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## 51° SEMINÁRIO BRASILEIRO DE ANÁLISE

### MINICURSO

# NONLINEAR WAVE PHENOMENA


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Universidade Federal de Santa Catarina



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Politics is for the moment.
An equation is for eternity.

A. EINSTEIN

Some calculus tricks are quite easy.
Some are enormously difficult. The fools
who write the textbooks of
advanced mathematics seldom take the trouble
to show you how easy the easy
calculations are.

SILVANUS P THOMPSON *Calculus Made Easy*, Macmillan (1910)

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# NONLINEAR WAVE PHENOMENA

JERRY L. BONA[1]

These notes are meant as a supplement to a Minicurso offered in the $51^{st}$ Seminário Brasileiro de Analíse. The lectures will be concerned with some aspects of theoretical fluid mechanics, particularly wave propagation. The lectures will be somewhat discursive, and the notes aim to fill in some of the more interesting technical details the underly the oral presentations.

The modelling and analysis are treated in some detail in these notes. The analysis calls upon methods from modern functional analysis and the theory of partial differential equations. A general familiarity with these ideas is assumed in the later sections of the notes.

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## Chapter 1. Introduction and a Brief Review of the History

Model equations for waves that take account of both nonlinearity and dispersion have their genesis in the discovery of the solitary wave by John Scott Russell. The story of Scott Russell's encounter with the solitary wave in 1834 has been retold many times.

While working as a consultant for a shipping firm on the Edinburgh-Glasgow canal, Scott Russell witnessed a heavily laden barge drawn by a pair of horses come suddenly to rest, owing to an obstruction in the canal. This sudden cessation of forward motion created various disturbances on the water's surface, including a long-crested wave some 18 inches in elevation that went rolling off down the canal in the direction the barge had been traveling. Scott Russell gave chase by horse and observed the wave, which was more or less uniform in the spanwise direction, propagated with constant speed and without change of shape. Fascinated, Scott Russell went on to conduct a set of laboratory experiments on this phenomenon which he reported in 1841 and 1844 to the British Association (see Scott Russell 1845). Among other appellations he called such waves solitary waves.

The more theoretically inclined scientists interested in fluid mechanics soon took Scott Russell to task. The Astronomer Royal, Sir George Airy addressed the issue of whether or not it was possible to have a steadily propagating wave of permanent form on the surface of water. He concluded such waves were not possible on the basis of analysis to be described presently.

Stokes, who was later also accorded the title Sir George, analyzed waves on the surface of water, concluding on the basis of forthcoming analysis that such wave motion was not possible.

Despite the mathematical theory, the experimental evidence in favor of solitary waves was convincing. The issue lay unresolved until the seminal work of Boussinesq in the 1870's. With the hindsight derived from Boussinesq's work, one sees clearly that both Airy and Stokes were on the right track, and both had part of the issue in hand, as will become apparent in the next section.

Lord Rayleigh also addressed the issue of existence of solitary waves, and concluded in a long article on waves published in 1876 that there were such motions. He was unaware of Boussinesq's work, but later commented that the credit for settling the issue of whether or not there are solitary waves went to Boussinesq.

In 1895, the famous paper of Korteweg and de Vries appeared. These Dutch scientists were apparently ignorant of the work of Boussinesq, for they refer to Stokes' much earlier work. In a clear account which is very readable more than 100 years after it was written, Korteweg and de Vries lay out the essential modelling and mathematical issues that go into the 19th century analysis of Scott Russell's solitary waves.

At the turn of the century, it seems fair to say that Scott Russell was vindicated in his view that single-crested, traveling waves of elevation exist on the surface of water. It is



worth note that Stokes reversed himself in print regarding whether or not solitary waves exist.

In the first half of the 20th century, solitary waves and related evolution equations were not a major topic of scientific conversation. The notion of a solitary wave was used in a descriptive manner, but it does not appear as a central issue in theoretical discussion. For example, Lamb's rendering of solitary waves accords Boussinesq a footnote, does not mention the Korteweg-de Vries equation, but centers around Lord Rayleigh's development, which in retrospect was probably the least interesting approach since he did not derive an evolution equation which could countenance a range of disturbances, but rather passed directly to a traveling-wave description.

The oceanographer Keulegan pioneered the use of the idea of a solitary wave, particularly solitary internal waves, in geophysical applications. Keulegan with Patterson wrote an article in 1941 that reviewed some of Boussinesq's ideas. As the original was somewhat inaccessible, this proved to be a very helpful endeavor.

