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ON THE STRUCTURE OF SINGULARITIES IN SOLUTIONS OF THE NONLINEAR SCHRÖDINGER EQUATION FOR THE CRITICAL CASE, $p = 4/n$

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1. Introduction

This paper is concerned with nonlinear dispersive wave equations whose solutions sometimes develop singularities in finite time. Here, attention is given principally to the initial-value problem in the *critical case* for the focussing nonlinear Schrödinger equation (NLS-equation henceforth),

$$\begin{cases} iu_t + \Delta u + \tau |u|^p u = 0, & x \in \mathbb{R}^n,\, t \geqq 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where Δ is the Laplace operator on $\mathbb{R}^n$, $\tau > 0$, and $u : [0,T) \times \mathbb{R}^n \to \mathbb{C}$ for some $T > 0$.

For $p < \frac{4}{n}$, it is well known that the problem (1.1) has global solutions no matter what the size of the initial data $u_0(x)$ in $H^1(\mathbb{R}^n)$ (cf. [C], Chapter 6). Moreover, for any p with $0 < p < \frac{4}{n-2}$ (any $p > 0$ if $n = 1$ or 2), the NLS-equation admits standing-wave solutions of the form

$$u_\lambda(x,t) = G(x)e^{i\lambda t}, \tag{1.2}$$

where $\lambda > 0$ and $G = G_\lambda$ is a real-valued, positive, radially symmetric function (called a ground state) which is rapidly decreasing to zero at infinity [BL1], [BL2], [BLP], [S]. When $p < \frac{4}{n}$, these standing waves are orbitally stable in the following sense (see [C], [CL], [W4]). For every $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that if $u_0 \in H^1(\mathbb{R}^n)$ and $\|u_0 - G\|_1 \leqq \delta$,

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then there are maps $\theta : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}^n$ such that if u is the solution of (1.1) with initial data u_0, then

$$\|u(\cdot, t) - e^{i\theta(t)} G(\cdot - \sigma(t))\|_1 \leqq \epsilon \tag{1.3}$$

for all $t > 0$, where $\|h\|_1^2 = \|h\|^2 + \|\nabla h\|^2$ connotes the standard norm of a function $h \in H^1(\mathbb{R}^n)$ and the unadorned symbol $\| \cdot \|$ is the $L^2(\mathbb{R}^n)$-norm. It is worth remark also that G is unique up to spatial translations and phase shifts.

If the initial data u_0 is small enough in the $H^1(\mathbb{R}^n)$-norm and $p \geqq \frac{4}{n}$, there still exist global solutions for (1.1), but the ground states are known to be unstable in this case (cf. [BC]).

The case $p = \frac{4}{n}$ is called the *critical case* for the NLS-equation because solutions of (1.1) with $\tau > 0$, $p \geqq \frac{4}{n}$ and initial data $u_0 \in H^1(\mathbb{R}^n)$ may blow up in finite time, whereas those for $p < \frac{4}{n}$ do not. This singular behavior is an indication of certain physical phenomena, for instance, the energy transfer to particles via the "Langmuir collapse" in plasma physics [Z], and, in nonlinear optics, the collapse of solutions is the "self-focusing" of light pulses in a nonlinear and dispersive medium [CGT].

The existence of solutions and stability or instability of standing waves for the NLS-equation has been considered in detail by many scientists. A small sample of papers that emphasize especially the aspects of interest here might include [BL1], [BL2], [C], [CL], [GSS], [GV], [LBSK], [S], [W1], [W4] and the references cited in these works.

As mentioned, our focus is on the critical case $p = \frac{4}{n}$ where light will be shed on how the blow-up occurs. (Some of our results are still valid for $p > \frac{4}{n}$.) To understand the blow-up phenomenon, consider a ground state $G = G_\lambda$ which defines a solution of the NLS-equation as in (1.2). The function G necessarily satisfies

$$\Delta G - \lambda G + G^{p+1} = 0, \tag{1.4}$$

where τ has been set equal to 1. (As long as $\tau > 0$, we can always assume $\tau = 1$ by rescaling the dependent variable.) If one assumes in (1.1) with $p = \frac{4}{n}$ that $u_0 \in H^1(\mathbb{R}^n)$

has $||u_0|| < ||G||$, then the corresponding solution is globally defined (cf. [W2]). This result is sharp in the sense that for any $R \geqq ||G||$, there are initial data u_0 with $||u_0|| = R$ such that there is a $t^* < \infty$ for which the solution of the NLS-equation corresponding to u_0 belongs to the class $C([0,t^*); H^1(\mathbb{R}^n))$ and

$$\lim_{t \to t^{*-}} ||\nabla u(\cdot, t)|| = +\infty. \tag{1.5}$$

It is well understood that for (1.5) to hold, it must be the case that the energy H, which is a conserved quantity under the NLS-flow, satisfies

$$H(u_0) = ||\nabla u_0||^2 - \frac{2}{p+2}|u_0|_{p+2}^{p+2} < 0.$$

As above and henceforth, the unadorned symbol $||\cdot||$ is the $L^2(R^n)$-norm while the symbol $|\cdot|_q$ represents the $L^q(R^n)$-norm, $1 \leqq q \leqq \infty$ and, incidentally, the inner-product in $L^2(R^n)$ will be indicated by $< \cdot, \cdot >$.

The first result describing the asymptotic behavior of blowing-up solutions of the NLS-equation appears to have been that of M. Weinstein in [W3] and [W5]. Later, Laedke *et al.* in [LBSK] studied Weinstein's result and attempted to extend it to a larger class of initial data. Our purpose here is to provide some details related to the last cited work, and to add precision to some of the resulting theory. Roughly speaking, the result in view is that for suitable, negative energy data that are close to an unstable ground state, the corresponding solutions blow-up in finite time. Moreover, it will be shown that the blow-up near ground-state solutions for the case $p = \frac{4}{n}$ is *stable* in a sense made precise in Theorem 2.2 below. We also give a detailed description of the evolution of the stability parameters θ and σ, analogous to those in (1.3), that appear in the proof of Theorem 2.2. These results are modelled on those provided by Bona and Soyeur (see [BS], Theorem 7) in the case $p < \frac{4}{n}$.

This note is organized as follows. Section 2 is devoted to the proof of the stability result and Section 3 is concerned with the evolution of the stability parameters θ and σ.

2. Main Result

Henceforth, it is assumed in (1.1) that $\tau = 1$, $p = \frac{4}{n}$ and $u_0 \in H^1(\mathbb{R}^n)$. If it is further assumed that either

$$(*) \quad \begin{cases} a) & n = 1 \text{ and } H(u_0) < 0, \\ b) & n \geqq 2, \; H(u_0) < 0, \text{ and } u_0 \text{ is radially symmetric, or} \\ c) & n \geqq 1, \; |x|u_0 \in L^2(R^n) \text{ and } H(u_0) \leqq 0, \end{cases}$$

then it is known that (1.5) holds for the corresponding solution u of (1.1), (see [N] and [OT]).

