Zero-Dissipation Limit for Nonlinear Waves *

Jerry L. Bona
Department of Mathematics and the
Texas Institute for Computational and Applied Mathematics
The University of Texas at Austin
and
Jiahong Wu
Department of Mathematics
The University of Texas at Austin

Abstract

Evolution equations featuring nonlinearity, dispersion and dissipation are considered here. For classes of such equations that include the Korteweg-de Vries-Burgers equation and the BBM-Burgers equation, the zero dissipation limit is studied. Uniform bounds independent of the dissipation coefficient are derived and zero dissipation limit results with optimal convergence rates are established.

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1 Introduction

The incorporation of dissipative effects is often crucial in obtaining good agreement between experimental observations and the prediction of theoretical models describing the propagation of waves in nonlinear dispersive media (cf. Bona *et al.* [14] for an example from water-wave theory). To take account of dissipative mechanisms, a Burgers-type term is often appended to nonlinearity and dispersion in these models (cf. Johnson [22], [23] for an early suggestion in this direction). Two such models are the well-known BBM-Burgers equation

$$u_t + u_x + u^p u_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0 \tag{1.1}$$

and the (generalized) Korteweg-de Vries-Burgers equation (GKdV-Burgers equation)

$$u_t + u_x + u^p u_x - \nu u_{xx} + u_{xxx} = 0, \tag{1.2}$$

where $u = u(x,t)$ is a real-valued function of two real variables x and t, $p \geq 1$ is an integer, $\nu \geq 0$ and $\alpha > 0$ are real numbers. Numerous numerical simulations and analytical studies have been carried out to determine the effect of such a term in these models (cf. [4], [7], [8], [12], [13], [15], [16], [21], [27], [28], [29]). Laboratory studies show (1.1) with $p = 1$ and a suitably chosen value of ν has good predictive power in cases where nonlinear effects are not too strong (e.g. the Stokes number is not too large in a water-wave context [14]).

It is the purpose of this article to investigate theoretically aspects of the dissipative effects inherent in these two models when $\nu > 0$. Consideration will also be given to a more general class of models of the form

$$u_t + (P(u))_x + \nu M u - (Lu)_x = 0, \tag{1.3}$$

where M and L are Fourier multiplier operators with non-negative symbols and P is a polynomial, say

$$P(u) = \sum_{k=1}^{p+1} a_k u^k$$

with $a_k \in \mathbb{R}$, $k = 1, 2, \cdots, p$ (see Bona [5] and Dix [21]). Interest will mainly focus on the pure initial-value problem (IVP) for these equations wherein

$$u(x,0) = u_0(x), \qquad \text{is specified for } x \in \mathbb{R};$$

however, the initial- and boundary-value problem (IBVP)

$$u(x,0) = u_0(x), \qquad \text{for } x \in \mathbb{R}^+,$$

$$u(0,t) = g(t), \qquad \text{for } t \in \mathbb{R}^+,$$

for the BBM-Burgers equation will also be examined. In this article, particular interest is directed toward the behavior of solutions in the zero dissipation limits.

In the limit as ν tends to zero, Equations (1.1), (1.2) and (1.3) formally reduce to the BBM equation, the GKdV equation and a class of equations of KdV-type in generalized form,

$$u_t + u_x + u^p u_x - u_{xxt} = 0,$$

$$u_t + u_x + u^p u_x + u_{xxx} = 0,$$

$$u_t + (P(u))_x - (Lu)_x = 0,$$

respectively. This suggests comparing solutions u to one of these equations with dissipation to the solution v of the corresponding equation without dissipation. It is expected that for various spatial norms $\|\cdot\|$,

$$\|u(\cdot,t) - v(\cdot,t)\| \to 0 \tag{1.4}$$

as $\nu \to 0$, uniformly for $t \geq 0$. Theory will be developed showing (1.4) is valid in certain circumstances. Moreover, we will be able to determine the rate at which $\|u(\cdot,t) - v(\cdot,t)\|$ approaches zero. A crucial step in proving such convergence results is to obtain $\nu-$independent bounds on solutions to the dissipative equations and very often these are not available in the literature. Precise statements are provided presently.

The paper is organized as follows. Section 2 contains the relatively straightforward analysis of the zero-dissipation limits for the IVP and the IBVP for the BBM-Burgers equation. In Section 3 we establish ν-independent bounds on solutions to the GKdV-Burgers equation in H^k for all integers $k \geq 0$ (The Hilbert space $H^k = H^k(\mathbb{R})$ is the L^2-based Sobolev class of functions whose derivatives to order k are all square integrable.). This result is interesting in its own right and crucial in obtaining the zero-dissipation limit results for the GKdV-Burgers equation in Section 4. The relation (1.4) is determined to hold in $\|\cdot\|_{H^k}$ and the convergence is shown to be $O(\nu)$ as $\nu \to 0$. Section 5 is devoted to the equation of general type depicted in (1.3). Zero-dissipation limit theory in this section relies upon growth conditions on the symbols of the dispersion and dissipation operators L and M, respectively.

2 Zero-dissipation limit for the BBM-Burgers equation

This section is divided into two parts. The first part is devoted to the zero-dissipation limit for the IVP for the BBM-Burgers equation while the second part deals with the zero-dissipation limit for the associated IBVP. Consider first the IVP

$$u_t + u_x + u^p u_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{2.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{2.2}$$

where $p \geq 1$ is an integer, $\nu > 0$ and $\alpha > 0$. As noted before, upon setting $\nu = 0$, Equation (2.1) formally reduces to

$$u_t + u_x + u^p u_x - \alpha^2 u_{xxt} = 0. \tag{2.3}$$

There is an adequate theory of well-posedness for the IVP (2.1)-(2.2) and the IVP (2.2)-(2.3) (cf. Bona *et al.* [2], [3]). For our purpose, it suffices to have the following proposition, in which $C_b(I, X)$ denotes the bounded continuous mappings u: $I \to X$, $I = [0,T] \subset \mathbb{R}^+$, with its usual norm.

Proposition 2.1 *Let $u_0 \in H^s$ with $s \geq 1$. Then there exists a unique solution u to the IVP (2.1)-(2.2) such that, for each $T > 0$,*

$$u \in C_b([0,\infty); H^1) \cap C([0,T]; H^s) \quad \text{and} \quad \partial_t^k u \in C([0,T]; H^s)$$

for each $k > 0$. Furthermore, for each $T > 0$, the solution map from u_0 to u is analytic from H^s to $C^k([0,T]; H^s)$.

The preceding results hold for the IVP (2.2)-(2.3), but in this case $\partial_t^k u \in C([0,T]; H^{s+1})$ for each $k > 0$ and $T > 0$.

We shall use u and v to denote the solution to the IVP (2.1)-(2.2) and the IVP (2.2)-(2.3), with initial data u_0 and v_0, respectively. The following lemma provides $\nu-$independent bounds and other helpful inequalities for a solution u to the IVP (2.1)-(2.2).

Lemma 2.2 *Assume that $p \geq 1$ and $s \geq 1$.*

(i) *If u is a solution of the IVP (2.1)-(2.2) with $u_0 \in H^s$, then for all $t \geq 0$,*

$$\|u(\cdot,t)\|_{L^2}^2 + \alpha^2\|u_x(\cdot,t)\|_{L^2}^2 + 2\nu\int_0^t\int_{-\infty}^{\infty} u_x^2 dxds = \|u_0\|_{L^2}^2 + \alpha^2\|u_{0x}\|_{L^2}^2, \tag{2.4}$$

$$u_{xx} \in L^2(\mathbb{R}\times\mathbb{R}^+), \qquad \textit{and} \quad \|u(\cdot,t)\|_{L^\infty} \leq C(\alpha)\|u_0\|_{H^1}$$

where $C(\alpha) = \max\{\alpha^2, \alpha^{-2}\}$.

(ii) *If v is a solution of the IVP (2.2)-(2.3) with initial data $v_0 \in H^s$, then*

$$\|v(\cdot,t)\|_{H^1} \leq C(\alpha)\|v_0\|_{H^1}, \qquad \|v(\cdot,t)\|_{L^\infty} \leq C(\alpha)\|v_0\|_{H^1} \tag{2.5}$$

and, if $s \geq 2$,

$$\int_{-\infty}^{\infty}\left(v_x^2(x,t) + \alpha^2 v_{xx}^2(x,t)\right)dx \leq e^{\frac{\|v_0\|_{H^1}^p}{\alpha}t}\int_{-\infty}^{\infty}\left(v_{0x}^2(x) + \alpha^2 v_{0xx}^2(x)\right)dx \tag{2.6}$$

$$\int_0^t \|v_x(\cdot,s)\|_{L^\infty}ds \leq 2\sqrt{2}\alpha\|v_0\|_{H^1}^{1-p}\|v_0\|_{H^2}\left(e^{\frac{\|v_0\|_{H^1}^p}{2\alpha}t} - 1\right) \tag{2.7}$$

Remark. In the proof that follows, and frequently in the rest of the paper, intermediate calculations are made that use regularity in excess of that assumed on the data and hence in excess of that which the solution possesses. The final inequalities do not suffer from this defect, however. Such calculations are easy to justify in the presence of a strong continuous dependence result. Simply regularize the initial data, make the calculation securely for the resulting smooth solution, and then in the final inequality pass to the limit as the regularization weakens to the identity. This standard procedure underlies much of the theory developed here, but we will not constantly remind the reader of its invocation.

Proof. The formula (2.4) is obtained by multiplying (2.1) by u, integrating over $\mathbb{R} \times [0,t]$ and integrating by parts appropriately. To show that $u_{xx} \in L^2(\mathbb{R} \times \mathbb{R}^+)$, multiply (2.1) by u_{xx} and integrate. To finish (i), it suffices to remark that

$$\|u(\cdot,t)\|_{L^\infty}^2 \le 2\|u(\cdot,t)\|_{L^2}\|u_x(\cdot,t)\|_{L^2} \le C(\alpha)^2\|u_0\|_{H^1}^2.$$

The proof of (2.5) is similar. To establish (2.6), multiply (2.3) by v_{xx} and integrate over $\mathbb{R}$ to obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty}(v_x^2(x,t) + \alpha^2 v_{xx}^2(x,t))dx = 2\int_{-\infty}^{\infty}(v_x v_{xx} v^p)(x,t)dx$$
$$\le \|v(\cdot,t)\|_{L^\infty}^p \alpha^{-1}\int_{-\infty}^{\infty}(v_x^2(x,t) + \alpha^2 v_{xx}^2(x,t))dx, \tag{2.8}$$

which leads to (2.6) after integration over $[0,t]$. The inequality (2.7) follows by combining (2.6) and the estimates

$$\|v_x(\cdot,s)\|_{L^\infty} \le \sqrt{2}\|v_x(\cdot,s)\|_{L^2}\|v_{xx}(\cdot,s)\|_{L^2} \le C(\alpha)\|v_0\|_{H^1}\|v_{xx}(\cdot,s)\|_{L^2},$$

where the constants depending on α may be different from line to line.