The linear heat equation features infinite speed of propagation. In principle, a candle lit in Austin, Texas could be detected immediately in Florianopolis with sufficiently accurate instruments. In fact, heat does not propagate at infinite speed. Enrico Fermi was looking for a model for heat conduction that featured finite speed of propagation. With John Pasta and Stanislaw Ulam, he put forward a discrete spring and mass model such as one encounters in elementary physics courses. The difference was the springs were not Hooksian, but instead the restoring force had a quadratic dependence on the extension. Gravity is ignored, and so Newton's laws lead to a coupled system of nonlinear ordinary differential equations. Exact solutions were not available, so they resorted to numerical simulation using Los Alamos Laboratory's ENIAC computer. What they found did not correspond well to heat conduction; it seems this simple mass and spring system features near recurrence of initial states, and not the kind of thermalization one expects. A Los Alamos report was duly constructed and the issue then lay dormant. Fermi died in the late 1950's holding the opinion that these numerical simulations were somehow important, but not knowing exactly why.

A few years later, Gardner and Morikawa studied the stability of a cold collisionless plasma as it arises in a putative description of nuclear fusion. Starting from the full Magneto-Hydrodynamic equations, and making simplifying assumptions about the motion of the plasma, they derived the same equations as had Boussinesq and Korteweg - de Vries, although the physical context was different. Their work appeared initially as an NYU report, but was published in the permanent literature only many years later.

At the Plasma Physics Laboratory in Princeton University, Martin Kruskal knew of the work of Gardner and Morikawa. He also knew about the work of Fermi, Pasta and Ulam and at a certain stage, in collaboration with Norman Zabusky, he revisited their model. Kruskal and Zabusky took a continuum limit of the original discrete system. The system of ordinary differential equations goes over to a partial differential equation in this limit, and the equation in question was the Boussinesq-Korteweg-de Vries equation again. A well-conceived sequence of numerical experiments for the spatially-periodic initial-value problem was carried out and reported in 1963. These experiments showed some of the same fascinating properties that Fermi, Pasta and Ulam had seen earlier. The Korteweg-de Vries equation had now arisen as a description of three, distinct physical systems.

Further study of the Korteweg-de Vries equation led to the inverse-scattering theory for the initial-value problem. This imaginative leap was first described by Gardner, Greene, Miura and Kruskal in 1967, and later amplified in a series of papers. Peter Lax made a fundamental step forward in 1968 by providing a mathematical framework in which to consider the inverse-scattering theory as it applies to initial-value problems for partial differential equations.

Shortly afterward, the subject began to assume industrial proportions and it quickly becomes difficult to trace the developments. Indeed, many areas of mathematics, physics and mechanics have been influenced by the elaboration and extension of the ideas just mentioned.

## Chapter 2. Derivation of Model Equations for Waves in Dispersive Media

We begin by considering a body of water of finite depth under the influence of gravity, bounded below by an impermeable surface. Ignoring the effects of viscosity and assuming the flow is incompressible and irrotational, the fluid motion is taken to be governed by the Euler equations together with suitable boundary conditions on the rigid surface and on the water-air interface. After briefly explaining the Euler equations, further approximations are introduced and analyzed, leading to a set of model equations formally valid for small-amplitude long wavelength motion.

The Euler equations are written in a right-handed Cartesian coordinate system $ox_1x_2y$ with $oy$ pointing in the direction opposite to that of gravity, $ox_1$ to the right, and $ox_2$ toward us from the page. Let $\mathbf{u}(x_1, x_2, y, t) = (u_1, u_2, v)$ denote the velocity vector for the fluid motion. Since the fluid is incompressible and has constant density $\rho$, it follows from conservation of mass that

$$\nabla \cdot \mathbf{u} = 0. \tag{2.1}$$

Conservation of momentum is expressed mathematically by the relation,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P - gj, \tag{2.2}$$

where $P = P(x_1, x_2, y, t)$ is the pressure, $\mathbf{j} = (0, 0, 1)$ is the unit vector in the direction opposite gravitation, and $g$ is the gravity constant. The lack of swirl in an irrotational flow is expressed as the vanishing of the curl of the velocity vector,

$$curl\, \mathbf{u} = \nabla \times \mathbf{u} = 0. \tag{2.3}$$



In consequence, there is a velocity potential $\phi = \phi(x_1, x_2, y, t)$ such that

$$\mathbf{u} = \nabla \phi. \tag{2.4}$$

Because of (2.3) and $\nabla(\mathbf{u} \cdot \mathbf{u}) = 2(\mathbf{u} \cdot \nabla)\mathbf{u} + 2\mathbf{u} \times (\nabla \times \mathbf{u})$, conservation of momentum (2.2) may be rewritten as

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) = -\frac{1}{\rho} \nabla P - Gk.. \tag{2.5}$$