Following [LBSK] and [W2], introduce the functions

$$\begin{cases} \phi(x,t) = \mu(t)^{-\frac{n}{2}} u(\frac{x}{\mu(t)}, t), \text{ with} \\ \mu(t) = \frac{\|\nabla u(\cdot,t)\|}{\|\nabla G\|}, \qquad 0 \leqq t < t^*, \text{ and } \mu(0) = 1, \end{cases} \tag{2.1}$$

where t^* is the maximal time of existence of the solution of (1.1) under consideration. Note that, unless u is the zero-solution, $0 < \mu(t) < \infty$ for $0 < t < t^*$. The normalization $\mu(0) = 1$ is a temporary one made to simplify the presentation of the argument. It will be dispensed with later. It is easy to check that the function ϕ verifies the identities

$$i) \; \|\phi(\cdot,t)\| = \|u(\cdot,t)\| = \|u_0\|, \tag{2.2}$$

$$ii) \; \|\nabla\phi(\cdot,t)\| = \|\nabla G\|, \tag{2.3}$$

$$iii) \; H(\phi(\cdot,t)) = \|\nabla\phi(\cdot,t)\|^2 - \frac{2}{p+2}|\phi(\cdot,t)|_{p+2}^{p+2} = \frac{1}{\mu^2(t)} H(u(\cdot,t)). \tag{2.4}$$

The identity $i)$ follows from the fact that the charge $N(u) = \|u\|^2$ is a conserved quantity for the NLS-equation. Our first lemma states that the function ϕ is in the same class as u.

Lemma 2.1. *If $u \in C([0,t^*); H^1(\mathbb{R}^n))$, then $\phi \in C([0,t^*); H^1(\mathbb{R}^n))$.*

Proof. This follows immediately since $\mu \in C([0,t^*); \mathbb{R})$ and $0 < \mu(t) < \infty$ for $0 \leqq t < t^*$. ■

As the stability considered here is with respect to form, i.e., up to translation in space and phase, it is propitious to introduce the orbit

$$\mathcal{O}(G_\lambda) \equiv \{g | g(x) = G_\lambda(x + \alpha_0)e^{i\alpha_1}, (\alpha_0, \alpha_1) \in \mathbb{R}^n \times [0, 2\pi)\}$$

of G_λ. An induced metric on the space $H^1(\mathbb{R}^n)$ factored by the closed subset $\mathcal{O}(G_\lambda)$ provides a pseudo-metric on $H^1(\mathbb{R}^n)$ (see [B], [Bo], [CL] and [W4]), namely

$$\rho_\lambda(\phi(\cdot,t),G_\lambda)^2 \equiv \operatorname*{Inf}_{\substack{\alpha_0\in\mathbb{R}^n\\ \alpha_1\in[0,2\pi)}} \{\|\nabla\phi(\cdot+\alpha_0,t)e^{i\alpha_1}-\nabla G(\cdot)\|^2 + \lambda\|\phi(\cdot+\alpha_0,t)e^{i\alpha_1}-G(\cdot)\|^2\}. \tag{2.5}$$

Define the set $\mathcal{S}$ to be

$$\mathcal{S}=\{u_0 | u_0\in H^1(\mathbb{R}^n) \text{ and condition (*) holds for } u_0\}. \tag{2.6}$$

Observe that for initial data $u_0\in\mathcal{S}$, the conditions (*) implies the corresponding solution u of (1.1) obeys (1.5) for some $t^*>0$.

Theorem 2.2. *Let $p=\frac{4}{n}$, $\lambda>0$, and let $G=G_\lambda$ be a ground-state solution of (1.2). For any $\epsilon>0$, there is a $\delta=\delta(\epsilon)$ such that if $u_0\in\mathcal{S}$ with $\rho_\lambda(u_0,G)<\delta$, and u is a solution of (1.1) corresponding to u_0 whose blow-up time is t^*, say, then $u\in C([0,t^*);H^1(\mathbb{R}^n))$ and*

$$\operatorname*{Inf}_{\substack{\alpha_0\in\mathbb{R}^n\\ \alpha_1\in[0,2\pi)}} \Big\{\lambda\|u(\cdot,t)-\mu^{\frac{n}{2}}(t)G(\mu(t)\cdot+\alpha_0)e^{-i\alpha_1}\|^2 + \mu^{-2}(t)\|\nabla u(\cdot,t)-\mu^{\frac{n}{2}}(t)\nabla G(\mu(t)\cdot+\alpha_0)e^{-i\alpha_1}\|^2\Big\}<\epsilon \tag{2.7}$$

for all $t\in[0,t^*)$, where $\mu(t)$ is as in (2.1).

Proof. Suppose at the outset that $\mu(0)=1$. The proof is based on the explicit time-dependent functional (see [LBSK], [LBS] and [LS])

$$L_t[u]=\frac{1}{\mu^2(t)}H(u(\cdot,t))+\lambda\Big(\frac{\|u(\cdot,t)\|}{\|G\|}\Big)^{2k}(\|u(\cdot,t)\|^2-\|G\|^2), \tag{2.8a}$$

where $k\in\mathbb{N}$ is a parameter that will be determined later. Observe that if u is a solution of (1.1), then at any time $t\in[0,t^*)$, $L_t[u]=L_t[u_0]$ since both the energy H and the charge $\|\cdot\|$ are preserved by the NLS-flow. The functional $L_t[u]$ may be written as a functional defined on ϕ for which the explicit dependence on μ disappears, viz.

$$\widetilde{L}_t[\phi]\equiv L_t[u]=H(\phi(\cdot,t))+\lambda\Big(\frac{\|\phi(\cdot,t)\|}{\|G\|}\Big)^{2k}(\|\phi(\cdot,t)\|^2-\|G\|^2). \tag{2.8b}$$

Suppose we are able to establish the inequalities

$$i)\ \Delta\widetilde{L}_t \leqq c_0\|u_0 - G\|_1 \quad \text{and} \tag{2.9}$$

$$ii)\ \Delta\widetilde{L}_t \geqq c_1\|\phi(\cdot,t) - G\|_1^2 - c_2\sum_{j=1}^{4}\|\phi(\cdot,t) - G\|_1^{j+2} - \sum_{j=1}^{2k} c_{k,j}\|\phi(\cdot,t) - G\|_1^{j+2}, \tag{2.10}$$

for $\Delta\widetilde{L}_t = \widetilde{L}_t[\phi] - \widetilde{L}_t[G]$, modulo translations and phase, where $c_0, c_i, c_{k,j}$ are fixed constants. The result in Theorem 2.2 will then follow. Indeed, if $\|u_0 - G\|_1 < \delta_0$, we have from (2.9) that

$$\Delta\widetilde{L}_t \leqq c_0\delta_0. \tag{2.11}$$

As will be argued in more detail later, (2.10), (2.11) and the identity

$$\|\phi(\cdot,t) - G\|_1^2 = \|u(\cdot,t) - \mu^{\frac{n}{2}}(t)G(\mu(t)\cdot)\|^2 + \mu^{-2}(t)\|\nabla u(\cdot,t) - \mu^{\frac{n}{2}}(t)\nabla G(\mu(t)\cdot)\|^2$$

lead to the desired result.

