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EXISTENCE AND ASYMPTOTIC PROPERTIES OF SOLITARY-WAVE SOLUTIONS OF BENJAMIN-TYPE EQUATIONS

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Abstract. Benjamin recently put forward a model equation for the evolution of waves on the interface of a two-layer system of fluids in which surface tension effects are not negligible. It is our purpose here to investigate the solitary-wave solutions of Benjamin's model. For a class of equations that includes Benjamin's model featuring conflicting contributions to dispersion from dynamic effects on the interface and surface tension, we establish existence of travelling-wave solutions. Using the recently developed theory of Li and Bona, we are also able to determine rigorously the spatial asymptotics of these solutions.

1. Introduction. Considered here is the evolution of waves on the interface of an idealized, incompressible, two-fluid system consisting of a light fluid of density ρ_1 and depth h_1, resting on a heavier fluid of density ρ_2 and depth h_2, bounded above and below by rigid horizontal planes. We assume that $\rho_2 > \rho_1, h_1 >> h_2$ (or $h_2 >> h_1$), and focus attention on wave motion that doesn't vary in the direction perpendicular to the primary direction of propagation. The fluid domain is described quantitatively by a standard x-y-z-Cartesian coordinate system so oriented that gravity acts in the $-z$-direction, the interface between the two fluids at rest is located at $z = 0$, the primary direction of wave propagation is along the x-axis, and so the dependent variables describing the fluid motion do not depend on the independent variable y.

If the surface tension is neglected and the waves are uni-directional, of small amplitude and long wavelength, the fluid motion can be described approximately by the Benjamin–Ono equation

$$\eta_t + \eta_x + \eta\eta_x - L\eta_x = 0,$$

where $L = H\partial_x$ is the composition of the Hilbert transform and the spatial derivative in the direction of primary propagation, or, equivalently, L is a Fourier multiplier operator with symbol $|\xi|$, which is to say $\widehat{Lv}(\xi) = |\xi|\hat{v}(\xi)$. Here, $\eta = \eta(x, t)$ is the vertical displacement of the interface between the two fluids at the spatial point x at time t, and the equation is written in a nondimensional, scaled form (see Benjamin 1992, Benjamin 1996, Albert *et al.* 1997).

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If surface tension cannot be safely ignored, the interfacial waves are described at the same level of approximation by the Benjamin equation,

$$\eta_t + \eta_x + \eta\eta_x - \alpha L\eta_x \pm \beta\eta_{xxx} = 0,$$

where L is as above and α and β are nonnegative constants. For a detailed analysis of the circumstances under which this equation is likely to be physically relevant, see Albert *et al.* (1997, Section 2). Define the new dependent variable u by the transformation $\eta(x,t) = u(x+t,t)$ if the plus-sign appears and $u(x,t) = -\eta(-x-t,t)$ if the minus-sign holds; in terms of u, Benjamin's equation becomes

$$u_t + uu_x \pm \alpha L u_x + \beta u_{xxx} = 0. \tag{1.1}$$

As written in (1.1), the equation reduces to the Korteweg–de Vries-equation (KdV-equation henceforth) when $\alpha = 0$ (dispersive effects are dominated by surface tension), and to the Benjamin–Ono equation (BO-equation henceforth) when $\beta = 0$ (negligible surface tension). In this paper we are interested in solitary-wave solutions of (1.1) in case both α and β are nonzero. Unlike the situation that obtains for the KdV-equation, the solitary-wave solutions of (1.1) feature algebraically—rather than exponentially—decaying outskirts. Moreover, depending on the sign in front of $\alpha L u_x$, the tails of the solitary waves may feature a finite number of oscillations as they evanesce.

Provided $\beta > 0$, a rescaling of x and t allows us to assume $\beta = 1$ in (1.1) without loss of generality. To study solitary-wave solutions, it is natural to substitute the travelling-wave form $u(x,t) = \phi(x-ct)$ into (1.1). After a few simple manipulations, this format for u leads to the operator equation

$$\phi = A\phi, \tag{1.2}$$

where the constant $c > 0$ is the propagation velocity of the wave and the operator A is defined by

$$A\phi(x) = \frac{1}{2}(c+\mathcal{L})^{-1}\phi(x) = \int_{-\infty}^{\infty} k(x-y)\phi^2(y)\,dy, \tag{1.3a}$$

with the kernel k given explicitly in terms of its Fourier symbol:

$$\hat{k}(\xi) = \frac{1}{2(\xi^2 \pm \alpha|\xi| + c)}. \tag{1.3b}$$

Our purpose here is several-fold. In Section 2, we bring to bear techniques from nonlinear functional analysis, notably positive-operator ideas and concentrated-compactness methods to establish a satisfactory theory of existence for solitary-wave solutions of (1.2). Section 3 is devoted to a rigorous analysis of the decay of these solitary waves. There are conflicting formal analyses of this aspect by Benjamin (1992, 1996), and our theory settles a couple of aspects of the spatial asymptotics of these waves. In Sections 4, 5 and 6, we take up a natural generalization of equation (1.2) which features at the same time more general nonlinearity and more general forms of competing dispersion. The theory is broadened to include existence and decay results for solitary-wave solutions of this general class of nonlinear, dispersive wave equations.

2. Existence of solitary waves. The discussion of existence of solitary waves is broken into two parts. First, consideration is given to the case of a plus sign in front of the term $\alpha L u_x$. For this situation, positive-operator methods come to the fore and existence is seen as a consequence of extant theory. Section 2.2 treats the more interesting case where a minus sign appears, and the dispersion arising from surface tension competes with the usual frequency dispersion brought on by finite-wavelength effects. Here, methods of concentrated-compactness (see P.-L. Lions 1984) win the day.

2.1. The symbol of $\mathcal{L}$ is $\xi^2 + \alpha|\xi|$. To establish existence in case the kernel k has Fourier symbol $\frac{1}{2(\xi^2+\alpha|\xi|+c)}$ where $\alpha > 0$, we rely upon the theory of solutions of nonlinear convolution equations of the form (1.3) put forward in Benjamin *et al.* (1990). The conclusion of the theory is that, provided the kernel $k = k_c$ satisfies appropriate hypotheses, equation (1.3) has a solution $\phi(x) = \phi_c(x)$ which is an even function, decaying monotonically to zero as $x \to \pm\infty$. Moreover, ϕ and all its derivatives lie in $L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$. In fact, more is true as will be seen presently.

The following technical lemma demonstrates that the kernel satisfies the hypotheses needed for the validity of the principle results in the last-quoted reference.

Lemma 2.1. *Let k be defined via its Fourier transform as in* (1.3b) *for the case of a plus sign. Then k has the following properties*:

(1) *k is a real-valued, even, bounded, continuous function and $k(x) \to 0$ as $|x| \to \infty$,*
(2) *k is positive on $\mathbb{R}$, $k \in L_1(\mathbb{R})$,*
(3) *k is monotone decreasing on $(0, \infty)$, and k is strictly convex for $x > 0$.*

Proof. Write k as a Fourier integral thusly:

$$k(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \widehat{k}(\xi)\, d\xi = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{\cos(x\xi)}{c + \alpha\xi + \xi^2}\, d\xi.$$

From this representation, it is clear that k is real-valued and even. Since k is the Fourier transform of an $L_1(\mathbb{R})$-function, the Riemann–Lebesque lemma assures that k is bounded, continuous and tends to zero at $\pm\infty$. To establish the rest of the properties, it is convenient to represent k in another way.

Consider the complex function

$$f(z) = \frac{e^{iwz}}{c + \alpha z + z^2},$$

where $w \geq 0$, and $z = x + iy$. Since $c, \alpha > 0$, f is analytic in the closed region

$$\Omega = \{z = \rho e^{i\theta} : 0 \leq \rho \leq R,\ \ 0 \leq \theta \leq \tfrac{\pi}{2}\}.$$

By Cauchy's theorem, for any $R > 0$,

$$\int_{\partial\Omega} f(z)\, dz = 0.$$

Parametrizing $\partial\Omega$ in the obvious way reveals that

$$\int_0^R \frac{e^{iwx}}{c + \alpha x + x^2}\, dx + \int_0^{\frac{\pi}{2}} \frac{e^{iwRe^{i\theta}}}{c + \alpha Re^{i\theta} + R^2 e^{2i\theta}} iRe^{i\theta}\, d\theta + \int_R^0 \frac{e^{-wy}}{c + i\alpha y - y^2} i\, dy = 0. \tag{2.1}$$

As in the usual proof of Jordan's lemma,

$$\left| \int_0^{\frac{\pi}{2}} \frac{e^{iwRe^{i\theta}}}{c + \alpha Re^{i\theta} + R^2 e^{2i\theta}} iRe^{i\theta}\, d\theta \right| \to 0 \text{ as } R \to \infty.$$

Taking the real part of the relation (2.1) and then passing to the limit as $R \to \infty$ thus yields

$$k(w) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\cos(wx)}{c + \alpha x + x^2}\, dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\alpha y e^{-wy}}{(c - y^2)^2 + \alpha^2 y^2}\, dy,$$

for $w \geq 0$. Because k is even, it follows that for all $x \in \mathbb{R}$,

$$k(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{\alpha \xi e^{-|x|\xi}}{(c - \xi^2)^2 + \alpha^2 \xi^2}\, d\xi. \tag{2.2}$$

Clearly, the representation (2.2) entails that $k(x) > 0$ for all $x \in \mathbb{R}$. Moreover, for $x > 0$, it is easily seen that

$$k'(x) = -\frac{1}{\sqrt{2\pi}} \int \frac{\alpha \xi^2 e^{-x\xi}}{(c - \xi^2)^2 + \alpha^2 \xi^2}\, d\xi < 0$$

and that

$$k''(x) = \frac{1}{\sqrt{2\pi}} \int \frac{\alpha \xi^3 e^{-x\xi}}{(c - \xi^2)^2 + \alpha^2 \xi^2}\, d\xi > 0.$$

Thus k is monotone decreasing and strictly convex on $(0, \infty)$, and the lemma is proved.□

Theorem 2.2. *For $\alpha, \beta > 0$, the Benjamin equation with the $+$-sign in front of the term $\alpha L u_x$ has a nontrivial, solitary-wave solution $u(x,t) = \phi_c(x - ct)$ for each $c > 0$. The solitary-wave ϕ_c may be chosen to be an even, positive function, strictly monotone decreasing on $(0, \infty)$, and such that ϕ_c and all its derivatives are bounded, continuous $L_1(\mathbb{R})$-functions.*

Remark. Theorem 2.2 follows immediately from Benjamin *et al.* (1990, Theorems 3.7 and 3.9), because the hypotheses of this result are a consequence of Lemma 2.1.

2.2. The symbol of $\mathcal{L}$ is $\xi^2 - \alpha|\xi|$. To assure the operator A has a kernel k that is free of certain types of singular behavior, the parameter α must be less than $2\sqrt{c}$. This restriction is already discussed in detail in Benjamin (1992). Proceeding as in Section 2.1, but using the Residue Theorem instead of Cauchy's Theorem, the symbol k is determined to be

$$k(x) = -\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\alpha y e^{-|x|y}}{(c - y^2)^2 + \alpha^2 y^2}\, dy + \frac{\sqrt{2\pi}}{\sqrt{4c - \alpha^2}} e^{-\frac{\sqrt{4c-\alpha^2}}{2}|x|} \cos \frac{\alpha}{2} x.$$

Because the kernel k fails to be nonnegative, an existence proof using positive operator theory is not available. We turn instead to the concentration-compactness theory (see Lions 1984, Weinstein 1987, Albert 1997).

Consider the equation

$$(c + \mathcal{L})\phi = \frac{1}{2}\phi^2. \tag{2.3}$$

Following Weinstein, we introduce the functional Λ for our variational analysis, namely

$$\Lambda(f) = \frac{\int f(c + \mathcal{L})f}{(\int f^3(x)\, dx)^{\frac{2}{3}}}.$$

Throughout, an unadorned integral will always mean an integral over $\mathbb{R}$. Define also the functional J on $H^1(\mathbb{R})$ by

$$J(u) = \int u(c + \mathcal{L})u,$$

and for $\lambda > 0$, let

$$\Theta(\lambda) = \inf\left\{ J(u) : u \in H^1(\mathbb{R}), \int u^3(x)\, dx = \lambda \right\}. \tag{2.4}$$

Remark. The restriction $\lambda > 0$ can be replaced by $\lambda \neq 0$ because if u is a solution of (2.4), then so is $-u$.

Lemma 2.2. (1) *The equation* (2.3) *is solvable if* $\min\{\Lambda(f) : f \in H^1, f \neq 0\}$ *is solvable;* (2) *Problem* (2.4) *is equivalent to* $\min\{\Lambda(f) : f \in H^1, f \neq 0\}$; *more precisely, if* f *is a minimizer of* $\Theta(\lambda)$ *for some* $\lambda > 0$, *then a simple rescaling* $\phi = \frac{2\Theta(\lambda)}{\lambda} f = 2\Theta(1)\frac{f}{\lambda^{\frac{1}{3}}}$ *solves* (2.3).