Combining (2.5) with (2.4), we come to the conclusion

$$\nabla \Big[ \frac{\partial \phi}{\partial t} + \frac{1}{2} \nabla \phi \cdot \nabla \phi + \frac{1}{\rho} P + gy \Big] = 0,$$

since $\nabla y = \mathbf{j}$. The gradient of the quantity in square brackets vanishes in the flow domain, and assuming the latter is simply connected, it follows that

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \nabla \phi \cdot \nabla \phi + \frac{1}{\rho} P + gy = B(t),$$

where $B(t)$ is a constant independent of the spatial coordinates $(x_1, x_2, y)$. The latter expression may be written in another form, namely

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \frac{1}{\rho}(P - P_0) + gy = B(t), \tag{2.6}$$

where $P_0$ is the pressure in the air near to the surface of the liquid. This quantity will be taken to be constant in the present consideration. Let $\tilde{\phi}(x_1, x_2, y, t) = \phi(x_1, x_2, y, t) - \int_0^t B(s)\, ds$, and rewrite (2.6) in terms of $\tilde{\phi}$, *viz.*

$$\frac{\partial \tilde{\phi}}{\partial t} + \frac{1}{2} \nabla \tilde{\phi} \cdot \nabla \tilde{\phi} + \frac{1}{\rho}(P - P_0) + gy = 0.$$

Dropping the tilde from $\tilde{\phi}$ and rearranging gives

$$\frac{P - P_0}{\rho} = -\frac{\partial \phi}{\partial t} - \frac{1}{2} \nabla \phi \cdot \nabla \phi - gy. \tag{2.7}$$

Since $\nabla \cdot \mathbf{u} = 0$, it follows that $\phi$ satisfies Laplace's equation

$$\Delta \phi = 0. \tag{2.8}$$

in the flow domain. Thus we are reduced to solving (2.8) with the appropriate boundary conditions and then the velocity field $\mathbf{u}$ and the pressure $P$ may be read off from (2.4) and (2.7), respectively.

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in the flow domain. Thus we are reduced to solving (2.8) with the appropriate boundary conditions and then the velocity field $\mathbf{u}$ and the pressure $P$ may be read off from (2.4) and (2.7), respectively.

*Remark*: At first sight this doesn't look very wavy, and indeed there would be no waves if it were not for the effects of the free surface, which are discussed next.

Suppose the free surface of the liquid is described by an equation of the form

$$f(x_1, x_2, y, t) = 0.$$

Since the fluid doesn't cross this surface, the velocity of the fluid at the surface must be the velocity of the surface. It is therefore straightforward to ascertain that the normal velocity of the surface can be reprsented by both

$$\frac{-f_t}{(f_{x_1}^2 + f_{x_2}^2 + f_y^2)^{\frac{1}{2}}}$$

and

$$\mathbf{u} \cdot \mathbf{n} = \frac{u_1 f_{x_1} + u_2 f_{x_2} + v f_y}{(f_{x_1}^2 + f_{x_2}^2 + f_y^2)^{\frac{1}{2}}}.$$

Setting them equal leads to the kinematic boundary condition

$$f_t + u_1 f_{x_1} + u_2 f_{x_2} + v f_y = 0. \tag{2.9}$$

If the free surface can be described by a single-valued function of $(x_1, x_2)$ for some time interval, say,

$$f(x_1, x_2, y, t) = \eta(x_1, x_2, t) - y,$$

then the boundary condition (2.9) above becomes

$$\eta_t + u_1 \eta_{x_1} + u_2 \eta_{x_2} - v = 0,$$

or what is the same,

$$\eta_t + \phi_{x_1} \eta_{x_1} + \phi_{x_2} \eta_{x_2} = \phi_y. \tag{2.10}$$

There is also a dynamical condition on the free surface. Since the surface has no mass, and if surface tension is neglected, the pressure in the water and the air pressure must be equal on the free surface. Of course a disturbance on the surface imparts some motion to the air. We argue, because of the small density of air relative to the density of the water, that the air pressure is not changed significantly, and so may be approximated by its undisturbed value. Hence the second boundary condition on the free surface is

$$P = P_0 \quad \text{at} \quad y = \eta(x_1, x_2, t), \tag{2.11}$$



where $P = P(x_1, x_2, \eta, t)$ is the pressure at the surface. Using (2.7) in conjunction with (2.11), the Bernoulli condition