The upper bound (2.9) is a straightforward consequence of $H(u_0) < 0$ and $H(G) \geqq 0$, where the constant c_0 depends on $\|G\|$.

To prove (2.10), write a renormalized version of ϕ as G plus a remainder w, thusly;

$$\phi(x + \alpha_0, t)e^{i\alpha_1} = G(x) + w(x,t), \qquad w = a + ib, \tag{2.12}$$

where $\alpha_0 = \alpha_0(t)$ and $\alpha_1 = \alpha_1(t)$ will be chosen later and where a and b are real functions. Thus w depends on α_0 and α_1. Using the representation (2.12) of the translated and phase-shifted ϕ, we calculate

$$\begin{aligned}\Delta\widetilde{L}_t &= \widetilde{L}_t[\phi(\cdot + \alpha_0)e^{i\alpha_1}] - \widetilde{L}_t[G] = \widetilde{L}_t[G + w] - \widetilde{L}_t[G]\\ &= H(G+w) - H(G) + \lambda\Big(\frac{\|G+w\|}{\|G\|}\Big)^{2k}(\|G+w\|^2 - \|G\|^2)\\ &=< -\Delta a, a > + < -\Delta b, b > +2 < a, \Delta G > -2 < a, G^{p+1} > -(p+1) < G^p a, a >\\ &- < G^p b, b > +2\lambda < a, G > +\lambda < a, a > +\lambda < b, b > +4\frac{k\lambda}{\|G\|^2} < G, a >^2 + R.\end{aligned}$$

The remainder R can be bounded below in terms of powers of $\|w\|_1$ of cubic order and higher by using the Sobolev imbedding of $H^1(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, which is valid since $p = \frac{4}{n}$.

The upshot of these ruminations is the inequality

$$\Delta\widetilde{L}_t \geqq < \mathcal{L}^+ b, b > + < \mathcal{L}^- a, a > + \frac{4k\lambda}{||G||^2}(< a, G >)^2 - c_2(\lambda)\sum_{j=1}^{4}||w||_1^{j+2}$$

$$-\sum_{j=1}^{2k} c_{k,j}(\lambda)||w||_1^{j+2}. \tag{2.13}$$

In (2.13) we have followed [LBSK] in introducing the naturally occurring differential operators

$$\mathcal{L}^+ = -\Delta - G^p + \lambda, \tag{2.14}$$

$$\mathcal{L}^- = -\Delta - (p+1)G^p + \lambda, \tag{2.15}$$

which are, respectively, the real and imaginary parts of the operator $\widetilde{L}_t$ linearized about the ground state.

The first step to obtaining a suitable lower bound on the quadratic form comprised of the first three terms on the right-hand side of (2.13) is to show there exist maps $\alpha_0 = \alpha_0(t)$ and $\alpha_1 = \alpha_1(t)$ minimizing the function

$$\Omega_t(\alpha_0, \alpha_1) = ||\nabla\phi(\cdot + \alpha_0, t)e^{i\alpha_1} - \nabla G(\cdot)||^2 + \lambda||\phi(\cdot + \alpha_0, t)e^{i\alpha_1} - G(\cdot)||^2.$$

The following lemma is the analog of [Bo], Lemma 1, in the present context.

Lemma 2.3. *Suppose that for some $t_0 \in [0, t^*)$ and some $(\widetilde{\alpha}_0, \widetilde{\alpha}_1) \in \mathbb{R} \times [0, 2\pi)$, it is the case that*

$$\Omega_{t_0}(\widetilde{\alpha}_0, \widetilde{\alpha}_1) < ||(-\Delta + \lambda)^{1/2} G||^2. \tag{2.16}$$

Then, it follows that

$$Inf\,\{\Omega_{t_0}(\alpha_0, \alpha_1)|\ (\alpha_0, \alpha_1) \in \mathbb{R}^n \times [0, 2\pi)\} \tag{2.17}$$

is attained at least once in $\mathbb{R}^n \times [0, 2\pi)$.

Proof. It is immediate that $\Omega_{t_0}(\alpha_0, \alpha_1)$ is a continuous function of (α_0, α_1) on $\mathbb{R}^n \times [0, 2\pi)$. Moreover, for any $\alpha_1 \in [0, 2\pi)$, we have

$$\lim_{|\alpha_0| \to \infty} \Omega_{t_0}(\alpha_0, \alpha_1) = ||\nabla\phi(\cdot, t_0)||^2 + ||\nabla G||^2 + \lambda(||\phi(\cdot, t_0)||^2 + ||G||^2)$$

$$= ||(-\Delta + \lambda)^{1/2} G||^2 + \lambda||u_0||^2 + ||\nabla\phi(\cdot, t_0)||^2. \tag{2.18}$$

The hypothesis (2.16), the continuity of Ω_{t_0}, and (2.18) imply the result. ■

Next, it is established that the infimum in (2.17) is attained at points (α_0, α_1) at least for t_0 in some interval of the form $[0, T]$. To this end, it is sufficient to obtain condition (2.16) in such an interval. Let $\epsilon > 0$ be such that

$$\epsilon^2 < \frac{1}{2}\max\{1, \lambda\}\|(-\Delta + \lambda)^{\frac{1}{2}} G\|^2.$$

The standing-wave solution $G(x)e^{it\lambda}$ is globally defined, and hence by the continuous dependence theory for (1.1) (see [C], Chapter 4 or [GV]), for the value of ϵ just specified and $T > 0$, there exists a $\delta > 0$ such that if $\|u_0 - G\|_1 < \delta$, then the solution u of (1.1) corresponding to u_0 exists at least for $0 \leqq t \leqq T$ and, in addition,

$$\|u(\cdot, t) - G(\cdot)e^{i\lambda t}\|_1 < \epsilon/2,$$

for all $t \in [0, T]$. Of course, if $u_0 \in \mathcal{S}$ as well, it follows that there exists $t^* > T$ such that $u \in C([0, t^*); H^1(\mathbb{R}^n))$ and u blows up at t^* in the sense of (1.5).

