\gamma_1 + \gamma_2) H u_{xx} + T_1 u_x + S_1 v = 0, \\ v_t + \alpha_2 v v_x + (\gamma_3 + \gamma_4) H v_{xx} + T_2 v_x + S_2 u = 0, \end{aligned}\right\} \tag{2.10}$$

where H is again the Hilbert transform, T_1 and T_2 are operators of order zero, and S_1 and S_2 are smoothing operators.

Concerning the Cauchy problem associated to (2.10), one derives exactly the same results as those stated in theorem 2.1. To see this, first observe that the total 'energy' $\int (\gamma_4 u^2 + \gamma_2 v^2)$ of the system is conserved (multiply $(2.7)_1$ by $\gamma_4 u$, $(2.7)_2$ by $\gamma_2 v$, add, and integrate the resulting equality over $\mathbb{R}$). Then treat each equation in (2.10) as a perturbed Benjamin–Ono equation, as in the case of the general ILW equation. The coupling terms are harmless, due to the smoothing properties of S_1 and S_2. The following result emerges from these considerations.

Theorem 2.3. *Let u_0 and v_0 be initial data for the problem (2.7). For $j = 0$, 1, 2, 3, if $u_0, v_0 \in H^{j/2}$, then there is a solution pair u, v of (2.7), both of which lie in $C(0,T; H^{j/2}) \cap L_2(0,T; H_{\text{loc}}^{(j+1)/2})$ for each $T > 0$. If $u_0, v_0 \in H^s$ for $s \geqslant \frac{3}{2}$, then the solution pair is unique and lies in $C(0,T;H^s) \cap L_2(0,T; H_{\text{loc}}^{(s+1/2)})$. In this case, the correspondence that associates to the initial data (u_0, v_0) the solution (u, v) is continuous as a mapping between the associated spaces.*

3. Existence of solitary waves

In the context of equations such as (2.1), the term 'solitary wave' usually refers to a localized solution which propagates unchanged in form at constant velocity. The goal of this section is to prove that such solutions exist for equation (2.1). More precisely, it will be shown that solutions to (2.1) exist which are of the form $u(x,t) = \phi(x - Ct)$, where C is a positive constant and the profile function $\phi(\xi)$ is even and decreases rapidly in both directions away from its maximum point at $\xi = 0$. For brevity, in what follows we will refer to such a function as a 'decreasing function of $|x|$'. Note that if a decreasing function of $|x|$ is in a class such as L_p for some $p \in [1, \infty)$, then the function must be non-negative and tend to zero as $|x| \to \infty$.

For sufficiently smooth functions, ϕ, that are appropriately evanescent at infinity, it is easily seen that $u(x,t) = \phi(x - Ct)$ will satisfy (2.1) if, and only if, ϕ satisfies the equation

$$(C + \beta_1 M_1 + \beta_2 M_2)\phi = \tfrac{1}{2}\phi^2. \tag{3.1}$$

Therefore, to prove the existence of solitary-wave solutions of (2.1) it suffices to establish the following result.

Theorem 3.1. *For every $C > 0$, equation (3.1) has a solution $\phi(x) \in H^\infty$ which is a decreasing function of $|x|$.*

The gist of our argument in favour of theorem 3.1 is taken in large part from Weinstein (1987), where the outline of a proof of existence of solitary waves is 9
presented for a general class of equations. In the present context, the argument which appears in Weinstein (1987) requires supplementation at two important points. The first is at the passage from equation (3.4) to equation (3.6) of Weinstein (1987), which is only valid under certain special conditions on the dispersion operator, and the second is at inequality (3.20) of Weinstein (1987), which requires additional justification in the case when the dispersion operator is nonlocal. Because addressing these two issues affects the structure of the entire argument, it has been thought proper to present here a complete and self-contained exposition.

We begin with some general commentary about equation (3.1) and its solutions. Let $m(k)$ be the function defined by

$$m(k) = \beta_1 m_1(k) + \beta_2 m_2(k).$$

Lemma 3.2. *Let $\mu > 0$ be given, and define the function $K = K_\mu$ by $\widehat{K}(k) = (\mu + m(k))^{-1}$. Then $K(x) > 0$ for all $x \in \mathbb{R}$. Moreover, K is a decreasing function of $|x|$, and is a member of $L_p(\mathbb{R})$ for every $p \in [1, \infty)$.*

Lemma 3.3. *Suppose $\phi \in L_2$ solves equation (3.1) in the sense of distributions on $\mathbb{R}$. Then ϕ is in $H^\infty(\mathbb{R})$, and (3.1) holds in the pointwise sense.*

Proof. A proof of lemmas 3.2 and 3.3, in the case when one of β_1 or β_2 is equal to zero, appears in lemmas 1 and 4 of Albert (1995). The proof in the general case is quite similar, and we therefore content ourselves with an indication of the changes required to make the earlier line of argument decisive. An elementary but tedious computation shows that, as the complex variable z ranges over the upper half plane, the function $[\mu + m(z)]^{-1}$ has poles only at the purely imaginary points $z = \mathrm{i}\nu_j$, $(j = 0, 1, 2, \ldots)$, where $0 < \nu_0 < \nu_1 < \cdots$. Because $[\mu + m(z)]^{-1}$ decays like $|z|^{-1}$ as $|z| \to \infty$, Jordan's lemma (cf. Whittaker & Watson (1952), ch. 6) and the Residue Theorem imply that the integral $K(x) = (1/2\pi) \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}[\mu + m(k)]^{-1}\,\mathrm{d}k$ is equal to $-\mathrm{i}$ times the sum of the residues of the integrand in the upper half plane, whence

$$K(x) = \sum_{j=0}^{\infty} 2\pi\gamma_j \mathrm{e}^{-\nu_j|x|}, \tag{3.2}$$

where

$$\gamma_j = \left[\sum_{k=1}^{2} \beta_k \frac{(2\nu_j H_k - \sin 2\nu_j H_k)}{2\sin^2(\nu_j H_k)}\right]^{-1}.$$

From this point on, the proofs proceed exactly as in the case when one of β_1 or β_2 is zero. ∎

Remarks. From lemma 3.2 and the observation that (3.1) may be rewritten in the form $\phi = (1/2)K * \phi^2$, it follows immediately that any non-trivial L_2-solution of (3.1) must be positive everywhere.

A closer examination of the parameters ν_j and γ_j, $j = 0, 1, \ldots$, appearing in (3.2) shows that the sequence $\{\gamma_j\}_{j=0,1,\ldots}$ is bounded and tends to zero as j becomes unboundedly large like c/j for some constant c, while the sequence $\{\nu_j\}_{j=0,1,\ldots}$ grows about linearly with j as j tends to infinity. In consequence, it is seen that for any $\nu < \nu_0$, $\mathrm{e}^{-\nu|x|}K(x) \in L_p$ for any $p \in [1, \infty)$. This point will be useful later in this section when the rate of decay of solitary-wave solutions of (2.1) is addressed.

Now define a nonlinear functional J on $H^{1/2}$ by

$$J(f) = \int_{-\infty}^{\infty} (Lf)^2\,\mathrm{d}x,$$

where L is the Fourier multiplier operator defined by

$$\widehat{Lf}(k) = (C + m(k))^{1/2}\hat{f}(k).$$

Since the symbol $(C + m(k))^{1/2}$ is everywhere positive and is comparable to $|k|^{1/2}$ for large values of $|k|$, it follows from the identity

$$J(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (C + m(k))|\hat{f}(k)|^2\,\mathrm{d}k, \tag{3.3}$$

that the quantity $J(f)$ is equivalent to the square of the $H^{1/2}$-norm of f.

Consideration is given to two constrained minimization problems for the functional J which will be referred to below as problems (P1) and (P2).

where their method of proof is attributed to Lieb.

Problem (P1) is that of minimizing the functional $J(f)$ over the set

$$S_1 = \left\{ f \in H^{1/2} : \int_{-\infty}^{\infty} f(x)^3 \, \mathrm{d}x = 1 \right\},$$

while problem (P2) is that of minimizing $J(f)$ over the set

$$S_2 = \left\{ f \in H^{1/2} : \int_{-\infty}^{\infty} |f(x)|^3 \, \mathrm{d}x = 1 \right\}.$$

It will be seen below that theorem 3.1 follows readily once it has been shown that a minimizing function for the problem (P1) exists in $H^{1/2}$. For technical reasons, however, it is preferable to work with problem (P2) instead. If the existence argument is to be based on problem (P2), then it is necessary to establish a relation between the solutions of the two problems. This will be accomplished with the aid of the next two lemmas, which are essentially taken from the Weinstein (1987), ~~who attributes the method of their proof to Lieb)~~.

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Lemma 3.4. *For every $f \in H^{1/2}$, one has $|f| \in H^{1/2}$ and $J(|f|) \leqslant J(f)$.*

Proof. If $g = |f|$, then it follows from lemma 3.2 that $K * g(x) \geqslant K * f(x)$ for all $x \in \mathbb{R}$ and every $\mu > 0$. In consequence, one has that

$$\begin{aligned} \int_{-\infty}^{\infty} (\mu + m(k))^{-1} |\widehat{g}(k)|^2 \, \mathrm{d}k &= 2\pi \int_{-\infty}^{\infty} g(x)(K * g)(x) \, \mathrm{d}x \\ &\geqslant 2\pi \int_{-\infty}^{\infty} f(x)(K * f)(x) \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} (\mu + m(k))^{-1} |\widehat{f}(k)|^2 \, \mathrm{d}k. \end{aligned} \tag{3.4}$$

Since $\int_{-\infty}^{\infty} |\widehat{g}|^2 \, \mathrm{d}k = \int_{-\infty}^{\infty} |\widehat{f}|^2 \, \mathrm{d}k$ by Parseval's identity, it follows that

$$\int_{-\infty}^{\infty} \mu \left[1 - \left(1 + \frac{m(k)}{\mu} \right)^{-1} \right] |\widehat{f}(k)|^2 \, \mathrm{d}k \geqslant \int_{-\infty}^{\infty} \mu \left[1 - \left(1 + \frac{m(k)}{\mu} \right)^{-1} \right] |\widehat{g}(k)|^2 \, \mathrm{d}k.$$

Now taking the limit as $\mu \to \infty$ on both sides of the preceding inequality and using the monotone convergence theorem gives

$$\int_{-\infty}^{\infty} m(k) |\widehat{f}(k)|^2 \, \mathrm{d}k \geqslant \int_{-\infty}^{\infty} m(k) |\widehat{g}(k)|^2 \, \mathrm{d}k,$$

which together with the definition of J in (3.3) yields the desired result. ∎

Recall that for each f in $L_2(\mathbb{R})$, one may define the *symmetric decreasing rearrangement* of f to be the unique function f^* with domain $\mathbb{R}$ which is a decreasing function of $|x|$ and has the property that the sets $\{x : |f(x)| > a\}$ and $\{x : f^*(x) > a\}$ have the same measure for every $a \geqslant 0$. In particular, one has $|f^*|_p = |f|_p$ for $1 \leqslant p \leqslant \infty$ (cf. Hardy *et al.* (1934), ch. X).