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + g\eta = 0 \quad \text{for} \quad y = \eta \tag{2.12}$$

is derived. Because the lower, containing boundary is impermeable, the velocity normal to the bottom must be zero, which is to say, there is no flow through the bottom. If the bottom profile is $y = -h_0(x_1, x_2)$, then $u \cdot n = 0$, where $n = (h_{0x_1}, h_{0x_2}, 1)$ is the normal direction to the bottom; hence

$$\phi_{x_1} h_{x_1} + \phi_{x_2} h_{x_2} + \phi_y = 0 \quad \text{at} \quad y = -h_0(x_1, x_2). \tag{2.13}$$

Now it might appear that our system is a little overdetermined since we have $\Delta\phi = 0$ inside the flow domain, one boundary condition on the bottom, but two on the free surface. This is contrary to what we know about elliptic equations. The resolution of this conondrum lies in the free surface not being prescribed in advance, but instead constituting part of solution of the problem. So, in summary, assuming the free surface and the bottom profile can be described as single-valued function of $(x_1, x_2, t)$, the motion of the perfect liquid may be described by the system:

$$\begin{aligned} &\Delta\phi = 0 && \text{in the flow domain} - h_0 < y < \eta. \\ &\left.\begin{aligned} &\eta_t + \phi_{x_1}\eta_{x_1} + \phi_{x_2}\eta_{x_2} = \phi_y \\ &\phi_t + \frac{1}{2}(\nabla\phi)^2 + g\eta = 0 \end{aligned}\right\} && \text{at the free surface} \quad y = \eta, \\ &\phi_{x_1} h_{0x_1} + \phi_{x_2} h_{0x_2} + \phi_y = 0 && \text{on the bottom} \quad y = -h_0(x_1, x_2). \end{aligned} \tag{2.14}$$

It is often interesting and sometimes appropriate to specialize to the case of two-dimensional flow; i.e. motions which are independent of $x_2$, say. Let $x$ denote $x_1$ and suppose additionally that $h_0$ is constant, so the bottom is flat and horizontal. Then the system (2.14) above reduces to

$$\begin{aligned} &\phi_{xx} + \phi_{yy} = 0 && \text{in the domain} \quad -h_0 < y < \eta. \\ &\left.\begin{aligned} &\eta_t + \phi_x \eta_x = \phi_y \\ &\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta = 0 \end{aligned}\right\} && \text{at the free surface} \quad y = \eta. \\ &\phi_y = 0 && \text{on the bottom} \quad y = -h_0. \end{aligned} \tag{2.15}$$

together with appropriate initial conditions and other boundary conditions if lateral surfaces intrude.

If the propagation of infinitesimal waves is considered, then it is warranted to linearize the equations of motion around the rest state. In this case (2.15) is reduced to

$$(2.16)\qquad \begin{aligned} &\Delta\phi = 0 && \text{in} \quad 0 < y < h_0, \\ &\left.\begin{aligned}\eta_t &= \phi_y \\ \phi_t + g\eta &= 0\end{aligned}\right\} && \text{on} \quad y = h_0, \\ &\phi_y = 0 && \text{at} \quad y = 0. \end{aligned}$$

We start by looking for a travelling-wave solution of the form $\phi(x,y,t) = \psi(y)e^{i(kx-\omega t)}$. Substituting this form into (2.16) and simplifying gives

$$(2.17)\qquad \begin{cases} \psi'' - k^2\psi = 0, \\ \psi'(0) = 0, \\ \dfrac{\omega^2}{g}\psi(h_0) - \psi'(h_0) = 0. \end{cases}$$

It follows that

$$\psi(y) = c\,\sinh(ky) + d\,\cosh(ky).$$

As $\psi'(0) = 0$, $c = 0$ and $\psi(y) = d\,\cosh(ky)$. Applying the second boundary condition leads to the dispersion relation

$$\omega^2 = g\frac{\psi'(h_0)}{\psi(h_0)} = gk\frac{\sinh(kh_0)}{\cosh(kh_0)} = gk\,\tanh(kh_0).$$

Thus the frequency $\omega$ is $\omega(k) = \sqrt{gk\,\tanh(kh_0)}$, while the phase speed is

$$c(k) = \frac{\omega(k)}{k} = \sqrt{gh_0}\sqrt{\frac{\tanh(kh_0)}{kh_0}}.$$

*Remark:* The quantity $c(k)$ is the speed of individual crests. The quantity $\sqrt{gh_0}$ is the so-called kinematic wave velocity, the velocity of extremely long waves. According to the linearized theory, long waves travel faster than short wavelength disturbances.

### 2.2 The Nonlinear Problem

To make progress in case the waves are not infinitesimally small, further assumptions are needed. The free surface in its undisturbed state will be relocated to $y = 0$. Let $a = sup_{x\in\mathbb{R},t\geq 0}|\eta(x,t)|$ be the maximum amplitude of the contemplated wave motion, $l$ a typical wavelength in the wave motion, and $c_0 = \sqrt{gh_0}$ the kinematic wave velocity.