Because both ϕ and u are continuous mappings of the time-axis into $H^1(\mathbb{R}^n)$ over the interval $[0, t^*)$, it follows that there is a $T_1 > 0$ such that for all $t \in [0, T_1]$,

$$\|\phi(\cdot, t) - \phi(\cdot, 0)\|_1 \leqq \frac{\epsilon}{4} \qquad \text{and} \qquad \|u(\cdot, t) - u(\cdot, 0)\|_1 \leqq \frac{\epsilon}{4}.$$

Then, for $0 \leqq t \leqq \widetilde{T} = min\{T, T_1\}$, we have

$$\begin{aligned} \|\phi(\cdot, t) - Ge^{i\lambda t}\|_1 &\leqq \|\phi(\cdot, t) - u(\cdot, t)\|_1 + \|u(\cdot, t) - Ge^{i\lambda t}\|_1 \\ &\leqq \|\phi(\cdot, t) - \phi(0)\|_1 + \|u(0) - u(\cdot, t)\|_1 + \frac{\epsilon}{2} \leqq \epsilon. \end{aligned} \tag{2.19}$$

Thus, the infimum (2.17) is taken on at values $(\alpha_0(t), \alpha_1(t))$ throughout the time-interval $[0, \widetilde{T}]$. We take such values of α_0 and α_1 as providing a meaning for the definition of w in (2.12), at least for $t \in [0, \widetilde{T}]$.

The result of Lemma 2.3 together with (1.4) provide us with compatibility relations on a and b, namely

$$\int_{\mathbb{R}^n} a(x, t) G^p(x) G_{x_i}(x)\, dx = 0, \qquad i = 1, .., n, \tag{2.20a}$$

and,

$$\int_{\mathbb{R}^n} b(x,t)G^{p+1}(x)\,dx = 0. \tag{2.20b}$$

for all $t \in [0, t^*)$. These relations are obtained by differentiating Ω_t with respect to α_0 and α_1 and evaluating at values that minimize Ω_t.

The issue of obtaining the lower bound advertised in (2.13) is addressed in the next several lemmas.

Lemma 2.4. *If $\mathcal{L}^+$ is as defined in (2.14), then there is a positive value C such that*

$$\mathop{Inf}_{\substack{\|f\|=1 \\ f\perp G^{p+1}}} \quad < f, \mathcal{L}^+ f > \;\geqq\; C. \tag{2.21}$$

Proof. Because of the variational characterization of the ground state G as the minimizer of the action (see [BL1], [BL2]), it follows that $\mathcal{L}^+$ is a nonnegative definite, self-adjoint operator on $L^2(\mathbb{R}^n)$ with null space spanned by G. Hence, the infimum on the left-hand side of (2.21) is non-negative. (cf. [C]).

Supposing that this infimum is zero, let $\{f_j\}$ be a sequence of $H^1(\mathbb{R}^n)$-functions with $\|f_j\| = 1$, $f_j \perp G^{p+1}$ and

$$\lim_{j\to\infty} < f_j, \mathcal{L}^+ f_j > = \; 0.$$

Then, for any $\eta > 0$, there is a J such that for $j > J$,

$$0 < \lambda \leqq \int |\nabla f_j|^2\,dx + \lambda \int |f_j|^2\,dx \leqq \int G^p |f_j|^2\,dx + \eta. \tag{2.22}$$

Since $\|G\|_\infty < \infty$, (2.22) implies $\|f_j\|_1$ to be uniformly bounded as j varies. By standard arguments, it follows that there is a subsequence of the $\{f_j\}$, which we denote again by $\{f_j\}$, and an $f^* \in H^1(\mathbb{R}^n)$ such that $f_j \to f^*$ weakly in $H^1(\mathbb{R}^n)$, pointwise almost everywhere, and in $L^2_{loc}(\mathbb{R}^n)$. By virtue of the weak convergence, f^* satisfies the condition $f^* \perp G^{p+1}$. It is a straightforward consequence of the just mentioned properties of the sequence f_j and the exponential decay of G to 0 as $|x| \to \infty$ that

$$\int G^p |f_j|^2\,dx \to \int G^p |f^*|^2\,dx$$

as $j \to \infty$. Taking the limit in (2.22) as $j \to \infty$ thus yields

$$0 < \lambda \leqq \int G^p |f^*|^2 \, dx + \eta.$$

As $\eta > 0$ is arbitrary, it must be the case that $f^* \neq 0$.

It is now shown that the infimum is achieved. Indeed, weak convergence is lower semi-continuous, so

$$\|\nabla f^*\| \leqq \liminf_{j \to \infty} \|\nabla f_j\|_2.$$

Since, also, $< G^p f_j, f_j > \to < G^p f^*, f^* >$ as $j \to \infty$, it is adduced that

$$< f^*, \mathcal{L}^+ f^* > \leqq \liminf_{j \to \infty} < f_j, \mathcal{L}^+ f_j >= 0.$$

Since $f^* \neq 0$, define $g^* = \frac{f^*}{\|f^*\|}$. Then, we have $\|g^*\| = 1, g^* \perp G^{p+1}$, and $< g^*, \mathcal{L}^+ g^* >= 0$.

A consequence of the last reasoning is that there exist non-trivial critical points (g^*, β, θ) for the Lagrange multiplier problem,

$$\begin{cases} \mathcal{L}^+ f = \beta f + \theta G^{p+1}, & \text{subject to} \\ \|f\| = 1 & \text{and} \\ f \perp G^{p+1}. \end{cases} \tag{2.23}$$

Using (2.23) and the fact that $< \mathcal{L}^+ g^*, g^* >= 0$, it is easily seen that $\beta = 0$. It is thereby concluded that

$$\mathcal{L}^+ f = \theta G^{p+1} \tag{2.24}$$

has nontrivial solutions (g^*, θ) satisfying the side constraints. Taking the inner product of (2.24) with G it is determined that $\theta = 0$, and therefore $g^* = \nu G$ for some $\nu \neq 0$, a contradiction since G is not orthogonal to G^{p+1}. Therefore, the minimum in (2.21) is positive and the proof of the lemma is complete. ■

It follows readily from (2.21) that for some $\widetilde{D}_1 > 0$,

$$< f, \mathcal{L}^+ f > \geqq \widetilde{D}_1 \|f\|_1^2, \qquad \text{if} \qquad f \perp G^{p+1}. \tag{2.25}$$

The next lemma will be used to prove an estimate similar to (2.25) for the operator $\mathcal{L}^-$.

Lemma 2.5. *Let A be a self-adjoint operator on $L^2(\mathbb{R}^n)$ having exactly one negative eigenvalue β with corresponding ground-state eigenfunction $f_\beta \geqq 0$ and let $g \in N^\perp(A)$. Assume $< g, f_\beta > \neq 0$ and that*

$$-\infty < \alpha \equiv \underset{\substack{\|f\|=1 \\ <f,g>=0}}{Min} < Af, f > .$$

If $< A^{-1}g, g > \leqq 0$, then it must be the case that $\alpha \geqq 0$.