In the following theorem, explicit estimates are established for the difference between a solution u to the IVP (2.1)-(2.2) and v to the IVP (2.3)-(2.2). As a consequence of these estimates, u converges to v with the sharp rate of order ν if the initial difference is maintained at order ν.

Theorem 2.3 *Assume that $p \ge 1$ and $s \ge 2$. Let u be the solution of the IVP (2.1)-(2.2) with $u_0 \in H^s$ and let v be the solution of the IVP (2.3)-(2.2) with initial data $v_0 \in H^s$. Then the difference $w = u - v$ satisfies the inequality*

$$\|w\|_{L^2}^2 + (1+\alpha^2)\|w_x\|_{L^2}^2 + \alpha^2\|w_{xx}\|_{L^2}^2$$
$$\le e^{\mathcal{A}(t)}\left(\|w_0\|_{L^2}^2 + (1+\alpha^2)\|w_{0x}\|_{L^2}^2 + \alpha^2\|w_{0xx}\|_{L^2}^2\right) + \nu^2 e^{\mathcal{A}(t)}\mathcal{B}(t), \tag{2.9}$$

for all $t \geq 0$, *where* $w_0 = u_0 - v_0$,

$$\mathcal{A}(t) = \max\{1, \alpha^2\}\Big(t + \|u_0\|_{H^1}^p t$$

$$+6\sqrt{2}\alpha p \max\{\|u_0\|_{H^1}^{p-1}, \|v_0\|_{H^1}^{p-1}\}\|v_0\|_{H^1}^{1-p}\|v_0\|_{H^2}\left(e^{\frac{\|v_0\|_{H^1}^p}{2\alpha}t} - 1\right)\Big),$$

and

$$\mathcal{B}(t) = \alpha\|v_0\|_{H^1}^{-p}\|v_0\|_{H^2}^2\left(e^{\frac{\|v_0\|_{H^1}^p}{\alpha}t} - 1\right).$$

If we consider a one-parameter family $\{u_0^\nu\}_{\nu>0}$ *of initial data such that* $\|u_0^\nu - v_0\|_{H^2} = O(\nu)$ *as* $\nu \to 0$ *(in particular if* $u_0^\nu \equiv v_0$*), then for any* $T > 0$ *and* $t \leq T$

$$\|w(\cdot,t)\|_{L^2}^2 + (1+\alpha^2)\|w_x(\cdot,t)\|_{L^2}^2 + \alpha^2\|w_{xx}(\cdot,t)\|_{L^2}^2 = O(\nu^2)$$

as $\nu \to 0$.

Proof. The difference w satisfies

$$w_t + w_x + (u^p u_x - v^p v_x) - \nu u_{xx} - \alpha^2 w_{xxt} = 0. \tag{2.10}$$

Multiplying (2.10) by $2(w - w_{xx})$ and integrating over $\mathbb{R}$ yields

$$\frac{d}{dt}\left(\|w\|_{L^2}^2 + (1+\alpha^2)\|w_x\|_{L^2}^2 + \alpha^2\|w_{xx}\|_{L^2}^2\right) + 2\nu\int_{-\infty}^{\infty}(w_x^2 + w_{xx}^2)dx$$

$$= 2\nu\int_{-\infty}^{\infty}(w - w_{xx})v_{xx}dx - 2\int_{-\infty}^{\infty}(w - w_{xx})((u^p w_x + (u^p - v^p)v_x)dx. \tag{2.11}$$

The first term on the right-hand side of (2.11) may be bounded by

$$\|w(\cdot,t)\|_{L^2}^2 + \|w_{xx}(\cdot,t)\|_{L^2}^2 + 2\nu^2\int_{-\infty}^{\infty}v_{xx}^2dx.$$

Using the results of Lemma 2.2, there obtains

$$\left|\int_{-\infty}^{\infty}w(u^p - v^p)v_x dx\right| \leq p\max\{\|u_0\|_{H^1}^{p-1}; \|v_0\|_{H^1}^{p-1}\}\|v_x(\cdot,t)\|_{L^\infty}\int_{-\infty}^{\infty}w^2dx,$$

$$\left|\int_{-\infty}^{\infty} wu^p w_x dx\right| \le \frac{1}{2}\|u_0\|_{H^1}^p \int_{-\infty}^{\infty} w^2 dx + \frac{1}{2}\|u_0\|_{H^1}^p \int_{-\infty}^{\infty} w_x^2 dx,$$

$$\left|\int_{-\infty}^{\infty} w_{xx}(u^p - v^p)v_x dx\right| \le \frac{p}{2}\max\{\|u_0\|_{H^1}^{p-1}, \|v_0\|_{H^1}^{p-1}\}\|v_x(\cdot,t)\|_{L^\infty} \int_{-\infty}^{\infty} w^2 dx$$

$$+\frac{p}{2}\max\{\|u_0\|_{H^1}^{p-1}, \|v_0\|_{H^1}^{p-1}\}\|v_x(\cdot,t)\|_{L^\infty} \int_{-\infty}^{\infty} w_{xx}^2 dx,$$

and

$$\left|\int_{-\infty}^{\infty} w_{xx}u^p w_x dx\right| \le \frac{1}{2}\|u_0\|_{H^1}^p \int_{-\infty}^{\infty} w_x^2 dx + \frac{1}{2}\|u_0\|_{H^1}^p \int_{-\infty}^{\infty} w_{xx}^2 dx.$$

These estimates are combined to give

$$\frac{d}{dt}Y(t) \le A(t)Y(t) + B(t) \tag{2.12}$$

where

$$Y(t) = \|w(\cdot,t)\|_{L^2}^2 + (1+\alpha^2)\|w_x(\cdot,t)\|_{L^2}^2 + \alpha^2\|w_{xx}\|_{L^2}^2, \tag{2.13}$$

$$A(t) = \max\{1,\alpha^{-2}\}\Big(1 + \|u_0\|_{H^1}^p + 3p\max\{\|u_0\|_{H^1}^{p-1}, \|v_0\|_{H^1}^{p-1}\}\|v_x(\cdot,t)\|_{L^\infty}\Big) \tag{2.14}$$

and

$$B(t) = 2\nu^2 \int v_{xx}^2(x,t)dx. \tag{2.15}$$

By Gronwall's inequality to applied to (2.12), there is derived the upper bound

$$Y(t) \le \left(Y(0) + \int_0^t B(s)ds\right) e^{\int_0^t A(\tau)d\tau}$$

which is (2.9) after reintroducing Y, A and B as in (2.13), (2.14) and (2.15), respectively, and using the bounds in Lemma 2.2 .

Next, attention is given to the zero-dissipation limit of solutions to the initial- and boundary-value problem (IBVP)

$$u_t + u_x + u^p u_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0, \qquad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{2.16}$$

$$u(0,t) = g_1(t), \qquad t \in \mathbb{R}^+, \tag{2.17}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^+, \tag{2.18}$$

where $p \geq 1$ is an integer, $\nu > 0$ and $\alpha > 0$ and the consistency condition $g_1(0) = u_0(0)$ is always assumed. Our approach is to compare the solution u of the IBVP (2.16)-(2.17)-(2.18) with the solution v to the IBVP for the BBM-equation

$$v_t + v_x + v^p v_x - \alpha^2 v_{xxt} = 0, \qquad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{2.19}$$

$$v(0,t) = g_2(t), \qquad t \in \mathbb{R}^+, \tag{2.20}$$

$$v(x,0) = v_0(x), \qquad x \in \mathbb{R}^+, \tag{2.21}$$

in which $g_2(0) = v_0(0)$.

The well-posedness of both the IBVP (2.16)-(2.17)-(2.18) and the IBVP (2.19)-(2.17)-(2.18) has been established by Bona, Bryant and Luo (cf. [6], [10]). The following result suffices for our purposes.

Proposition 2.4 *Let $T > 0$, $1 \leq p < 4$, $u_0 \in C_b^2(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$ and $g_1 \in C^1(0,T)$ with $g_1(0) = u_0(0)$ (respectively, $v_0 \in C_b^2(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$ and $g_2 \in C^1(0,T)$ with $g_2(0) = v_0(0)$). Then the IBVP (2.16)-(2.17)-(2.18) (respectively, the IBVP (2.19)-(2.20)-(2.21)) has a unique solution u such that, for any finite $T > 0$, $u \in \mathcal{B}_T^{2,1}(\mathbb{R}^+) \cap C([0,T]; H^2(\mathbb{R}^+))$ (respectively, $v \in \mathcal{B}_T^{2,1}(\mathbb{R}^+) \cap C([0,T]; H^2(\mathbb{R}^+))$). Furthermore, the bound for $\|u\|_{H^2}$ is independent of ν for small ν.*

In Proposition 2.4 $\mathcal{B}_T^{k,l}(\mathbb{R}^+)$ stands for the functions u defined on $\mathbb{R}^+ \times [0,T]$ such that $\partial_x^i \partial_t^j u$ are continuous and bounded over $\mathbb{R}^+ \times [0,T]$ for $0 \leq i \leq k$ and $0 \leq j \leq l$. The principal zero-dissipation limit result for solutions to the IBVP (2.16)-(2.17)-(2.18) is as follows.

Theorem 2.5 *Let $T > 0$, $1 \leq p < 4$, $u_0, v_0 \in C_b^2(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$ and $g_1, g_2 \in C^1(0,T)$ with $g_1(0) = u_0(0)$ and $g_2(0) = v_0(0)$. Consider the difference*

$$w(x,t) = u(x,t) - v(x,t)$$

between a solution u to the IBVP (2.16)-(2.17)-(2.18) with data u_0 and g_1 and a solution v to the IBVP (2.19)-(2.20)-(2.21) with data v_0 and g_2. Then for any $t \in [0,T]$,

$$\|w\|_{L^2}^2 + (1+\alpha^2)\|w_x\|_{L^2}^2 + \alpha^2\|w_{xx}\|_{L^2}^2$$

$$\leq C_1(t)\Big[||w_0||_{L^2}^2 + (1+\alpha^2)||w_{0x}||_{L^2}^2 + \alpha^2||w_{0xx}||_{L^2}^2\Big]$$
$$+C_2(t)\nu^2 + C_3||g_1 - g_2||_{C^1(0,T)} + C_4||g_1 - g_2||_{C^1(0,T)}^2$$

where $w_0 = u_0 - v_0$, C_1, C_2 are functions of t and C_3, C_4 are constants, all of which depend only on α, p, T, $||u_0||_{H^2}$,$||v_0||_{H^2}$, $||g_1||_{C^1(0,T)}$ and $||g_2||_{C^1(0,T)}$.