Assume that $a << h_0$ and that $h_0 << l$. It is natural to non-dimensionalize the variables to bring these assumptions to the fore: let

$$x' = lx, \quad y' = h_0(y-1), \quad t' = \frac{lt}{c_0}, \quad \eta' = a\eta, \quad \phi' = g\frac{la}{c_0}\phi.$$

Here, the primed variables connote the original coordinates, while the unprimed quantities are the new dimesionless variables. In the new variables, the system (2.15) becomes

$$(2.18)\qquad \begin{cases} \beta\phi_{xx} + \phi_{yy} = 0 & \text{in} \quad 0 < y < 1 + \alpha\eta, \\ \phi_y = 0 & \text{at} \quad y = 0, \\ \left.\begin{aligned}\eta_t + \alpha\phi_x\eta_x - \frac{1}{\beta}\phi_y &= 0 \\ \eta + \phi_t + \frac{1}{2}\alpha\phi_x^2 + \frac{1}{2}\frac{\alpha}{\beta}\phi_y^2 &= 0\end{aligned}\right\} & \text{at} \quad y = 1 + \alpha\eta, \end{cases}$$

where $\alpha = \dfrac{a}{h_0}$ and $\beta = \dfrac{h_0^2}{l^2}$.

A formal expansion of $\phi$ in a power series in $y$ is posited:

$$\phi(x,y,t) = \sum_{m=0}^{\infty} f_m(x,t)y^m.$$

¿From the Laplace equation in (2.18), there follows

$$\begin{aligned} 0 &= \beta\phi_{xx} + \phi_{yy} \\ &= \beta\sum_{m=0}^{\infty} f_m'' y^m + \sum_{m=2}^{\infty} m(m-2)f_m y^{m-2} \\ &= \sum_{m=0}^{\infty}\left(\beta f_m'' + (m+2)(m+1)f_{m+2}\right)y^m, \end{aligned}$$

whence

$$\beta f_m'' = -(m+2)(m+1)f_{m+2} \qquad \text{for} \quad m \geq 0.$$

Since $\phi_y(x,0) = 0$ is specified in the first boundary condition in (2.18), $f_1(x,t) = 0$, and so by recursion $f_3 = f_5 = f_7 = \cdots = f_{2n+1} \cdots = 0$ for all $n \geq 0$. If we write $f(x,t)$ for $f_0(x,t)$, then $f_2 = -\frac{\beta}{2!}f''$, $f_4 = -\frac{\beta}{4\cdot 3}f_2'' = \frac{\beta^2}{4!}f''''$ and so forth. Thus, the Laplace equation together with the boundary condition at the bottom leads to

$$(2.19)\qquad \phi(x,y,t) = \sum_{m=0}^{\infty} f_{2m}(x,t)y^{2m} = \sum_{m=0}^{\infty}(-1)^m\beta^m\frac{y^{2m}}{(2m)!}f^{(2m)}(x,t).$$

*Remark*: The variable $f_x$ is the horizontal velocity of the fluid at the bottom.

Substituting (2.19) into the non-dimensional version of the Euler equations in (2.18), the kinematic boundary condition on the free surface yields

$$(2.20)\quad \eta_t + \alpha\eta_x[f_x - \beta\frac{(1+\alpha\eta)^2}{2}f_{xxx}] - \frac{1}{\beta}[-\beta(1+\alpha\eta)f_{xx} + \frac{\beta^2(1+\alpha\eta)^3}{6}f_{xxx}] + O(\beta^2) = 0.$$

Ignoring terms quadratic in $\alpha$ and $\beta$, this simplifies first to

$$\eta_t + ((1+\alpha\eta)f_x)_x - \left\{\frac{1}{6}(1+\alpha\eta)^3 f_{xxxx} + \frac{1}{2}(1+\alpha\eta)^2\eta_x f_{xxx}\right\}\beta + O(\beta^2) = 0$$

and then even further to

$$(2.21)\qquad \eta_t + ((1+\alpha\eta)f_x)_x - \frac{\beta}{6}f_{xxxx} + O(\beta^2, \alpha\beta) = 0.$$

The Bernoulli condition on the free surface gives, after simplifying,

$$(2.22)\qquad \eta + f_t + \frac{1}{2}\alpha f_x^2 - \frac{1}{2}\beta f_{xxt} + O(\beta^2, \alpha\beta) = 0.$$