Proof. See Lemma E.1 in [W2]. ■

Corollary 2.6. *There exists $\gamma < 0$ such that if $g = G + \gamma \Delta G$, then*

$$\underset{\substack{\|f\|=1 \\ \langle f,g\rangle=0}}{Min} \langle \mathcal{L}^- f, f\rangle = 0. \tag{2.26}$$

Proof. The operator $\mathcal{L}^-$ has a unique negative eigenvalue with eigenfunction $f_\lambda > 0$ from Sturm-Liouville theory in dimension $n = 1$ (see [CLe], [ABH]), and variational methods and the mini-max principle in dimension $n \geqq 2$ (see [W4]). Notice that whatever the value of γ, it follows from the equation (1.4) for G that

$$< f_\lambda, g > = < f_\lambda, G > + \gamma < f_\lambda, \lambda G - G^{p+1} > .$$

Since both f_λ and G are everywhere positive, it follows that for small values of γ, the inner product $< f_\lambda, g >$ is non-zero. On the other hand, for any given γ, define the function f by

$$f(x) = -\frac{n}{4\lambda}G(x) - \frac{\lambda\gamma + 1}{2\lambda} x \cdot \nabla G(x)$$

for $x \in \mathbb{R}^n$. Short calculations show that $\mathcal{L}^- f = G + \gamma \Delta G = g$ and that

$$< f, g > = \left(\frac{n}{4}||G||^2 + \frac{1}{2\lambda}||\nabla G||^2\right)\gamma + \left(\frac{2-n}{4}||\nabla G||^2\right)\gamma^2.$$

It thus becomes obvious that for small, negative values of γ, it is possible to have both $< f_\lambda, g > \neq 0$ and $< (\mathcal{L}^-)^{-1} g, g > = < f, g > < 0$.

Since $\mathcal{N}(\mathcal{L}^-) = \text{Span}\{G_{x_i} | i = 1, ..., n\}$ (see [W2], [K] and [M]), it is clear from its definition that $g \in \mathcal{N}^\perp(\mathcal{L}^-)$. It follows immediately from Lemma 2.5 that the minimum in (2.26) is non-negative. On the other hand, G_{x_i}, suitably normalized, is orthogonal to g, has norm 1 and $\mathcal{L}^-$ vanishes there, so the minimum cannot be positive. ■

Lemma 2.7. *If* $g \equiv G + \gamma\Delta G$, *with* $\gamma < 0$ *as in the last Corollary. then,*

$$Inf\{ < \mathcal{L}^- f, f > \; : \; \|f\| = 1, < f, g >= 0, f \perp G^p G_{x_i}, 1 \leqq i \leqq n\} \; = \; D_2 > 0. \qquad (2.27)$$

Proof. Because of the Corollary, we know that $D_2 \geqq 0$. Suppose that $D_2 = 0$. Following the proof in Lemma 2.4, one ascertains that there exists f^* such that $\|f^*\| = 1$, $< \mathcal{L}^- f^*, f^* >= 0$, $< f^*, g >= 0$ and $< f^*, G^p G_{x_i} >= 0$, $i = 1, ..., n$. Thus there exists ξ, θ, χ such that

$$\mathcal{L}^- f^* = \xi f^* + \theta(G + \gamma\Delta G) + \chi G^p G_{x_i}, \qquad i = 1, ..., n. \qquad (2.28)$$

In consequence of the conditions satisfied by f^*, it must be that $< \mathcal{L}^- f^*, f^* >= \xi = 0$ and $< \mathcal{L}^- f^*, G_{x_i} >=< f^*, \mathcal{L}^- G_{x_i} >= 0 = \chi \int \, G^p (G_{x_i})^2 \, dx$, which implies $\chi = 0$. Therefore, $\mathcal{L}^- f^* - \theta g = 0$. But, if f is the auxiliary function arising in the proof of Corollary 2.6, then $\mathcal{L}^- f = g$, whence $\mathcal{L}^-(f^* - \theta f) = 0$, and therefore $f^* - \theta f \in \mathcal{N}(\mathcal{L}^-)$. From the property $< f, g > \neq 0$ established in Corollary 2.6, it follows from the preceding that $\theta = 0$. Therefore, for some non-zero $l \in \mathbb{R}^n$, it is true that $f^* = l \cdot \nabla G$, which is a contradiction since such a function cannot be orthogonal to $G^p G_{x_i}$ for $1 \leqq i \leqq n$. This completes the proof. ■

Remark 2.8. Comparing the results of Corollary 2.6 and Lemma 2.7 with similar results of Weinstein (the case $\gamma = 0$ in [W4]), it is worth pointing out that in his work, the identity (2.26) is true for $p \leqq \frac{4}{n}$, but (2.27) is only true for $p < \frac{4}{n}$.

Attention is now turned to estimating the term $< \mathcal{L}^- a, a > + \frac{4k\lambda}{\|G\|^2}(< a, G >)^2$ in (2.13), where a satisfies the compatibility relations (2.20a). We continue to carry over the notation from Corollary 2.6 and Lemma 2.7. In particular, γ is chosen so that the conclusions of Corollary 2.6 are valid. Let $a_{\shortparallel} = \frac{<a,\ g>}{\|\ g\|^2}\ g$ and let $a_\perp = a - a_{\shortparallel}$. It follows from the properties of a and $g = G + \gamma\Delta G$ that $< a_\perp,\ g >= 0$ and $\int G^p G_{x_i} a_\perp\, dx = 0$, $i = 1, \ldots, n$. Without loss of generality, take it that $< a,\ g > < 0$. Thus, from Lemma 2.7, the Cauchy-Schwarz inequality and the properties of $a, a_\perp, a_{\shortparallel}$ and g, it follows that

$$\begin{cases} < a_\perp, \mathcal{L}^- a_\perp > \ \geqq\ D_2\|a_\perp\|^2 \\ < a_{\shortparallel}, \mathcal{L}^- a_{\shortparallel} >= \frac{\|a_{\shortparallel}\|^2}{\|\ g\|^2} < g, \mathcal{L}^-\ g > \\ < a_\perp, \mathcal{L}^- a_{\shortparallel} >= \frac{<a,\ g>}{\|\ g\|^2} < a_\perp, \mathcal{L}^-\ g > \geqq -D_3\|a_\perp\|\|a_{\shortparallel}\|, \end{cases} \tag{2.29}$$

with $D_i > 0, i = 2, 3$. Identity (2.3) and the elementary properties of Hilbert spaces imply that

$$2 < a, \Delta G >= \|\nabla a\|^2 + \|\nabla b\|^2.$$

Thus, from the Cauchy-Schwartz inequality, we obtain (remember, γ and $< a, g >$ are both negative)

$$\begin{aligned} \frac{4k\lambda}{\|G\|^2}(< a, G >)^2 &\geqq \frac{4k\lambda}{\|G\|^2} < a,\ g >^2 - \gamma\frac{4k\lambda}{\|G\|^2} < a,\ g > (\|\nabla a\|^2 + \|\nabla b\|^2) \\ &\geqq \frac{4k\lambda\|\ g\|^2}{\|G\|^2}\|a_{\shortparallel}\|^2 + \gamma\frac{4k\lambda\|\ g\|}{\|G\|^2}\|a\|(\|\nabla a\|^2 + \|\nabla b\|^2) \\ &\geqq \frac{4k\lambda\|\ g\|^2}{\|G\|^2}\|a_{\shortparallel}\|^2 + 4k\lambda\gamma D_4\|a + ib\|_1^3, \end{aligned} \tag{2.30}$$

with $D_4 > 0$ also.