Lemma 3.5. *For every $f \in H^{1/2}$, one has $f^* \in H^{1/2}$ and $J(f^*) \leqslant J(f)$.*

Proof. A lemma of Riesz (1930) states that if g is any even function on $\mathbb{R}$ which decreases with increasing values of $|x|$, then

$$\int_{-\infty}^{\infty} f^*(x) \, (g * f^*) \, (x) \mathrm{d}x \geqslant \int_{-\infty}^{\infty} f(x) \, (g * f) \, (x) \mathrm{d}x.$$

In particular, this inequality holds when g is replaced by $K = K_\mu$ (for any $\mu > 0$). Also, by Parseval's identity, $\int_{-\infty}^{\infty} |\widehat{f^*}|^2 \,\mathrm{d}k = \int_{-\infty}^{\infty} |\widehat{f}|^2 \,\mathrm{d}k$. The result then follows exactly as in the proof of the preceding lemma. ∎

Lemma 3.6. *If f_0 is a minimizer for problem (P2), then $|f_0|^*$ is a minimizer for problem (P1).*

Proof. Suppose f_0 is a minimizer for problem (P2). Since rearrangement preserves the L_3-norm, it follows immediately from lemmas 3.4 and 3.5 that $|f_0|^*$ is a minimizer for problem (P2). If $|f_0|^*$ is not a minimizer for problem (P1), then there exists $g \in H^{1/2}$ such that $\int_{-\infty}^{\infty} g^3 \,\mathrm{d}x = 1$ and $J(g) < J(|f_0|^*)$. Letting

$$h = \frac{g}{\left(\displaystyle\int_{-\infty}^{\infty} |g|^3 \,\mathrm{d}x\right)^{1/3}},$$

one obtains

$$J(h) = \frac{1}{\left(\displaystyle\int_{-\infty}^{\infty} |g|^3 \,\mathrm{d}x\right)^{2/3}} J(g) \leqslant J(g),$$

since $\int_{-\infty}^{\infty} |g|^3 \,\mathrm{d}x \geqslant \int_{-\infty}^{\infty} g^3 \,\mathrm{d}x = 1$. Therefore $J(h) < J(|f_0|^*)$. But $\int_{-\infty}^{\infty} |h|^3 \,\mathrm{d}x = 1$, so this contradicts the fact that $|f_0|^*$ is a minimizer for problem (P2). ∎ Close up

Let $\{f_j\}_{j=1,2,\ldots}$ be a minimizing sequence for problem (P2), so that $\int_{-\infty}^{\infty} |f_j|^3 \,\mathrm{d}x = 1$ for all j, while $\lim_{j\to\infty} J(f_j) = \inf_{f\in S_2} J(f)$. In general, because the inclusion of $H^{1/2}(\mathbb{R})$ into $L_3(\mathbb{R})$ is not compact, one cannot extract a subsequence of the sequence $\{f_j\}$ which converges in L_3-norm. As in Weinstein (1987), this difficulty will be circumvented by means of Lions' 'concentration compactness' principle. Briefly put, Lions' principle provides a method for proving that a subsequence of $\{f_j\}$ can be found such that, after being suitably translated, each function in the subsequence is 'concentrated' on a fixed bounded interval. This then enables one to bring into play the compactness of the inclusion of $H^{1/2}(\Omega)$ into $L_3(\Omega)$ for bounded sets Ω.

For each j, define a function $Q_j(t)$ on $\mathbb{R}^+$ by

$$Q_j(t) = \sup_{y\in\mathbb{R}} \int_{y-t}^{y+t} |f_j(x)|^3 \,\mathrm{d}x.$$

Then $Q_j(t)$ is a non-decreasing non-negative function on $\mathbb{R}^+$, with $\lim_{t\to\infty} Q_j(t) = 1$ for every j. Now, for any finite interval $[a, b] \in \mathbb{R}^+$, the space of non-decreasing functions from $[a, b]$ to $[0, 1]$ with the topology of pointwise convergence is compact (by Tychonoff's compactness theorem) and metrizable (since on this space, the topology of pointwise convergence is equivalent to that of uniform convergence). Therefore, arguing first on finite subintervals of $\mathbb{R}^+$ and then using a Cantor diagonalization argument to pass to all of $\mathbb{R}^+$, one can extract from the sequence $\{Q_j\}_{j=1,2,\ldots}$ a subsequence $\{Q_{j_n}\}_{n=1,2,\ldots}$ which converges pointwise on $\mathbb{R}^+$ to a limit function $Q(t)$. (For brevity, the notations $\{Q_j\}$ and $\{f_j\}$ will henceforth be used to refer to the subsequence $\{Q_{j_n}\}$ and the corresponding subsequence $\{f_{j_n}\}$.) Since $Q(t)$ is also non-decreasing, non-negative, and bounded above by one, the number

$$\alpha = \lim_{t\to\infty} Q(t)$$

exists and satisfies $0 \leqslant \alpha \leqslant 1$. The concentration-compactness principle rests on the fact that the three possibilities $\alpha = 0$, $0 < \alpha < 1$ and $\alpha = 1$ correspond to quite distinct types of limiting behavior of the sequence $\{f_j\}$ as $j \to \infty$, which are suggestively labeled by Lions as 'vanishing', 'dichotomy' and 'compactness', respectively, (see Lions 1984). Typically, one proves compactness by ruling out the first two possibilities. We begin by ruling out 'vanishing' using an argument of Brezis which appears in Lieb (1983). .

remove comma

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Lemma 3.7. *Let $\phi \in C^\infty(\mathbb{R})$ be such that $0 \leqslant \phi \leqslant 1$ everywhere, $\phi(x) = 0$ for $x \notin [-2, 2]$, and $\sum_{j\in\mathbb{Z}} \phi(x-j) = 1$ for all $x \in \mathbb{R}$. Then there exists $C_0 > 0$ such that for all $f \in H^{1/2}$,*

$$\sum_{j\in\mathbb{Z}} \|\phi(x-j)f(x)\|_{1/2}^2 \leqslant C_0\|f\|_{1/2}^2.$$

Proof. For any $s \in \mathbb{R}$, let $l_2(H^s)$ denote the Hilbert space of all sequences $\{g_j\}_{j\in\mathbb{Z}}$ such that $g_j \in H^s$ for each j and $\sum_{j\in\mathbb{Z}} \|g_j\|_s^2 < \infty$. For each $f \in H^s$, define Tf to be the sequence of functions $\{\phi(x-j)f(x)\}_{j\in\mathbb{Z}}$. Clearly $T : H^s \to l_2(H^s)$ boundedly for $s = 0$ and $s = 1$. It then follows by interpolation (cf. Bergh & Löfstrom (1976), § 5.6) that $T \in B(H^{1/2}, l_2(H^{1/2}))$. ■

Lemma 3.8. *Let ϕ be as in lemma 3.7 and let A be any positive real number. Then there exists $C_1 > 0$ depending only on ϕ and A with the property that for every $f \in H^{1/2}$ which satisfies $\|f\|_{1/2} \leqslant A$ and which is not the zero function, there exists an integer k such that*

$$\|\phi(x-k)f(x)\|_{1/2}^2 \leqslant (1 + C_1|f|_3^{-3})|\phi(x-k)f(x)|_3^3.$$

Proof. Since $\sum_{j\in\mathbb{Z}} |\phi(x-j)| = 1$, with no more than four of the terms in the sum being non-zero at any given value of x, it follows that there exists a constant $C_2 > 0$ such that $\sum_{j\in\mathbb{Z}} |\phi(x-j)|^3 \geqslant C_2$ for all $x \in \mathbb{R}$. We claim that the statement of the lemma is satisfied by the constant $C_1 = (C_0A^2)/C_2$, where C_0 is the constant in lemma 3.7. To prove this, assume to the contrary that there exists a non-zero f such that $\|f\|_{1/2} \leqslant A$ and

$$\|\phi(x-j)f(x)\|_{1/2}^2 \geqslant (1 + C_1|f|_3^{-3})|\phi(x-j)f(x)|_3^3$$

holds for every $j \in \mathbb{Z}$. Summing over j and applying lemma 3.7, one obtains

$$C_0\|f\|_{1/2}^2 \geqslant (1 + C_1|f|_3^{-3}) \sum_{j\in\mathbb{Z}} |\phi(x-j)f(x)|_3^3,$$

from which it follows that

$$C_0A^2 \geqslant (1 + C_1|f|_3^{-3})C_2|f|_3^3 = C_2|f|_3^3 + C_0A^2,$$

contradicting the fact that f is not zero. ■

Lemma 3.9. *There exists $\eta > 0$ such that*

$$\sup_{y\in\mathbb{R}} \int_{y-2}^{y+2} |f_j(x)|^3 \, \mathrm{d}x \geqslant \eta$$

for all $j = 1, 2, 3, \ldots$.