Since $f_x$ is the horizontal velocity at the bottom, it is a variable with a direct physical interpretation. Writing $w$ for $f_x$ and combining (2.21) and (2.22) gives one version of the Boussinesq system of equations, namely

$$(2.23)\qquad \begin{cases} \eta_t + [(1+\alpha\eta)w]_x - \frac{1}{6}\beta w_{xxx} = 0, \\ w_t + \eta_x + \alpha w w_x - \frac{1}{2}\beta w_{xxt} = 0. \end{cases}$$

If $\alpha << \beta$, we would be tempted to drop the nonlinear terms and thereby arrive at the linear system

$$(2.24)\qquad \begin{cases} \eta_t + w_x - \frac{1}{6}\beta w_{xxx} = 0, \\ w_t + \eta_x - \frac{1}{2}\beta w_{xxt} = 0. \end{cases}$$

The behavior of solutions of such linear systems is determined by their dispersion relation. This is obtained in a straigtforward way by first eliminating $\eta$ to reach the single equation

$$(2.25)\qquad w_{tt} - w_{xx} + \frac{1}{6}\beta w_{xxxx} - \frac{1}{2}\beta w_{xxtt} = 0.$$



We refer to this as the linear Boussinesq equation. Substituting the form $w(x,t) = w_0 e^{i(kx-\omega t)}$ into (2.25) leads to

$$-\omega^2 + k^2 + \frac{\beta}{6}k^4 - \frac{\beta}{2}\omega^2 k^2 = 0,$$

so

$$\omega^2 = k^2\left[\frac{1+\frac{\beta}{6}k^2}{1+\frac{\beta}{2}k^2}\right] \quad\text{and}\quad c(k) = \frac{\omega(k)}{k} = \pm\left[\frac{1+\frac{\beta}{6}k^2}{1+\frac{\beta}{2}k^2}\right]^{\frac{1}{2}}$$

This agrees with the dispersion relation

$$\sqrt{\frac{\tanh(\beta^{\frac{1}{2}}k)}{\beta^{\frac{1}{2}}k}}$$

for the full, linearized Euler equations to the fourth order in $k$. But there is a difficulty associated with large wavenumbers (small wavelengths), which will be discussed presently.

### *2.3 One-way Propagation*

Here, the Boussinesq system of equations is specialized to the description of waves propagating just to the right. At the very lowest order where even the terms of order $\alpha$ and $\beta$ are dropped, there appears a factored version of the one-dimensional wave equation, *viz.*

$$(2.26)\qquad \begin{cases} \eta_t + w_x = 0, \\ w_t + \eta_x = 0, \\ \eta(x,0) = f(x), \\ w(x,0) = g(x), \end{cases}$$

posed with initial conditons on both $\eta$ and $w$. The solution of (2.26) is

$$\eta(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}[g(x+t) - g(x-t)] \quad\text{and}$$
$$w(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}[f(x+t) - f(x-t)].$$

As the left-propagating component must vanish, it is required that $f = g$, whence $\eta(x,t) = f(x-t) = w(x,t)$. Thus, at the lowest order, we have $w = \eta$ and $\eta_t + \eta_x = 0$.

The next step is to extend the relations just obtained via the linear wave equation to obtain a model correct to order $\alpha$ and $\beta$ while still maintaining one-way propagation. Whatever the extensions are that lead to the next order, it seems clear they will involve terms of order $\alpha$ and $\beta$. It is therefore natural to try the Ansatz $w = \eta + \alpha A + \beta B$, where

$A = A(\eta, \eta_x, \eta_t, \cdots)$, and $B = B(\eta, \eta_x, \eta_t, \cdots)$. Putting this relation into the Boussinesq system (2.23) results in the pair of equations

$$\begin{aligned} &\eta_t + \eta_x + \alpha A_x + \beta B_x + \alpha\{\eta[\eta + \alpha A + \beta B]\}_x - \frac{1}{6}\beta[\eta + \alpha A + \beta B]_{xxx} = 0, \\ (2.27) \quad &\eta_t + \alpha A_t + \beta B_t + \eta_x + \alpha(\eta + \alpha A + \beta B)(\eta_x + \alpha A_x + \beta B_x) \\ &\quad - \frac{1}{2}\beta(\eta_{xxt} + \alpha A_{xxt} + \beta B_{xxt}) = 0. \end{aligned}$$