If $\theta > 0$ is chosen so that $D_2 - 2\theta D_3 = D_5 > 0$ and k is fixed in such a way that

$$\frac{4k\lambda\|\ g\|^2}{\|G\|^2} + \frac{< g, \mathcal{L}^-\ g >}{\|\ g\|^2} - 2\frac{D_3}{\theta} \equiv D_6 > 0,$$

it follows from (2.29), (2.30) and Young's inequality that

$$\begin{aligned} < \mathcal{L}^- a, a > + \frac{4k\lambda}{\|G\|^2}(< a, G >)^2 &\geqq D_6\|a_{\shortparallel}\|^2 + D_5\|a_\perp\|^2 + 4k\lambda\gamma D_4\|w\|_1^3 \\ &\geqq D'\|a\|^2 - D''\|w\|_1^3, \end{aligned} \tag{2.31}$$

for some positive constants D' and D''. With (2.31) in hand, it follows easily from the specific form of the operator $\mathcal{L}^-$ that,

$$< \mathcal{L}^- a, a > + \frac{4k\lambda}{\|G\|^2}(< a, G >)^2 \geqq \widetilde{D}_2\|a\|_1^2 - \widetilde{D}_3\|w\|_1^3, \tag{2.32}$$

with $\widetilde{D}_2, \widetilde{D}_3 > 0$.

Finally, collecting the results (2.25), (2.32) and substituting them in (2.13), there obtains

$$\Delta\widetilde{L}_t \geqq \widetilde{D}_1\|b\|_1^2 + \widetilde{D}_2\|a\|_1^2 - \widetilde{D}_3\|w\|_1^3 - c_2\sum_{j=1}^{4}\|w\|_1^{j+2} - \sum_{j=1}^{2k} c_{k,j}\|w\|_1^{j+2}$$

$$\geqq c_1\|w\|_1^2 - c_2\sum_{j=1}^{4}\|w\|_1^{j+2} - \sum_{j=1}^{2k} c_{k,j}\|w\|_1^{j+2}, \tag{2.33}$$

where $c_1, c_2, c_{k,j}$ are constants that depend only on λ and the dimension of the space.

Now we are in position to prove Theorem 2.2. Suppose first that $u_0 \in \mathcal{S}$ (defined in (2.5)) and $\|u_0 - G\|_1 = \delta$. Then, for $\widetilde{T}$ as in (2.19) and at least for $t \in [0, \widetilde{T}]$, it follows from (2.9) and (2.10) or (2.33) that

$$q(\rho_\lambda(\phi(\cdot,t),G)) \leqq \Delta\widetilde{L}_t \leqq c_0\delta \tag{2.34}$$

where $q(x) = c_1x^2 - c_2\sum_{j=1}^{4} x^{j+2} - \sum_{j=1}^{2k} c_{k,j}x^{j+2}$ and $\rho_\lambda(\phi(\cdot,t),G)$ is as in (2.5).

Since $\|w(t)\|_1^2 = \rho_\lambda(\phi(\cdot,t),G)^2$ is a continuous function of $t \in [0, t^*)$ (see Lemma 2 in [Bo]), it follows from the inequality

$$q(\rho(\phi(\cdot,0),G)) \leqq c_0\delta \tag{2.35}$$

and from (2.34) that for $t \in [0, \widetilde{T}]$ we have

$$\rho(\phi(\cdot,t),G) \leqq \epsilon. \tag{2.36}$$

provided the δ is chosen small enough at the outset.

To finish the proof, we show that the inequality (2.36) is still true for $t \in [0, t^*)$. Following the ideas in [Bo], let

$$\mathcal{A} = \{t : \text{the infimum in (2.17) is attained at finite values of } (\alpha_0, \alpha_1)\}.$$

As shown above, $[0, \widetilde{T}] \subset \mathcal{A}$. Let T_1 be the largest value such that $[0, T_1) \subset \mathcal{A}$ and suppose that $T_1 < t^*$. Then from (2.36) we obtain that

$$\text{Inf}\ \Omega_t = \rho_\lambda(\phi(\cdot,t),G)^2 \leqq \epsilon^2 \leqq \frac{\|(-\Delta+\lambda)^{1/2}G\|^2}{2}. \tag{2.37}$$

Since Inf Ω_t is a continuous function of t for all $t \in [0, t^*)$, there is a $T > 0$ such that

$$\text{Inf } \Omega_t < \|(-\Delta + \lambda)^{1/2} G\|^2$$

for $t \in [T_1, T_1 + T]$. But then Lemma 2.3 implies that the infimum in (2.17) is taken at finite values of (α_0, α_1) and this contradicts the choice of T_1. Therefore $T_1 = t^*$ and the stability Theorem 2.2 is established for $\mu(0) = 1$.

Now we discuss the general case where we do not necessarily start with $\mu(0) = 1$. First, remark that if G_λ is a solution of (1.4), then $G_\lambda(x) = \lambda^{1/p} R(\lambda^{1/2} x)$ where R satisfies,

$$\Delta R - R + R^{p+1} = 0.$$

Let $u_0 \in \mathcal{S}$ obey the restriction

$$\lambda \|u_0 - G\|^2 + \|\nabla u_0 - \nabla G\|^2 < \delta^2, \tag{2.38}$$

where δ will be determined presently. Corresponding to $\lambda_1 > 0$, a solution G_{λ_1} of (1.4) has $\|\nabla G_{\lambda_1}\|^2 = \lambda_1 \|\nabla R\|^2$. It is therefore possible to choose λ_1 such that $\|\nabla G_{\lambda_1}\|^2 = \|\nabla u_0\|^2$. Then if

$$\mu_{\lambda_1}(t) = \frac{\|\nabla u(\cdot, t)\|}{\|\nabla G_{\lambda_1}\|},$$

it is obviously the case that $\mu_{\lambda_1}(0) = 1$.