Proof. Observe that since $\{f_j\}$ is a minimizing sequence for problem (P2) and

$J(f_j)$ is comparable to $\|f_j\|_{1/2}$, then $\|f_j\|_{1/2}$ is bounded independently of j, and hence lemma 3.8 can be applied to f_j with a constant C_2 that is independent of j. Thus for each j one can find $k_j \in \mathbb{Z}$ such that

$$(1 + C_2|f_j|_3^{-3})|\phi(x-k_j)f_j(x)|_3^3 \geqslant \|\phi(x-k_j)f_j(x)\|_{1/2}^2 \geqslant \frac{1}{C_3^2}|\phi(x-k_j)f_j(x)|_3^2,$$

where C_3 is the constant in the Sobolev inequality $|f|_3 \leqslant C_3\|f\|_{1/2}$ (valid for all $f \in H^{1/2}$). Since $|f_j|_3 = 1$ for all j, it transpires that $|\phi(x-k_j)f_j(x)|_3 \geqslant (1/((1+C_2)C_3^2)$, and hence

$$\int_{k_j-2}^{k_j+2} |f_j(x)|^3 \, \mathrm{d}x \geqslant |\phi(x-k_j)f_j(x)|_3^3 \geqslant \eta,$$

where $\eta = ((1+C_2)C_3^2)^{-3}$. ∎

From the preceding lemma it follows that $\alpha \neq 0$, so that the sequence $\{f_j\}$ does not 'vanish' in the sense of Lions. Next we rule out the possibility of 'dichotomy'. To do this we use a procedure which is an analogue for non-local operators of the method used in lemma III.1 of Lions (1984).

Lemma 3.10. *There exists a constant $c > 0$ such that if $\theta \in W_\infty^1$ and $f \in H^{1/2}$, then*

$$|[L,\theta]f|_2 \leqslant c|\theta'|_\infty|f|_2,$$

where $[L,\theta]f$ denotes the commutator $L(\theta f) - \theta(Lf)$.

Proof. Write $L = \sqrt{C}\cdot I + \tilde{L}$, where I is the identity operator on L_2 and $\tilde{L}$ is the Fourier multiplier operator with symbol $\tilde{m}(k) = (C + m(k))^{1/2} - C^{1/2}$. Since $[L,\theta] = [\sqrt{C}\cdot I, \theta] + [\tilde{L},\theta] = [\tilde{L},\theta]$, it suffices to prove the lemma with L replaced by $\tilde{L}$.

The operator $\tilde{L}$ can be written as $\tilde{L} = (\mathrm{d}/\mathrm{d}x)T = T(\mathrm{d}/\mathrm{d}x)$, where T is the Fourier multiplier operator with symbol $\sigma(k) = (-\tilde{m}(k))/(\mathrm{i}k)$. Since $\tilde{m}(0) = 0$ and $\tilde{m}$ is differentiable at $k = 0$, then $\sigma(k)$ is bounded and differentiable on $\mathbb{R}$ (when suitably defined at $k = 0$). Hence T is a bounded operator on L_2. Moreover, it is easily verified that for every integer $j \geqslant 0$, one has $\sup_{k\in\mathbb{R}} |k|^j|(\mathrm{d}/\mathrm{d}k)^j\sigma(k)| < \infty$. It therefore follows from theorem 35 of Coifman & Meyer (1978) that there exists a constant c such that

$$|[T,\theta](f')|_2 \leqslant c|\theta'|_\infty|f|_2 \tag{3.5}$$

for all functions θ and f in $C_0^\infty(\mathbb{R})$. A standard density argument then shows that (3.5) holds with the same constant c for all $\theta \in W_\infty^1$ and all $f \in H^{1/2}$. Hence the desired result may be validated by writing

$$|[\tilde{L},\theta]f|_2 = \left|T\frac{\mathrm{d}}{\mathrm{d}x}(\theta f) - \theta T\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\right|_2 \leqslant |T(\theta' f)|_2 + |[T,\theta](f')|_2$$

and using (3.5) and the fact that T is bounded on L_2. ∎

Let ϕ and ψ be smooth functions on $\mathbb{R}$ such that $\phi(x) = 1$ for $x \in [-1,1]$, $\phi(x) = 0$ for $x \notin [-2,2]$, $\psi(x) = 1$ for $x \notin [-2,2]$, $\psi(x) = 0$ for $x \in [-1,1]$, and $\phi^2(x) + \psi^2(x) = 1$ for all $x \in \mathbb{R}$. For each $R > 0$ let $\phi_R(x) = \phi(x/R)$ and $\psi_R(x) = \psi(x/R)$.

Lemma 3.11. *Let $\epsilon > 0$, $A > 0$, and $a \in \mathbb{R}$ be given, and suppose $f \in H^{1/2}$ is such that $\|f\|_{1/2} \leqslant A$. Then there exists $R_0 = R_0(\epsilon, A) > 0$ (which depends on ϵ and A but not on a or f) such that for every $R \geqslant R_0$,*

$$|J(f) - J(g) - J(h)| < \epsilon,$$

where $g(x) = \phi_R(x-a)f(x)$ and $h(x) = \psi_R(x-a)f(x)$.

Proof. Since J is invariant under translations, we may assume $a = 0$, so that $g = \phi_R f$ and $h = \psi_R f$. For all $R > 0$, one has

$$J(g) = \int_{-\infty}^{\infty} (L(\phi_R f))^2 = \int_{-\infty}^{\infty} \phi_R^2 (Lf)^2 \\ +2\int_{-\infty}^{\infty} \phi_R (Lf)([L,\phi_R]f) + \int_{-\infty}^{\infty} ([L,\phi_R]f)^2. \tag{3.6}$$

Since $|\phi_R|_\infty = 1$ and $|(\phi_R)'|_\infty \leqslant |\phi'|_\infty / R$, it follows from lemma 3.10 that

$$\int_{-\infty}^{\infty} |\phi_R(Lf)([L,\phi_R]f)| \leqslant |\phi_R|_\infty |Lf|_2 |[L,\phi_R]f|_2 \leqslant c\|f\|_{1/2} |(\phi_R)'|_\infty |f|_2 \leqslant cA^2/R$$

and

$$\int_{-\infty}^{\infty} |[L,\phi_R]f|^2 \leqslant c|(\phi_R)'|_\infty^2 |f|_2^2 \leqslant cA^2/R^2,$$

where c is independent of a, f, A and R. Hence one obtains from (3.6) that

$$\left| J(g) - \int_{-\infty}^{\infty} \phi_R^2 (Lf)^2 \right| < \tfrac{1}{2}\epsilon$$

for $R > \max(\sqrt{(4/c)A^2}\epsilon, (4cA^2/\epsilon))$. Similarly, one has

$$\left| J(h) - \int_{-\infty}^{\infty} \psi_R^2 (Lf)^2 \right| < \tfrac{1}{2}\epsilon$$

for R sufficiently large. Since $J(f) = \int_{-\infty}^{\infty} (Lf)^2 = \int_{-\infty}^{\infty} \phi_R^2 (Lf)^2 + \int_{-\infty}^{\infty} \psi_R^2 (Lf)^2$, the lemma follows. ■

Lemma 3.12. *For every $\epsilon > 0$, there exists a natural number N and sequences $\{g_j\}_{j=N,N+1,\ldots}$ and $\{h_j\}_{j=N,N+1,\ldots}$ in $H^{1/2}$ such that for all $j \geqslant N$,*

(i) $\left| \left(\int_{-\infty}^{\infty} |g_j(x)|^3 \, dx \right) - \alpha \right| < \epsilon,$

(ii) $\left| \left(\int_{-\infty}^{\infty} |h_j(x)|^3 \, dx \right) - (1-\alpha) \right| < \epsilon,$ *and*

(iii) $J(f_j) \geqslant J(g_j) + J(h_j) - \epsilon.$

Proof. For a given ϵ, it follows from the definition of α that there exists $R_1 > 0$ such that for all $R \geqslant R_1$,

$$\alpha - \epsilon < Q(R) \leqslant Q(2R) \leqslant \alpha.$$

Let A be such that $\|f_j\|_{1/2} \leqslant A$ for all $j \geqslant 1$, and let $R_0 = R_0(\epsilon, A)$ be as in lemma 3.11. Finally fix $R = \max(R_0, R_1)$. Since Q_j tends pointwise to Q on $\mathbb{R}^+$, then a number N can be found such that for all $j \geqslant N$,

$$\alpha - \epsilon < Q_j(R) \leqslant Q_j(2R) < \alpha + \epsilon.$$

Hence for each $j \geqslant N$ one can find y_j such that

$$\int_{y_j-R}^{y_j+R} |f_j|^3 \, dx > \alpha - \epsilon,$$

and

$$\int_{y_j-2R}^{y_j+2R} |f_j|^3 \, dx < \alpha + \epsilon.$$

Now for each $j \geqslant N$ define $g_j(x) = \phi_R(x - y_j) f_j(x)$ and $h_j(x) = \psi_R(x - y_j) f_j(x)$, where ϕ_R and ψ_R are as in the last lemma. Since $\int_{-\infty}^{\infty} |f_j|^3 = 1$, parts (i) and (ii) of lemma 3.12 follow easily from the preceding two inequalities and the properties of the functions ϕ and ψ, while part (iii) follows immediately from lemma 3.11. ■

Lemma 3.13. *For each $y > 0$, define*

$$I(y) = \inf \left\{ J(f) : f \in H^{1/2} \quad \text{and} \quad \int_{-\infty}^{\infty} |f(x)|^3 \, dx = y \right\}.$$

Then, for all $y \in (0, 1)$,

$$I(y) + I(1 - y) > I(1).$$

Proof. The Sobolev embedding theorem implies that $\int_{-\infty}^{\infty} |f(x)|^3 \, dx \leqslant c \|f\|_{1/2}^3$ for all $f \in H^{1/2}$, where c is independent of f. Since $J(f)$ is comparable to $\|f\|_{1/2}^2$, it follows that $I(y) > 0$ for all $y > 0$, and in particular $I(1) > 0$. Also, one clearly has $I(y) = y^{2/3} I(1)$ for all $y > 0$. The result then follows from the subadditivity of the function $y^{2/3}$. ■