Proof. (1) If f is a minimizer of $\Lambda(f)$, then the Fréchet derivative of Λ at f must vanish. For $h \in H^1$, we have

$$0 = \Lambda'(f)h =$$
$$\frac{2\left(\int f^3(x)\,dx\right)^{\frac{2}{3}} \int f(c+\mathcal{L})h\,dx - 2\int f(c+\mathcal{L})f\,dx \int f^2 h\,dx \left(\int f^3(x)\,dx\right)^{-\frac{1}{3}}}{\left(\int f^3\,dx\right)^{\frac{4}{3}}},$$

whence

$$\int f(c+\mathcal{L})h\,dx = \frac{\int f(c+\mathcal{L})f\,dx}{\int f^3\,dx} \int f^2 h\,dx$$

for any $h \in H^1$. It follows that

$$(c+\mathcal{L})f = \frac{\int f(c+\mathcal{L})f\,dx}{\int f^3\,dx} f^2,$$

and this equation is the same as (2.3) except for the coefficient. If we define ϕ by

$$\phi = \frac{2\int f(c+\mathcal{L})f\,dx}{\int f^3\,dx} f = 2\Lambda(f)\frac{f}{\left(\int f^3\,dx\right)^{\frac{1}{3}}} = 2\Theta(1)\frac{\bar{u}}{\lambda^{\frac{1}{3}}},$$

then ϕ does not depend on λ and satisfies equation (2.3). □

The next two results are taken from Lions (1984) and repeated here for the readers' convenience.

Lemma 2.3. *Let* $\{\rho_n\}_{n\geq 1}$ *be a sequence of nonnegative functions in* $L_1(\mathbb{R}^N)$ *satisfying* $\int_{\mathbb{R}} \rho_n(x)\,dx = \lambda$ *for all* n *and some* $\lambda > 0$. *Then there exists a subsequence* $\{\rho_{n_k}\}_{k\geq 1}$ *satisfying one of the following three conditions*:

(1) (*Compactness*) *there are* $y_k \in \mathbb{R}^N$ *for* $k = 1, 2, \ldots,$ *such that* $\rho_{n_k}(\cdot + y_k)$ *is tight, which is to say that for any* $\epsilon > 0$, *there is an* R *large enough that*

$$\int_{|x-y_k|\leq R} \rho_{n_k}(x)\,dx \geq \lambda - \epsilon;$$

(2) (*Vanishing*) *for any* $R > 0$, $\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq R} \rho_{n_k}(x)\,dx = 0$;

(3) (*Dichotomy*) *there exists* $\alpha \in (0, \lambda)$ *such that for any* $\epsilon > 0$, *there exists* $k_0 \geq 1$ *and* $\rho_k^1, \rho_k^2 \in L_1(\mathbb{R}^N)$, $\rho_k^1, \rho_k^2 \geq 0$ *such that for* $k \geq k_0$,

$$\begin{cases} \|\rho_{n_k} - (\rho_k^1 + \rho_k^2)\|_{L_1} \leq \epsilon, & \\ \left|\int_{\mathbb{R}^N} \rho_k^1\,dx - \alpha\right| \leq \epsilon, & \left|\int_{\mathbb{R}^N} \rho_k^2\,dx - (\lambda-\alpha)\right| \leq \epsilon, \\ \operatorname{supp}\rho_k^1 \cap \operatorname{supp}\rho_k^2 = \emptyset, & \operatorname{dist}\{\operatorname{supp}\rho_k^1, \operatorname{supp}\rho_k^2\} \to \infty \text{ as } k \to \infty. \end{cases}$$

Remark. In Lemma 2.3 above, the condition $\int \rho_n(x)\,dx = \lambda$ can be replaced by $\int \rho_n(x)\,dx = \lambda_n$ where $\lambda_n \to \lambda > 0$. To see this simply replace ρ_n by $\frac{\rho_n}{\lambda_n}$ and apply the lemma.

Lemma 2.4. *Let $1 < p \le \infty$ and $1 \le q < \infty$. Assume for $n = 1, 2, \ldots$, that u_n is bounded in $L_q(\mathbb{R})$, u_n' is bounded in $L_p(\mathbb{R})$, and for some $R > 0$,*

$$\sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} |u_n(x)|^q\,dx \to 0,$$

as $n \to \infty$. Then it transpires that for any $r > q$,

$$u_n \to 0 \text{ as } n \to \infty \text{ in } L_r(\mathbb{R}).$$

Theorem 2.5. *Let $c > 0$ be a given wave-speed such that $\min_{x\in\mathbb{R}}\{x^2 - \alpha|x| + c\} > 0$, and let λ be any positive number. Then every minimizing sequence $\{u_n\}_{n\ge1}$ of the problem (2.4) is, up to a translation in the underlying spatial domain, relatively compact in $H^1(\mathbb{R})$. Hence, there is a solution of the problem (2.4), and therefore there exists a nontrivial solution of problem (2.3) by Lemma 2.2. Thus the Benjamin equation has solitary-wave solutions corresponding to the wave speed c.*

Proof. We begin with a couple of observations about the functional J. Since $\alpha < 2\sqrt{c}$, it follows that

$$J(u) = \int u(c + \mathcal{L})u = \int \hat{u}(k^2 - \alpha|k| + c)\bar{\hat{u}}$$

$$\ge \max\{(c - \frac{\alpha^2}{4})\|u\|_{L_2}^2, (1 - \frac{\alpha^2}{4c})\|u'\|_{L_2}^2\} \ge \underline{\gamma}\|u\|_{H^1}^2 \ge \underline{\gamma}\|u\|_{L_3}^2$$

and, in any event,

$$J(u) \le \int (k^2 + c)|\hat{u}|^2 \le \bar{\gamma}\|u\|_{H^1}^2,$$

where $\underline{\gamma} = \frac{1}{2}\left(1 - \frac{\alpha^2}{4c}\right)\min\{1, c\}$, $\bar{\gamma} = \max\{1, c\}$. This means that $0 < \Theta(\lambda) < \infty$ and that the minimizing sequence $\{u_n\}_{n\ge1}$ is bounded in $H^1(\mathbb{R})$.

Denote by ρ_n the quantity $|u_n|^2 + |u_n'|^2$, and let $\int \rho_n(x)\,dx = \mu_n$. Then μ_n is bounded, and furthermore, $\mu_n = \|u_n\|_{H^1}^2 \ge \|u\|_{L_3}^2 \ge (\int u^3(x)\,dx)^{\frac{2}{3}} = \lambda^{\frac{2}{3}}$, because $H^1 \subset L_3$ with an embedding constant less than one.

Without loss of generality, suppose $\mu_n \to \mu$ as $n \to \infty$. To prove the theorem, we apply Lemma 2.3 to the sequence $\{\rho_n\}_{n\ge1}$, after ruling out the possibilities of Vanishing and Dichotomy. Suppose there is a subsequence $\{\rho_{n_k}\}_{k\ge1}$ of $\{\rho_n\}_{n\ge1}$ which satisfies either Vanishing or Dichotomy. If Vanishing occurs, which is to say for any $R > 0$,

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\le R} \rho_{n_k}(x)\,dx = 0,$$

then

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\le R} |u_{n_k}(x)|^2\, dx = 0;$$

so, by Lemma 2.4,

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} |u_{n_k}(x)|^3\, dx = 0.$$

This leads to a contradiction since

$$0 < \lambda = \int_{-\infty}^{\infty} (u_{n_k}(x))^3\, dx \le \int_{-\infty}^{\infty} |u_{n_k}(x)|^3\, dx \to 0.$$

If Dichotomy occurs, there is a $\bar{\mu} \in (0, \mu)$ such that for any $\epsilon > 0$ there corresponds a k_0 and $\rho_k^1, \rho_k^2 \in L_1(\mathbb{R})$, $\rho_k^1, \rho_k^2 \ge 0$ such that for $k \ge k_0$,

$$\begin{cases} \|\rho_{n_k} - (\rho_k^1 + \rho_k^2)\|_{L_1} \le \epsilon, \\ |\int_{\mathbb{R}} \rho_k^1\, dx - \bar{\mu}| \le \epsilon, \qquad |\int_{\mathbb{R}} \rho_k^2\, dx - (\mu - \bar{\mu})| \le \epsilon, \\ \text{supp } \rho_k^1 \cap \text{supp } \rho_k^2 = \emptyset \text{ and } \operatorname{dist}\{\text{supp } \rho_k^1, \text{supp } \rho_k^2\} \to \infty \text{ as } k \to \infty. \end{cases} \tag{2.5}$$

Without loss of generality, it may be supposed that the supports of ρ_k^1 and ρ_k^2 are separated as follows: supp $\rho_k^1 \subset (y_k - R_0, y_k + R_0)$, supp $\rho_k^2 \subset (-\infty, y_k - 2R_k) \cup (y_k + 2R_k, \infty)$ for some fixed $R_0 > 0$, for some sequence $\{y_k\}_{k\ge1} \subset \mathbb{R}$ and $R_k \to \infty$ (see the construction in Lions 1984).

To obtain splitting functions u_k^1 and u_k^2 of u_{n_k}, $k = 1, 2, \dots$, let $\zeta, \psi \in C_b^\infty$ with $0 \le \zeta, \psi \le 1$ be such that

$$\zeta(x) = 1 \text{ when } |x| \le 1, \qquad \zeta(x) = 0 \text{ when } |x| \ge 2,$$
$$\psi(x) = 0 \text{ when } |x| \le 1, \qquad \psi(x) = 1 \text{ when } |x| \ge 2.$$

Denote by $\zeta_k(x) = \zeta(\frac{x-y_k}{R_1})$, $\psi_k(x) = \psi(\frac{x-y_k}{R_k})$, for $x \in \mathbb{R}$, where $R_1 > R_0$ is chosen large enough that

$$|\int \left(|\zeta_k u_{n_k}|^2 + |\partial_x(\zeta_k u_{n_k})|^2 - \rho_k^1\right) dx| \le \epsilon \tag{2.6a}$$

and

$$|\int \left|\psi_k u_{n_k}|^2 + |\partial_x(\psi_k u_{n_k})|^2 - \rho_k^2\right) dx| \le \epsilon. \tag{2.6b}$$

To see this is possible, first note that from (2.5),

$$\begin{aligned} \epsilon &\ge \|\rho_{n_k} - (\rho_k^1 + \rho_k^2)\|_{L_1} \\ &= \int_{|x-y_k|\le R_0} |\rho_{n_k} - \rho_k^1|\, dx + \int_{|x-y_k|\ge 2R_k} |\rho_{n_k} - \rho_k^2|\, dx + \int_{R_0\le|x-y_k|\le 2R_k} \rho_{n_k}\, dx, \end{aligned}$$

whence

$$\int_{|x-y_k|\le R_0} |\rho_{n_k} - \rho_k^1|\,dx \le \epsilon, \qquad \int_{|x-y_k|\ge 2R_k} |\rho_{n_k} - \rho_k^2|\,dx \le \epsilon,$$

$$\int_{R_0\le|x-y_k|\le 2R_k} \rho_{n_k}\,dx \le \epsilon.$$

In consequence of these relations, it transpires that

$$\begin{aligned}
&|\int \left(|\zeta_k u_{n_k}|^2 + |\partial_x(\zeta_k u_{n_k})|^2 - \rho_k^1\right)\,dx| \\
&= |\int_{|x-y_k|\le 2R_1} \left(|\zeta_k u_{n_k}|^2 + |\partial_x(\zeta_k u_{n_k})|^2 - \rho_k^1\right)dx| \\
&= |\int_{|x-y_k|\le R_0} \left(|u_{n_k}|^2 + |u'_{n_k}|^2 - \rho_k^1\right)dx| \\
&\quad + \int_{R_0\le|x-y_k|\le 2R_1} \left(|\zeta_k u_{n_k}|^2 + |\zeta_k'|^2|u_{n_k}|^2 + |\zeta_k|^2|u'_{n_k}|^2\right)dx \\
&\le \int_{|x-y_k|\le R_0} |\rho_{n_k} - \rho_k^1|\,dx + \max_{x\in\mathbb{R}}\left\{|\zeta_k(x)|^2 + |\zeta_k'(x)|^2\right\} \int_{R_0\le|x-y_k|\le 2R_1} \rho_{n_k}\,dx \\
&\le \epsilon + \epsilon = O(\epsilon)
\end{aligned}$$

as $\epsilon \to 0$, and

$$\begin{aligned}
&|\int \left(|\psi_k u_{n_k}|^2 + |\partial_x(\psi_k u_{n_k})|^2 - \rho_k^2\right)\,dx| \\
&= |\int_{|x-y_k|\ge R_k} \left(|\psi_k u_{n_k}|^2 + |\partial_x(\psi_k u_{n_k})|^2 - \rho_k^2\right)\,dx| \\
&\le |\int_{R_k\le|x-y_k|\le 2R_k} \left(|\psi_k u_{n_k}|^2 + |\partial_x(\psi_k u_{n_k})|^2 - \rho_k^2\right)dx| \\
&\quad + |\int_{|x-y_k|\ge 2R_k} \left(|u_{n_k}|^2 + |u'_{n_k}|^2 - \rho_k^2\right)dx| \\
&\le \max_{x\in\mathbb{R}}\left\{|\psi_k(x)|^2 + |\psi_k'(x)|^2\right\} \int_{R_k\le|x-y_k|\le 2R_k} \rho_{n_k}\,dx + \int_{|x-y_k|\ge 2R_k} |\rho_{n_k} - \rho_k^2|\,dx \\
&= O(\epsilon)
\end{aligned}$$

as $\epsilon \to 0$. Thus if we set $u_k^1 = \zeta_k u_{n_k}$, $u_k^2 = \psi_k u_{n_k}$ and define w_k by $u_{n_k} = u_k^1 + u_k^2 + w_k$, then $u_k^1, u_k^2, w_k \in H^1$, $\int |u_k^1(x)|^3\,dx$ is bounded, and there exists a subsequence of $\{u_k^1\}_{k\ge1}$, still denoted $\{u_k^1\}_{k\ge1}$, for which there is a $k_0 > 0$ and $\alpha, \beta \in \mathbb{R}$ such that for $k \ge k_0$,

$$|\int u_k^1(x)^3\,dx - \beta| \le \epsilon, \qquad |\int u_k^2(x)^3\,dx - (\lambda - \beta)| \le \epsilon,$$