Collecting terms featuring the same power of $\alpha$ and $\beta$ leads to the relations

$$\begin{cases} \eta_t + \eta_x + \alpha(A_x + 2\eta\eta_x) + \beta(B_x - \frac{1}{6}\eta_{xxx}) = \text{terms of order } \alpha^2, \alpha\beta, \beta^2, \\ \eta_t + \eta_x + \alpha(A_t + \eta\eta_x) + \beta(B_t - \frac{1}{2}\eta_{xxt}) = \text{terms of order } \alpha^2, \alpha\beta, \beta^2, \end{cases}$$

or, dropping the terms quadratic in $\alpha$ and $\beta$,

$$(2.28) \qquad \begin{cases} \eta_t + \eta_x + \alpha(A_x + 2\eta\eta_x) + \beta(B_x - \frac{1}{6}\eta_{xxx}) = 0, \\ \eta_t + \eta_x + \alpha(-A_x + \eta\eta_x) + \beta(-B_x - \frac{1}{2}\eta_{xxt}) = 0. \end{cases}$$

This pair of equations can be made consistent by choosing $A_x = -\frac{1}{2}\eta\eta_x$, or $A = -\frac{1}{4}\eta^2$, and $B_x = \frac{1}{12}\eta_{xxx} - \frac{1}{4}\eta_{xxt}$, or $B = \frac{1}{12}\eta_{xx} - \frac{1}{4}\eta_{xt}$. It is worthwhile noting that from the lowest-order theory, $\eta_t = -\eta_x + O(\alpha, \beta)$ as $\alpha, \beta \to 0$. In consequence, we may use $\eta_t$ and $-\eta_x$ interchangeably in terms whose formal order is $\alpha$ or $\beta$ without affecting the overall level of the approximation. Thus, at the formal level,

$$(2.29) \qquad B = \frac{1}{12}\eta_{xx} - \frac{1}{4}\eta_{xt} = \frac{1}{3}\eta_{xx} + O(\alpha, \beta) = -\frac{1}{3}\eta_{xt} + O(\alpha, \beta)$$

as $\alpha, \beta \to 0$. Because $B$ appears in (2.28) multiplied by $\beta$, the dispersive terms in (2.28) could have either of the forms $\frac{1}{3}\eta_{xxx}$ or $-\frac{1}{3}\eta_{xxt}$, or, indeed, any convex combination of theses two forms. Taking only the pure forms $\eta_{xxx}$ or $-\eta_{xxt}$, we come to

$$(2.30) \qquad \begin{cases} w = \eta - \frac{1}{4}\alpha\eta^2 + \frac{1}{3}\beta\eta_{xx} = \text{terms quadratic in } \alpha, \beta, \\ \eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} = \text{terms quadratic in } \alpha, \beta, \end{cases}$$

or

$$(2.31) \qquad \begin{cases} w = \eta - \frac{1}{4}\alpha\eta^2 - \frac{1}{3}\beta\eta_{xt} + \text{terms quadratic in } \alpha, \beta, \\ \eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{1}{6}\eta_{xxt} = \text{terms quadratic in } \alpha, \beta. \end{cases}$$



We thus have two separate model equations for unidirectional propagation of long waves of small amplitude. In fact, more models could be constructed using the observation that $\partial_t = -\partial_x + \text{order}(\alpha, \beta)$, namely

$$(2.32) \qquad \eta_t + \eta_x + \begin{Bmatrix} +\frac{3}{2}\eta\eta_x \\ -\frac{3}{2}\eta\eta_t \end{Bmatrix} + \begin{Bmatrix} +\frac{1}{6}\beta\eta_{xxx} = 0 \\ -\frac{1}{6}\beta\eta_{xxt} \\ +\frac{1}{6}\beta\eta_{xtt} = 0 \\ -\frac{1}{6}\beta\eta_{ttt} = 0 \end{Bmatrix} = 0.$$

There are eight different model equations here, without doing anything more complicated (like changing the dependent variable or allowing convex combinations of the individual nonlinear and dispersive terms).

Omitting the nonlinear terms yields four possibilities,

$$(2.33a) \qquad \eta_t + \eta_x + \frac{1}{6}\beta \begin{Bmatrix} +\eta_{xxx} \\ -\eta_{xxt} \\ +\eta_{xtt} \\ -\eta_{ttt} \end{Bmatrix} = 0.$$

Trying $\eta = e^{i(kx - \omega t)}$ leads to the linearized dispersion relations

$$(2.33b) \qquad \omega(k) = \begin{cases} k(1 - \frac{\beta}{6}k^2), \\ \dfrac{k}{1 + \frac{\beta}{6}k^2}, \\ \dfrac{3}{\beta k}\left[\pm\sqrt{1 + \frac{2}{3}\beta k^2} - 1\right], \\ \text{solution of a cubic equation.} \end{cases}$$

The associated phase speeds are

$$(2.33c) \qquad c(k) = \begin{cases} 1 - \frac{\beta}{6}k^2, \\ \dfrac{1}{1 + \frac{\beta}{6}k^2}, \\ \dfrac{3}{\beta k^2}\left[\pm\sqrt{1 + \frac{2}{3}\beta k^2} - 1\right], \\ \text{solution of a cubic equation.} \end{cases}$$

The first two, the third with a + sign and the fourth if the right branch is taken, all agree to order $k^4$ with the linearized dispersion relation for the full two-dimensional Euler equations.