(6) It follows from lemmas 3.12 and 3.13 that the number α cannot lie in the range $(0, 1)$. To see this, let $\epsilon > 0$ be given and choose a natural number N and sequences $\{g_j\}$ and $\{h_j\}$ as in lemma 3.12. If, for all $j \geqslant N$, we set $\tilde{g}_j = (\alpha^{1/3}/|g_j|_3) g_j$ and $\tilde{h}_j = ((1 - \alpha)^{1/3}/|h_j|_3) h_j$, then $\int_{-\infty}^{\infty} |\tilde{g}_j|^3 \, dx = \alpha$ and $\int_{-\infty}^{\infty} |\tilde{h}_j|^3 \, dx = (1 - \alpha)$, from which it follows that $J(\tilde{g}_j) \geqslant I(\alpha)$ and $J(\tilde{h}_j) \geqslant I(1 - \alpha)$. Therefore

insert parenthesis ~~[illegible]~~

$((1-\alpha)^{1/3}/|h_j|_3)h_j$

should read like this

$$J(g_j) \geqslant \frac{|g_j|_3^2}{\alpha^{2/3}} I(\alpha),$$

and

$$J(h_j) \geqslant \frac{|h_j|_3^2}{(1 - \alpha)^{2/3}} I(1 - \alpha),$$

and so from lemma 3.12 it follows that

$$J(f_j) \geqslant J(g_j) + J(h_j) - \epsilon \geqslant \left[\frac{(\alpha - \epsilon)^{2/3}}{\alpha^{2/3}} \right] I(\alpha) + \left[\frac{((1 - \alpha) - \epsilon)^{2/3}}{(1 - \alpha)^{2/3}} \right] I(1 - \alpha) - \epsilon.$$

Now taking the limit first as $j \to \infty$ (for fixed ϵ) and then as $\epsilon \to 0$, one obtains that

$$I(1) \geqslant I(\alpha) + I(1 - \alpha).$$

But if α were in the range $(0, 1)$ then this result would contradict lemma 3.13.

Since we have already ruled out the possibility that $\alpha = 0$, it follows from the above that α must equal one, and so the 'compactness' alternative of Lions ~~is obtained~~ prevails.

Lemma 3.14. (Lions 1984). *Suppose $\alpha = 1$. Then there exists a sequence of* *real numbers* $\{y_j\}$ *with the following property: for every* z *in the interval* $(\frac{1}{2}, 1)$, *there exists a real number* $R = R(z)$ *such that for all* j *sufficiently large,*

$$\int_{y_j-R}^{y_j+R} |f_j|^3 > z.$$

Proof. Since $\alpha = 1$, for every $z \in (\frac{1}{2}, 1)$ we can find numbers $R_1(z)$ and $N_1(z)$ such that for all $j \geqslant N_1(z)$

$$Q_j(R_1(z)) = \sup_{y\in\mathbb{R}} \int_{y-R_1(z)}^{y+R_1(z)} |f_j|^3 > z.$$

Hence, for each z there exists a sequence $\{y_j(z)\}$ such that

$$\int_{y_j(z)-R_1(z)}^{y_j(z)+R_1(z)} |f_j|^3 > z.$$

Now define $y_j = y_j(\frac{1}{2})$ for each $j \geqslant N_1(\frac{1}{2})$. Since $\int_{-\infty}^{\infty} |f_j|^3 = 1$ for all j, it follows that

$$|y_j(z) - y_j| \leqslant R_1(z) + R_1(\tfrac{1}{2}),$$

for all $z > \frac{1}{2}$. Then, taking $R(z) = 2R_1(z) + R_1(\frac{1}{2})$ and $N(z) = \max(N_1(z), N_1(\frac{1}{2}))$, we have

$$\int_{y_j-R(z)}^{y_j+R(z)} |f_j(x)|^3 > z,$$

for all $j \geqslant N(z)$. ■

Let $\tilde{f}_j$ be the function obtained by translating f_j by y_j, so that

$$\tilde{f}_j(x) = f_j(x + y_j) \quad \text{for all } x \in \mathbb{R}.$$

For each natural number k, let $z = (1 - (1/k))^3$. Then by lemma 3.14, there exists a number R_k such that for all sufficiently large j, the L_3-norm of $\tilde{f}_j$ on the interval $[-R_k, R_k]$ is greater than $1 - (1/k)$. Hence, since the sequence $\{\tilde{f}_j\}$ is uniformly bounded in $H^{1/2}$, it follows from the compactness of the embedding of $H^{1/2}(\Omega)$ into $L_3(\Omega)$ on bounded intervals Ω that some subsequence of $\{\tilde{f}_j\}$ converges, weakly in $H^{1/2}$ and strongly in $L_3([-R_k, R_k])$, to a limit function f_0 whose norm in $L_3([-R_k, R_k])$ is greater than or equal to $(1 - (1/k))$. Now, by a Cantor diagonalization argument, we can find a subsequence of $\{\tilde{f}_j\}$ which converges, weakly in $H^{1/2}$ and strongly in L_3 on every compact subset of $\mathbb{R}$, to a function f_0 defined on $\mathbb{R}$ whose norm in $L_3(\mathbb{R})$ is equal to one. But from the lower semicontinuity of the norm defined by $\sqrt{J}$ on $H^{1/2}$, we have

$$J(f_0) \leqslant \lim_{j\to\infty} J(\tilde{f}_j) = I(1).$$

Therefore f_0 is a solution of the variational problem (P2).

Now from lemma 3.6 it follows that $g_0 = |f_0|^*$ is a minimizer for problem (P1). Hence, by a standard result in the calculus of variations (see Luenberger (1968), theorem 2 of § 7.7), g_0 satisfies the Lagrange-multiplier equation

$$\delta J(g_0) = \lambda \cdot \delta K(g_0) \tag{3.7}$$

for some $\lambda \in \mathbb{R}$. Here K is the functional defined by $K(f) = \int_{-\infty}^{\infty} f^3\,\mathrm{d}x$, whilst $\delta J(g_0)$

and $\delta K(g_0)$ are the Fréchet derivatives of J and K at f_0, given as maps from $H^{1/2}$ to $\mathbb{R}$ by

$$\delta J(g_0)[h] = 2\int_{-\infty}^{\infty} Lg_0(x)Lh(x)\,\mathrm{d}x \quad \text{and} \quad \delta K(g_0)[h] = 3\int_{-\infty}^{\infty} g_0^2(x)h(x)\,\mathrm{d}x. \tag{3.8}$$

From (3.7) it follows that $L^2(g_0) = (\frac{3}{2}\lambda)g_0^2$, in the sense of equality between distributions on $\mathbb{R}$. Taking $\phi = 3\lambda g_0$, one then obtains a distributional solution of (3.1), which must in fact be an H^∞-solution by lemma 3.3. Since ϕ is a decreasing function of $|x|$, this completes the proof of theorem 3.1.

Remarks. As noted by Weinstein (1987), the method used above for proving existence of solitary waves can be applied to more general equations of the form

$$u_t + u^p u_x - (Mu)_x = 0,$$

where $p \geqslant 1$ is an integer and M is the Fourier multiplier operator defined by $\widehat{Mu}(k) = m(k)\widehat{u}(k)$, with $m(k)$ now denoting an arbitrary measurable function of k. The associated equation for solitary waves is

$$(C + M)\phi = \left(\frac{1}{p+1}\right)\phi^{p+1}. \tag{3.9}$$

The proof of theorem 3.1 goes through essentially unchanged for equation (3.9) provided that:

(i) lemma 3.2 holds for the new choice of $m(k)$; and

(1/ik) (ii) the operator T with symbol $\sigma(k) = -(1/ik)(\sqrt{C+m(k)} - \sqrt{C})$ satisfies the commutator estimate (3.5).

Of course the Hilbert space in which the variational problems analogous to those below (3.3) are set will depend on the growth at infinity of m.

We continue our analysis of the solitary-wave solutions of the general ILW equation. Suppose ϕ to be a solution of (3.1) as obtained via theorem 3.1, say. Interest now focuses on the spatial asymptotics of $\phi(x)$ as $|x| \to \infty$. Of course, since ϕ lies in L_2 and is a decreasing function of $|x|$, $\phi(x) \to 0$ as $x \to \pm\infty$. However, more precise information may be obtained by using the recent theory for decay of solitary waves worked out by Bona & Li (1997). details?

First, observe that if $\beta_1 = 0$, we are faced with the ILW equation itself. As mentioned earlier, in this case solitary-wave solutions are known explicitly (and known to be unique to within a translation (Albert 1995; Albert & Toland 1994). These solitary waves have the form

$$\phi(x) = \left[\frac{2a\beta_2 \sin aH_2}{\cosh a(x + x_0) + \cos aH_2}\right], \tag{3.10}$$

where $a \in (0, \pi/H_2)$ satisfies $aH_2 \cot aH_2 = (1 - CH_2/\beta_2)$, and it is obvious by inspection that they decay exponentially to zero at infinity. It is natural to conjecture the same property for solitary-wave solutions of the general ILW equation (2.1), though one must keep in mind that not all such model equations feature exponentially decaying solitary waves (e.g. the Benjamin–Ono equation possesses solitary waves that decay algebraically to zero at infinity).

In fact, the following general result holds for solutions of the convolution equation (3.1).

Theorem 3.15. *Let $C > 0$ be specified and let ϕ be a solution of equation (3.1) corresponding to this value of C which is an L_∞-function with the property that $\lim_{|x|\to\pm\infty} \phi(x) = 0$. Let $z_0 = \mathrm{i}\nu_0$ be the zero of $C + \beta_1 m_1(z) + \beta_2 m_2(z)$ in the upper half-plane with smallest imaginary part. Then it follows that:*

(1) $\phi(x)\mathrm{e}^{\sigma|x|} \in L_\infty$ for any $\sigma \leqslant \nu_0$;

(2) there is a finite constant $\lambda > 0$ such that $\lim_{|x|\to\infty} \phi(x)\mathrm{e}^{\nu_0|x|} = \lambda$; and

(3) ϕ is the restriction to the real axis of a function Φ which is analytic in the strip $\{z : -\nu_0 < \mathrm{Im}(z) < \nu_0\}$.