$$\|w_k\|_{H^1} = \|(1 - \zeta_k - \psi_k)u_{n_k}\|_{H^1} = C(\zeta, \psi)\left(\int_{R_1\le|x-y_k|\le 2R_k} \rho_{n_k}\,dx\right)^{\frac{1}{2}} = O(\epsilon)$$

as $\epsilon \to 0$, where $C(\zeta, \psi) = \max_{x\in\mathbb{R}}\{|1-\zeta_k(x)-\psi_k(x)|+|(1-\zeta_k(x)-\psi_k(x))'|\}$ is a constant dependent only on ζ and ψ, and $\mathrm{dist}\{\mathrm{supp}\, u_k^1, \mathrm{supp}\, u_k^2\} \to \infty$ as $k \to \infty$. A simple calculation shows that

$$J(u_{n_k}) = J(u_k^1 + u_k^2 + w_k) = J(u_k^1) + J(u_k^2) + J(w_k) - 2\alpha \int u_k^1 H(u_k^2)'\, dx$$
$$+ 2\int u_k^1 (c+\mathcal{L}) w_k\, dx + 2\int u_k^2 (c+\mathcal{L}) w_k\, dx,$$

where $Hu' = \int \frac{u'(y)}{x-y}\, dy$ is the nonlocal operator whose Fourier symbol is $|\xi|$. Notice that, as $\epsilon \to 0$,

$$J(w_k) \le \frac{1+c}{2}\|w_k\|_{H^1}^2 = O(\epsilon),$$
$$\left|\int u_k^1 (c+\mathcal{L}) w_k\, dx\right| \le \gamma \|u_k^1\|_{H^1}\|w_k\|_{H^1} = O(\epsilon)$$

and

$$\left|\int u_k^2 (c+\mathcal{L}) w_k\, dx\right| \le \gamma \|u_k^2\|_{H^1}\|w_k\|_{H^1} = O(\epsilon),$$

and that

$$\begin{aligned}
\left|\int u_k^1 H(u_k^2)'\, dx\right| &= \left|\int u_k^1(x) \int \frac{\partial_y u_k^2(y)}{x-y}\, dy\, dx\right| \\
&= \left|\int_{-2R_1}^{2R_1} u_k^1(x) \int_{|y-y_k|\ge R_k} \frac{\partial_y u_k^2(y)}{x-y}\, dy\, dx\right| \\
&= \left|\int_{y_k-2R_1}^{y_k+2R_1} u_k^1(x) \int_{|y-y_k|\ge R_k} \frac{\partial_y u_k^2(y)}{x-y}\, dy\, dx\right| \\
&= \left|\int_{y_k-2R_1}^{y_k+2R_1} u_k^1(x) \int_{|y-y_k|\ge R_k} \frac{-u_k^2(y)}{(x-y)^2}\, dy\, dx\right| \\
&\le \max_{x\in\mathbb{R}}\{|\zeta(x)|, |\psi(x)|\}\|u_{n_k}\|_{L_2}^2 \left\{\int_{-2R_1}^{2R_1}\int_{|y|\ge R_k} \frac{1}{(x-y)^4}\, dx\, dy\right\}^{\frac{1}{2}} \to 0
\end{aligned}$$

as $k \to \infty$, because $R_k \to \infty$ and R_1 is fixed. In consequence, it is seen that

$$\begin{aligned}
\Theta(\lambda) &= \liminf_n \{J(u_n)\} = \liminf_k \{J(u_{n_k})\} \\
&= \liminf_k \{J(u_k^1) + J(u_k^2) + J(w_k) \\
&\quad - 2\alpha \int u_k^1 H(u_k^2)' + 2\int u_k^1(c+\mathcal{L})w_k + 2\int u_k^2 (c+\mathcal{L}) w_k\} \\
&\ge \liminf_k J(u_k^1) + \liminf_k J(u_k^2) + O(\epsilon)
\end{aligned}$$

as $\epsilon \to 0$. If $\int u_k^1(x)^3\,dx \to \beta = 0$, then by (2.6a),

$$\liminf_k J(u_k^1) \geq \liminf_k \underline{\gamma} \|u_k^1\|_{H^1}^2 \geq \liminf_k \underline{\gamma} \|\rho_k^1\|_{L_1} + O(\epsilon) \geq \underline{\gamma}\bar{\mu} + O(\epsilon)$$

as $\epsilon \to 0$, and therefore

$$\Theta(\lambda) > \underline{\gamma}\bar{\mu} + \liminf_k J(u_k^2) + O(\epsilon)$$

as $\epsilon \to 0$. Letting $\epsilon \to 0$ in the last relation leads to

$$\Theta(\lambda) \geq \underline{\gamma}\bar{\mu} + \Theta(\lambda) > \Theta(\lambda).$$

If, on the other hand, $\int u_k^1(x)^3\,dx \to \beta \neq 0$, then

$$\Theta(\lambda) \geq \Theta(\beta) + \Theta(\lambda - \beta) + O(\epsilon),$$

and letting $\epsilon \to 0$ gives

$$\Theta(\lambda) \geq \Theta(\beta) + \Theta(\lambda - \beta).$$

But, for $\theta \in \mathbb{R}$, $\Theta(\theta\lambda) = |\theta|^{\frac{2}{3}}\Theta(\lambda)$. If we write $\beta = \theta\lambda$, then

$$\begin{aligned} 0 > \Theta(\lambda) \geq \Theta(\theta\lambda) + \Theta((1-\theta)\lambda) &= |\theta|^{\frac{2}{3}}\Theta(\lambda) + |1-\theta|^{\frac{2}{3}}\Theta(\lambda) \\ &= \{|\theta|^{\frac{2}{3}} + |1-\theta|^{\frac{2}{3}}\}\Theta(\lambda) > \Theta(\lambda), \end{aligned}$$

another contradiction. Thus Dichotomy is seen to be impossible.

Since Vanishing and Dichotomy have been ruled out, it is concluded that there is a sequence $\{y_n\}_{n\geq 1} \subset \mathbb{R}$ such that for any $\epsilon > 0$, there are $R < \infty$ and $n_0 > 0$ such that for $n > n_0$,

$$\int_{|x-y_n|\leq R} \rho_n(x)\,dx \geq \mu - \epsilon, \qquad \int_{|x-y_n|\geq R} \rho_n(x)\,dx \leq \epsilon,$$

$$\Big|\int_{|x-y_n|\geq R} u_n(x)^3\,dx\Big| \leq \int_{|x-y_n|>R} |u_n(x)|^3\,dx \leq \|u_n\|_{H^1} \int_{|x-y_n|\geq R} \rho_n(x)\,dx \leq O(\epsilon)$$

as $\epsilon \to 0$. It follows that

$$\Big|\int_{|x-y_n|\leq R} u_n(x)^3\,dx - \lambda\Big| \leq \epsilon.$$

Letting $\tilde{u}_n(x) = u_n(x - y_n)$ for $x \in \mathbb{R}$, the above property means that $\tilde{u}_n$ (or a subsequence) converges weakly in H^1, almost everywhere on $\mathbb{R}$, and strongly in $L_3(\mathbb{R})$ to some H^1-function $\tilde{u}$, say, and

$$\int \tilde{u}(x)^3\,dx = \lim \int \tilde{u}_n(x)^3\,dx = \lambda.$$

Furthermore,

$$\Theta(\lambda) = \liminf_n \int u_n(c+\mathcal{L})u_n\,dx \geq \int \tilde{u}(c+\mathcal{L})\tilde{u}\,dx.$$

Thus the function $\tilde{u}$ solves the variational problem (2.4) and therefore $\phi = \frac{2\Theta(\lambda)}{\lambda}\tilde{u}$ solves the problem (2.3). Theorem 2.5 is proved.□

The uniqueness up to translations of these solutions appears likely. For both the KdV-equation and the BO-equation, solitary waves corresponding to a given, supercritical value of the wave-speed c are known to be unique up to translations in the underlying spatial domain. For the KdV-equation, this follows from phase-plane analysis, but for the BO-equation or its near relative the ILW-equation, the issue is far more delicate (see Amick and Toland 1991 and Albert and Toland 1994).

While having nothing to report about uniqueness, we can at least assert that corresponding to a given value of c, there is at least one solitary-wave solution of Benjamin's equation having positive Fourier transform.[1]

Corollary 2.6. *Let $c > 0$ be a wave-speed such that* $\min_{x\geq 0}\{x^2 - \alpha|x| + c\} > 0$. *There is a minimizer ϕ of the variational problem* (2.4) *such that $\hat{\phi}(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. It follows that ϕ is an even function.*

Proof. This follows directly from the first part of Theorem 5.1 in Albert (1992). Indeed, the idea of Albert's proof is take any minimizer ψ of (2.4) and let ϕ be the inverse Fourier transform $\phi = \mathcal{F}^{-1}(|\hat{\psi}|)$. The function ϕ is real-valued since $|\hat{\psi}|$ is real-valued and even. Then the numerator in $\Lambda(\phi)$ is the same as the numerator of $\Lambda(\psi)$ and the denominator in $\Lambda(\phi)$ is greater than or equal to that of $\Lambda(\psi)$. Thus ϕ also minimizes Λ and so $a\phi$ is a solitary-wave solution for some constant a. On the other hand,

$$\begin{aligned}\hat{\phi}(\xi) = |\hat{\psi}(\xi)| &= \frac{1}{\sqrt{2\pi}}\left|\int \psi(x)\cos(x\xi)\,dx - i\int \psi(x)\sin(x\xi)\,dx\right| \\ &= \frac{1}{\sqrt{2\pi}}\sqrt{\left|\int \psi(x)\cos(x\xi)\,dx\right|^2 + \left|\int \psi(x)\sin(x\xi)\,dx\right|^2}\end{aligned}$$

is an even function; hence,

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2\pi}}\int \hat{\phi}(\xi)\cos(x\xi)\,d\xi + \frac{i}{\sqrt{2\pi}}\int \hat{\phi}(\xi)\sin(x\xi)\,d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int \hat{\phi}(\xi)\cos(x\xi)\,d\xi\end{aligned}$$

is an even function. The corollary is proved. □

[1]The authors thank John Albert for pointing this out.

3. Asymptotic decay of solitary-wave solutions. In the previous section, we discussed existence of solitary-wave solutions of the Benjamin equation. In this section, attention is turned to the asymptotic properties of such solutions. According to a recent result reported in Bona and Li (1997) they resemble those of the kernel k. Here is the relevant theorem.

Lemma 3.1. *Suppose that $f \in L_\infty$ with $\lim_{|x|\to\infty} f(x) = 0$ is a solution of the convolution equation*

$$f(x) = \int k(x-y)G(f(y))\,dy,$$

where the kernel k is a measurable function satisfying $\hat{k} \in H^s$ for some $s > \frac{1}{2}$ and G is a function such that $|G(x)| \le C|x|^r$ for all $x \in \mathbb{R}$, and for some constants $C > 0$ and $r > 1$. Then $f \in L_1$ and there is a constant l with $0 < l < s$ such that $|x|^l f(x) \in L_1 \cap L_\infty$. Furthermore,

(1) *if there is a constant $m > 1$ such that $\lim_{x\to\pm\infty} |x|^m k(x) = C_\pm$ for some constants $C_\pm \in \mathbb{C}$ corresponding to limits at $+\infty$ and $-\infty$, respectively, then*

$$\lim_{x\to\pm\infty} |x|^m f(x) = C_\pm \int G(f(t))\,dt,$$

(2) *and if $\lim_{x\to\pm\infty} e^{\sigma_0|x|}k(x) = C_\pm$, then $\sup e^{\sigma_0|x|}|f(x)| < \infty$ and*

$$\lim_{x\to\pm\infty} e^{\sigma_0|x|} f(x) = C_\pm \int e^{\pm\sigma_0 t} G(f(t))\,dt$$

for some constants $C_\pm$ corresponding to limits at $+\infty$ and $-\infty$, respectively.

In the present context, based on Lemma 3.1, the following may be concluded.

Theorem 3.2. *For the problem* (1.2) *where $\alpha > 0$ is as restricted previously,*

$$\lim_{x\to\pm\infty} x^2\phi(x) = C$$

for some constant $C \in \mathbb{R}$, $C \neq 0$.

Proof. Obviously, $\hat{k} \in H^s$ for any $s < \frac{3}{2}$. Hence it will suffice to find a suitable expression for k. For the case $\hat{k}(\xi) = \frac{1}{2(c+\alpha|\xi|+\xi^2)}$, it was found earlier that

$$k(x) = \mathcal{F}^{-1}\hat{k}(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\alpha y e^{-|x|y}}{(c-y^2)^2 + \alpha^2 y^2}\,dy.$$

The change of variables $\xi = xy$ transforms the right-hand side to

$$k(x) = \frac{1}{\sqrt{2\pi}x^2} \int_0^\infty \frac{\alpha\xi e^{-\xi}}{(\frac{\xi}{x})^4 + (\alpha^2 - 2c)(\frac{\xi}{x})^2 + c^2}\,d\xi,$$

and thus

$$\lim_{x\to\pm\infty} x^2 k(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\alpha\xi e^{-\xi}}{0+c^2}\, d\xi = \frac{\alpha}{\sqrt{2\pi}c^2}$$

by the Dominated Convergence Theorem.