As a consequence, if $\{u_0^\nu\}_{\nu>0}$ and $\{g_1^\nu\}_{\nu>0}$ are families of initial and boundary data for which $||u_0 - v_0||_{H^2} = O(\nu)$ and $||g_1 - g_2||_{C^1(0,T)} = O(\nu^2)$, as $\nu \to 0$, then

$$||w||_{L^2}^2 + (1+\alpha^2)||w_x||_{L^2}^2 + \alpha^2||w_{xx}||_{L^2}^2 = O(\nu^2)$$

as $\nu \to 0$.

Proof. The difference $w = u - v$ satisfies Equation 2.10 with initial value $u_0 - v_0$. Upon multiplying this equation by $w - w_{xx}$, integrating over $[0,\infty)$ and integrating by parts appropriately, there appears

$$\frac{d}{dt}\Big(||w||_{L^2}^2 + (1+\alpha^2)||w_x||_{L^2}^2 + \alpha^2||w_{xx}||_{L^2}^2\Big) + 2\nu\int_0^\infty (w_x^2 + w_{xx}^2)dx \quad (2.22)$$

$$= 2\nu\int_0^\infty (w - w_{xx})v_{xx}dx - 2\int_0^\infty (w - w_{xx})(u^p u_x - v^p v_x)dx \quad (2.23)$$

$$-w_x^2(0,t) - 2\nu(g_1 - g_2)u_x(0,t) + (g_1 - g_2)^2 \quad (2.24)$$

$$-2(g_1 - g_2)_t w_x(0,t) - 2\alpha^2(g_1 - g_2)w_{xt}(0,t). \quad (2.25)$$

The terms in line (2.23) may be estimated as in the proof of Theorem 2.3 and, due to the bounds for $||u||_{H^2}$ and $||v||_{H^2}$ (see Proposition 2.4),

$$2\nu\int_0^\infty (w - w_{xx})v_{xx}dx - 2\int_0^\infty (w - w_{xx})(u^p u_x - v^p v_x)dx$$

$$\leq C_5(t)\Big(||w||_{L^2}^2 + (1+\alpha^2)||w_x||_{L^2}^2 + \alpha^2||w_{xx}||_{L^2}^2\Big) + 2\nu^2\int_0^\infty v_{xx}^2 dx \quad (2.26)$$

for $0 \leq t \leq T$, where $C_5(t)$ is a function of t with dependence only on p, α, $||u_0||_{H^2}$, $||v_0||_{H^2}$, $||g_1||_{C^1(0,T)}$ and $||g_2||_{C^1(0,T)}$.

For $0 \leq t \leq T$, the temporal integrals in lines (2.24) and (2.25) are estimated as follows:

$$\int_0^t w_x^2(0,\tau)d\tau \leq \int_0^t \|w_x(\cdot,\tau)\|_{L^\infty}^2 d\tau$$

$$\leq \int_0^t \|w_x(\cdot,\tau)\|_{L^2}\|w_{xx}(\cdot,\tau)\|_{L^2}d\tau \leq \frac{1}{2}\int_0^t \left(\|w_x(\cdot,\tau)\|_{L^2}^2 + \|w_{xx}(\cdot,\tau)\|_{L^2}^2\right) d\tau,$$

$$-2\nu\int_0^t (g_1(\tau)-g_2(\tau))u_x(0,\tau)d\tau \leq 2\nu T\|g_1-g_2\|_{C(0,T)}\|u\|_{H^2},$$

$$\int_0^t (g_1(\tau)-g_2(\tau))^2 d\tau \leq T\|g_1-g_2\|_{C(0,T)}^2,$$

$$-2\int_0^t (g_1'-g_2')w_x(0,\tau)d\tau \leq 2\|g_1-g_2\|_{C^1(0,T)}\int_0^t \|w_x\|_{L^2}\|w_{xx}\|_{L^2}d\tau$$

$$\leq \|g_1-g_2\|_{C^1(0,T)}^2 + \frac{1}{2}\int_0^t (\|w_x\|_{L^2}^2 + \|w_{xx}\|_{L^2}^2)d\tau,$$

and

$$-2\alpha^2\int_0^t (g_1-g_2)w_{xt}(0,t)d\tau = 2\alpha^2(g_1(0)-g_2(0))w_x(0,0)$$

$$-2\alpha^2(g_1(t)-g_2(t))w_x(0,t) + 2\alpha^2\int_0^t (g_1'-g_2')w_x d\tau$$

$$\leq 2\alpha^2\|g_1-g_2\|_{C(0,T)}(\|u\|_{H^2}+\|v\|_{H^2}) + \frac{\alpha^2}{2}\int_0^t (\|w_x\|_{L^2}^2 + \|w_{xx}\|_{L^2}^2)d\tau.$$

Integrating equation (2.22) over $[0,t)$ and combining the outcome with (2.26) and the last set of estimates for the terms arising from lines (2.24) and (2.25), the inequality

$$\Gamma(w)(t) \leq C_6(t)\int_0^t \Gamma(w)(\tau)d\tau + 2\nu^2\int_0^t\int_0^\infty v_{xx}^2 dx d\tau$$
$$+(C_7\nu + C_8)\|g_1-g_2\|_{C^1(0,T)} + C_9\|g_1-g_2\|_{C^1(0,T)}^2 \tag{2.27}$$

obtains, where $\Gamma(w)(t) = \|w(\cdot,t)\|_{L^2}^2 + (1+\alpha^2)\|w_x(\cdot,t)\|_{L^2}^2 + \alpha^2\|w_{xx}(\cdot,t)\|_{L^2}^2$. The desired result follows after application of Gronwall's inequality to (2.27).

3 $\nu-$independent H^k-bounds for the GKdV-Burgers equation

This section focuses on the IVP of the GKdV-Burgers equation

$$u_t + u^p u_x - \nu u_{xx} + u_{xxx} = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{3.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{3.2}$$

where $p \geq 1$ and $\nu > 0$. The GKdV-Burgers equation and its dissipationless counterpart

$$u_t + u^p u_x + u_{xxx} = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{3.3}$$

have been the subject of numerous investigations (cf. Bona *et al.* [8], Kenig *et al.* [24], [25]). There is an adequate theory of well-posedness for both the IVP (3.1)-(3.2) and the IVP (3.3)-(3.2). The following results of Bona *et al.* [8] and Kenig *et al.* [24] serve our purpose nicely.

Proposition 3.1 *Let $\nu > 0$ and $u_0 \in H^s(\mathbb{R})$ with $s \geq 2$.*

(1) *If $p < 4$, then there is a unique global solution u of (3.1)-(3.2) such that*

$$u \in C([0,T]; H^s), \quad \text{for every } T > 0$$

and $\|u(\cdot,t)\|_{H^1}$ is uniformly bounded in t.

(2) *If $p \geq 4$, then there is a $T_0 = T_0(\|u_0\|_{H^1}) > 0$ independent of $\nu \geq 0$, and a unique solution $u \in C([0,T_0); H^s)$. If $\|u_0\|_{H^1}$ is sufficiently small, T_0 may be taken to be $+\infty$ and the solution is global.*

Moreover, for $t > 0$, $u(\cdot,t)$ is an $H^\infty(\mathbb{R})$-function of its spatial variables and consequently u is a C^∞-function in the domain $\{(x,t) : x \in \mathbb{R}, \quad 0 < t < T_0\}$ where $T_0 = \infty$ in case (1) or in case (2) if the data is small. In all the above cases, the solution u depends continuously on u_0 in the exhibited function classes.

Proposition 3.2 *Let $p \geq 1$ be an integer and s satisfy*

$$\begin{cases} s > 3/4, & \text{if } p = 1; \\ s \geq 1/4, & \text{if } p = 2; \\ s \geq 1/12, & \text{if } p = 3; \\ s \geq (p-4)/(2p), & \text{if } p \geq 4. \end{cases}$$

Then for any $u_0 \in H^s(\mathbb{R})$ *there exists* $T = T(\|u_0\|_{H^s}) > 0$ *and a unique solution* u *of the IVP (3.3),(3.2) satisfying*

$$u \in C([0,T]; H^s).$$

When $p = 1$ *and* $s \geq 1$ *or* $p < 4$ *and* $s \geq 2$ *or when* u_0 *is small enough, the solution* u *extends globally in time. In any event,* u *depends continuously on* u_0 *in the exhibited function classes.*

Remark. The situation for KdV-Burgers is different from that arising with BBM-Burgers in the following respect. At least for the pure initial-value problem, the BBM-Burgers equation is globally well-posed regardless of how large p is. It is otherwise with the (generalized) KdV-Burgers equations where the indications are that large solutions may blow up in finite time if $p \geq 4$ (see Bona *et al.* [8], [9] and Bona and Weissler [20]) even when $\nu > 0$.

However, bounds on solutions of (3.1)-(3.2) which do not depend upon ν seems not to have been derived. It is the goal of this section to provide such bounds. More precisely, it will be shown that for each positive integer k, there is a constant C_k depending only on $\|u_0\|_{H^k}$ such that the solution u to the IVP (3.1)-(3.2) obeys

$$\|u(\cdot,t)\|_{H^k} \leq C_k$$

for all $t \geq 0$ if $p = 1$ or 2 and for all t in bounded intervals $[0,T]$ if $p \geq 3$, where $T < T^*$, the existence time for the solution in question. The proof is made via an induction argument. Attention is concentrated on the cases $k = 1$ and $k = 2$. When $k \geq 3$, the argument simplifies because, with $k = 2$ in hand, it follows that u_x is bounded, independent of t in the relevant interval.

Theorem 3.3 *Let* $p \geq 1$ *and* $u_0 \in H^1(\mathbb{R})$. *Then solutions* u *to the IVP (3.1)-(3.2) for the GKdV-Burgers equation have the following properties.*

(i) *If* $p \in [1,4)$, *then there is a constant* C_1 *depending only on* p *and* $\|u_0\|_{H^1}$ *such that for any* $t \in [0,\infty)$,

$$\|u(\cdot,t)\|_{H^1} \leq C_1. \tag{3.4}$$

(ii) *If $p \geq 4$, suppose that $\epsilon = \|u_0\|_{H^1}$ is such that*

$$(1+\mu_p)\epsilon^{2+\frac{p}{2}} < 1 \quad \text{and} \quad \epsilon^2(1+\mu_p\epsilon^p) < \left(1-\frac{4}{p}\right)\left(\frac{4}{p}\right)^{\frac{4}{p-4}},$$

where $\mu_p = 2/(p+1)(p+2)$. Then for any $t \in [0,\infty)$, there is a constant C_2 depending only on p and ϵ such that

$$\|u(\cdot,t)\|_{H^1} \leq C_2.$$

Remark. These bounds are not only independent of ν, but also uniform with respect to t, regardless of the value of p.