Proof. Part (1) for $\sigma < \nu_0$ and part (3) follow directly from theorem 4.1.7 of Bona & Li (1997, see also Li & Bona 199~~7~~ 6) and the earlier remark that the kernel K defined by 6

$$\widehat{K}(k) = [C + \beta_1 m_1(k) + \beta_2 m_2(k)]^{-1}$$

has the property

$$\int_{-\infty}^{\infty} \mathrm{e}^{2\sigma|x|}|K(x)|^2 \,\mathrm{d}x < +\infty,$$

for any $\sigma < \nu_0$.

Part (2) is a consequence of theorem 3.1.6 of Bona & Li (1997). Indeed, from the representation (3.2) for K, it follows that

$$\lim_{x\to\pm\infty} \mathrm{e}^{\nu_0|x|}K(x) = 2\pi\gamma_0.$$

The just-mentioned theorem therefore allows one to conclude that

$$\lim_{x\to\pm\infty} \phi(x) = 2\pi\gamma_0 \int_{-\infty}^{\infty} \mathrm{e}^{\pm\nu_0 y}\phi(y)^2 \,\mathrm{d}y.$$

Since ϕ is even in this case,

$$\int_{-\infty}^{\infty} \mathrm{e}^{\nu_0 y}\phi(y)^2 \,\mathrm{d}y = \int_{-\infty}^{\infty} \mathrm{e}^{-\nu_0 y}\phi(y)^2 \,\mathrm{d}y.$$

The result claimed in part (2) now follows if we choose λ to be $2\pi\gamma_0$ times the quantity in the last display. ■

We conclude this section with a result about the stability of solitary waves as solutions of the initial-value problem (2.1). A solitary-wave solution is said to be (orbitally) stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that when $u_0 \in H^s$ (where $s > 3/2$) and $\|u_0 - \phi\|_{1/2} < \delta$, the solution u of (2.1) described in theorem 2.1 satisfies

$$\inf_{\substack{y\in\mathbb{R}\\ 0\leqslant t<\infty}} \|u(\cdot + y, t) - \phi\|_{1/2} < \epsilon.$$

The result to be proved here is that stable solitary-wave solutions of (2.1) exist when H_1 is close to H_2. For general values of H_1 and H_2, not necessarily close together, see the remarks at the end of this section.

Theorem 3.16. *Let H and C be fixed positive numbers. Then there exists $\alpha_1 > 0$ such that for each $\alpha \in (-\alpha_1, \alpha_1)$, equation (3.1) with $H_1 = H$ and $H_2 = H + \alpha$ has a solution ϕ_α which is a stable solitary-wave solution of (2.1).*

The proof of theorem 3.16 will proceed by verifying the sufficient conditions for the stability of a solitary wave given in Bona *et al.* (1987). Since these conditions are stated as conditions on the spectrum of a certain operator associated with the solitary wave, it will be necessary to quote some results from the spectral theory of closed linear operators on Hilbert spaces. A good reference for this material is ch. IV–V of Kato (1976) from which our notation and terminology have been drawn.

Lemma 3.17. *Let $s \geqslant 0$ be an integer, and suppose $\phi \in H^{s+1}$. Let $T = C + \beta_1 M_1 + \beta_2 M_2$ (where H_1 and H_2 are arbitrary), and let Q be the operation of multiplication by ϕ. Then $\mathcal{L} = T - Q$ is a self-adjoint operator on H^s with domain H^{s+1}. The essential spectrum of $\mathcal{L}$ is the interval $[C, \infty)$ on the real axis, while the remainder of the spectrum of $\mathcal{L}$ consists of isolated eigenvalues of finite multiplicity. In case ϕ is even, the same statements also hold if H^s is replaced by the closed subspace H^s_e consisting of all even functions in H^s.*

Proof. The operator T is a self-adjoint operator on H^s with domain H^{s+1} because it is unitarily equivalent (via the Fourier transform) to the self-adjoint maximal multiplication operator defined by the multiplier $C + \beta_1 m_1(k) + \beta_2 m_2(k)$ on the space of Fourier transforms of functions in H^s. The operator Q is bounded on H^s because of the estimates

$$\|\phi\eta\|_s \leqslant (\sup_{j \leqslant s} |(\mathrm{d}/\mathrm{d}x)^j \phi|_\infty)\|\eta\|_s \leqslant \|\phi\|_{s+1}\|\eta\|_s, \tag{3.11}$$

which follow easily from the fact that the H^s-norm of a function is equivalent to the sum of the L_2-norms of its derivatives up to order s. Therefore, by theorem V.4.3 of Kato (1976), $\mathcal{L}$ is a self-adjoint operator on H^s with domain H^{s+1}.

From the definition of T it follows easily that its spectrum consists only of the interval $[C, \infty)$, which is also its essential spectrum. As explained in § V.5.3 of Kato (1976), the remaining statements of the lemma will then follow once it is shown that Q is relatively compact with respect to T.

Suppose therefore that $\{f_n\}$ and $\{Tf_n\}$ are bounded sequences in H^s; it is required to show that the sequence $\{Qf_n\}$ has a convergent subsequence in H^s. The assumptions on the sequence $\{f_n\}$ imply that it is bounded in H^{s+1}. Since the inclusion of $H^{s+1}(\Omega)$ into $H^s(\Omega)$ is compact on bounded domains Ω, one may conclude (passing to subsequences if necessary and using a Cantor diagonalization argument) that there exists a function $g \in H^s$ to which $\{f_n\}$ converges in $H^s(\Omega)$-norm on any bounded interval Ω in $\mathbb{R}$. Let ψ be a smooth function with compact support on $\mathbb{R}$ which equals one on a neighbourhood of the origin, and define $\psi_R(x) = \psi(x/R)$ for $x \in \mathbb{R}$. Then for each fixed R, one has $\|\psi_R\phi(f_n - g)\|_s \to 0$ as $n \to \infty$. Moreover, applying (3.11), with ϕ replaced by $\psi_R\phi$, and using the fact that $\phi(x)$ and its derivatives up to order s tend to zero as $|x| \to \infty$, one obtains that $\|(1 - \psi_R)\phi(f_n - g)\|_s \to 0$ uniformly in n as $R \to \infty$. Hence, writing

$$\|\phi(f_n - g)\|_s \leqslant \|\psi_R\phi(f_n - g)\|_s + \|(1 - \psi_R)\phi(f_n - g)\|_s,$$

one sees that $\|\phi(f_n - g)\|_s$ can be made arbitrarily small by first choosing R sufficiently large in the second term on the right-hand side, and then choosing n sufficiently large in the first term. This proves that Qf_n converges to Qg in H^s norm, as desired.

The proof of the statements in the lemma for H^s_e is exactly the same as for H^s, once it is noted that when ϕ is even, $\mathcal{L}$ carries even functions to even functions. ∎

Lemma 3.18. *Let $s \geqslant 1$ be an integer. Then there exists $\alpha_0 > 0$ such that for every $\alpha \in (-\alpha_0, \alpha_0)$, equation (3.1) with $H_1 = H$ and $H_2 = H + \alpha$ has a solution $\phi_\alpha \in H^{s+1}_e$, and the correspondence $\alpha \mapsto \phi_\alpha$ defines a continuous map from $(-\alpha_0, \alpha_0)$ to H^{s+1}_e.*

Proof. Define a map $F : \mathbb{R} \times H_e^{s+1} \to H_e^s$ by

$$F(\alpha, \phi) = T_\alpha(\phi) - \tfrac{1}{2}\phi^2,$$

where T_α denotes the operator obtained from T by setting $H_1 = H$ and $H_2 = H + \alpha$. Using the fact that $\|fg\|_s \leqslant c\|f\|_s\|g\|_s$ for all f and g in H^s (with the constant c independent of f and g), one obtains easily that the Fréchet derivative F_ϕ of F with respect to ϕ exists at any point $(\alpha, \phi) \in (-H, \infty) \times H_e^{s+1}$, and coincides with the operator $\mathcal{L} = T_\alpha - Q$.

We claim that the maps $F : \mathbb{R} \times H_e^{s+1} \to H_e^s$ and $F_\phi : \mathbb{R} \times H_e^{s+1} \to B(H_e^{s+1}, H_e^s)$ are continuous. To see this, note first that a simple computation using the mean-value theorem shows that the inequality

$$|k \coth k\theta_1 - k \coth k\theta_2| \leqslant |\theta_1 - \theta_2| \max(1/\theta_1^2, 1/\theta_2^2) \tag{3.12}$$

holds for all $\theta_1, \theta_2 > 0$ and all $k \in \mathbb{R}$. Hence if M_2 and M_2' are the Fourier multiplier operators corresponding to $H_2 = H + \alpha$ and $H_2 = H + \alpha'$, respectively, then $M_2 - M_2'$ is a bounded operator on H_e^{s+1} with $\|M_2 - M_2'\|_{B(H_e^{s+1}, H_e^{s+1})} \leqslant c|\alpha - \alpha'|$, where c can be chosen independently of α and α' provided neither $H + \alpha$ nor $H + \alpha'$ approach zero. The claimed continuity properties of F and F_ϕ now follow easily from this observation and the estimate (3.11).

When $\alpha = 0$ and $H_1 = H_2 = H$, equation (3.1) has the explicit solution (3.10) discovered by Joseph (1977), which is denoted by ϕ_0. Let $\mathcal{L}_0 = F_\phi(0, \phi_0) = T_0 - Q_0$, where Q_0 denotes the operation of multiplication by ϕ_0. In Albert & Bona (1991) it is shown that the only solutions $\eta \in L_2$ of the equation $\mathcal{L}_0\eta = 0$ are the functions in the subspace spanned by $(\mathrm{d}\phi_0/\mathrm{d}x)$. Since these functions are odd, it follows that zero is not an eigenvalue of $\mathcal{L}_0$ in H_e^s. Therefore, by lemma 3.17, zero is not in the spectrum of $\mathcal{L}_0$, and hence $\mathcal{L}_0$ is a linear isomorphism of H_e^{s+1} onto H_e^s. Application
F/T of the implicit function theorem (cf. Deimling (1985), theorem 15.1) now yields the caps
existence of a continuous map $\alpha \mapsto \phi_\alpha$, taking some interval $(-\alpha_0, \alpha_0)$ into H_e^{s+1}, such that $F(\alpha, \phi_\alpha) = 0$ for $\alpha \in (-\alpha_0, \alpha_0)$. This is the advertised result. ∎

Remarks. Because the mapping F introduced in the proof of the last lemma depends analytically on α in the appropriate sense, it is adduced from the relevant version of the Implicit Function Theorem that the correspondence $\alpha \to \phi_\alpha$ is an analytic mapping of a neighbourhood of zero in $\mathbb{R}$ to H^{s+1}.