For the case $\hat{k}(\xi) = \frac{1}{2(c-\alpha|\xi|+\xi^2)}$, it was seen previously that

$$k(x) = -\frac{1}{\sqrt{2\pi}} \int \frac{\alpha y e^{-|x|y}}{(c-y^2)^2+\alpha^2 y^2}\, dy + \frac{\sqrt{2\pi}}{\sqrt{4c-\alpha^2}} e^{-\frac{\sqrt{4c-\alpha^2}}{2}x} \cos\frac{\alpha}{2}x.$$

The same transformation $\xi = xy$ gives

$$k(x) = -\frac{\alpha}{\sqrt{2\pi}x^2} \int_0^\infty \frac{\xi e^{-\xi}}{(c-(\frac{\xi}{x})^2)^2+\alpha^2\xi^2}\, d\xi + \frac{\sqrt{2\pi}}{\sqrt{4c-\alpha^2}} e^{-\frac{\sqrt{4c-\alpha^2}}{2}x} \cos\frac{\alpha}{2}x.$$

Applying the Dominated Convergence Theorem again leads to the conclusion

$$\lim_{x\to\pm\infty} x^2 k(x) = -\frac{\alpha}{\sqrt{2\pi}c^2}.$$

By Lemma 3.1, the theorem is proved. □

Remark. Notice that this rigorous result shows clearly that Benjamin (1992) is incorrect in his assertion that the tails of solitary-wave solutions $\phi(x)$ of his model equation possess an infinite number of oscillations (zero-crossings) as $x \to \pm\infty$. The present theorem does square with Benjamin's second formal analysis of these solitary waves in his 1996 paper. We expect that a more refined analysis will show that these solutions approach zero monotonically as $x \to \pm\infty$, but that they do feature a finite number of oscillations as one sees from the numerical approximations reported in Albert *et al.* (1997).

4. More general nonlinearity and dispersion. In this section and the next, attention is turned to an extension of the theory developed in Sections 2 and 3. Consideration is given to the situation where the effects of dispersion are modelled by a competing pair of homogeneous terms and the nonlinearity is a pure power. Thus the operator $\mathcal{L}$ is defined by

$$\widehat{\mathcal{L}u}(\xi) = \left(-\alpha|\xi|^{2r} + |\xi|^{2m}\right)\hat{u}(\xi), \tag{4.1}$$

and the nonlinearity has the form

$$F(\phi) = \frac{1}{p-1}\phi^{p-1}, \tag{4.2}$$

where m and p are positive integers, $p > 2$, and r is a real number with $0 \le r < m$.

We aim to establish existence and asymptotic decay rates for solitary-wave solutions of the nonlinear, dispersive wave equation

$$u_t + F(u)_x - \mathcal{L}u_x = 0, \tag{4.3}$$

where F and $\mathcal{L}$ are defined in (4.1)–(4.2). Some of the development can be abbreviated because it parallels the discussion of Benjamin's equation in Sections 2 and 3.

As before, the assumption that $u(x,t) = \phi(x - ct)$ is a solitary-wave solution of (4.3) implies that ϕ is a solution of

$$(c + \mathcal{L})\phi = \frac{1}{p-1}\phi^{p-1}. \tag{4.4}$$

The proof of the following theorem is the goal of the present section. In Section 5, the issue of the large-space asymptotics of solutions of (4.4) is taken up.

Theorem 4.1. *Let $c > 0$ be a given wave-speed and suppose $m, p \in \mathbb{Z}^+$, $p > 2$, and $0 \le r < m$. If α lies in the range $\alpha < \frac{c^{\frac{m-r}{m}}}{\frac{r}{m}(\frac{m}{r}-1)^{\frac{m-r}{m}}}$, equation* (4.4) *has a solution $\phi \in H^\infty$.*

Remark. Notice that if $\phi \in H^1(\mathbb{R})$ satisfies (4.4), then necessarily $\phi \in H^\infty(\mathbb{R})$. This follows because the assumption on the range of α implies that the symbol of $c + \mathcal{L}$ has the property

$$c - \alpha|\xi|^{2r} + |\xi|^{2m} \ge \beta > 0$$

for all ξ. In consequence, there are positive constants $\underline{\gamma}$ and $\bar{\gamma}$ such that

$$\underline{\gamma}(1 + |\xi|^{2m}) \le c - \alpha|\xi|^{2r} + |\xi|^{2m} \le \bar{\gamma}(1 + |\xi|^{2m}), \tag{4.5}$$

for all ξ. Hence $c + \mathcal{L}$ is an isomorphism of $H^{s+2m}(\mathbb{R})$ onto $H^s(\mathbb{R})$ for any $s \ge 0$, say. Therefore, if ϕ solves (4.4) and $\phi \in H^1(\mathbb{R})$, then $\phi^{p-1} \in H^1(\mathbb{R})$, and hence

$$\phi = \frac{1}{p-1}(c + \mathcal{L})^{-1}\phi^{p-1} \in H^{1+2m}(\mathbb{R}).$$

It then follows that $\phi^{p-1} \in H^{1+2m}(\mathbb{R})$, whence

$$\phi = \frac{1}{p-1}(c + \mathcal{L})^{-1}\phi^{p-1} \in H^{1+4m}(\mathbb{R}).$$

Continuing this argument inductively demonstrates that $\phi \in H^\infty(\mathbb{R})$. An application of the theory of Li and Bona (1996) shows also that these solitary waves are the restriction to the real axis of a function holomorphic in a strip $\{z : |\Im(z)| < \sigma\}$ for some $\sigma > 0$.

To prove Theorem 4.1, introduce the functional

$$\Lambda(f) = \frac{\int f(c+\mathcal{L})f}{(\int f^p\,dx)^{\frac{2}{p}}}, \tag{4.6}$$

and, in direct analogy with the developments in Section 2, define J for $u \in H^m(\mathbb{R})$ by

$$J(u) = \int u(c+\mathcal{L})u\,dx$$

and Θ for $\lambda > 0$ by

$$\Theta(\lambda) = \inf\Big\{J(u) : u \in H^m(\mathbb{R}), \int u^p(x)\,dx = \lambda\Big\}. \tag{4.7}$$

Remark. Actually, $\Theta(\lambda) = \Theta(-\lambda)$ if p is an odd number, and so we need only require $\lambda \neq 0$ instead of $\lambda > 0$. Also, p need not be an integer in our theory. The development is unchanged if $p = \frac{m}{n}$ where m and n are relatively prime and n is odd, if we choose the branch of $z \to z^{\frac{1}{n}}$ that is real-valued on the real axis.

By the same argument as that in Lemma 2.2, the following result appears.

Lemma 4.2. (1) *The problem* (4.4) *is solvable if* $\min\{\Lambda(f) : f \in H^m, f \neq 0\}$ *is solvable;* (2) *minimization of* $\Lambda(f)$ *in* (4.6) *is equivalent to* (4.7); *namely, any solution of* (4.7) *is a minimizer of* (4.6), *and if* f *is a minimizer of* (4.6), *then the rescaling* $f \mapsto \frac{\lambda^{\frac{1}{p}}}{(\int f^p\,dx)^{\frac{1}{p}}} f$ *is a solution of* (4.4). *Therefore if the problem* (4.7) *has a nontrivial solution* $\bar{u}$, *then* (4.4) *is solved by the function* ϕ *defined by the rescaling*

$$\phi = \Big(\frac{(p-1)\Theta(\lambda)}{\lambda}\Big)^{\frac{1}{p-2}}\bar{u} = \big((p-1)\Theta(1)\big)^{\frac{1}{p-2}}\lambda^{-\frac{1}{p}}\bar{u}, \tag{4.8}$$

which is independent of λ.

Proof of Theorem 4.1. It is first shown that any minimizing sequence $\{u_n\}_{n\geq 1}$ of the variational problem (4.7) is, up to translations in the underlying spatial domain, relatively compact in $H^m(\mathbb{R})$. Because of the restriction on α, (4.5) applies and it is concluded that any minimizing sequence $\{u_n\}_{n\geq 1}$ is bounded in $H^m(\mathbb{R})$ and that $0 < \Theta(\lambda) < \infty$ if $\lambda > 0$.

The following representations will be useful in the arguments presented presently.

Sublemma. *Let* M *be the operator defined by* $\widehat{Mu}(\xi) = |\xi|^r\hat{u}(\xi)$ *for* $u \in H^m$, *where* $0 < r \leq m$, m *a positive integer. It follows that*

(1) *if* $r = 2n$ *for* $n \in \mathbb{Z}^+$, *then*

$$Mu(x) = (-1)^n\partial_x^{2n}u(x);$$

(2) *if* $r = 2n + 1$ *for* $n \in \mathbb{Z}^+$, *then*

$$Mu(x) = (-1)^n \sqrt{\frac{2}{\pi}} \int \frac{\partial_y^{2n+1} u(y)}{x - y}\, dy;$$

(3) *if* $r = 2n + \delta$ *for* $n \in \mathbb{Z}^+, 0 < \delta < 1$, *then*

$$Mu(x) = (-1)^n \sqrt{\frac{\pi}{2}} \Big(\cos(\frac{\delta\pi}{2}) \Gamma(1 - \delta) \Big)^{-1} \int \operatorname{sign}(x - y) \frac{\partial_y^{2n+1} u(y)}{|x - y|^\delta}\, dy,$$

where Γ *connotes the usual gamma-function;*

(4) *and if* $r = 2n + 1 + \delta$ *for* $n \in \mathbb{Z}^+, 0 < \delta < 1$, *then*

$$Mu(x) = (-1)^{n-1} \sqrt{\frac{\pi}{2}} \Big(\sin(\frac{\delta\pi}{2}) \Gamma(1 - \delta) \Big)^{-1} \int \frac{\partial_y^{2n+2} u(y)}{|x - y|^\delta}\, dy.$$

Proof. Part (1) is obvious. Part (2) is proved by taking the Fourier Transform of the right-hand side of the expected equality and using the formula $\int \frac{\sin x}{x}\, dx = \pi$ to obtain

$$\begin{aligned}
&(-1)^n \sqrt{\frac{2}{\pi}}\, \mathcal{F}^{-1}\{\partial_x^{2n+1} u(x)\}\, \mathcal{F}^{-1}\{\frac{1}{x}\} \\
=&(-1)^n \sqrt{\frac{2}{\pi}}\, (i\xi)^{2n+1} \hat{u}(\xi)\, \frac{1}{\sqrt{2\pi}} \int \frac{e^{-ix\xi}}{x}\, dx \\
=&\frac{1}{\pi} i \xi^{2n+1} \hat{u}(\xi) \big(-i \operatorname{sign}(\xi) \pi \big) = |\xi|^{2n+1} \hat{u}(\xi).
\end{aligned}$$

(3) When $r = 2n + \delta$, the Fourier Transform of the right-hand side of the desired relation is

$$\begin{aligned}
&(-1)^n \sqrt{\frac{\pi}{2}} \Big(\cos(\frac{\delta\pi}{2}) \Gamma(1 - \delta) \Big)^{-1} \{\widehat{\partial^{2n+1} u}\} \widehat{\operatorname{sign}(x) \frac{1}{|x|^\delta}} \\
=&\frac{(-1)^n}{2} \Big(\cos\frac{\delta\pi}{2} \Gamma(1 - \delta) \Big)^{-1} (i\xi)^{2n+1} \hat{u}(\xi) \int \operatorname{sign}(x) \frac{e^{-i\xi x}}{|x|^\delta}\, dx \\
=&\frac{(-1)^n}{2} \Big(\cos(\frac{\delta\pi}{2}) \Gamma(1 - \delta) \Big)^{-1} i^{2n+1} \xi^{2n+1} \hat{u}(\xi) (-2i) \int_0^\infty \frac{\sin \xi x}{x^\delta}\, dx \\
=& |\xi|^{2n+\delta} \hat{u}(\xi),
\end{aligned}$$

where the formula $\int_0^\infty \frac{\sin \xi x}{x^\delta}\, dx = \cos(\frac{\delta\xi}{2}) \Gamma(1 - \delta)$ can be found in Oberhettinger (1957, p.5).

When $r = 2n + 1 + \delta$, then the Fourier Transform of the right-hand side of the advertised equality in (4) is

$$
\begin{aligned}
&= (-1)^{n-1}\sqrt{\frac{\pi}{2}}\Big(\sin(\frac{\delta\pi}{2})\Gamma(1-\delta)\Big)^{-1}\widehat{\partial^{2n+2}u}\,\widehat{\frac{1}{|x|^{\delta}}} \\
&= \frac{(-1)^{n-1}}{2}\Big(\sin(\frac{\delta\pi}{2})\Gamma(1-\delta)\Big)^{-1}(i\xi)^{2n+2}\int_{-\infty}^{\infty}\frac{e^{-i\xi x}}{|x|^{\delta}}\,dx \\
&= \frac{(-1)^{n-1}}{2}\Big(\sin(\frac{\delta\pi}{2})\Gamma(1-\delta)\Big)^{-1}(i\xi)^{2n+2}2\int_{0}^{\infty}\frac{\cos\xi x}{x^{\delta}}\,dx \\
&= |\xi|^{2n+1+\delta}\hat{u}(\xi),
\end{aligned}
$$

where the elementary formula $\int_0^\infty \frac{\cos \xi x}{x^\delta}\,dx = \sin(\frac{\delta\xi}{2})\Gamma(1-\delta)$ can be found in Oberhettinger (1957, p.116). The sublemma is established. □

Attention is now given to finishing the proof of Lemma 4.2. Denote by ρ_n and μ_n the quantities $\rho_n(x) = |u_n(x)|^2 + |\partial_x^{2m} u_n(x)|^2$ and $\mu_n = \int \rho_n(x)\,dx$. Then $\{\mu_n\}_{n\ge1}$ is bounded, and $\mu_n = \|\rho_n\|_{L_1} \ge \|u\|_{L_p}^2 \ge \big(\int u_n(x)^p\,dx\big)^{\frac{2}{p}} = \lambda^{\frac{2}{p}}$, because $H^m(\mathbb{R}) \subset L_p(\mathbb{R})$ with an embedding constant less than one for any $p \ge 2$. Without loss of generality, suppose $\mu_n \to \mu$ as $n \to \infty$. To prove the theorem, apply the concentration-compactness principle Lemma 2.3 to the sequence $\{\rho_n\}_{n\ge1}$ and aim to rule out the possibilities of Vanishing and Dichotomy. Suppose there is a subsequence $\{\rho_{n_k}\}_{k\ge1}$ of $\{\rho_n\}_{n\ge1}$ which is either Vanishing or Dichotomous. If Vanishing occurs, which is to say for any $R > 0$,

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}}\int_{|x-y|\le R}\rho_{n_k}(x)\,dx = 0,$$

then

$$\int_{|x-y_n|\le R}|u_{n_k}(x)|^2\,dx < \epsilon.$$

On the other hand, we already know that $\{u'_{n_k}\}_{k\ge1}$ is bounded in $L_2(\mathbb{R})$, so by Lemma 2.4,

$$\int_{-\infty}^{\infty}|u_{n_k}(x)|^q\,dx \to 0 \qquad \text{for any } q > 2,$$

as $n \to \infty$. This leads to a contradiction since

$$0 < \lambda < \int |u_{n_k}(x)|^p\,dx \to 0.$$

Thus Vanishing does not occur.