Proof. For notational simplicity in the calculations here, references to the measures dx and dt are omitted when we write integrals. First, recall that

$$\int_{-\infty}^{\infty} u^2 + 2\nu\int_0^t\int_{-\infty}^{\infty} u_x^2 = \int_{-\infty}^{\infty} u_0^2. \tag{3.5}$$

Multiplying (3.1) by $u_{xx} + u^{p+1}/(p+1)$ and integrating on $\mathbb{R}\times[0,t]$ leads to

$$\int_{-\infty}^{\infty} u_x^2 - \frac{2}{(p+1)(p+2)}\int_{-\infty}^{\infty} u^{p+2} + 2\nu\int_0^t\int_{-\infty}^{\infty} u_{xx}^2$$

$$= \int_{-\infty}^{\infty} u_{0x}^2 - \frac{2}{(p+1)(p+2)}\int_{-\infty}^{\infty} u_0^{p+2} - \frac{2\nu}{p+1}\int_0^t\int_{-\infty}^{\infty} u_{xx}u^{p+1}. \tag{3.6}$$

This formula constitutes the base for our further estimates. Clearly, we have

$$\int_{-\infty}^{\infty} u^{p+2}(x,t) \leq \|u(\cdot,t)\|_{L^\infty}^p\|u(\cdot,t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^{2+\frac{p}{2}}\|u_x(\cdot,t)\|_{L^2}^{\frac{p}{2}}. \tag{3.7}$$

To simplify the presentation, define

$$\epsilon = \|u_0\|_{H^1} \quad \text{and} \quad \sigma(t) = \sup_{0\leq s\leq t}\|u_x(\cdot,s)\|_{L^2}.$$

Integrating by parts and using (3.5) gives

$$-\frac{2\nu}{p+1}\int_0^t\int_{-\infty}^{\infty} u_{xx}u^{p+1} = 2\nu\int_0^t\int_{-\infty}^{\infty} u^p u_x^2$$

$$\leq \sup_{0\leq s\leq t} \|u(\cdot,s)\|_{L^\infty}^{p} \int_0^t \int_{-\infty}^{\infty} u_x^2$$

$$\leq \epsilon^2 \sup_{0\leq s\leq t} \|u(\cdot,s)\|_{L^2}^{p/2} \|u_x(\cdot,s)\|_{L^2}^{p/2} \leq \epsilon^{2+\frac{p}{2}} \sup_{0\leq s\leq t} \|u_x(\cdot,s)\|_{L^2}^{p/2}. \tag{3.8}$$

Putting (3.6), (3.7) and (3.8) together yields

$$\sigma^2(t) - C_3\sigma^{\frac{p}{2}}(t) + 2\nu \int_0^t \int_{-\infty}^{\infty} u_{xx}^2 \leq C_4 \tag{3.9}$$

for all $t \geq 0$, where

$$C_3 = \left(1 + \frac{2}{(p+1)(p+2)}\right) \epsilon^{2+\frac{p}{2}} \quad \text{and} \quad C_4 = \epsilon^2 + \frac{2}{(p+1)(p+2)}\epsilon^{2+p}$$

depend only on p and ϵ. Formula (3.9) suggests a natural trichotomy.

(i) If $p \in [1,4)$, then $\frac{p}{2} < 2$ and we can apply Lemma 3.4 below to inequality (3.9), whereafter the desired result (3.4) follows.

(ii) When $p = 4$, we insist that ϵ is such that

$$\left(1 + \frac{2}{(p+1)(p+2)}\right) \epsilon^{2+\frac{p}{2}} < 1, \quad \text{i.e.,} \quad C_3 < 1,$$

and then (3.8) implies $\sigma^2(t) \leq (1 - C_3)^{-1}C_4$.

(iii) For $p > 4$, if ϵ is small enough that

$$C_4 < \left(1 - \frac{4}{p}\right) \left(\frac{4}{pC_3}\right)^{\frac{4}{p-4}},$$

and since $\sigma(t)$ is a continuous function of t, it follows from (3.8) that

$$\sigma(t) \leq \gamma(\epsilon), \quad \text{for all } t \geq 0,$$

where $\gamma(\epsilon)$ is the smallest positive root of

$$\sigma^2(t) - C_3\sigma^{\frac{p}{2}}(t) = C_4.$$

The proof of Theorem 3.3 is thereby completed.

Lemma 3.4 *Let P, Q and $\beta < 2$ be positive numbers. If $Y \geq 0$ satisfies*

$$Y^2 - PY^\beta \leq Q,$$

then Y is bounded by

$$Y \leq \max\left\{(2P)^{\frac{1}{2-\beta}}, \sqrt{2Q}\right\}.$$

A simple proof of this lemma is provided in [31].

We now proceed to the case $k = 2$. A crucial step in establishing the uniform bound in this case is the derivation of a particular integral identity valid for smooth solutions of (3.1)-(3.2). This result is the subject of the next proposition.

Proposition 3.5 *Let $\nu > 0$ (respectively, $\nu = 0$) and $u_0 \in H^s$ with $s \geq 2$. Then the associated solution u of the GKdV-Burgers (respectively, GKdV) equation with initial data u_0 satisfies the formula*

$$\begin{aligned}\int_{-\infty}^{\infty}\left[u_{xx}^2(x,t) - \frac{5}{3}u_x^2u^p(x,t)\right]dx + 2\nu\int_0^t\int_{-\infty}^{\infty}u_{xxx}^2\,dxds \\ = \int_{-\infty}^{\infty}\left[u_{xx}^2(x,0) - \frac{5}{3}u_x^2u^p(x,0)\right]dx \\ + \int_0^t\int_{-\infty}^{\infty}\left[\frac{1}{12}p(p-1)(p-2)u_x^5u^{p-3} + \frac{5}{3}pu_x^3u^{2p-1}\right]dxds \\ + \frac{5}{3}\nu\int_0^t\int_{-\infty}^{\infty}\left[2u_{xx}^2u^p - \frac{1}{3}p(p-1)u_x^4u^{p-2}\right]dxds,\end{aligned} \tag{3.10}$$

for all $t \geq 0$ for which it exists.

Proof. We write $\int\int$ for $\int_0^t\int_{-\infty}^{\infty}$ and omit dx and ds for simplicity of reading and writing. The proof of this proposition involves two steps. The first step is to derive the identity

$$\int_{-\infty}^{\infty} u_{xx}^2(x,t) + 2\nu\int\int u_{xxx}^2(x,s)dxds + 5p\int\int u_xu_{xx}^2u^{p-1}$$

$$= \int_{-\infty}^{\infty} u_{xx}^2(x,0) + \frac{1}{2}p(p-1)(p-2)\int\int u_x^5 u^{p-3} \tag{3.11}$$

and the second is to establish that

$$\int_{-\infty}^{\infty} \left[u_{xx}^2(x,t) - u_x^2 u^p(x,t)\right] + 2\nu\int\int u_{xxx}^2 + 2p\int\int u_x u_{xx}^2 u^{p-1}$$

$$= \int_{-\infty}^{\infty} \left[u_{xx}^2(x,0) - u_x^2 u^p(x,0)\right] + \int\int \left[\frac{1}{4}p(p-1)(p-2)u_x^5 u^{p-3} + p u_x^3 u^{2p-1}\right]$$

$$+\nu\int\int\left[2u_{xx}^2 u^p - \frac{1}{3}p(p-1)u_x^4 u^{p-2}\right], \tag{3.12}$$

provided u is the solution of (3.1)-(3.2) with initial data u_0.

The purpose of deriving these two identities is to use them jointly, but at the same time eliminate the troublesome term

$$\int\int u_x u_{xx}^2 u^{p-1},$$

after which (3.10) follows easily.

For (3.11), differentiate the GKdV-Burgers equation with respect to x, multiply the result by u_{xxx} and integrate over $(-\infty,\infty)\times[0,t]$, so coming to

$$\int_{-\infty}^{\infty} u_{xx}^2(x,t) + 2\nu\int\int u_{xxx}^2 = \int_{-\infty}^{\infty} u_{xx}^2(x,0) + 2\int\int u_{xxx}(u^p u_x)_x. \tag{3.13}$$

The last term may be treated as follows:

$$\int\int u_{xxx}(u^p u_x)_x = \int\int u_{xxx}(u^p u_{xx} + p u^{p-1}u_x^2)$$

$$= \frac{1}{2}\int\int (u_{xx}^2)_x u^p - p\int\int u_{xx}(u^{p-1}u_x^2)_x$$

$$= -\frac{5p}{2}\int\int u_x u_{xx}^2 u^{p-1} - p(p-1)\int\int u_x^3 u_{xx} u^{p-2}$$

$$= -\frac{5p}{2}\int\int u_x u_{xx}^2 u^{p-1} - \frac{1}{4}p(p-1)\int\int (u_x^4)_x u^{p-2}$$

$$= -\frac{5p}{2}\int\int u_x u_{xx}^2 u^{p-1} + \frac{1}{4}p(p-1)(p-2)\int\int u_x^5 u^{p-3}. \qquad (3.14)$$

Equation (3.11) follows from (3.13) and (3.14).