We now turn to the proof of theorem 3.16. For each $\alpha \in (-\alpha_0, \alpha_0)$, let $\mathcal{L}_\alpha = T_\alpha - Q_\alpha$, where Q_α is the operation of multiplication by ϕ_α. The results of Bona *et al.* (1987) (see in particular the proof of lemma 5.1 of that paper) imply that the solitary wave ϕ_α will be stable if $\mathcal{L}_\alpha$ has the following two properties:

(1) as an operator on L_2, $\mathcal{L}_\alpha$ has one simple negative eigenvalue, a simple eigenvalue at zero, and no other points of its spectrum on the non-positive real axis; and

(2) there exists $\chi \in L_2$ such that $\mathcal{L}_\alpha(\chi) = \phi_\alpha$ and the inner product $\langle \chi, \phi_\alpha \rangle_0$ is negative.

Our task is to verify that conditions (1) and (2) hold for α near zero.

Note first that (3.11) and (3.12) together imply that $\mathcal{L}_\alpha - \mathcal{L}_{\alpha'}$ is a bounded operator on L_2 for every pair of numbers α and that α' in $(-\alpha_0, \alpha_0)$ and that $\|\mathcal{L}_\alpha - \mathcal{L}_{\alpha'}\|_{B(L_2, L_2)} \to 0$ as $\alpha \to \alpha'$. Therefore, the correspondence $\alpha \mapsto \phi_\alpha$ defines a continuous map from $(-\alpha_0, \alpha_0)$ to the space of closed operators on L_2 when the latter space is endowed with the topology of generalized convergence defined in ch. IV.2 of Kato (1976).

In the proof of theorem 11 of Albert & Bona (1991) it is shown that condition (1) holds for the operator $\mathcal{L}_0$. Moreover, differentiation of the identity $F(\alpha, \phi_\alpha) = 0$ with respect to x yields that $\mathcal{L}_\alpha(\mathrm{d}\phi_\alpha/\mathrm{d}x) = 0$, so that $\mathcal{L}_\alpha$ has zero as an eigenvalue (with eigenfunction $\mathrm{d}\phi_\alpha/\mathrm{d}x$) for every $\alpha \in (-H, \infty)$. Therefore, since $\mathcal{L}_\alpha$ varies continuously with α in the space of closed operators on L_2, it follows from standard theorems on the continuity of eigenvalues of linear operators with respect to perturbations (cf. Kato (1976), § V.4.3), that condition (1) holds for all α in some neighbourhood of zero. (For details of the argument we refer the reader to the proof of theorem 5 of Albert *et al.* (1987) where an exactly analogous argument appears.)

Next recall that, as noted in the proof of lemma 3.18, zero is not an eigenvalue for $\mathcal{L}_0$ in H_e^0. Hence, when considered as an operator on H_e^0, $\mathcal{L}_\alpha$ does not have zero in its spectrum for values of α sufficiently near to zero. In particular, for such values of α, the inverse $\mathcal{L}_\alpha^{-1}$ is well-defined as an operator on H_e^0; and from theorem IV.2.25 of Kato (1976) and lemma 3.18, it then follows that the function χ_α defined by $\chi_\alpha = \mathcal{L}_\alpha^{-1}(\phi_\alpha)$ depends continuously on α in the L_2-norm. Therefore the L_2-inner product $\langle \chi_\alpha, \phi_\alpha \rangle_0$ also depends continuously on α, for α near zero. Now in Albert & Bona (1991) it is shown that there exists a function $\chi \in L_2$ such that $\mathcal{L}_0(\chi) = \phi_0$ and $\langle \chi, \phi_0 \rangle_0 < 0$ (see the proof of theorem 11 and the remarks following theorem 2 in Albert & Bona (1991)). Since the nullspace of $\mathcal{L}_0$ is spanned by $(\mathrm{d}\phi_0/\mathrm{d}x)$, which is orthogonal to ϕ_0 in L_2, then $\langle \chi_0, \phi_0 \rangle_0$ must equal $\langle \chi, \phi_0 \rangle_0$, and hence $\langle \chi_0, \phi_0 \rangle_0 < 0$. We conclude by continuity that $\langle \chi_\alpha, \phi_\alpha \rangle_0 < 0$ for α in some neighbourhood of zero.

It has now been shown that conditions (1) and (2) both hold for α in some neighbourhood $(-\alpha_1, \alpha_1)$ contained in $(-\alpha_0, \alpha_0)$, and so the proof of theorem 3.16 is complete. ■

C/F/T

Remarks. Notice that for H_1 near H_2, both the concentration-compactness theory and the implicit function theorem arguments are valid and yield existence of solitary-wave solutions. Of course, the branch of solutions obtained via the Implicit Function Theorem comprises the unique even solutions near the explicit solitary-wave solution (3.10) of the ILW equation. However, in the absence of a uniqueness result analogous to that appearing in Albert & Toland (1994) for the ILW equation, the solutions obtained from the two different approaches might not coincide, even when the non-uniqueness due to the translation invariance of the evolution equation is taken into account. In consequence, even for H_1 near H_2, the solitary waves obtained from the concentration-compactness approach may not be the same as those for which the orbital stability result holds.

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By a separate argument to be reported elsewhere (see Albert 1997), one of us has been able to use concentration compactness to establish a stability result, valid for all H_1 and H_2, which is related to, ~~but not as satisfactory as, an orbital stability result.~~ Roughly speaking, the theorem in view states that if initial data sufficiently near to a ~~solitary wave solution~~ is posed, then the resulting solution of (1.1), (1.2) remains forever in a neighbourhood of the set $\mathcal{M}$ of all minimizers of ~~a~~ variational problem ~~closely related to (P1)~~. In case ~~the solitary wave~~ is unique, the set $\mathcal{M}$ would amount to the set of all translates of ~~the/given~~ a particular solitary wave, and the result of Albert (1997) would generalize the orbital stability result established via the Implicit Function Theorem for H_2 near H_1.

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(solitary wave that minimizes a variational problem closely related to (P1))

this

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It also deserves remark that the orbital stability result put forward in theorem 3.16 may be improved by an application of the theory developed by Bona & Soyeur (1994). Theorem 2 of the last-quoted reference allows us to conclude that there is a smooth function $a(t)$ such that

$$\|\phi(x - a(t)) - u(x,t)\|_{1/2} \leqslant \epsilon, \quad \text{for all } t \geqslant 0, \quad a(0) = 0,$$

and

$$|a'(t) - C| \leqslant C_1 \epsilon,$$

where C_1 is a constant depending only on the phase speed C of the given solitary wave ϕ.

4. Existence of coupled solitary waves

A 'coupled solitary-wave' solution of the system (2.7) is a solution of the form $(u(x,t), v(x,t)) = (\phi(x - Ct), \psi(x - Ct))$, where ϕ and ψ are localized profiles and C is a constant. Substitution of this form into (2.7) reveals that (u, v) is a solution if and only if ϕ and ψ satisfy the equations

1. c.

$$\left.\begin{aligned} (C + \gamma_1 M_1 + \gamma_2 M_2)\phi - \gamma_2 N\psi &= \tfrac{1}{2}\alpha_1\phi^2, \\ (C + \Delta C + \gamma_3 M_3 + \gamma_4 M_2)\psi - \gamma_4 N\phi &= \tfrac{1}{2}\alpha_2\psi^2. \end{aligned}\right\} \qquad (4.1)$$

ΔC

The following existence result for (4.1) is the analogue of theorem 3.1 above.

1. c.

Theorem 4.1. *Suppose that the parameters $\alpha_1, \alpha_2, \gamma_1, \gamma_2, \gamma_3, \gamma_4$ are positive, and suppose $C > 0$ is given such that $C(C + \Delta C) > (\gamma_2\gamma_4)/H_2^2$. Then the system (4.1) has a solution (ϕ, ψ) such that ϕ and ψ are in H^∞ and are decreasing functions of $|x|$.*

The proof of theorem 4.1 presented here uses essentially the same method as that of theorem 3.1, and so it will be possible here to refer to § 3 for most of the technical points that emerge in the discussion.

Define functions $a_{ij}(k)$ for $i, j = 1, 2$ by

$$\begin{aligned} a_{11}(k) &= \gamma_4(\gamma_1 m_1(k) + \gamma_2 m_2(k)), \\ a_{22}(k) &= \gamma_2(\gamma_3 m_3(k) + \gamma_4 m_2(k)), \\ a_{12}(k) &= a_{21}(k) = -\gamma_2\gamma_4 n(k), \end{aligned}$$

(1.3) / (2.9) where $m_i(k)$, $i = 1, 2, 3$, and $n(k)$ are as defined in (2.3) and (2.9). Let $A(k)$ be the 2×2 matrix-valued function given by 3 / 9

$$A(k) = \begin{bmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{bmatrix},$$

and for each pair of numbers $\mu > 0$, $\nu > 0$ define

$$\Delta_{\mu\nu} = \begin{bmatrix} \mu & 0 \\ 0 & \nu \end{bmatrix}.$$

If $\mu\nu > (\gamma_2\gamma_4/H_2)^2$, then the determinant $d_{\mu\nu}(k)$ of the matrix $\Delta_{\mu\nu} + A(k)$ satisfies

$$d_{\mu\nu}(k) \geqslant d_{\mu\nu}(0) = \mu\nu - \left(\frac{\gamma_2\gamma_4}{H_2}\right)^2 > 0, \qquad (4.2)$$

for all $k \in \mathbb{R}$. The eigenvalues of $\Delta_{\mu\nu} + A(k)$ are given by

$$\begin{aligned} \lambda_1^{\mu\nu}(k) &= \tfrac{1}{2}(\mu + \nu + a_{11}(k) + a_{22}(k) - S), \\ \lambda_2^{\mu\nu}(k) &= \tfrac{1}{2}(\mu + \nu + a_{11}(k) + a_{22}(k) + S), \end{aligned}$$