If Dichotomy occurs, then for any $\epsilon > 0$, there is a $k_0 > 0$ and $\rho_k^1, \rho_k^2 \in L_1(\mathbb{R})$, $\rho_k^1, \rho_k^2 \geq 0$ such that, for $k \geq k_0$,

$$\begin{cases} \|\rho_{n_k} - (\rho_k^1 + \rho_k^2)\|_{L_1} \leq \epsilon, \\ \Big|\int \rho_k^1(x)\,dx - \bar{\mu}\Big| \leq \epsilon, \quad \Big|\int \rho_k^2 - (\mu - \bar{\mu})\Big| \leq \epsilon, \\ \text{supp } \rho_k^1 \cap \text{supp } \rho_k^2 = \emptyset \text{ and } \operatorname{dist}\{\text{supp } \rho_k^1, \text{supp } \rho_k^2\} \to \infty \text{ as } k \to \infty. \end{cases} \tag{4.9}$$

As in the proof of Theorem 2.5, the supports of ρ_k^1 and ρ_k^2 may be taken so that

$$\text{supp } \rho_k^1 \subset (y_k - R_0, y_k + R_0), \quad \text{supp } \rho_k^2 \subset (-\infty, y_k - 2R_k) \cup (y_k + 2R_k, \infty),$$

for some fixed $R_0 > 0$ and sequences $\{y_k\}_{k\geq 1}, \{R_k\}_{k\geq 1} \subset \mathbb{R}$, where $R_k \to \infty$.

To construct the splitting functions u_k^1 and u_k^2 of u_{n_k} for $k = 1, 2, \ldots$, let $\zeta, \psi \in C_b^\infty$ be as defined in Section 2.2, and $\zeta_k(x) = \zeta(\frac{x-y_k}{R_1})$, $\psi_k(x) = \psi(\frac{x-y_k}{R_k})$ for $R_1 > R_0$ chosen large enough that

$$\int \big||\zeta_k u_{n_k}|^2 + |\partial^m(\zeta_k u_{n_k})|^2 - \rho_k^1\big|\,dx \leq \epsilon \tag{4.10a}$$

and

$$\int \big||\psi_k u_{n_k}|^2 + |\partial^m(\psi_k u_{n_k})|^2 - \rho_k^2\big|\,dx \leq \epsilon. \tag{4.10b}$$

The reason (4.10a) and (4.10b) obtain for large R_1 is the same as argued earlier in Section 2.2.

Let $u_k^1 = \zeta_k u_{n_k}$, $u_k^2 = \psi_k u_{n_k}$, $w_k = u_{n_k} - u_k^1 - u_k^2$ or $u_{n_k} = u_k^1 + u_k^2 + w_k$. Then $u_k^1, u_k^2, w_k \in H^1$ and the supports of u_k^1 and u_k^2 lie in $(y_k - 2R_1, y_k + 2R_1)$ and $(-\infty, y_k - 2R_1) \cup (y_k + 2R_k, \infty)$, respectively. Moreover, $\int u_k^1(x)^p\,dx$ is bounded, so there is a subsequence of $\{u_k^1\}_{k\geq 1}$, still denoted by $\{u_k^1\}_{k\geq 1}$, such that $\int u_k^1(x)^p\,dx$ converges to some number β, say. Then for any $\epsilon > 0$ and k sufficiently large, we have

$$\Big|\int (u_k^1(x))^p\,dx - \beta\Big| \leq \epsilon, \qquad \Big|\int (u_k^2(x))^p\,dx - (\lambda - \beta)\Big| \leq \epsilon,$$

and

$$\begin{aligned} \|w_k\|_{H^m}^2 &= \|(1 - \zeta_k - \psi_k)u_{n_k}\|_{H^m}^2 \\ &= \int \Big|(1 - \zeta_k - \psi_k)u_{n_k}\Big|^2 + \Big|\partial^m\Big((1 - \zeta_k - \psi_k)u_{n_k}\Big)\Big|^2\,dx \\ &\leq C(\zeta, \psi)\int_{R_1 \leq |x-y_k| \leq 2R_k} \left(|u_{n_k}|^2 + |\partial^m u_{n_k}|^2\right)dx \\ &= C(\zeta, \psi)\int_{R_1 \leq |x-y_k| \leq 2R_k} \rho_{n_k}\,dx \leq O(\epsilon) \end{aligned}$$

as $\epsilon \to 0$, by the first inequality in (4.9), where $C(\zeta, \psi)$ is a constant dependent only on ζ, ψ. As before

$$J(u) = J(u_k^1) + J(u_k^2) + J(w_k) - 2\alpha \int |\xi|^{2r} \widehat{u_k^1}(\xi)\widehat{u_k^2}(\xi)\, d\xi$$
$$+ 2\int u_k^1 (c + \mathcal{L}) w_n + 2 \int u_k^2 (c + \mathcal{L}) w_n.$$

Here, there appears the nonlocal operator defined by the Fourier symbol $|\xi|^{2r}$ if r is not an integer. In any event, we always have

$$J(w_k) \le \bar{\gamma} \|w_k\|_{H^m}^2 = O(\epsilon),$$
$$|\int u_k^1 (c + \mathcal{L}) w_k|\, dx \le \int (c - \alpha|\xi|^{2r} + \xi^{2m}) |\widehat{u_k^1}(\xi)|\, |\widehat{w_k}(\xi)|\, d\xi$$
$$\le \bar{\gamma} \|u_k^1\|_{H^m} \|w_k\|_{H^m} = O(\epsilon)$$

and

$$|\int u_k^2 (c + \mathcal{L}) w_k|\, dx \le \int (c - \alpha|\xi|^{2r} + \xi^{2m}) |\widehat{u_k^2}(\xi)|\, |\widehat{w_k}(\xi)|\, d\xi$$
$$\le \bar{\gamma} \|u_k^2\|_{H^m} \|w_k\|_{H^m} = O(\epsilon)$$

as $\epsilon \to 0$. To deal with the integral $\int |\xi|^{2m} \widehat{u_k^1}(\xi)\widehat{u_k^2}(\xi)\, d\xi$, use is made of the Sublemma.

(1) If $2r = 2n$ for some $n \in \mathbb{Z}^+$, ξ^{2r} is the symbol of a differential operator, and therefore

$$\int |\xi|^{2r} \widehat{u_k^1}(\xi)\widehat{u_k^2}(\xi)\, d\xi = (-1)^n \int \partial_x^n u_k^1(x) \partial_x^n u_k^2(x)\, dx = 0,$$

due to supp $u_k^1 \cap$ supp $u_k^2 = \emptyset$.

(2) If $2r = 2n + 1$ for some $n \in \mathbb{Z}^+$, then $m \ge n + 1$,

$$\int |\xi|^{2r} \widehat{u_k^1}(\xi)\overline{\widehat{u_k^2}}(\xi)\, d\xi = \int \big((i\xi)^n \widehat{u_k^1}(\xi)\big)\big((i\xi)^{n+1}(-i \ \mathrm{sign}(\xi))\widehat{u_k^2}(\xi)\big)\, d\xi$$
$$= \int \partial_x^n u_k^1(x)\, \frac{1}{\pi} \int \frac{\partial_y^{n+1} u_k^2(y)}{x - y}\, dy\, dx,$$

and hence

$$\left|\int |\xi|^{2r} \widehat{u_k^1}(\xi)\widehat{u_k^2}(\xi)\, d\xi\right| \le \frac{1}{\pi} \left| \int_{|x-y_k| \le 2R_1} \partial_x^n u_k^1(x) \int_{|x-y_k| \ge 2R_k} \frac{\partial_y^n u_k^2(y)}{(x-y)^2}\, dy\, dx \right|$$
$$\le \frac{1}{\pi} \|u_{n_k}\|_{H^n}^2 \left\{ \int_{|x-y_k| \le 2R_1} \int_{|x-y_k| \ge 2R_k} \frac{1}{(x-y)^4}\, dx\, dy \right\}^{\frac{1}{2}} \to 0,$$

since $R_k \to \infty$ with $R_1 > 0$ fixed.

If $2r = 2n + \delta$ for $n \in \mathbb{Z}^+$, $0 < \delta < 1$, then

$$\begin{aligned}\int |\xi|^{2r}\widehat{u_k^1}(\xi)\widehat{u_k^2}(\xi)\,d\xi &= \int (-1)^n\big((i\xi)^n\widehat{u_k^1}(\xi)\big)\big((i\xi)^n\widehat{u_k^2}(\xi)\big)|\xi|^\delta\,d\xi\\ &= (-1)^n\int \widehat{\partial^n u_k^1}(\xi)\widehat{\partial^n u_k^2}(\xi)|\xi|^\delta\,d\xi\\ &= C(\delta)\int \partial_x^n u_k^1(x)\int \operatorname{sign}(x-y)\frac{\partial_y^{n+1}u_k^2(y)}{|x-y|^\delta}\,dy\,dx\end{aligned}$$

where $C(\delta) = (-1)^n\big(2\cos(\frac{\delta\pi}{2})\int_0^\infty y^{-\delta}e^{-y}\,dy\big)^{-1}$, so

$$\begin{aligned}\Big|\int |\xi|^{2r}\widehat{u_k^1}(\xi)\widehat{u_k^2}(\xi)\,d\xi\Big| &\le C(\delta)\Big|\int_{|x-y_k|\le 2R_1}\partial_x^n u_k^1(x)\int_{|x-y_k|\ge 2R_k}\frac{\partial_y^n u_k^2(y)}{|x-y|^{1+\delta}}\,dy\,dx\Big|\\ &\le C(\delta)\|u_{n_k}\|_{H^r}\left\{\int_{|x-y_k|\le 2R_1}\int_{|x-y_k|\ge 2R_k}\frac{1}{|x-y|^{2+2\delta}}\,dx\,dy\right\}^{\frac12}\to 0,\end{aligned}$$

as $R_k \to \infty$.

(3) $2r = 2n + 1 + \delta$ for $n \in \mathbb{Z}^+$, $0 < \delta < 1$, then as $R_k \to \infty$,

$$\begin{aligned}\int |\xi|^{2r}\widehat{u_k^1}(\xi)\widehat{u_k^2}(\xi)\,d\xi &= \int i^{2n-1}\big((i\xi)^{n+1}\widehat{u_k^1}(\xi)\big)\big((i\xi)^n\widehat{u_k^2}(\xi)\big)|\xi|^\delta\,d\xi\\ &= C_1(\delta)i^{2n-1}\int \widehat{\partial^{n+1}u_k^1}(\xi)\widehat{\partial^n u_k^2}(\xi)|\xi|^\delta\,d\xi\\ &= C_1(\delta)i^{2n-1}\int \partial_x^{n+1}u_k^1(x)\int \operatorname{sign}(x-y)\frac{\partial_y^{n+1}u_k^2(y)}{|x-y|^\delta}\,dy\,dx\end{aligned}$$

where $C_1(\delta) = [2\sin\frac{\delta\pi}{2}\int_0^\infty y^{-\delta}e^{-y}\,dy]^{-1}$ and

$$\begin{aligned}\Big|\int |\xi|^{2r}\widehat{u_k^1}(\xi)\widehat{u_k^2}(\xi)\,d\xi\Big| &\le C_1(\delta)\Big|\int_{\operatorname{supp}\,u_k^1}\partial_x^{n+1}u_k^1(x)\int_{\operatorname{supp}\,u_k^2}\frac{\partial_y^n u_k^2(y)}{(x-y)^{1+\delta}}\,dy\,dx\Big|\\ &\le C_1(\delta)\|u_{n_k}\|_{H^r}^2\left\{\int_{|x-y_k|\le 2R_1}\int_{|x-y_k|\ge 2R_k}\frac{1}{(x-y)^{2+2\delta}}\,dx\,dy\right\}^{\frac12}\to 0.\end{aligned}$$

Thus in all cases, it is observed that

$$\begin{aligned}\Theta(\lambda) = \lim_n\{J(u_n)\} = \lim_k\{J(u_{n_k})\} &= \liminf_k\Big\{J(u_k^1) + J(u_k^2) + J(w_n)\\ &- 2\alpha\int u_k^1 M u_k^2 + 2\int u_k^1(c+\mathcal{L})w_k + 2\int u_k^2(c+\mathcal{L})w_k\Big\}\\ &\ge \liminf_k J(u_k^1) + \liminf_k J(u_k^2) + O(\epsilon)\end{aligned}$$

as $\epsilon \to 0$. If $\int u_k^1(x)^p\,dx \to \beta = 0$, then

$$\liminf_k J(u_k^1) \geq \liminf_k \bar{\gamma}\|u_k^1\|_{H^m} = \liminf_k \bar{\gamma}\|\rho_k^1\|_{L_1} > 0,$$

so

$$\Theta(\lambda) \geq \underline{\gamma}\bar{\mu} + \liminf_k J(u_k^2) + O(\epsilon) \geq \underline{\gamma}\bar{\mu} + \Theta(\lambda) + O(\epsilon)$$

as $\epsilon \to 0$, and therefore we reach the contradiction $\Theta(\lambda) > \Theta(\lambda)$ for sufficiently small values of ϵ. If $\int u_k^1(x)^p\,dx \to \beta \neq 0$, then

$$\Theta(\lambda) \geq \Theta(\beta) + \Theta(\lambda - \beta) + O(\epsilon),$$

and letting $\epsilon \to 0$ gives

$$\Theta(\lambda) \geq \Theta(\beta) + \Theta(\lambda - \beta).$$

As before, $\Theta(\theta\lambda) = \theta^{\frac{2}{p}}\Theta(\lambda)$ for $\theta > 0$. Writing $\beta = \theta\lambda$, we have

$$\begin{aligned} 0 > \Theta(\lambda) &\geq \Theta(\theta\lambda) + \Theta((1-\theta)\lambda) = \theta^{\frac{2}{p}}\Theta(\lambda) + (1-\theta)^{\frac{2}{p}}\Theta(\lambda) \\ &= \{\theta^{\frac{2}{p}} + (1-\theta)^{\frac{2}{p}}\}\Theta(\lambda) > \Theta(\lambda), \end{aligned}$$

another contradiction. Thus Dichotomy is seen to be impossible.