For (3.12), multiply the GKdV-Burgers equation by $u_{xxxx} + (u^p u_x)_x$ and integrate over $(-\infty, \infty) \times [0, t]$. After suitable integrations by parts, we obtain

$$\int_{-\infty}^{\infty} u_{xx}^2(x,t) + 2\nu\int\int u_{xxx}^2 + \int\int (u^p u_x)_x u_t$$

$$= \int_{-\infty}^{\infty} u_{xx}^2(x,0) + \nu\int\int u_{xx}^2 u^p + p\nu\int\int u_{xx} u_x^2 u^{p-1}. \qquad (3.15)$$

In (3.15), the two terms

$$\int\int u_{xx} u_x^2 u^{p-1} \quad \text{and} \quad \int\int (u^p u_x)_x u_t$$

need further elucidation. First of all, note that

$$\int\int u_{xx} u_x^2 u^{p-1} = \frac{1}{2}\int\int (u_x^2)_x u_x u^{p-1}$$

$$= -\frac{1}{2}\int\int u_{xx} u_x^2 u^{p-1} - \frac{p-1}{2}\int\int u_x^4 u^{p-2},$$

and therefore

$$\int\int u_{xx} u_x^2 u^{p-1} = -\frac{p-1}{3}\int\int u_x^4 u^{p-2}. \qquad (3.16)$$

On the other hand, it is clear that

$$\int\int (u^p u_x)_x u_t = -\frac{1}{2}\int\int (u_x^2)_t u^p = -\frac{1}{2}\int\int (u_x^2 u^p)_t + \frac{p}{2}\int\int u_x^2 u^{p-1} u_t.$$

Use the evolution equation itself to represent u_t, so obtaining

$$\int\int (u^p u_x)_x u_t = -\frac{1}{2}\int_{-\infty}^{\infty} [u_x^2 u^p(x,t) - u_x^2 u^p(x,0)]$$

$$-\frac{p}{2}\int\int u_x^3 u^{2p-1} + \frac{p\nu}{2}\int\int u_x^2 u_{xx} u^{p-1} - \frac{p}{2}\int\int u_x^2 u^{p-1} u_{xxx}, \qquad (3.17)$$

while the last term in (3.17) can be further expressed as

$$-\frac{p}{2}\int\int u_x^2 u^{p-1} u_{xxx} = p\int\int u_x u_{xx}^2 u^{p-1} + \frac{1}{2}p(p-1)\int\int u_x^3 u_{xx} u^{p-2}$$

$$= p\int\int u_x u_{xx}^2 u^{p-1} + \frac{1}{8}p(p-1)(p-2)\int\int u_x^5 u^{p-3}.$$

In summary, there obtains

$$\int\int (u^p u_x)_x u_t = -\frac{1}{2}\int_{-\infty}^{\infty}\left[u_x^2 u^p(x,t) - u_x^2 u^p(x,0)\right]$$

$$-\frac{p}{2}\int\int u_x^3 u^{2p-1} + \frac{p\nu}{2}\int\int u_x^2 u_{xx} u^{p-1}$$

$$+p\int\int u_x u_{xx}^2 u^{p-1} + \frac{p(p-1)(p-2)}{8}\int\int u_x^5 u^{p-3}. \tag{3.18}$$

Collect the estimates (3.15),(3.16), (3.18) and the desired identity (3.12) follows.

This completes the proof of Proposition 3.5.

The $\nu-$independent bounds in H^2 are now stated and proved.

Theorem 3.6 *Let $p \geq 1$ and $\nu > 0$ (respectively, $\nu = 0$). Assume that the initial data $u_0 \in H^2$ and for $p \geq 4$, that $\|u_0\|_{H^1}$ is sufficiently small. Then for all $t \geq 0$, the solution u of the GKdV-Burgers (respectively, GKdV) equation with data u_0 obeys*

$$\|u(\cdot,t)\|_{H^2} + \nu\int_0^t\int_{-\infty}^{\infty} u_{xxx}^2(x,s)dxds \leq C_5 e^{C_6 t}, \tag{3.19}$$

for some constants C_5 and C_6 depending only on p, α and $\|u_0\|_{H^2}$. For $p = 1$ or 2, we may take $C_6 = 0$ and thus the bounds are uniform in both t and ν.

Proof. The argument is first made for general values of p. Recall the already established uniform bounds

$$\nu\int_0^t\int_{-\infty}^{\infty} u_x^2(x,s)dxds + \|u(\cdot,t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2,$$

(3.19) with $C_6 = 0$. The case $p = 1$ is worked out here, but the case $p = 2$ is entirely similar. The crux of the matter is to use rather than (3.10), the more specific identity

$$\int_{-\infty}^{\infty} \left[\frac{9}{5}u_{xx}^2(x,t) - 3u(x,t)u_x^2(x,t) + \frac{1}{4}u^4(x,t)\right] dx$$

$$+\nu \int_0^t \int_{-\infty}^{\infty} \left[\frac{18}{5}u_{xxx}^2(x,s) + 6u(x,s)u_{xx}^2(x,s) + 3u^2(x,s)u_x^2(x,s)\right] dxds$$

$$= \int_{-\infty}^{\infty} \left[\frac{9}{5}u_{xx}^2(x,0) - 3u(x,0)u_x^2(x,0) + \frac{1}{4}u^4(x,0)\right] dx,$$

which holds for H^2−solutions of the initial-value problem (3.1)-(3.2). This relation is obtained by multiplying (3.1) by $u^3 + 3u_x^2 - 6uu_{xx} + \frac{18}{5}u_{xxxx}$, integrating the result over $\mathbb{R} \times [0,t]$ and integrating by parts suitably. This identity implies that

$$\frac{9}{5}\int_{-\infty}^{\infty} u_{xx}^2(x,t)dx + \nu\frac{18}{5}\int_0^t \int_{-\infty}^{\infty} u_{xxx}^2(x,s)dxds$$

$$\leq 6\nu\|u(\cdot,t)\|_{L^\infty} \int_0^t \int_{-\infty}^{\infty} u_{xx}^2(x,s)dxds + 3\|u(\cdot,t)\|_{L^\infty}\|u_x(\cdot,t)\|_{L^2}^2 + C_{11},$$

where C_{11} depends only on the H^2−norm of the initial data. Because of the prior results in (3.20), it follows that

$$\int_{-\infty}^{\infty} u_{xx}^2(x,t)dx + \nu\int_0^t \int_{-\infty}^{\infty} u_{xxx}^2(x,s)dxds \leq 6C_7^3 + 3C_7^3 + C_{11} = 9C_7^3 + C_{11}$$

is bounded, independent of t and ν, solely in terms of p, α and $\|u_0\|_{H^2}$ only.

Attention is now turned to the inductive step which corresponds to the cases $k \geq 3$.

Theorem 3.7 *Let $p \geq 1$ and $\nu > 0$ (respectively, $\nu = 0$). Assume that the initial data $u_0 \in H^k$ with $k \geq 3$ and if $p \geq 4$, that $\|u_0\|_{H^1}$ is sufficiently small. Then the solution u of the GKdV-Burgers (respectively, GKdV) equation with* *initial data u_0 is uniformly bounded in H^k. That is, for any $T > 0$, there exists a constant C_k depending only on p, α, T and $\|u_0\|_{H^k}$ for which*

$$\|u(\cdot,t)\|_{H^k} + \nu \int_0^t \int_{-\infty}^{\infty} (\partial_x^{k+1} u)^2 dxds \le C_k \tag{3.22}$$

for all $t \in [0,T]$. If $p = 1$ or 2, C_k can be taken to be independent of T.

Proof. The argument for $k = 3$ is representative. Multiply the GKdV-Burgers equation (3.1) by u_{xxxxxx} and integrate over $(-\infty,\infty) \times [0,t]$; after integrations by parts, we have

$$\int_{-\infty}^{\infty} u_{xxx}^2(x,t) + 2\nu \int\int u_{xxxx}^2 = \int_{-\infty}^{\infty} u_{xxx}^2(x,0) + 2\int\int u^p u_x u_{xxxxxx}.$$

Only the last term needs attention. Integrate by parts further to obtain

$$\int\int u^p u_x u_{xxxxxx} = -\int\int (u^p u_x)_{xxx} u_{xxx}$$

$$= -p(p-1)(p-2)\int\int u^{p-3} u_x^4 u_{xxx} + 7p(p-1)\int\int u^{p-2} u_x^2 u_{xx} u_{xxx}$$

$$+4p\int\int u^{p-1} u_{xx}^2 u_{xxx} + \frac{9}{2}p\int\int u^{p-1} u_x u_{xxx}^2.$$

The last two identities enable us to argue successfully for the bound (3.22) as in the proof of Theorem 3.6. The argument for arbitrary k is similar.

As in Theorem 3.6, for the cases $p = 1$ and 2, a more elaborate argument can be mounted which leads to bounds that are independent of both ν and t. The argument relies upon the hierarchy of conservation laws that obtain in case $\nu = 0$. Briefly, for each $k = 1, 2, \cdots$, a sufficiently smooth solution of the KdV-equation ($p = 1$), or the mKdV-equation ($p = 2$) satisfies a sequence of identities of the form

$$\frac{\partial}{\partial t} I_k(u) = \frac{\partial}{\partial x} F_k(u), \tag{3.23}$$

where, I_k and F_k are polynomials in u and the partial derivatives $\partial_x^j u$, which we write as $u_{(j)}$ for convenience, $j = 1, 2, \cdots$. In more detail, for KdV, I_k

depends on u, $\partial_x u$, $\cdots$, $\partial_x^k u$ and F_k depends on u, $\partial_x u$, $\cdots$, $\partial_x^{k+2} u$. Moreover, suitably normalized, I_k has the form

$$I_k(u) = 2u_{(k)}^2 + auu_{(k-1)}^2 + \cdots, \tag{3.24}$$

which is a finite sum of terms of index $k+2$ where the *index* of a monomial

$$u_{(\beta_1)}^{\alpha_1} \cdots u_{(\beta_r)}^{\alpha_r} \tag{3.25}$$

is

$$\sum_{i=1}^{r} \alpha_i + \frac{1}{2} \sum_{i=1}^{r} \beta_i. \tag{3.26}$$

The fluxes F_k have a similar form except that their general term, which is also of the form (3.25), has index $k+3$. The formulae in (3.23) are derived by multiplying the KdV-equation by a factor $A_k(u)$, where $A_k(u)$ is a polynomial in u, u_x, $\cdots$, $u_{(2k)}$ composed of monomials of index $k+1$. In general, $A_k(u)$ may be normalized to have the form

$$A_k(u) = (-1)^k u_{(2k)} + \cdots + au^{k+1}. \tag{3.27}$$

These facts follow directly from the original analysis of the KdV- and mKdV-conservation laws given by Miura *et al.* [26].