$a_1(k) + a_{22}(k) - S)$

a_{22}^2 a_{22}

where

$$S = \sqrt{[(\mu + a_{11}) - (\nu + a_{12})]^2 + 4a_{12}^2} = \sqrt{(\mu + a_{11} + \nu + a_{12})^2 - 4d_{\mu\nu}}.$$

It follows that $0 < \lambda_1^{\mu\nu}(k) < \lambda_2^{\mu\nu}(k)$ for all $k \in \mathbb{R}$. Moreover, one finds that as $|k| \to \infty$,

$$\lambda_1^{\mu\nu}(k) \sim \min(\delta_1, \delta_2)|k|,$$
$$\lambda_2^{\mu\nu}(k) \sim \max(\delta_1, \delta_2)|k|,$$

where $\delta_1 = (\gamma_1 + \gamma_2)\gamma_4$ and $\delta_2 = (\gamma_3 + \gamma_4)\gamma_2$. Hence there exist positive constants $C_1^{\mu\nu}$ and $C_2^{\mu\nu}$ (independent of k) such that

$$C_1^{\mu\nu}(1 + |k|) \leqslant \lambda_1^{\mu\nu}(k) < \lambda_2^{\mu\nu}(k) \leqslant C_2^{\mu\nu}(1 + |k|), \tag{4.3}$$

for all $k \in \mathbb{R}$.

Lemma 4.2. *Suppose $\mu\nu > ((\gamma_2\gamma_4)/H_2)^2$; define functions $P_{ij}^{\mu\nu}(x)$ for $i, j = 1, 2$ by*

$$\begin{bmatrix} \widehat{P_{11}^{\mu\nu}} & \widehat{P_{12}^{\mu\nu}} \\ \widehat{P_{21}^{\mu\nu}} & \widehat{P_{22}^{\mu\nu}} \end{bmatrix} = [\Delta_{\mu\nu} + A(k)]^{-1}.$$

Then for $i, j = 1, 2$, $P_{ij}^{\mu\nu}(x)$ is a decreasing function of $|x|$ that lies in L_p for every $p \in [1, \infty)$.

Proof. Since

$$[\Delta_{\mu\nu} + A(k)]^{-1} = \frac{1}{d_{\mu\nu}(k)} \begin{bmatrix} \nu + a_{22}(k) & -a_{12}(k) \\ -a_{21}(k) & \mu + a_{11}(k) \end{bmatrix},$$

then

$$\widehat{P_{11}^{\mu\nu}}(k) = \frac{1}{(\mu + a_{11}(k))}\left(1 + \frac{g(k)}{1 - g(k)}\right),$$

where

$$g(k) = \frac{a_{12}^2(k)}{(\mu + a_{11}(k))(\nu + a_{22}(k))}.$$

The function $1 - g(k)$ is bounded away from zero on $\mathbb{R}$ as seen clearly in (4.2), and so $g(k)/(1 - g(k))$ is a C^∞-function that decays exponentially to zero as $|k| \to \infty$. Therefore $g/(1 - g) = \widehat{\theta}$, where θ is a smooth rapidly decaying function on $\mathbb{R}$. Also, because of lemma 3.2, the function K_1 defined by $\widehat{K_1} = (\mu + a_{11})^{-1}$ is in L_p for $1 \leqslant p < \infty$. Hence $P_{11}^{\mu\nu} = K_1 + K_1 * \theta$ is in L_p for $1 \leqslant p < \infty$.

Next observe that the function $h_1(x)$ defined by $\widehat{h_1} = g$ can be written as $h_1 = K_1 * K_2 * K_3 * K_3$, where K_1 is as above, $\widehat{K_2} = (\nu + a_{22})^{-1}$ and $\widehat{K_3} = -a_{12}$. By lemma 3.2, K_1 and K_2 are integrable, decreasing functions of $|x|$, while a table of Fourier transforms enables one to ascertain that $K_3(x) = (\gamma_2\gamma_4\pi/(4H_2^2)) \operatorname{sech}^2(\pi x/(2H_2))$. The convolution of an integrable decreasing function of $|x|$ and a smooth rapidly decreasing function of $|x|$ is again a smooth rapidly decreasing function of $|x|$ (this follows readily from the identity $(p * q)'(x) = \int_0^\infty q'(z)[p(x - z) - p(x + z)]\,\mathrm{d}z$ which is valid for even functions p and q). Hence h_1 is a decreasing function of $|x|$, and an inductive argument shows that the same is true of the functions $h_2, h_3, \ldots$ defined by

$(\gamma_2\gamma_4\pi/(4H_2^2))\operatorname{sech}^2(\pi x/2H_2)$

$\widehat{h}_j(k) = (g(k))^j$. Since

$$\widehat{P_{11}^{\mu\nu}} = \widehat{K}_1 + \sum_{j=1}^{\infty} \widehat{K}_1 \widehat{h}_j,$$

with the series on the right-hand side converging uniformly on $\mathbb{R}$, then

$$P_{11}^{\mu\nu} = K_1 + \sum_{j=1}^{\infty} (K_1 * h_j).$$

On the other hand, the series on the right-hand side of the last display converges in L_2 and, since it is a series of positive functions, pointwise almost everywhere on $\mathbb{R}$. It follows that $P_{11}^{\mu\nu}(x)$ is a positive function decreasing with increasing values of $|x|$. This proves the assertions of the lemma for $P_{11}^{\mu\nu}$; the proofs for $P_{12}^{\mu\nu}$ and $P_{22}^{\mu\nu}$ are similar and hence omitted. ■

As in § 3, standard arguments allow one to deduce from the preceding lemma the following consequence.

Lemma 4.3. *Suppose ϕ and ψ are L_2-functions which satisfy (4.1) in the sense of distributions on $\mathbb{R}$. Then ϕ and ψ are in $H^\infty(\mathbb{R})$, and (4.1) holds in the pointwise sense.*

Next, variational problems analogous to those considered in § 3 are introduced. Let X and Y denote the product spaces $L_2 \times L_2$ and $H^{1/2} \times H^{1/2}$, respectively, with inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$ defined by

$$\langle (f_1, f_2), (g_1, g_2) \rangle_X = \langle f_1, f_2 \rangle_0 + \langle g_1, g_2 \rangle_0,$$
$$\langle (f_1, f_2), (g_1, g_2) \rangle_Y = \langle f_1, f_2 \rangle_{1/2} + \langle g_1, g_2 \rangle_{1/2},$$

and norms given by $\|f\|_X = \langle f, f \rangle_X^{1/2}$, $\|f\|_Y = \langle f, f \rangle_Y^{1/2}$. If

Δc

$$\Delta = \begin{bmatrix} \gamma_4 C & 0 \\ 0 & \gamma_2 (C + \Delta C) \end{bmatrix},$$ J. c.

then from the assumption in theorem 4.1 that $C(C + \Delta C) > (\gamma_2 \gamma_4)/H_2^2$ and the discussion preceding lemma 4.2, it follows that the matrix $\Delta + A(k)$ is positive definite, and hence has a positive-definite square root $B(k) = [\Delta + A(k)]^{1/2}$. One may therefore define a matrix Fourier multiplier operator $L : Y \to X$ by I. c.

$$\widehat{Lf}(k) = B(k) \widehat{f}(k),$$

where the Fourier transform $\widehat{f}$ of a vector-valued function $f = (f_1, f_2)$ is defined componentwise by $\widehat{f} = (\widehat{f}_1, \widehat{f}_2)$. Finally, define a functional $J : Y \to \mathbb{R}$ by

$$J(f) = \|Lf\|_X^2.$$

From (4.3) it follows that the functional $\sqrt{J}$ defines a norm on Y which is equivalent to $\| \cdot \|_Y$.

Let (P1) denote the variational problem of minimizing $J(f)$ over the set

$$S_1 = \left\{ f \in Y : \int_{-\infty}^{\infty} [\alpha_1 \gamma_4 f_1(x)^3 + \alpha_2 \gamma_2 f_2(x)^3] \, dx = 1 \right\},$$

and let (P2) be the problem of minimizing $J(f)$ over the set

$$S_2 = \left\{ f \in Y : \int_{-\infty}^{\infty} [\alpha_1\gamma_4|f_1(x)|^3 + \alpha_2\gamma_2|f_2(x)|^3]\,\mathrm{d}x = 1 \right\}.$$

Lemma 4.4. *For every* $f = (f_1, f_2) \in Y$, *one has* $(|f_1|, |f_2|) \in Y$ *and* $J(|f_1|, |f_2|) \leqslant J(f_1, f_2)$.