Since Vanishing and Dichotomy have been ruled out, it is concluded that there is a sequence $\{y_n\}_{n\geq 1} \subset \mathbb{R}$ such that for any $\epsilon > 0$, there is an $R < \infty$ satisfying

$$\int_{|x-y_n|\leq R} \rho_n(x)\,dx > \mu - \epsilon,$$

or, what is the same,

$$\int_{|x-y_n|\geq R} \rho_n(x)\,dx < \epsilon,$$

for n sufficiently large. Reinterpreting in terms of u_n, this amounts to

$$\int_{|x-y_n|\geq R} |u_n(x)|^p\,dx < \epsilon,$$

whence

$$\left|\int_{|x-y_n|\leq R} u_n(x)^p\,dx - \lambda\right| < \epsilon.$$

Denote by $\tilde{u}_n$ the translated function $\tilde{u}_n(\cdot) = u_n(\cdot - y_n)$. The above estimates mean that the sequence $\tilde{u_n}$ (or a subsequence) converges weakly in H^m and strongly in L_p to some function $\tilde{u} \in H^m$, and

$$\int \tilde{u}(x)^p\,dx = \lim_{n\to\infty} \int \tilde{u}_n(x)^p\,dx = \lambda.$$

Furthermore,

$$\Theta(\lambda) = \liminf_{n\to\infty} \int u_n(c + \mathcal{L})u_n\,dx \geq \int \tilde{u}(c + \mathcal{L})\tilde{u}\,dx.$$

Thus the limiting function $\tilde{u}$ solves the variational problem (4.3), and therefore $\phi = (\frac{(p-1)\Theta(\lambda)}{\lambda})^{\frac{1}{p-2}}\tilde{u}$ solves the problem (4.1). Lemma 4.2 is proved and with it Theorem 4.1. □

Theorem 4.3. *Problem* (4.1) *has a solution in* $H^\infty(\mathbb{R})$ *when* $\mathcal{L}$ *is defined by the Fourier symbol* $\alpha|\xi|^{2r} + \xi^{2m}$ *with* m *a positive integer,* $0 \le r < m$ *and any* $\alpha \ge 0$.

The same proof as that put forward in Lemma 4.2 concludes Theorem 4.3.

Remark. As mentioned before, the nonlinear term ϕ^p may have p noninteger provided $p = \frac{m}{n}$, m, n relatively prime and n odd so that $y^p \in \mathbb{R}$ if $y \in \mathbb{R}$. For the convolution equation (4.4), the theory would still be available if the nonlinear term had the form $\phi^q|\phi|^\sigma$ where $q \ge 0$ is an integer, $\sigma > 0$ is real, and $q + \sigma \ge 2$. In Section 6, we offer brief commentary on a more general class of dispersion operators than those considered thus far.

5. Asymptotic decay of solitary-wave solutions. To determine the asymptotic property of the solitary-wave solutions discussed in Section 4, write problem (4.2) in the form

$$\phi = \frac{1}{p-1}(c+\mathcal{L})^{-1}\phi^{p-1} = \frac{1}{p-1}\int k(x-y)\phi^{p-1}(y)\,dy, \qquad (5.1)$$

as in (4.4). The kernel k is the inverse Fourier transform of $\hat{k}(\xi) = \frac{1}{c \pm \alpha|\xi|^{2r} + \xi^{2m}}$, where $m > 0$ is an integer and r is a real number in the range $0 < r < m$ with $\alpha \ge 0$ such that $\min_{x\in\mathbb{R}}\{c \pm \alpha|x|^{2r} + |x|^{2m}\} > 0$.

Theorem 5.1. *Suppose* ϕ *is a solitary-wave solution of* (5.1).

(i) *If* r *is a positive integer or if* $\alpha = 0$, *then there is a* $\sigma_0 > 0$ *such that for any* $\sigma < \sigma_0$,

$$e^{\sigma|x|}\phi(x) \to 0,$$

as $x \to \pm\infty$.

(ii) *Otherwise, there is a constant* μ *such that*

$$\frac{1}{|x|^{2r+1}}\phi(x) \to \mu$$

as $x \to \pm\infty$.

Proof. To prove the theorem, it is sufficient to determine the spatial asymptotics of the kernel k in (5.1) and then apply Lemma 3.1. To this end, it is useful to write k in a more suitable form. Let $f_w(z) = \frac{e^{izw}}{z^{2m} - \alpha z^{2r} + c}$, and without loss of generality, suppose $w \ge 0$. The proof breaks naturally into three parts.

(1) If $2r$ is an even positive integer or $\alpha = 0$, then f_w is analytic in the entire z plane except for $2m$ poles. Since α and c are real, the $2m$ poles divide evenly between the upper- and lower-half planes. Thus we may order the poles $\{z_j\}_{j=1}^{2m}$ so that $Im z_j = y_j > 0$ for $j = 1, \ldots, m$. The Residue Theorem leads to

$$k(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_w(x)\,dx = \frac{1}{\sqrt{2\pi}} Re\Big\{2\pi i \sum_j C_j e^{-y_j w + i x_j w}\Big\}, \qquad (5.2)$$

for $w > 0$. Since k is an even function, it follows readily from (5.4) that

$$\lim_{x\to\pm\infty} e^{\sigma|x|}k(x) = 0$$

for any $\sigma < \sigma_0 = \min_{1\le j\le m}\{y_j\} > 0$.

(2) If $2r$ is an odd positive number and $\alpha \ne 0$, let Ω be the closed quarter-disc

$$\Omega = \{z : z = \rho e^{i\theta} : 0 \le \rho \le R,\ \ 0 \le \theta \le \frac{\pi}{2}\}.$$

Then the function f_w is analytic in Ω with finitely many simple poles, $\{z_j = x_j + iy_j : j = 1, 2, \dots, k\}$, say. Appealing again to the Residue Theorem, it is determined that

$$\begin{aligned} k(w) &= \frac{2}{\sqrt{2\pi}}\int_0^\infty f_w(x)\,dx \qquad (5.3)\\ &= \frac{2}{\sqrt{2\pi}}Re\Big\{\int_0^\infty \frac{e^{-xy}}{(-1)^m y^{2m} - \alpha(-1)^{\frac{2r-1}{2}} i y^{2r} + c} i\,dy + 2\pi i\sum_j C_j e^{-y_j w + i x_j w}\Big\}\\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{(-1)^{\frac{2r+1}{2}} y^{2r} e^{-xy}}{(c + (-1)^m y^{2m})^2 + \alpha^2 y^{4r}}\,dy + 2\sqrt{2\pi}\,Re\Big\{i\sum_j C_j e^{-y_j w + i x_j w}\Big\}. \end{aligned}$$

Since k is an even function, (5.3) implies

$$k(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{(-1)^{\frac{2r+1}{2}} y^{2r} e^{-xy}}{(c + (-1)^m y^{2m})^2 + \alpha^2 y^{4r}}\,dy + 2\sqrt{2\pi}\,\Re\Big\{i\sum_j C_j e^{-y_j|x| + i x_j |x|}\Big\} \qquad (5.4)$$

for all $x \in \mathbb{R}$. Changing variables in (5.4) gives the representation

$$k(x) = \frac{2}{|x|^{2r+1}}\int_0^\infty \frac{(-1)^{\frac{2r+1}{2}} y^{2r} e^{-y}}{(c + (-1)^m (\frac{y}{x})^{2m})^2 + \alpha^2 (\frac{y}{x})^{4r}}\,dy + Re\Big\{4\pi i\sum_j C_j e^{-y_j w + i x_j w}\Big\}$$

of k. Since the y_j are all positive, it is straightforward to determine that

$$\lim_{x\to\pm\infty} |x|^{2r+1}k(x) = (-1)^{\frac{2r+1}{2}}\frac{2}{c^2}\int_0^\infty y^{2r}e^{-y}\,dy.$$

(3) Lastly, if $2r > 0$ is not an integer, the origin of the x-y plane is a branch point. Cut the plane through the negative x-axis, and consider the branch of the logarithm which makes $1^{2r} = 1$. Define the domain

$$\Omega = \{z = \rho e^{i\theta} :\ \epsilon \le \rho \le R,\ \ 0 \le \theta \le \frac{\pi}{2}\},$$

where $\epsilon > 0$ is sufficiently small and $R > 0$ is sufficiently large that there is no number outside Ω and inside the first quadrant which makes $z^{2m} - \alpha z^{2r} + c$ equal to 0. As before, an application of the Residue Theorem leads to

$$
\begin{aligned}
k(w) &= \frac{2}{\sqrt{2\pi}} \int_0^\infty f_w(x)\,dx \qquad (5.5)\\
&= \frac{2}{\sqrt{2\pi}} Re\left\{\int_0^\infty f_w(iy)i\,dy\right\} + \frac{4\pi}{\sqrt{2\pi}} i \sum_j{}' resi\{f(z) : z_j\}\\
&= \frac{2}{\sqrt{2\pi}} Re\left\{\int_0^\infty \frac{i e^{iw(iy)}}{(iy)^{2m} - \alpha(iy)^{2r} + c}\,dy + \sum_j C_j e^{ix_j w - y_j w}\right.\\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{-\alpha \sin(r\pi) y^{2r} e^{-wy}}{(c - \alpha\cos(r\pi)y^{2r} + (-1)^m y^{2m})^2 + \alpha^2 \sin(r\pi) y^{4r}}\,dy\\
&\quad + Re \sum_j C_j e^{ix_j w - y_j w},
\end{aligned}
$$

where C_j are some complex constants dependent on the singular points of $f_w(z)$. Using the fact that k is even, and with the change of variables $y \mapsto |x|y$, there obtains

$$
\begin{aligned}
k(x) = {} & \frac{2}{|x|^{2r+1}} \int_0^\infty \frac{-\alpha \sin(r\pi) y^{2r} e^{-y}}{(c - \alpha\cos(r\pi)(\frac{y}{x})^{2r} + (-1)^m(\frac{y}{x})^{2m})^2 + \alpha^2 \sin(r\pi)(\frac{y}{x})^{4r}}\,dy\\
& + Re \sum_j C_j e^{ix_j|x| - y_j|x|}. \qquad (5.6)
\end{aligned}
$$

It is thus clear that

$$
\lim_{x\to\pm\infty} |x|^{2r+1} k(x) = -\frac{2\alpha\sin(r\pi)}{c^2} \int_0^\infty y^{2r} e^{-y}\,dy
$$

as $x \to \pm\infty$. Applying Lemma 4.1, Theorem 5.1 is proved. □

6. Further discussion. In the previous sections, we discussed the generalized version of Benjamin's equation where the nonlinear term is a pure power and the effects of dispersion are modelled by homogeneous terms in which the highest-order term corresponds to a differential operator. In this section, interest is turned to the situation where

$$
\widehat{\mathcal{L}u}(\xi) = \sum_{j=1}^{k} \alpha_j |\xi|^{2r_j} \hat{u}(\xi) \qquad (6.1)
$$

with the nonlinear term

$$
F(\phi) = \frac{1}{p-1}\phi^{p-1}; \qquad (6.2)
$$

as before, $p > 2$ is a positive integer and the parameters α_j are real numbers with $\alpha_k > 0$, $0 < r_1 < r_2 < \cdots < r_k$, but r_k is not an integer. This corresponds to the situation where the highest-order term in the dispersion relation is not local.

The goal is, as before, to establish existence and asymptotic decay rates for solitary-wave solutions of the nonlinear, dispersive wave equation

$$u_t + F(u)_x - \mathcal{L}u_x = 0. \tag{6.3}$$

This amounts to finding a suitably behaved solution of the convolution equation (5.1) where F and $\mathcal{L}$ are as above and $c > 0$ is the wave velocity. As the outline of the theory in this more general case parallels that developed already for the simpler situations considered earlier, the exposition in this section will be abbreviated, and concentrated on the points where additional argument is needed to bring the theory to completion.

As before, for $f \in H^{r_k}(\mathbb{R})$ define

$$\Lambda(f) = \frac{\int f(x)(c+\mathcal{L})f(x)\,dx}{\left(\int f(x)^p\,dx\right)^{\frac{2}{p}}}, \qquad J(f) = \int f(x)(c+\mathcal{L})f(x)\,dx,$$

and, for $\lambda > 0$, set

$$\Theta(\lambda) = \min\Big\{J(u) : u \in H^m,\ \int u^p(x)\,dx = \lambda\Big\}. \tag{6.4}$$

Then as in Lemmas 2.4 and 4.2, a solution of (6.4) yields a solitary wave.