The idea is to apply the preceding theory to the case $\lambda = \lambda_1$ and then use the triangle inequality to conclude the desired result for the given value of λ and u_0. Thus the program parallels that given first in the context of Korteweg-de Vries-type equations in [B] and [Bo].

An estimate of the quantity

$$I_\lambda(u(\cdot, t), G, \mu) \equiv \mu^{-2}(t) \|\nabla u(\cdot, t) - \mu^{\frac{n}{2}}(t) \nabla G(\mu(t)(\cdot))\|^2 + \lambda \|u(\cdot, t) - \mu^{\frac{n}{2}}(t) G(\mu(t)(\cdot))\|^2$$

will be helpful. Denoting G_{λ_1} and μ_{λ_1} by G_1 and μ_1, respectively, it follows from the definitions of μ and μ_1 that

$$\begin{aligned} I_\lambda(u(\cdot, t), G, \mu) \leqq 2\Big[\mu^{-2} \, \|\mu^{\frac{n}{2}} \nabla G(\mu(\cdot)) - \mu_1^{\frac{n}{2}} \nabla G_1(\mu_1(\cdot))\|^2 \\ + \lambda \|\mu^{\frac{n}{2}} \, G(\mu(\cdot)) - \mu_1^{\frac{n}{2}} \, G_1(\mu_1(\cdot))\|^2\Big] + \frac{2\lambda}{\lambda_1} \, I_{\lambda_1}(u(\cdot, t), G_1, \mu_1). \end{aligned} \tag{2.39}$$

Now, we estimate the right-hand side of (2.39). First observe that

$$\lambda\|\mu^{\frac{n}{2}}\, G(\mu(\cdot)) - \mu_1^{\frac{n}{2}}\, G_1(\mu_1(\cdot))\|^2 = 0$$

and

$$\mu^{-2}\, \|\mu^{\frac{n}{2}}\nabla G(\mu(\cdot)) - \mu_1^{\frac{n}{2}}\nabla G_1(\mu_1(\cdot))\|^2 = 0.$$

Thus we only need to estimate the term $I_{\lambda_1}(u(\cdot,t), G_1, \mu_1)$ in (2.39). For this, it suffices to show that $\rho_{\lambda_1}(u_0, G_{\lambda_1}) \leqq \widetilde{C}\delta$, where $\widetilde{C} = \widetilde{C}(\lambda, R) > 0$ and then apply the foregoing theory for the special case $\mu_1(0) = 1$. Because $\rho_{\lambda_1}(u_0, G_{\lambda_1}) \leqq \rho_{\lambda_1}(u_0, G) + \rho_{\lambda_1}(G, G_{\lambda_1})$, we may estimate $\rho_{\lambda_1}(u_0, G)$ and $\rho_{\lambda_1}(G, G_{\lambda_1})$ separately and still reach the desired inequality.

First, estimate the term $\rho_{\lambda_1}(G, G_{\lambda_1})$, viz.

$$\begin{aligned}
[\rho_{\lambda_1}(G, G_{\lambda_1})]^2 \;&\leqq\; \lambda_1\|G - G_{\lambda_1}\|^2 + \|\nabla G - \nabla G_{\lambda_1}\|^2 \\
&= \lambda_1 \int_{\mathbb{R}^n} \Big|R(x) - (\frac{\lambda_1}{\lambda})^{\frac{n}{4}} R((\frac{\lambda_1}{\lambda})^{1/2}x)\Big|^2\, dx \\
&+ \lambda \int_{\mathbb{R}^n} \Big|\nabla R(x) - (\frac{\lambda_1}{\lambda})^{\frac{n}{4}}\, \nabla R((\frac{\lambda_1}{\lambda})^{1/2}x)\Big|^2\, dx. \qquad (2.40)
\end{aligned}$$

The first integral in (2.40) can be bounded above as follows:

$$\begin{aligned}
&\lambda_1 \int_{\mathbb{R}^n} \Big|R(x) - (\frac{\lambda_1}{\lambda})^{\frac{n}{4}} R((\frac{\lambda_1}{\lambda})^{1/2}x)\Big|^2\, dx \\
&\leqq 2\lambda_1\Big(\frac{\lambda_1}{\lambda}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} |R(x) - R((\frac{\lambda_1}{\lambda})^{1/2}x)|^2\, dx + 2\, \frac{\lambda_1\Big|\lambda_1^{\frac{n}{4}} - \lambda^{\frac{n}{4}}\Big|^2}{\lambda^{\frac{n}{2}}}\, \|R\|^2.
\end{aligned}$$

Thus, the Fundamental Theorem of Calculus together with Minkowski's inequality yield

$$\begin{aligned}
\int_{\mathbb{R}^n} \Big|R(x) - R((\frac{\lambda_1}{\lambda})^{1/2}x)\Big|^2\, dx &\leqq \int_{\mathbb{R}^n} \Big(\int_{(\frac{\lambda_1}{\lambda})^{1/2}}^1 |\frac{d}{dt}R(tx)|\, dt\Big)^2\, dx \\
&\leqq \Big(\int_{(\frac{\lambda_1}{\lambda})^{1/2}}^1 \Big(\int_{\mathbb{R}^n} |\frac{d}{dt}R(tx)|^2\, dx\Big)^{1/2}\, dt\Big)^2 \\
&= \|x\cdot\nabla R\|^2 \Big(\int_{(\frac{\lambda_1}{\lambda})^{1/2}}^1 t^{-\frac{n+2}{2}}\, dt\Big)^2 = \frac{4|\lambda_1^{\frac{n}{4}} - \lambda^{\frac{n}{4}}|^2}{n^2\lambda_1^{\frac{n}{2}}}\|x\cdot\nabla R\|^2.
\end{aligned}$$

Consequently, it transpires that

$$\lambda_1 \int_{\mathbb{R}^n} \Big|R(x) - (\frac{\lambda_1}{\lambda})^{\frac{n}{4}} R((\frac{\lambda_1}{\lambda})^{1/2}x)\Big|^2\, dx \leqq \frac{2\lambda_1\, |\lambda_1^{\frac{n}{4}} - \lambda^{\frac{n}{4}}|^2}{\lambda^{\frac{n}{2}}}\Big[\|R\|^2 + \frac{4}{n^2}\|x\cdot\nabla R\|^2\Big]. \qquad (2.41)$$

Similarly, the second integral on the right-hand side of (2.40) may be bounded above thusly:

$$\lambda \int_{\mathbb{R}^n} \left|\nabla R(x) - (\frac{\lambda_1}{\lambda})^{\frac{n}{4}} \nabla R((\frac{\lambda_1}{\lambda})^{1/2} x)\right|^2 dx$$
$$\leqq \frac{2\lambda|\lambda_1^{\frac{n}{4}} - \lambda^{\frac{n}{4}}|^2}{\lambda^{\frac{n}{2}}} \sum_{i=1}^{n} \left[\|\partial_{x_i} R\|^2 + \frac{4}{n^2}\|x \cdot \nabla \partial_{x_i} R\|^2\right]. \tag{2.42}$$