Proof. Define $g = (|f_1|, |f_2|)$. For every $\rho > 0$, taking $\mu = \rho + \gamma_4 C$ and $\nu =$ $\rho + \gamma_2(C + \Delta C)$ in lemma 4.2 allows one to conclude that l.c. Δc

$$\begin{aligned}
\langle \widehat{g}, (\rho I + \Delta + A)^{-1}\widehat{g}\rangle_X &= \sum_{i,j} \int_{-\infty}^{\infty} g_i(x)(P_{ij}^{\mu\nu} * g_j)(x)\,\mathrm{d}x \\
&\geqslant \sum_{i,j} \int_{-\infty}^{\infty} f_i(x)(P_{ij}^{\mu\nu} * f_j)(x)\,\mathrm{d}x \\
&= \langle \widehat{f}, (\rho I + \Delta + A)^{-1}\widehat{f}\rangle_X.
\end{aligned}$$

Since $\|\widehat{g}\|_X = \|\widehat{f}\|_X$, it follows that

$$\langle \widehat{f}(k), R_\rho(k)\widehat{f}(k)\rangle_X \geqslant \langle \widehat{g}(k), R_\rho(k)\widehat{g}(k)\rangle_X, \tag{4.4}$$

where

$$R_\rho(k) = \rho\left[I - \left(I + \frac{1}{\rho}[\Delta + A(k)]\right)^{-1}\right].$$

$\lambda_i(k)/(1+\cdots$ The eigenvalues of R_ρ are $\lambda_i(k)/(1 + (1/\rho)\lambda_i(k))$, $i = 1, 2$, where $\lambda_1(k)$ and $\lambda_2(k)$ are the eigenvalues of $\Delta + A(k)$. From the estimates (4.3) for ~~the~~ $\lambda_i(k)$, $i = 1, 2$, it follows that the eigenvalues of R_ρ are less than $C_2(1 + |k|)$, where C_2 is independent of ρ. Therefore the integrands of the integrals represented in (4.4) are dominated by the integrable functions $\widehat{g}$ $\widehat{g}/\widehat{g}$

$$C_2(1 + |k|)(|\widehat{f}_1(k)|^2 + |\widehat{f}_2(k)|^2),$$

and

$$C_2(1 + |k|)(|\widehat{g}_1(k)|^2 + |\widehat{g}_2(k)|^2),$$

so the dominated convergence theorem allows passage to the limit in the integrands as $\rho \to \infty$. This yields

$$\langle \widehat{f}(k), (\Delta + A(k))\widehat{f}(k)\rangle_X \geqslant \langle \widehat{g}(k), (\Delta + A(k))\widehat{g}(k)\rangle_X,$$

or $J(f) \geqslant J(g)$, which is the desired result. ∎

The same argument as in the proof of the preceding lemma, together with the lemma of Riesz cited in the proof of lemma 3.5, yields the following result.

Lemma 4.5. *For every* $f = (f_1, f_2) \in Y$, *one has* $(f_1^*, f_2^*) \in Y$ *and* $J(f_1^*, f_2^*) \leqslant J(f_1, f_2)$.

Then from lemmas 4.4 and 4.5 and the argument used to prove lemma 3.6, with the obvious substitution of $\int_{-\infty}^{\infty}(\alpha_1\gamma_4|g_1|^3 + \alpha_2\gamma_2|g_2|^3)\,\mathrm{d}x$ for $\int_{-\infty}^{\infty}|g|^3\,\mathrm{d}x$, one obtains the following analogue of lemma 3.6.

Lemma 4.6. *If* $f_0 = (f_{01}, f_{02})$ *is a minimizer for problem (P2), then* $(|f_{01}|^*, |f_{02}|^*)$ *is a minimizer for problem (P1).*

Now let $\{f_j = (f_{j1}, f_{j2})\}_{j=1,2,\ldots}$ be a minimizing sequence in Y for problem (P2), and make the definition

$$Q_j(t) = \sup_{y\in\mathbb{R}} \int_{y-t}^{y+t} (\alpha_1\gamma_4(f_{j1}(x))^3 + \alpha_2\gamma_2(f_{j2}(x))^3)\,\mathrm{d}x.$$

Then, as in § 3, $Q_j(t)$ converges to a limit function $Q(t)$ as $j \to \infty$ and $\alpha = \lim_{t\to\infty} Q_j(t)$ exists in the interval $[0,1]$. Analogues of lemmas 3.7, 3.8 and 3.9, in which the $H^{1/2}$-norm is replaced by the Y-norm and integrals of $|f|^3$ are replaced by integrals of $(\alpha_1\gamma_4|f_1|^3 + \alpha_2\gamma_2|f_2|^3)$, are easily established by applying the foregoing arguments to each of the components f_1 and f_2. It then follows that $\alpha \neq 0$.

We continue with the numbering of the lemmas in this section so they correspond to their counterparts in the preceding section. The following is an analogue of lemma 3.10.

Lemma 4.10. *There exists a constant $c > 0$ such that if $\theta \in W^1_\infty(\mathbb{R})$ and $f \in Y$, then*

$$\|[L,\theta]f\|_X \leqslant c|\theta'|_\infty\|f\|_X.$$

Proof. The matrix $B(k) = (\Delta + A(k))^{1/2}$ which is the symbol of L is given explicitly by the formula

$$B(k) = \begin{bmatrix} b_{11}(k) & b_{12}(k) \\ b_{21}(k) & b_{22}(k) \end{bmatrix} = \frac{1}{\sqrt{\lambda_1} + \sqrt{\lambda_2}} \begin{bmatrix} a_{11} + S & a_{12} \\ a_{21} & a_{22} + S \end{bmatrix},$$

Δc

where λ_i are the eigenvalues of $\Delta + A(k)$ and

$$S = \sqrt{\gamma_4 + a_{11} + \gamma_2(C + \Delta C) + a_{22} - 4\ \det(\Delta + A(k))}.$$

l. c.

If $L_{ij} : H^{1/2} \to L_2$ is the Fourier multiplier operator with symbol $b_{ij}(k)$, then one clearly has

$$[L,\theta]f = \begin{bmatrix} [L_{11},\theta] & [L_{12},\theta] \\ [L_{21},\theta] & [L_{22},\theta] \end{bmatrix},$$

for $f \in X$, whence

$$\|[L,\theta]f\|_X^2 \leqslant \sum_{i,j} |[L_{ij},\theta]f_j|_2^2.$$

Therefore it suffices to show that an estimate of the type given in lemma 3.10 holds for each of the operators L_{ij}. But such an estimate follows from the argument given in the proof of lemma 3.10 and the fact that, for every integer $l \geqslant 0$,

$$\sup_{k\in\mathbb{R}} |k|^l \left| \left(\frac{\mathrm{d}}{\mathrm{d}k}\right)^l \left[\frac{b_{ij}(k) - b_{ij}(0)}{k}\right] \right| < \infty.$$

■

Now from lemma 4.10 and the obvious analogues of lemmas 3.11–3.13, one arrives at the conclusion that α cannot lie in the interval $(0,1)$, and hence $\alpha = 1$. Further, the same argument as in the proof of lemma 3.14 then shows that there exists a sequence of real numbers $\{y_j\}_{j=1,2,\ldots}$ and, for each integer $k \geqslant 0$, a number $R_k > 0$

such that

$$\int_{-R_k}^{R_k} (\alpha_1\gamma_4|f_{j1}(x+y_j)|^3 + \alpha_2\gamma_2|f_{j2}(x+y_j)|^3)\,\mathrm{d}x > 1 - \frac{1}{k}$$

holds for all sufficiently large values of j. Letting $(\tilde{f}_{j1}(x), \tilde{f}_{j2}(x)) = (f_{j1}(x+y_j), f_{j2}(x+y_j))$, it follows as in § 3 that the sequences f_{j1} and f_{j2} have subsequences converging, weakly in $H^{1/2}(\mathbb{R})$ and strongly in L_3 on compact subsets of $\mathbb{R}$, to functions f_{01} and f_{02} satisfying

$$\int_{-\infty}^{\infty} (\alpha_1\gamma_4|f_{01}(x)|^3 + \alpha_2\gamma_2|f_{02}(x)|^3)\,\mathrm{d}x = 1.$$

Then $f_0 = (f_{01}, f_{02}) \in Y$ is a solution of the problem (P2), and so by lemma 4.6, the function $g_0 \in Y$ defined by $g_0 = (g_{01}, g_{02}) = (|f_{01}|^*, |f_{02}|^*)$ is a solution of (P1).

The Lagrange-multiplier equation satisfied by g_0 is

$$\delta J(g_0) = \lambda\,\delta K(g_0), \tag{4.5}$$

where K is defined by $K(f) = \int_{-\infty}^{\infty} (\alpha_1\gamma_4 f_1^3 + \alpha_2\gamma_2 f_2^3)\,\mathrm{d}x$. The Fréchet derivatives of J and K at g_0 are given as maps from Y to $\mathbb{R}$ by

$$\delta J(g_0)[h] = 2\langle Lg_0, Lh\rangle_X,$$

$$\delta K(g_0)[h] = 3\int_{-\infty}^{\infty} (\alpha_1\gamma_4 g_{01}^2 h_1 + \alpha_2\gamma_2 g_{02}^2 h_2)\,\mathrm{d}x.$$

In particular, if the components h_1 and h_2 of h are test functions on $\mathbb{R}$, one has

$$\delta J(g_0)[h] = 2\langle L^2 g_0, h\rangle_X = 2\int_{-\infty}^{\infty} ((L^2 g_0)_1 h_1 + (L^2 g_0)_2 h_2)\,\mathrm{d}x.$$

Since h_1 and h_2 may vary independently over the space $C_c^\infty(\mathbb{R})$, (4.5) implies that

$$(L^2 g_0)_1 = (\tfrac{3}{2}\lambda)\alpha_1\gamma_4 g_{01}^2,$$

$$(L^2 g_0)_2 = (\tfrac{3}{2}\lambda)\alpha_2\gamma_2 g_{02}^2,$$

as distributions on $\mathbb{R}$. Hence, taking $\phi = 3\lambda g_{01}$ and $\psi = 3\lambda g_{02}$ yields

$$L^2(\phi, \psi) = (\tfrac{1}{2}\alpha_1\gamma_4\phi^2, \tfrac{1}{2}\alpha_2\gamma_2\psi^2).$$

But from the definition of L, one has

Ac

$$L^2 = \begin{bmatrix} \gamma_4(C + \gamma_1 M_1 + \gamma_2 M_2) & -\gamma_2\gamma_4 N \\ -\gamma_2\gamma_4 N & \gamma_2(C + \Delta C + \gamma_3 M_3 + \gamma_2 M_2) \end{bmatrix},$$

l.c.

whence ϕ and ψ are seen to be distributional solutions of (4.1). An application of lemma 4.3 then completes the proof of theorem 4.1.

Remarks. A discussion of the physical meaning of the parameters α_i and γ_i, along with a description of the situations in which these parameters are expected to be positive may be found in Albert *et al.* (1997). Briefly put, the theorem above applies to the case in which all the fluid particles in a given vertical column, extending from one confining surface to the other, move up or down simultaneously. Solitary waves with more complicated vertical structure, for example the 'mode-two' type waves observed in Davis & Acrivos (1967) and Weidman & Johnson (1982), would in general correspond to the case when the coefficients α_1 and α_2 are of different signs. The topic of existence of solitary waves in this case will be the subject of a future report.

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