Lemma 6.1. *If u is a minimizer of the problem* (6.4), *then* $\phi = \left(\frac{(p-1)\Theta(\lambda)}{\lambda}\right)^{\frac{1}{p-2}}u = ((p-1)\Theta(1))^{\frac{1}{p-2}}\lambda^{-\frac{1}{p}}u$ *solves* (5.1).

Theorem 6.2. *For any wave-speed $c > 0$ and dispersive parameters α_j, r_j, $j = 1, \ldots, k$ satisfying*

(1) $r_k \geq \frac{1}{2} - \frac{1}{p}$ *and*

(2) $\min_{x\geq 0}\{c + \sum \alpha_j x^{2r_j}\} > 0$,

every minimizing sequence $\{u_n\}_{n\geq 1}$ of the variational problem (6.4) *is, up to a translation in the underlying spatial domain, relatively compact in $H^{r_k}(\mathbb{R})$. Therefore the problem* (5.1) *has a nontrivial solitary-wave solution $\phi = \phi_c \in H^{r_k}(\mathbb{R})$. Furthermore, if $r_k > \frac{1}{2} - \frac{1}{p}$, then $\phi \in H^\infty(\mathbb{R})$. Moreover, if $r_0 = \min\{r_j : r_j \notin \mathbb{Z}, j = 1, \ldots, k\}$, then ϕ_c satisfies*

$$\lim_{x\to\pm\infty} |x|^{2r_0+1}\phi_c(x) = C,$$

for some nonzero constant C.

Remark. In Theorem 6.2 above, if $r_j \in \mathbb{Z}$ for $j = 1, 2, \ldots$, then $\phi_c(x)$ decays exponentially, which is to say, there exists a $\sigma_0 > 0$ such that for all $\sigma < \sigma_0$, $\phi_c(x)$ satisfies

$$\lim_{x\to\pm\infty} e^{\sigma|x|}\phi(x_0) = 0.$$

The proof of this remark is the same as that of the first part of Theorem 5.1.

As in Theorems 2.5 and 4.2, there are two positive numbers $\underline{\gamma}$ and $\bar{\gamma}$ such that

$$\underline{\gamma}\|u\|_{H^{r_k}}^2 \le J(u) \le \bar{\gamma}\|u\|_{H^{r_k}}^2.$$

This means $0 < \Theta(\lambda) < \infty$ and any minimizing sequence $\{u_n\}_{n\ge 1}$ is thus bounded in $H^{r_k}(\mathbb{R})$.

To deal with the problem (6.4) via the concentration-compactness principle in the present, more general circumstances, it is useful for $s > 0$ and $s \notin \mathbb{Z}$ to endow H^s with a slightly unusual version of its norm. If $s = m + \delta$, where m is a nonnegative integer and $0 < \delta < 1$, then

$$\begin{aligned}\|u\|_{H^s}^2 &= \int (1 + |\xi|^2)^{m+\delta}|\hat{u}(\xi)|^2\, d\xi \\ &\sim \int (1 + |\xi|^{2m} + |\xi|^{2m+2\delta})|\hat{u}(\xi)|^2\, d\xi \sim \|u\|_{H^m}^2 + \int |\xi|^{2\delta}|\widehat{\partial^m u}|^2\, d\xi \\ &\sim \|u\|_{H^m}^2 + \iint \frac{|\partial^m u(x) - \partial^m u(y)|^2}{|x-y|^{1+2\delta}}\, dy\, dx,\end{aligned}$$

where $\sim$ stands for equivalence of norms.

Lemma 6.3. *Let $p > 2$ and f be a nonzero element of H^s, where $s \ge \frac{1}{2} - \frac{1}{p} = \frac{p-2}{2p}$. Then for any $R > 0$, there are $j_0 \in \mathbb{Z}$ and $\eta > 0$, dependent only on $\|f\|_{L^p}^p$ and R, such that*

$$\int_{(j_0-\frac{1}{2})R}^{(j_0+\frac{1}{2})R} |f(x)|^p\, dx \ge \eta.$$

Proof. It suffices to establish the result in the case $s < 1$. Using the equivalent norm above, we have

$$\int |f(x)|^2\, dx + \iint \frac{|f(x) - f(y)|^2}{|x-y|^{1+2s}}\, dy\, dx = \|f\|_{H^s}^2 = \frac{\|f\|_{H^s}^2}{\|f\|_{L^p}^p} \int |f(x)|^p\, dx,$$

or, what is the same, for any $R > 0$,

$$\begin{aligned}&\sum_{j=-\infty}^{\infty} \int_{(j-\frac{1}{2})R}^{(j+\frac{1}{2})R} \left(|f(x)|^2 + \int \frac{|f(x) - f(y)|^2}{|x-y|^{1+2s}}\, dy\right) dx \\ &\quad = \frac{\|f\|_{H^s}^2}{\|f\|_{L^p}^p} \sum_{j=-\infty}^{\infty} \int_{(j-\frac{1}{2})R}^{(j+\frac{1}{2})R} |f(x)|^p\, dx.\end{aligned}$$

Comparing both sides of the last equality, it is seen that there must be a $j_0 \in \mathbb{Z}$ for which

$$\int_{(j_0-\frac{1}{2})R}^{(j_0+\frac{1}{2})R} \left(|f(x)|^2 + \int \frac{|f(x) - f(y)|^2}{|x-y|^{1+2s}}\, dy\right) dx \le \frac{\|f\|_{H^s}^2}{\|f\|_{L^p}^p} \int_{(j_0-\frac{1}{2})R}^{(j_0+\frac{1}{2})R} |f(x)|^p\, dx,$$

and hence,

$$\int_{(j_0-\frac{1}{2})R}^{(j_0+\frac{1}{2})R} |f(x)|^2\,dx + \int_{(j_0-\frac{1}{2})R}^{(j_0+\frac{1}{2})R}\int_{(j_0-\frac{1}{2})R}^{(j_0+\frac{1}{2})R} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}}\,dy\,dx$$
$$\le \frac{\|f\|_{H^s}^2}{\|f\|_{L^p}^p}\int_{(j_0-\frac{1}{2})R}^{(j_0+\frac{1}{2})R} |f(x)|^p\,dx;$$

i.e.,

$$\|f\|^2_{H^s\left((j_0-\frac{1}{2})R,(j_0+\frac{1}{2})R\right)} \le \frac{\|f\|_{H^s}^2}{\|f\|_{L^p}^p}\int_{(j_0-\frac{1}{2})R}^{(j_0+\frac{1}{2})R} |f(x)|^p\,dx.$$

On the other hand, by Sobolev-imbedding theory,

$$H^s\big((j_0-\tfrac{1}{2})R,(j_0+\tfrac{1}{2})R\big) \subset L^p\big((j_0-\tfrac{1}{2})R,(j_0+\tfrac{1}{2})R\big)$$

for $s \ge \frac{1}{2}-\frac{1}{p}$, so there is a positive constant $k = k(R,p,s)$ such that

$$\|f\|_{H^s\left((j_0-\frac{1}{2})R,(j_0+\frac{1}{2})R\right)} \ge k\|f\|_{L^p\left((j_0-\frac{1}{2})R,(j_0+\frac{1}{2})R\right)},$$

whence,

$$\frac{\|f\|_{H^s}^2}{\|f\|_{L^p}^p}\int_{(j_0-\frac{1}{2})R}^{(j_0+\frac{1}{2})R} |f(x)|^p\,dx \ge k^2\Big(\int_{(j_0-\frac{1}{2})R}^{(j_0+\frac{1}{2})R} |f(x)|^p\,dx\Big)^{\frac{2}{p}},$$

and thus

$$\int_{(j_0-\frac{1}{2})R}^{(j_0+\frac{1}{2})R} |f(x)|^p\,dx \ge \eta = \Big(\frac{k^2\|f\|_{L^p}^p}{\|f\|_{H^s}^2}\Big)^{\frac{p}{p-2}},$$

as advertised in the statement of the lemma. □

Below is a slightly different version of Lion's Lemma 2.4 that will be used in the present, more general context.

Lemma 6.4. *Let $p > 2$ and $s \ge \frac{1}{2}-\frac{1}{p}$. Suppose $\{u_n\}_{n\ge1}$ is a bounded sequence in H^s. If there is an $R > 0$ for which*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}} \|u_n\|_{H^s(y-R,y+R)} = 0,$$

then it follows that

$$\lim_{n\to\infty}\int |u_n(x)|^p\,dx = 0.$$

Proof. We argue by contradiction. If $\int |u_n(x)|^p\,dx \to 0$ is not true, then there must be a subsequence of $\{u_n\}_{n\ge1}$, still denoted by $\{u_n\}_{n\ge1}$, and an $A > 0$ such that $\int |u_n(x)|^p\,dx \geq A$ for all n. Then, by Lemma 6.3, there exist, for each n, real numbers y_n and $\eta > 0$ such that

$$\int_{y_n-R}^{y_n+R} |u_n(x)|^p\,dx \geq \eta.$$

Note that η depends only on R, s and p. It follows from the Sobolev imbedding Theorem that

$$\int_{y_n-R}^{y_n+R} |u_n(x)|^p\,dx \leq k\|u_n\|^p_{H^s(y_n-R,y_n+R)},$$

where $k = k(R, s, p)$ does not depend on n. This contradicts the assumption. The lemma is proved. □

Sketch of a proof of Theorem 6.2. Denote by

$$\rho_n(x) = |u_n(x)|^2 + |\partial_x^m u_n(x)|^2 + \int \frac{|\partial_x^m u_n(x) - \partial_y^m u_n(y)|^2}{|x-y|^{1+2\delta}}\,dy,$$

and $\mu_n = \int \rho_n(x)\,dx$. Then μ_n is bounded, so it may be supposed that $\mu_n \to \mu$ as $n \to \infty$. Suppose there is a subsequence of $\{\rho_n\}_{n\geq 1}$, still denoted by $\{\rho_n\}_{n\geq 1}$, which satisfies either vanishing or dichotomy. If vanishing occurs, then, as before,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \|u_n\|_{H^s(y-R,y+R)} = 0.$$

By Lemma 6.4

$$0 < \lambda = \int u_n(x)^p\,dx \leq \int |u_n(x)|^p\,dx \to 0.$$

If dichotomy occurs, there is a $\bar{\mu} \in (0, \mu)$ such that for any $\epsilon > 0$, there corresponds an n_0 and functions $\rho_n^1, \rho_n^2 \in L_1(\mathbb{R})$, $\rho_n^1, \rho_n^2 \geq 0$ such that for $n \geq n_0$,

$$\begin{cases} \|\rho_n - (\rho_n^1 + \rho_n^2)\|_{L_1} \leq \epsilon, \\ |\int_{\mathbb{R}} \rho_n^1(x)\,dx - \bar{\mu}| \leq \epsilon, \qquad |\int_{\mathbb{R}} \rho_n^2(x)\,dx - (\mu - \bar{\mu})| \leq \epsilon, \qquad \text{and} \\ \text{supp } \rho_n^1 \cap \text{supp } \rho_n^2 = \emptyset \text{ and } \text{dist}\{\text{supp } \rho_n^1, \text{supp } \rho_n^2\} \to \infty \text{ as } n \to \infty. \end{cases}$$

As before, it may be assumed that

$$\text{supp } \rho_n^1 \subset (y_n - R_0, y_n + R_0) \quad \text{and} \quad \text{supp } \rho_n^2 \subset (-\infty, y_n - 2R_n) \cup (y_n + 2R_n, \infty)$$

for some fixed $R_0 > 0$, and real sequences $\{y_n\}_{n\geq 1}$, $\{R_n\}_{n\geq 1}$ with $R_n \to \infty$. Then, we have

$$\int_{R_0 \leq |x-y_k| \leq 2R_n} \rho_n(x)\,dx \leq \epsilon,$$

which is to say,

$$\|u_n\|^2_{H^m(R_0\le|x-y_k|\le 2R_n)} + \int_{R_0\le|x-y_k|\le 2R_n}\int \frac{|\partial_x^m u_n(x)-\partial_y^m u_n(y)|^2}{|x-y|^{1+2\delta}}\,dy\,dx \le \epsilon. \tag{6.5}$$