The inequalities (2.41) and (2.42) imply

$$\begin{aligned}[\rho_{\lambda_1}(G, G_{\lambda_1})]^2 &\leqq \frac{2\lambda_1|\lambda_1^{\frac{n}{4}} - \lambda^{\frac{n}{4}}|^2}{\lambda^{\frac{n}{2}}}\left[\|R\|^2 + \frac{4}{n^2}\|x \cdot \nabla R\|^2\right] \\ &+ \frac{2\lambda|\lambda_1^{\frac{n}{4}} - \lambda^{\frac{n}{4}}|^2}{\lambda^{\frac{n}{2}}} \sum_{i=1}^{n} \left[\|\partial_{x_i} R\|^2 + \frac{4}{n^2}\|x \cdot \nabla \partial_{x_i} R\|^2\right] \\ &\leqq C_1(R)\frac{\lambda + \lambda_1}{\lambda^{\frac{n}{2}}}|\lambda_1^{\frac{n}{4}} - \lambda^{\frac{n}{4}}|^2. \end{aligned} \tag{2.43}$$

It is now determined that there is a positive constant $C = C(\lambda, R)$ such that

$$|\lambda_1 - \lambda| \leqq C\delta \tag{2.44}$$

at least for small values of δ. In fact, from Young's inequality and (2.38),

$$\begin{aligned}|\lambda_1 - \lambda| = \frac{1}{\|\nabla R\|^2}\left|\|\nabla u_0\|^2 - \|\nabla G\|^2\right| &\leqq \frac{2\delta}{\|\nabla R\|^2}\|\nabla G\|^2 + \frac{1}{\|\nabla R\|^2}(1 + \frac{1}{2\delta})\|\nabla u_0 - \nabla G\|^2 \\ &\leqq \frac{1}{\|\nabla R\|^2}(2\lambda\|\nabla R\|^2 + \frac{3}{2})\ \delta.\end{aligned}$$

The inequality (2.44) certainly implies that, $|\frac{\lambda}{\lambda_1} - 1| \leqq 1$ and $|\lambda_1^{\frac{n}{4}} - \lambda^{\frac{n}{4}}| \leqq C_2\delta$, where $C_2 = C_2(\lambda, R) > 0$. From (2.43), it then follows that

$$\rho_{\lambda_1}(G, G_{\lambda_1}) \ \leqq C_3(R)\delta. \tag{2.45}$$

From (2.44) we have that $\frac{\lambda_1}{\lambda} \leqq 1 + \frac{C}{\lambda}\delta$. Therefore the assumption (2.38) implies

$$[\rho_{\lambda_1}(u_0, G)]^2 \leqq \max\left\{\frac{\lambda_1}{\lambda}, 1\right\} \left[\lambda\|u_0 - G\|^2 + \|\nabla u_0 - \nabla G\|^2\right] \leqq C_4\delta^2, \tag{2.46}$$

where $C_4 = C_4(\lambda, R) > 0$. Inequalities (2.45) and (2.46) lead to the conclusion $\rho_{\lambda_1}(u_0, G_{\lambda_1}) \leqq \widetilde{C}\delta$, and therefore that

$$I_{\lambda_1}(u(\cdot, t), G_1, \mu_1) \leqq \epsilon^2. \tag{2.47}$$

Theorem 2.2. is now established. ■

3. Behavior of the Stability Parameters $\alpha_0 = \alpha_0(t)$ and $\alpha_1 = \alpha_1(t)$

In the proof of Theorem 2.2, it was actually shown that there is a choice of $\alpha_0 = \alpha_0(t)$ and $\alpha_1 = \alpha_1(t)$ for which

$$\rho_\lambda(\phi(\cdot,t),G) = \Big(\|\nabla\phi(\cdot+\alpha_0,t)e^{i\alpha_1} - \nabla G(\cdot)\|^2 + \lambda\|\phi(\cdot+\alpha_0,t)e^{i\alpha_1} - G(\cdot)\|^2\Big)^{1/2} \leqq \epsilon \quad (3.1)$$

for all $t < t^*$, and that a choice of α_0 and α_1 for which (3.1) holds may be determined via the orthogonality conditions (see (2.20))

$$\mathrm{Im}\int_{\mathbb{R}^n} G^{p+1}(x)\Big[e^{i\alpha_1(t)}\phi(x+\alpha_0(t),t)\Big]\,dx = 0, \quad (3.2)$$

$$\mathrm{Re}\int_{\mathbb{R}^n} G^{p}(x)G_{x_i}(x)\Big[e^{i\alpha_1(t)}\phi(x+\alpha_0(t),t)\Big]\,dx = 0, \quad (3.3)$$

for $i = 1,..,n$ and $\phi(x,t) = \mu^{-\frac{n}{2}}(t)u(\mu(t)^{-1}x,t)$.

By an application of the implicit-function theorem as in [BS], it may be shown that as long as ϕ satisfies (3.1), there is a unique, continuously differentiable choice of the values $\alpha_0(t)$ and $\alpha_1(t)$ that achieve (3.2) and (3.3).

The principal result regarding the behavior of the parameters α_0 and α_1 is stated in the next theorem. Its proof will appear elsewhere.

Theorem 3.1. *Let $p = \frac{4}{n}$ and $G = G_\lambda$ be a ground-state solution of (1.4). For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $\|u_0 - G\|_1 < \delta$, then there are C^1-mappings $\alpha_0 : (-t^*,t^*) \to \mathbb{R}^n$ and $\alpha_1 : (-t^*,t^*) \to \mathbb{R}$ such that*

(i) $\|\phi(\cdot+\alpha_0(t),t)e^{i\alpha_1(t)} - G(\cdot)\|_1 \leqq \epsilon$, for $t \in (-t^*,t^*)$, and

(ii) for $\alpha_0(t) = (\alpha_{0,1}(t),...,\alpha_{0,n}(t))$ and $t \in [0,t^*)$,

$$\Big|\alpha_1(t) + \lambda\int_0^t \mu^2(s)ds\Big| \leqq C\epsilon\Big[\int_0^t \mu^2(s)ds + \int_0^t \frac{|\mu'(s)|}{\mu(s)}\,ds\Big],$$

$$\Big|\alpha_{0,i}(t)\Big| \leqq C\epsilon\mu(t)\Big[\int_0^t \mu(s)ds + \int_0^t \frac{|\mu'(s)|}{\mu^2(s)}\,ds\Big], \qquad i = 1,...,n,$$

where C depends only on G.

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