When $\nu > 0$, the formula

$$A_k(u)(u_t + uu_x + u_{xxx} - \nu u_{xx}) = 0$$

may be put into the form

$$\partial_t I_k(u) - \nu u_{xx} A_k(u) = \partial_x F_k(u). \tag{3.28}$$

The second term on the left-hand side of (3.28) may be written as

$$\begin{aligned} -\nu u_{xx} A_k(u) &= -\nu u_{xx}\Big((-1)^k 2u_{(2k)} + \cdots + au^{k+1}\Big) \\ &= 2\nu\Big[u_{(k+1)}^2 + Q_k(u)\Big] + \nu \partial_x G_k(u), \end{aligned} \tag{3.29}$$

where Q_k is a linear combination of the other monomials of index $k+3$. Integration of (3.28) with respect to x over $\mathbb{R}$, and after imposing zero boundary conditions on u, u_x, $\cdots$, at $\pm\infty$, leads to the relation

$$\frac{d}{dt}\int_{-\infty}^{\infty} I_k(u)dx + \nu\int_{-\infty}^{\infty} u_{(k+1)}^2 dx = \nu\int_{-\infty}^{\infty} Q_k(u)dx.$$

Integration with respect to t over the interval $[0, t_0]$ then yields

$$\int_{-\infty}^{\infty} u_{(k)}^2(x,t_0)dx + 2\nu\int_0^{t_0}\int_{-\infty}^{\infty} u_{(k+1)}^2(x,t)dxdt$$

$$= \int_{-\infty}^{\infty} I_k(g)dx - \int_{-\infty}^{\infty} \tilde{I}_k(u(x,t_0))dx + \nu\int_0^{t_0}\int_{-\infty}^{\infty} Q_k(u(x,t))dxdt, \quad (3.30)$$

where

$$\tilde{I}_k(u(\cdot,t)) = I_k(u(\cdot,t)) - \int_{-\infty}^{\infty} u_{(k)}^2(x,t)dx.$$

The stage is now set for an induction on k. Assume that (3.22) is valid for all $k \le m$ and that C_k does not depend on ν and t. We then use (3.30) to show that, provided $g \in H^{m+1}$, then (3.22) is valid for $k = m+1$. As it is already established that (3.22) is true for $k \le 2$, this will finish the proof. It suffices to bound the right-hand side of (3.30) for $k = m+1$, independent of ν and t. By the induction hypothesis, there is a constant C_m depending only on $\|g\|_{H^m}$ such that

$$\|u(\cdot,t)\|_{H^m} \le C_m \quad \text{and} \quad \nu\int_0^t\int_{-\infty}^{\infty} u_{(m+1)}^2(x,s)dxds \le C_m \qquad (3.31)$$

for all $\nu, t \ge 0$. It is easy to see that if $g \in H^{m+1}$, then $\int_{-\infty}^{\infty} I_{m+1}(g(x))dx$ is finite – a fixed constant independent of t and ν. Moreover, it is straightforward to determine that all the terms in $\int_{-\infty}^{\infty} I_{m+1}(u(x,t))dx$ except the top-order term $\int_{-\infty}^{\infty} u_{(m+1)}^2(x,t)dx$ are bounded by a suitable power of the constant C_m in (3.30) (c.f. Bona-Smith [17], §4). Thus $\int_{-\infty}^{\infty} \tilde{I}_{m+1}(u(x,t))dx$ is bounded independently of t and ν. A similar conclusion may be drawn about $\int_0^t\int_{-\infty}^{\infty} Q_k(u(x,t))dxdt$. Indeed, the only terms that might be troublesome are

$$\nu\int_0^t\int_{-\infty}^{\infty} uu_{(m+1)}^2 dxdt \quad \text{and} \quad \nu\int_0^t\int_{-\infty}^{\infty} u_x u_{(m)}^2 dxdt. \qquad (3.32)$$

Neither of these gives trouble since $||u(\cdot,t)||_{H^2}$ is already known to be bounded, independently of ν and t on account of Theorem 3.6, and so $||u(\cdot,t)||_{L^\infty}$ and $||u_x(\cdot,t)||_{L^\infty}$ are bounded, independently of ν and t. Thus the terms in (3.32) are bounded by $C_1C_m^2$ and $C_2C_{m-1}^2$, respectively. Thus, for $k = m+1$, the right-hand side of (3.30) is seen to be bounded, independently of $t \geq 0$ and $\nu \geq 0$. The inductive step is completed and the derived result follows.

4 Zero-dissipation limit for the GKdV-Burgers equation

The uniform bounds derived in Section 3 lead directly to the zero-dissipation limit results for the GKdV-Burgers equation. It is shown in this section that for each nonnegative integer k, the solution of the IVP (3.1)-(3.2) converges in H^k to the solution of the IVP (3.3)-(3.2) with the sharp rate of order ν. Our approach is again inductive and the focus is on the cases $k = 0$ and $k = 1$, which correspond to the results in L^2 and H^1.

The first result is the zero-dissipation limit in case $k = 0$.

Theorem 4.1 *Let $p \geq 1$ be a positive integer. Assume that v_0 and $\{u_0^\nu\}_{\nu>0}$ lie in $H^2(\mathbb{R})$ and consider the difference*

$$w(x,t) = u(x,t) - v(x,t)$$

between a solution $u = u_\nu$ to the IVP (3.1)-(3.2) with initial data u_0^ν and a solution v to the IVP (3.3)-(3.2) with initial data v_0. Let T_0 be the maximal existence time for v. By Proposition 3.1, the solutions $u = u_\nu$ all exist at least on the time interval $[0, T_0)$. Then, for any T with $0 < T < T_0$ and $t \in [0,T]$,

$$||u(\cdot,t) - v(\cdot,t)||_{L^2}^2 \leq e^{\int_0^t C_1(\tau)d\tau}||u_0^\nu - v_0||_{L^2}^2 + \nu^2 \int_0^t e^{\int_s^t C_1(\tau)d\tau} \int_{-\infty}^{\infty} v_{xx}^2(x,s)dxds \qquad (4.1)$$

where C_1 is a function of t with dependence on p, $\|u_0\|_{H^1}$ and $\|v_0\|_{H^2}$ only. In particular, if $\|u_0^\nu - v_0\|_{L^2} = O(\nu)$ as $\nu \to 0$, then

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^2} = O(\nu)$$

as $\nu \to 0$, uniformly for $t \in [0,T]$.

Proof. The difference w solves the equation

$$w_t + (u^p - v^p)v_x + u^p w_x - \nu w_{xx} - \nu v_{xx} + w_{xxx} = 0. \tag{4.2}$$

Multiplying (4.2) by w and integrating over $(-\infty,\infty)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2 + \nu\int_{-\infty}^{\infty} w_x^2 = I + II + III \tag{4.3}$$

where the three terms on the right-hand side may be estimated as follows:

$$I = \nu\int_{-\infty}^{\infty} w v_{xx} \le \frac{1}{2}\int_{-\infty}^{\infty} w^2 + \frac{\nu^2}{2}\int_{-\infty}^{\infty} v_{xx}^2, \tag{4.4}$$

$$\begin{aligned} II = -\int_{-\infty}^{\infty}(u^p - v^p)v_x w &= -\sum_{j=0}^{p-1}\int_{-\infty}^{\infty}(u^{p-1-j}v^j v_x)w^2 \\ &\le \sum_{j=0}^{p-1}\|u^{p-1-j}v^j v_x(\cdot,t)\|_{L^\infty}\int_{-\infty}^{\infty} w^2, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} III = -\int_{-\infty}^{\infty}(w+v)^p w w_x &= -\sum_{j=0}^{p}\binom{p}{j}\int_{-\infty}^{\infty}(w^{j+1}w_x)v^{p-j} \\ &= \sum_{j=0}^{p-1}\binom{p}{j}\frac{p-j}{j+2}\int_{-\infty}^{\infty} v^{p-j-1}v_x w^{j+2} \\ &\le \sum_{j=0}^{p-1}\binom{p}{j}\frac{p-j}{j+2}\|v^{p-j-1}v_x w^j(\cdot,t)\|_{L^\infty}\int_{-\infty}^{\infty} w^2. \end{aligned} \tag{4.6}$$

Noticing that

$$\|v_x(\cdot,t)\|_{L^\infty} \le \|v_x(\cdot,t)\|_{L^2}^{1/2} \|v_{xx}(\cdot,t)\|_{L^2}^{1/2}$$

and using the results in Theorem 3.3 and Proposition 3.2, we obtain from (4.5) and (4.6) that

$$II \le C_2(t) \int_{-\infty}^{\infty} w^2, \qquad III \le C_3(t) \int_{-\infty}^{\infty} w^2, \tag{4.7}$$

for some functions C_2 and C_3 which depend on p, $\|u_0\|_{H^1}$ and $\|v_0\|_{H^2}$ only. Combining (4.3), (4.4) and (4.7) gives

$$\frac{d}{dt}\int_{-\infty}^{\infty} w^2 + 2\nu \int_{-\infty}^{\infty} w_x^2 \le \nu^2 \int_{-\infty}^{\infty} v_{xx}^2 + C_4(t) \int_{-\infty}^{\infty} w^2, \tag{4.8}$$

where $C_4(t)$ is a function of t which depends only on p, $\|u_0\|_{H^1}$ and $\|v_0\|_{H^2}$. The desired result (4.1) follows from (4.8).

Remark. It seems likely that the result of Theorem 4.1 actually holds for any p of the form m/n where m and n have no common prime factors and n is odd, provided we interpret $y^{1/n}$ as that branch of the n-th root which is positive for $y > 0$.

A familiar bootstrap argument allows us to extend the convergence results to higher values of k. The $\nu-$independent H^k-bounds play an important role in obtaining this general result.

Theorem 4.2 *Let $p \ge 1$ be a positive integer. Assume that $\{u_0^\nu\}_{\nu>0}$ and v_0 lie in H^s with $s \ge 2$ and suppose that there is a constant C_5 such that $\|u_0^\nu - v_0\|_{H^s} \le C_5\nu$ as $\nu \to 0$. Then for any integer k with $1 \le k \le s-2$, the difference $u - v$ between the solution $u = u_\nu$ of the IVP (3.1)-(3.2) with initial data u_0^ν and the solution v of the IVP (3.3)-(3.2) with initial data v_0 has the property*

$$\|u(\cdot,t) - v(\cdot,t)\|_{H^k} \le \tilde{C}_5\nu \tag{4.9}$$

uniformly for $0 \le t \le T$, where $\tilde{C}_5$ is a constant depending only on C_5, p, T, $\|u_0\|_{H^s}$ and $\|v_0\|_{H^s}$ and $T > 0$ is any fixed time less than the existence time T_0 for v.

Proof. The proof of (4.9) is sketched for $k = 1$. The proof of (4.9) for $k \geq 2$ is similar. Differentiate the equation (4.2) for the difference $w = u - v$ with respect to x, multiply by w_x and integrate over $(-\infty, \infty) \times [0, t]$ to obtain

$$\int_{-\infty}^{\infty} w_x^2(x,t) + 2\nu \int\int w_{xx}^2 = \int_{-\infty}^{\infty} w_x^2(x,0) + 2\nu \int\int v_{xxx} w_x$$

$$- \int\int (u^p w_x)_x w_x - \int\int [(u^p - v^p)v_x]_x w_x.$$

Further integrations by parts show that

$$\int\int (u^p w_x)_x w_x = - \int\int u^p w_x w_{xx} = \frac{p}{2} \int\int u_x u^{p-1} w_x^2$$

$$\int\int [(u^p - v^p)v_x]_x w_x = p \int\int (u^{p-1}u_x - v^{p-1}v_x) v_x w_x + \int\int (u^p - v^p) v_{xx} w_x$$

$$= p \int\int (u^{p-1} - v^{p-1}) v_x^2 w_x + p \int\int u^{p-1} v_x w_x^2 + \int\int (u^p - v^p) v_{xx} w_x.$$

It is known from Section 3 that the H^2-bound on u is independent of ν. This in turn implies ν-independent L^∞-bounds for u and u_x. Thus, the terms above may be bounded as follows:

$$2\nu \int_{-\infty}^{\infty} v_{xxx} w_x \leq \nu^2 \int_{-\infty}^{\infty} v_{xxx}^2 + \int_{-\infty}^{\infty} w_x^2$$

$$\int\int (u^p w_x)_x w_x \leq \frac{p}{2} \|(u_x u^{p-1})(\cdot, t)\|_{L^\infty} \int\int w_x^2$$

$$\int\int (u^{p-1} - v^{p-1}) v_x^2 w_x \leq \sum_{k=0}^{p-1} \|(u^{p-1-k} v^k v_x^2)(\cdot, t)\|_{L^\infty} \int\int |w||w_x|$$

$$\leq \frac{1}{2} \sum_{k=0}^{p-1} \|(u^{p-1-k} v^k v_x^2)(\cdot, t)\|_{L^\infty} \int\int (w^2 + w_x^2)$$

$$\int\int u^{p-1} v_x w_x^2 \leq \|(u^{p-1} v_x)(\cdot, t)\|_{L^\infty} \int\int w_x^2$$

$$\int\int (u^p - v^p) v_{xx} w_x \leq \sum_{j=0}^{p-1} \|(u^{p-1-j} v^j v_{xx})(\cdot, t)\|_{L^\infty} \int\int (w^2 + w_x^2).$$

Combining these estimates, applying Gronwall's inequality and using the zero-dissipation limit result in L^2 establishes (4.9). This completes the proof of the theorem.