To construct the splitting functions u_n^1 and u_n^2 of $\{u_n\}_{n\ge 1}$ for $n = 1, 2, \dots$, let $\zeta \in C_b^\infty$ with $0 \le \zeta \le 1$ be such that $0 \le \zeta(x) \le 1$ for all x, $\zeta(x) = 1$ when $|x| \le 1$, $\zeta(x) = 0$ when $|x| \ge 2$, and define $\psi(x) = 1 - \zeta(x)$. For $R_1 > R_0$ sufficiently large, and for $n = 1, 2, \dots$, define

$$\zeta_n(x) = \zeta\Big(\frac{x-y_n}{R_1}\Big), \qquad \psi_n(x) = \psi\Big(\frac{x-y_n}{R_n}\Big),$$

and set

$$\eta_n^1(x) = \begin{cases} \zeta\big(\frac{x-y_n}{R_1}\big) - \zeta\big(\frac{x-y_n}{R_n}\big), & \text{if } y_n + R_1 \le x \le y_n + 2R_n, \\ 0, & \text{otherwise,} \end{cases}$$
$$\eta_n^2(x) = \begin{cases} \zeta\big(\frac{x-y_n}{R_1}\big) - \zeta\big(\frac{x-y_n}{R_n}\big) & \text{if } y_n - 2R_n \le x \le y_n - R_1, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that for any $x \in \mathbb{R}$, $\zeta_n(x) + \psi_n(x) + \eta_n^1(x) + \eta_n^2(x) = 1$, and supp $\eta_n^1 \subset (y_n - 2R_n, y_n - R_1)$, supp $\eta_n^2 \subset (y_n + R_1, y_n + 2R_n)$, supp $\zeta_n \subset (y_n - 2R_1, y_n + 2R_1)$, supp $\psi_n \subset (-\infty, y_k - 2R_n) \cup (y_n + 2R_n, \infty)$. The functions $\{u_n\}_{n\ge 1}$ can then be decomposed as

$$u_n = \zeta_n u_n + \psi_n u_n + \eta_n^1 u_n + \eta_n^2 u_n.$$

Notice that

$$\begin{aligned}\|\eta_n^1 u_n\|^2_{H^{r_k}} &\sim \|\eta_n^1 u_n\|_{H^m} + \iint \frac{|\partial_x^m(\eta_n^1 u_n)(x) - \partial_y^m(\eta_n^1 u_n)(y)|^2}{|x-y|^{1+2\delta}}\,dy\,dx \\ &\le C(\zeta)\|u_n\|^2_{H^m(R_0\le|x-y_n|\le 2R_n)} + \iint \frac{|\partial_x^m(\eta_n^1 u_n)(x) - \partial_y^m(\eta_n^1 u_n)(y)|^2}{|x-y|^{1+2\delta}}\,dy\,dx,\end{aligned}$$

where $C(\zeta)$ is a constant dependent only on ζ. The second term on the right-hand side of the last inequality may be bounded above thusly:

$$\begin{aligned}&\iint \frac{\big|\partial_x^m(\eta_n^1 u_n)(x) - \partial_y^m(\eta_n^1 u_n)(y)\big|^2}{|x-y|^{1+2\delta}}\,dy\,dx \\ &= \int_{y_n+R_1}^{y_n+2R_n}\int_{y_n+R_1}^{y_n+2R_n} \frac{\big|\partial_x^m(\eta_n^1 u_n)(x) - \partial_y^m(\eta_n^1 u_n)(y)\big|^2}{|x-y|^{1+2\delta}}\,dy\,dx \\ &\quad + 2\int_{y_n+R_1}^{y_n+2R_n}\Big(\int_{-\infty}^{y_n+R_1} + \int_{y_n+2R_n}^{\infty}\Big)\frac{\big|\partial_x^m(\eta_n^1 u_n)(x)\big|^2}{|x-y|^{1+2\delta}}\,dy\,dx\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{R_1}^{2R_n}\int_{R_1}^{2R_n} \frac{\left|\partial_x^m(\eta_n^1 u_n)(x+y_n) - \partial_y^m(\eta_n^1 u_n)(y+y_n)\right|^2}{|x-y|^{1+2\delta}}\,dy\,dx \\
&\quad + \frac{1}{\delta}\int_{R_1}^{2R_n} \frac{\left|\partial_x^m(\eta_n^1 u_n)(x+y_n)\right|^2}{(x-R_1)^{2\delta}}\,dx + \frac{1}{\delta}\int_{R_1}^{2R_n} \frac{\left|\partial_x^m(\eta_n^1 u_n)(x+y_n)\right|^2}{(2R_n-x)^{2\delta}}\,dx \\
&= \int_{R_1}^{2R_n}\int_{R_1}^{2R_n} \frac{\left|\sum_{j=0}^m \binom{m}{j}\left(\partial_x^{m-j}\eta_n^1\partial_x^j u_n(x+y_n) - \partial_y^{m-j}\eta_n^1\partial_y^j u_n(y+y_n)\right)\right|^2}{|x-y|^{1+2\delta}}\,dy\,dx \\
&\quad + \frac{1}{\delta}\int_{R_1}^{2R_n} \frac{\left|\sum_{j=0}^m \binom{m}{j}\partial_x^{m-j}\eta_n^1(x+y_n)\partial_x^j u_n(x+y_n)\right|^2}{|x-R_1|^{2\delta}}\,dx \\
&\quad + \frac{1}{\delta}\int_{R_1}^{2R_n} \frac{\left|\sum_{j=0}^m \binom{m}{j}\partial_x^{m-j}\eta_n^1(x+y_n)\partial_x^j u_n(x+y_n)\right|^2}{|2R_n-x|^{2\delta}}\,dx.
\end{aligned}
$$

We make detailed estimates of the summands in the terms on the right-hand side of the last inequality: for $j = 0, 1, \ldots, m$, use (6.5) repeatedly to conclude

$$
\begin{aligned}
&\int_{R_1}^{2R_n}\int_{R_1}^{2R_n} \frac{\left|\partial^{m-j}\eta_n^1\partial_x^j u_n(x+y_n) - \partial_y^{m-j}\eta_n^1\partial_y^j u_n(y+y_n)\right|^2}{|x-y|^{1+2\delta}}\,dy\,dx \\
&\leq \int_{R_1}^{2R_n}\int_{R_1}^{2R_n} \frac{\left|\partial_x^{m-j}\eta_n^1(x+y_n)\right|^2\left|\partial_x^j u_n(x+y_n) - \partial_y^j u_n(y+y_n)\right|^2}{|x-y|^{1+2\delta}}\,dy\,dx \\
&\quad + \int_{R_1}^{2R_n}\int_{R_1}^{2R_n} \frac{\left|\partial_x^{m-j}\eta_n^1(x+y_n) - \partial_y^{m-j}\eta_n^1(y+y_n)\right|^2\left|\partial_y^j u_n(y+y_n)\right|^2}{|x-y|^{1+2\delta}}\,dy\,dx \\
&\leq \|\eta_n^1\|_{C_b^m}^2\|u_n^1\|_{H^{m+\delta}(y_n+R_1,y_n+2R_n)}^2 + O(\epsilon) = O(\epsilon) \quad \text{as } \epsilon \to 0,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{R_1}^{2R_n} \frac{\left|\partial_x^{m-j}\eta_n^1(x+y_n)\partial_y^j u_n(x+y_n)\right|^2}{|x-R_1|^{2\delta}}\,dx \\
&\leq \max_{x\in\mathbb{R}}\left\{\frac{|\partial_x^{m-j}\eta_n^1(x+y_n)|^2}{|x-R_1|^{2\delta}}\right\}\|u_n\|_{H^j(y_n+R_1,y_n+2R_n)}^2 = O(\epsilon) \quad \text{as } \epsilon \to 0
\end{aligned}
$$

and, similarly,

$$
\int_{R_1}^{2R_n} \frac{\left|\partial_x^m(\eta_n^1 u_n)(x-y_n)\right|^2}{|2R_n-x|^{2\delta}}\,dx = O(\epsilon) \quad \text{as } \epsilon \to 0.
$$

Combining these last three estimates and using (6.5) once more yields

$$
\|\eta_n^1 u_n\|_{H^{r_k}}^2 = O(\epsilon) \quad \text{as } \epsilon \to 0. \tag{6.6}
$$

By the same argument, it appears that

$$\|\eta_n^2 u_n\|_{H^{r_k}}^2 = O(\epsilon) \quad \text{as } \epsilon \to 0. \tag{6.7}$$

If $u_n^1 = \zeta_n u_n$, $u_n^2 = \psi_n u_n$ and $w_n = \eta_n^1 u_n + \eta_n^2 u_n$, then it transpires that supp $u_n^1 \subset (y_n - 2R_1, y_n + 2R_1)$ and supp $u_n^2 \subset (-\infty, y_n - 2R_n) \cup (y_n + 2R_n, \infty)$. Using (6.6) and (6.7), it is deduced that

$$\begin{aligned}
\|u_n\|_{H^{r_k}}^2 &= \|u_n^1 + u_n^2\|_{H^{r_k}}^2 + O(\epsilon) = \|u_n^1 + u_n^2\|_{H^m}^2 \\
&+ \iint \frac{\left|\left(\partial_x^m u_n^1(x) + \partial_x^m u_n^2(x)\right) - \left(\partial_y^m u_n^1(y) + \partial_y^m u_n^2(y)\right)\right|^2}{|x-y|^{1+2\delta}}\, dy\, dx + O(\epsilon) \\
&= \|u_n^1\|_{H^m}^2 + \|u_n^2\|_{H^m}^2 + \int_{y_n-2R_1}^{y_n+2R_1}\int_{y_n-2R_1}^{y_n+2R_1} \frac{\left|\partial_x^m u_n^1(x) - \partial_y^m u_n^1(y)\right|^2}{|x-y|^{1+2\delta}}\, dy\, dx \\
&+ \int_{|x-y_n|\ge 2R_n}\int_{|x-y_n|\ge 2R_n} \frac{\left|\partial_x^m u_n^2(x) - \partial_y^m u_n^2(y)\right|^2}{|x-y|^{1+2\delta}}\, dy\, dx \\
&+ 2\int_{y_n-2R_1}^{y_n+2R_1}\int_{|x-y_n|\ge 2R_n} \frac{\left|\partial_x^m u_n^1(x) - \partial_y^m u_n^2(y)\right|^2}{|x-y|^{1+2\delta}}\, dy\, dx + O(\epsilon) \\
&= \|u_n^1\|_{H^{r_k}}^2 + \|u_n^2\|_{H^{r_k}}^2 + 2\int_{y_n-2R_1}^{y_n+2R_1}\int_{|x-y_n|\ge 2R_n} \frac{\left|\partial_x^m u_n^1(x) - \partial_y^m u_n^2(y)\right|^2}{|x-y|^{1+2\delta}}\, dy\, dx + O(\epsilon) \\
&= \|u_n^1\|_{H^{r_k}}^2 + \|u_n^2\|_{H^{r_k}}^2 + O(\epsilon)
\end{aligned}$$

as $\epsilon \to 0$. Then for large n and small ϵ, we have

$$0 \le \|u_n^1\|_{H^{r_k}}^2 \le \|u_n\|_{H^{r_k}}^2, \qquad 0 \le \|u_n^2\|_{H^{r_k}}^2 \le \|u_n\|_{H^{r_k}}^2,$$

and so there is $\mu_0 \in [0, \mu]$ such that $\|u_n^1\|_{H^{r_k}}^2 \to \mu_0$ and $\|u_n^2\|_{H^{r_k}}^2 \to \mu - \mu_0$.

From this information, we may derive the contradiction $0 > \Theta(\lambda) > \Theta(\lambda)$, as before. Thus Dichotomy is ruled out. By simply adapting the details of the proof of Theorems 2.4 or 4.2, the existence portion of theorem 6.3 is proved.

To prove the decay part of the theorem, it is only required to determine the inverse Fourier transform of $\left(c + \sum_{j=1}^k \alpha_j |\xi|^{2r_j}\right)^{-1}$. Similar calculation as in previous sections leads to the result

$$k(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sum_{j=1}^k \alpha_j (\sin r_j\pi) y^{2r_j} e^{-y|x|}}{\left(c + \sum_{j=1}^k (\cos r_j\pi) y^{2r_j}\right)^2 + \left(\sum_{j=1}^k \alpha_j \sin(r_j\pi) y^{2r_j}\right)^2}\, dy$$
$$+ \text{ terms with exponential decay in } x,$$

whence

$$\lim_{x\to\pm\infty} |x|^{2r_0+1} k(x) = C,$$

for some constant C. The theorem is complete. □

Theorem 6.5. *If the parameters c and α_j satisfy*

(1) $r_k > \frac{1}{2}$,
(2) $\min_{x\geq 0}\{c + \sum \alpha_j x^{2r_j}\} > 0$,
(3) $\sum \alpha_j (\sin r_j\pi) y^{2r_j} \geq 0$ *for all* $y \geq 0$, *and*
(4) *for any* $\rho \geq 0$ *and* $0 \leq \theta \leq \frac{\pi}{2}$,

$$c + \sum \alpha_j \rho_j^{2r_j} e^{i2r_j\theta} \neq 0,$$

then

(1) *the equation* (5.1) *has a nontrivial solitary-wave solution* $\phi = \phi_c \in H^\infty(\mathbb{R})$, *and* ϕ_c *may be chosen to be an even, positive function, strictly monotone decreasing on* $(0, \infty)$ *and such that* ϕ_c *and all its derivatives are bounded and continuous* L_1*-functions.*
(2) *Moreover, if* $r_0 = \min\{r_j : r_j \notin \mathbb{Z}, j = 1, \ldots, k\}$, *then* ϕ_c *decays at the rate*
$$\lim_{x\to\pm\infty} |x|^{2r_0+1}\phi_c(x) = C,$$
for some constant C.

Proof. Write (5.1) in the form

$$\phi = \frac{1}{p-1}(c+\mathcal{L})^{-1}\phi^p = \frac{1}{p-1}\int k(x-y)\phi^p(y)\,dy, \tag{6.8}$$

where k is defined by the Fourier symbol

$$\hat{k}(\xi) = \frac{1}{c + \sum \alpha_j |\xi|^{2r_j}}.$$

Computing as before, it is seen that

$$k(x) = k(-x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sum_j \alpha_j \sin r_j\pi\, y^{2r_j} e^{-|x|y}}{\left(c + \sum_j \alpha_j \cos r_j\pi\, y^{2r_j}\right)^2 + \left(\sum_j \alpha_j \sin r_j\pi\, y^{2r_j}\right)^2}\,dy,$$

and from this representation, it is obvious that k satisfies all three conditions in Lemma 2.1. Hence the existence part of the theorem is in hand. It is straightforward to check that

$$\lim_{x\to\pm\infty} |x|^{2r_0+1}k(x) = \sqrt{\frac{2}{\pi}}\frac{\alpha_0 \sin r_0\pi}{c^2}\int_0^\infty y^{2r_0}e^{-y}\,dy,$$

where α_0 is the coefficient corresponding to the power r_0 in the definition the symbol of $\mathcal{L}$. Applying the results in Section 3, the decay part of the theorem is thereby concluded. □

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