5 Zero-dissipation limit for the equation with more general forms of nonlinearity, dispersion and dissipation

This section is concerned with the more general IVP

$$\partial_t u + (P(u))_x + \nu M u - (Lu)_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{5.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{5.2}$$

where $\nu > 0$, $P : \mathbb{R} \to \mathbb{R}$ is of the form

$$P(u) = \sum_{k=1}^{p+1} a_k u^k \qquad \text{for some constants } a_k,\ 1 \leq k \leq p+1, \tag{5.3}$$

and L and M are Fourier multiplier operators defined in terms of the Fourier transform by

$$\widehat{Lu}(\xi) = \alpha(\xi)\widehat{u}(\xi), \qquad \widehat{Mu}(\xi) = \beta(\xi)\widehat{u}(\xi), \tag{5.4}$$

respectively. The symbols α and β are even, positive and are presumed to satisfy the growth conditions

$$C_1|\xi|^\lambda \leq \alpha(\xi) \leq C_2|\xi|^\mu, \tag{5.5}$$

$$C_3|\xi|^\gamma \leq \beta(\xi) \leq C_4|\xi|^\sigma, \tag{5.6}$$

for some numbers $C_i > 0$, $1 \leq i \leq 4$, where $0 < \lambda \leq \mu$ and $0 < \gamma \leq \sigma$.

The goal of this section is to establish zero-dissipation limits (the limit as $\nu \to 0$) of solutions to the IVP (5.1)-(5.2). Guided by what has gone before, the approach is to compare the solution to the IVP (5.1)-(5.2) with the solution to the IVP for the corresponding equation without dissipative effects, namely

$$\partial_t v + (P(v))_x - (Lv)_x = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{5.7}$$

$$v(x,0) = v_0(x), \qquad x \in \mathbb{R}. \tag{5.8}$$

The well-posedness of the two initial value problems (5.1)-(5.2) and (5.7)-(5.8) was developed by Saut [30] (see also Abdelouhab *et al.* [1]) and the following propositions will serve our purpose. In what follows $D(L^{1/2}) \subset L^2(\mathbb{R})$ denotes the completion of $C_0^\infty(\mathbb{R})$ in the norm induced by the inner product $[\cdot,\cdot]$ defined by

$$[u,v] = \int_{\mathbb{R}} \widehat{u}(\xi)\bar{\widehat{v}}(\xi)d\xi + \int_{\mathbb{R}} \alpha(\xi)\widehat{u}(\xi)\bar{\widehat{v}}(\xi)d\xi.$$

Thus $D(L^{1/2})$ is a Hilbert space and it follows from (5.5) that

$$H^{\mu/2} \subset D(L^{1/2}) \subset H^{\lambda/2}. \tag{5.9}$$

Proposition 5.1 *Assume that the symbols α of L and β of M are positive, even, satisfy (5.5) and (5.6), respectively, and that P is of the form (5.3) with*

$$\lambda + \gamma > \sigma \quad \text{and} \quad p < 2(\lambda + \gamma - \sigma).$$

If $u_0 \in D(L^{1/2})$, then for any $T > 0$, there is a solution to the IVP (5.1)-(5.2) such that

$$u \in C([0,T]; D(L^{1/2})) \cap L^2([0,T]; D((LM)^{1/2})).$$

Proposition 5.2 *Assume that the symbol α of the operator L is positive, even and satisfies (5.5), and that P is of the form (5.3) with $1 \le p < 2\lambda$. If $v_0 \in D(L^{1/2})$, then for any $T > 0$, there exists a $v \in C([0,T]; D(L^{1/2}))$ solving the IVP (5.7)-(5.8). Moreover, v is unique and $v_x \in L^\infty(\mathbb{R} \times (0,T))$ if $\lambda > 3$.*

In addition, if $v_0 \in H^s(\mathbb{R})$ with $s > 3/2$, then there is $T^* = T^*(\|v_0\|_{H^s})$ such that $v \in L^\infty([0,T^*); H^s)$. Moreover, the correspondence between initial data and the associated solution is an analytic mapping between the displayed function classes.

Remark. In Saut's original paper, solutions were obtained as weak* limits of solutions of the evolution equation with a strong parabolic regularization. Consequently, the function class obtained was only $L^\infty([0,T]; D(L^{1/2}))$. In [1], a limiting procedure was developed that featured strong convergence, and hence solutions were inferred to lie in $C([0,T]; D(L^{1/2}))$, and, moreover, they were inferred to depend continuously on the initial data. Using the techniques of Zhang (see [33]), the analyticity of the solution map may be adduced.

To establish zero-dissipation limit results, we need ν-independent bounds for the solutions to the IVP (5.1)-(5.2) . These are obtained in Theorem 5.3 below, following the developments of Saut [30].

Theorem 5.3 *Assume that the symbols α of L and β of M are positive, even and satisfy (5.5) and (5.6), respectively, and that P is of the form (5.3) with*

$$\lambda + \gamma > \sigma \quad \textit{and} \quad p < 2(\lambda + \gamma - \sigma).$$

If $u_0 \in D(L^{1/2})$, then a solution u to the IVP (5.1)-(5.2) with initial data u_0 is bounded as follows. For any $t \geq 0$,

$$\int_{\mathbb{R}} \alpha(\xi)|\widehat{u}(\xi,t)|^2 d\xi + \nu \int_0^t \int_{\mathbb{R}} \alpha(\xi)\beta(\xi)|\widehat{u}(\xi,\tau)|^2 d\xi d\tau \leq C_5 + C_6 \nu t$$

where C_5 and C_6 are constants depending only on $||u_0||_{D(L^{1/2})}$.

Remark. As an important consequence of this theorem and the Gagliardo-Nirenberg inequality, for $\lambda > 1$, there is inferred to exist a constant C_7 for which

$$||u||_{L^\infty} \leq C_7 ||u||_{L^2}^{1-\frac{1}{\lambda}} ||u||_{H^{\lambda/2}}^{\frac{1}{\lambda}},$$

which shows there is an L^∞-bound on u which is independent of the dissipation coefficient ν.

Proof of Theorem 5.3. For notational convenience, $\int$ will mean the spatial integral $\int_{-\infty}^{\infty}$. Multiplying (5.1) by u and integrating over $\mathbb{R} \times [0,t]$ yields the analog

$$||u||_{L^2}^2 + 2\nu \int_0^t \int uMu \leq ||u_0||_{L^2}^2 \tag{5.10}$$

of (3.5). Multiplying (5.1) by $P(u)-Lu$ and integrating by parts over $\mathbb{R}\times[0,t]$ gives

$$\int uLu - 2\int \Lambda(u) + 2\nu \int_0^t \int (Lu)(Mu)$$

$$= \int (uLu)(0) - 2\int \Lambda(u)(0) + 2\nu \int_0^t \int P(u)Mu, \qquad (5.11)$$

where $\Lambda'(u) = P(u)$ and $\Lambda(0) = 0$. The individual terms in (5.11) are now estimated. First, notice that

$$\left|\int \Lambda(u)\right| \leq \sum_{k=1}^{p+1} \frac{a_k}{k+1} \int |u|^{k+1} \leq \sum_{k=1}^{p+1} \frac{a_k}{k+1} ||u||_{L^2}^{k+1-\frac{k-1}{\lambda}} ||u||_{H^{\lambda/2}}^{\frac{k-1}{\lambda}}, \qquad (5.12)$$

and since $p < 2(\lambda+\gamma-\sigma) \leq 2\lambda$, it follows that $(k-1)/\lambda \leq p/\lambda < 2$. Hence, after applying Young's inequality to (5.12) and using (5.5), there obtains

$$\left|\int \Lambda(u)\right| \leq C_8 + \frac{C_1}{2} ||u||_{H^{\lambda/2}}^2 \leq C_8 + \frac{1}{2} \int \alpha(\xi)|\widehat{u}|^2 \qquad (5.13)$$

where C_1 is as in (5.5).

For the integral $\int P(u)M(u)$, it suffices to consider the leading order term

$$\int u^{p+1}Mu = \int \widehat{u^{p+1}}(\xi)\beta(\xi)\bar{\widehat{u}}(\xi) \leq ||u^{p+1}||_{H^{\sigma/2}}||u||_{H^{\sigma/2}}.$$

The Gagliardo-Nirenberg inequality implies that

$$||u||_{H^{\sigma/2}} \leq C_9 ||u||_{L^2}^{1-\frac{\sigma}{\lambda+\gamma}} ||u||_{H^{(\lambda+\gamma)/2}}^{\frac{\sigma}{\lambda+\gamma}}.$$

However, the term $||u^{p+1}||_{H^{\sigma/2}}$ requires a little more effort. The following standard lemma is helpful.

Lemma 5.4 *If f_1, f_2, $\cdots$, f_m lie in $H^\rho(\mathbb{R})$ with $m\rho > (m-1)/2$, then their product $f_1 f_2 \cdots f_m$ is in $H^\varrho(\mathbb{R})$ for any $\varrho < m\rho - (m-1)/2$ and*

$$||f_1 f_2 \cdots f_m||_{H^\varrho} \leq ||f_1||_{H^\rho} ||f_2||_{H^\rho} \cdots ||f_m||_{H^\rho}.$$

Since $\gamma+\lambda-\sigma>0$ and $p<2(\gamma+\lambda-\sigma)$, which is to say,

$$\frac{p+\sigma}{2}<\gamma+\lambda-\frac{\sigma}{2},$$

there is an $s>0$ such that

$$\frac{p+\sigma}{2}<(p+1)s<\gamma+\lambda-\frac{\sigma}{2}, \tag{5.14}$$

or what is the same, $\sigma/2<(p+1)s-p/2$. Applying Lemma 5.4 gives

$$\|u^{p+1}\|_{H^{\sigma/2}}\leq\|u\|_{H^s}^{p+1},$$

and then the Gagliardo-Nirenberg inequality leads to the inequality

$$\|u\|_{H^s}\leq C_{10}\|u\|_{L^2}^{1-\frac{2s}{\gamma+\lambda}}\|u\|_{H^{\frac{\gamma+\lambda}{2}}}^{\frac{2s}{\gamma+\lambda}}.$$

In summary, we obtain the inequality

$$\left|\int u^{p+1}Mu\right|\leq C_{11}\|u\|_{H^{\frac{\gamma+\lambda}{2}}}^{\frac{2s(p+1)+\sigma}{\gamma+\lambda}}. \tag{5.15}$$

From (5.14), the exponent in (5.15) has

$$\frac{2s(p+1)+\sigma}{\gamma+\lambda}<2,$$

and thus another application of Young's inequality yields

$$\left|\int u^{p+1}Mu\right|\leq C_{12}+\frac{C_1C_3}{2}\|u\|^2_{H^{\frac{\gamma+\lambda}{2}}}\leq C_{12}+\frac{1}{2}\int\alpha(\xi)\beta(\xi)|\widehat{u}|^2 \tag{5.16}$$

where C_1 and C_3 are as in (5.5) and (5.6).

Collecting the estimates (5.11), (5.13) and (5.16) and using the L^2-bound in (5.10), there obtains

$$\int_{\mathbb{R}}\alpha(\xi)|\widehat{u}(\xi,t)|^2d\xi+\nu\int_0^t\int_{\mathbb{R}}\alpha(\xi)\beta(\xi)|\widehat{u}(\xi,\tau)|^2d\xi d\tau\leq C_5+C_6\nu t$$

for some constants C_5 and C_6 depending on $\|u_0\|_{D(L^{1/2})}$.

These preparatory results set the stage for a proof of the following zero-dissipation limit result.

Theorem 5.5 *Assume that the symbols α of L and β of M are positive, even and satisfy (5.5) and (5.6), respectively, and that P is of the form (5.3) with*

$$\lambda > 1, \quad \lambda + \gamma > \sigma \quad \text{and} \quad p < 2(\lambda + \gamma - \sigma).$$

Let $u_0, v_0 \in D(L^{1/2})$. Consider the difference

$$w = u - v$$

between the solution $u = u_\nu$ to the IVP (5.1)-(5.2) with initial data u_0 and the solution v to the IVP (5.7)-(5.8) with initial data v_0. Then as long as v has the properties

$$v \in L^2([0,T]; D(M^{1/2})), \quad \text{and} \quad \mathcal{A}(T) = \int_0^T \|v_x(\cdot,\tau)\|_{L^\infty} d\tau < \infty, \tag{5.17}$$

for some $T > 0$, then

$$\sup_{0 \le t \le T} \|u(\cdot,t) - v(\cdot,t)\|_{L^2}^2 \le e^{C_{13}\mathcal{A}(t)} \|u_0 - v_0\|_{L^2}^2 + C_{14}\nu t e^{C_{13}\mathcal{A}(t)} \tag{5.18}$$

where C_{13} and C_{14} depend only on $\|u_0\|_{D(L^{1/2})}$ and $\|v_0\|_{D(L^{1/2})}$.

The condition (5.17) is fulfilled when either $\lambda > 3$, and then it holds for all $T > 0$, or when $v_0 \in H^s$ for some $s > 3/2$ and then it is valid for some $T = T^*$, where T^* is as in Proposition 5.2. If $\{u_0^\nu\}_{\nu > 0}$ is a one-parameter family of initial data for which $\|u_0 - v_0\|_{L^2}^2 = O(\nu)$ as $\nu \to 0$, then it follows from (5.18) that

$$\sup_{0 \le t \le T} \|u(\cdot,t) - v(\cdot,t)\|_{L^2}^2 = O(\nu)$$

as $\nu \to 0$.

Proof. The difference $w = u - v$ is a solution of the equation

$$\partial_t w + P'(u)w_x + [P'(u) - P'(v)]v_x - (Lw)_x + \nu M w + \nu M v = 0. \tag{5.19}$$

Multiplying (5.19) by w and integrating over $\mathbb{R}$ leads to

$$\frac{1}{2}\frac{d}{dt}\int |w|^2 + \int P'(u) w w_x + \int [P'(u) - P'(v)] w v_x$$

$$-\int w(Lw)_x + \nu \int wMw + \nu \int wMv = 0.$$

Since α is positive and even, L is self-adjoint and so

$$\int w(Lw)_x = i \int \xi\alpha(\xi)|\widehat{w}|^2 d\xi = 0.$$

For the remaining terms, argue as follows. First,

$$\nu \int wMv \leq \frac{\nu}{2} \int \beta(\xi)|\widehat{v}|^2 + \frac{\nu}{2} \int \beta(\xi)|\widehat{w}|^2,$$

and also

$$\int P'(u)ww_x = \sum_{k=0}^{p}(k+1)a_{k+1} \int (w+v)^k ww_x$$

$$= \sum_{k=1}^{p}\sum_{j=0}^{k}(k+1)a_{k+1} \begin{pmatrix} k \\ j \end{pmatrix} \int w^{j+1}v^{k-j}w_x$$

$$= -\sum_{k=1}^{p}\sum_{j=0}^{k}(k+1)a_{k+1} \begin{pmatrix} k \\ j \end{pmatrix} \frac{k-j}{j+2} \int \left(w^{j+2}v^{k-j-1}v_x\right) w^2.$$

As a consequence of Theorem 5.3, the L^∞-bound on u is independent of the dissipation coefficient ν. It then follows that

$$\left|\int P'(u)ww_x\right| \leq C_{15}\|v_x\|_{L^\infty} \int w^2.$$

Similarly, it is seen that

$$\left|\int [P'(u) - P'(v)]wv_x\right| = \left|\sum_{k=2}^{p+1} ka_k \int (u^{k-1} - v^{k-1})wv_x\right|$$

$$= \left|\sum_{k=2}^{p+1}\sum_{j=0}^{k-2} ka_k \int \left(u^{k-j-2}v^j v_x\right) w^2\right| \leq C_{16}\|v_x(\cdot,t)\|_{L^\infty} \int w^2.$$

Collecting the above estimates and letting $Y(t) = \int |w(x,t)|^2 dx$, there appears

$$Y' + \nu \int wMw \leq \nu \int \beta(\xi)|\widehat{v}|^2 + C_{13}\|v_x\|_{L^\infty} Y.$$

The desired result (5.18) now follows from Gronwall's lemma.

The convergence rate obtained in Theorem 5.5 can be improved if the solution v of the dissipationless equation is smoother, as the following theorem attests.

Theorem 5.6 *In addition to the assumptions made in Theorem 5.5, we further assume that $v_0 \in D(L^{1/2}) \cap H^s$ with $s \geq \max\{3/2, \sigma\}$. Then for any $t < T^*$,*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^2}^2 = O(\nu^2) \tag{5.20}$$

where T^* is the maximal existence time for v.

Proof. According to Proposition 5.2, the solution v of (5.7)-(5.8) remains in H^s over $[0, T^*)$. Thus for any $t < T^*$,

$$\left|\nu \int wMv\right| \leq \frac{\nu^2}{2} \int \beta^2(\xi)|\widehat{v}|^2 + \frac{1}{2}\int |\widehat{w}|^2 \leq \frac{\nu^2}{2}\|v\|_{H^s} + \frac{1}{2}\int |w|^2.$$

Consequently, the following inequality emerges:

$$\frac{d}{dt}\int |w|^2 dx + 2\nu \int wMw \leq \int |w|^2 dx + \nu^2 \|v\|_{H^s} + C_{14}\|v_x\|_{L^\infty} \int |w|^2 dx,$$

and this leads to the conclusion (5.20).

We illustrate the application of the zero-dissipation limit results obtained here for the equation in general form in the context of several well-known wave models. We start with the generalized KdV-Burgers equation

$$u_t + u_x + u^p u_x - \nu u_{xx} + u_{xxx} = 0.$$

In this example, the symbols of the operators are $\alpha(\xi) = \beta(\xi) = \xi^2$. The exponents $\lambda = \mu = \gamma = \sigma = 2$ satisfy the assumptions of Theorems 5.5 and 5.6. If $p < 2(\lambda + \gamma - \sigma) = 4$, $u_0 \in H^1$ and $v_0 \in H^1$, then by Theorem 5.5

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^2} = O(\nu^{\frac{1}{2}}).$$

If further $v_0 \in H^2$, then Theorem 5.6 indicates

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^2} = O(\nu),$$

where v is the solution of the corresponding equation without the dissipative term. This reproduces part of the results in Section 4.

Attention is now turned to the version of these types of wave equations originally proposed by Ott & Sudan [29] and Ostrovsky [28]. They have the form

$$u_t + u^p u_x + u_{xxx} + \frac{\nu}{\pi} p.v. \int_{-\infty}^{\infty} \frac{u_y(y,t)}{x-y} dy = 0 \tag{5.21}$$

and

$$u_t + u^p u_x + u_{xxx} + \nu u + \nu \int_{-\infty}^{\infty} \frac{sgn(y-x)}{\sqrt{|y-x|}} u_y(y,t) dy = 0, \tag{5.22}$$

respectively. These two equations with $p = 1$ describe ion-acoustic waves in a plasma with Landau damping. The symbols of operators are $\alpha(\xi) = \xi^2$ and $\beta(\xi) = |\xi|$ for (5.21) and $\alpha(\xi) = \xi^2$ and $\beta(\xi) = 1 + \sqrt{|\xi|}$ for (5.22).

The growth exponents are $\lambda = \mu = 2$, $\gamma = \sigma = 1$ for (5.21)and $\lambda = \mu = 2$, $\gamma = \sigma = 1/2$ for (5.22). These fall within the range of applicability of Theorems 5.5 and 5.6. That means, if $p < 4$, $u_0 \in H^1$ and $v_0 \in H^1$, then the solution of (5.21) or (5.22) with initial data u_0 converges in L^2 to the solution of the corresponding equation without dissipation,

$$v_t + v^p v_x + v_{xxx} = 0, \qquad v(\cdot, 0) = v_0(\cdot),$$

and the convergence rate is of order $\nu^{\frac{1}{2}}$. If further $v_0 \in H^2$, the estimate for convergence rate may be improved to order ν.

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