Consider the pure initial-value problem posed on $\mathbb{R}$ for the above models, namely

$$\text{(2.34)} \qquad \begin{cases} \eta_t + \eta_x + \frac{\beta}{6} Lu = 0, & x \in \mathbb{R}, t \geq 0, \\ \eta(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

where $L$ represents one or another of the dispersion operator $\partial_x^3$, $-\partial_x^2\partial_t$, $\partial_x\partial_t^2$, or $-\partial_t^3$, at least for small values of $\beta$ and order-one initial data. This should represent a well posed problem if one is to take the equation seriously as a model of physical phenomena. For the moment, attention is given over to the cases where $L$ is $\partial_x^3$ or $-\partial_x^2\partial_t$. The other two cases are interesting because there is apparently insufficient data to initiate the motion uniquely, but they will not be considered here.

Taking the Fourier transform in the spatial variable $x$ for the linearized Korteweg–de Vries–equation where $L = \partial_x^3$ gives

$$\hat{\eta}_t + i(\xi - \frac{\beta}{6}\xi^3)\hat{\eta} = 0,$$

whence

$$\hat{\eta}(\xi, t) = \hat{\eta}(\xi, 0)e^{-it(\xi - \frac{\beta}{6}\xi^3)}.$$

Computing similarly for the linear regularized long-wave equation (RLW equation or BBM equation where $L = -\partial_x^2\partial_t$ leads to

$$(1 + \frac{\beta}{6}\xi^2)\hat{\eta} + i\xi\hat{\eta} = 0,$$

and so

$$\hat{\eta}(\xi, t) = \hat{\eta}(\xi, 0)e^{-it\frac{\xi}{1+\frac{\beta}{6}\xi^2}}.$$

For these two models, the frequency dispersion $\omega = \omega(\xi)$ is modelled by $\xi - \frac{\beta}{6}\xi^3$ and $\frac{\xi}{1+\frac{\beta}{6}\xi^2}$, respectively. In terms of the phase speed $c = c(\xi) = \frac{\omega(\xi)}{\xi}$, there are the two alternatives

$$c(\xi) = \begin{cases} \dfrac{1}{1 + \frac{\beta}{6}\xi^2} & \text{RLW - } \quad L = -\partial_{xxt}, \\ 1 - \dfrac{\beta}{6}\xi^2 & \text{KdV - } \quad L = \partial_{xxx}. \end{cases}$$

For values of $\xi$ in the range $|\xi| \leq 1$, which is apropriate in the present scaling, these two dispersion relations differ by less than $\frac{\beta^2}{36}$. As for the nonlinear term $\eta\eta_x$ versus $\eta\eta_t$, the conservation laws

$$\text{(2.35)} \qquad \eta_t + \eta_x + \begin{cases} \frac{3}{2}\alpha\eta\eta_x = 0, \\ -\frac{3}{2}\alpha\eta\eta_t = 0, \end{cases}$$



correspond to the characteristic equations

$$\frac{dx}{dt}\Big|_{\eta=constant} = \begin{cases} 1 + \frac{3}{2}\alpha\eta, \\ \dfrac{1}{1 - \frac{3}{2}\alpha\eta}, \end{cases}$$

respectively. For values of $\eta$ with $|\eta| \leq \frac{1}{4}$, say, which is consistent with the small-amplitude presumption in force, these differ by less than $\alpha^2$.

Thus for small values of $\alpha$ and $\beta$, these models appear likely to present nearly identical outcomes. Nevertheless, there might be a marginal preference for the choices $-\eta_{xxt}$ and $\eta\eta_x$. As far as the preference for $-\eta_{xxt}$ goes, observe that short-wave components for the linear equation (2.33a) with $\eta_{xxx}$ can propagate in the direction of decreasing values of $x$ with arbitarily large phase velocities (and the group velocity is likewise unbounded), whereas the $\eta_{xxt}$ term has bounded (and positive) phase velocities (and bounded group velocity). Regarding the nonlinear term, whilst one cannot really distinguish between the two possibilities in (2.35) for $\eta$ small, as $\eta$ gets large, the $\eta\eta_t$ term has singular characteristics, whilst $\eta\eta_x$ just propagates larger amplitude waves faster.

On the basis of these arguments, the model equations

$$\text{(2.36)} \qquad \begin{cases} \eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{\beta}{6}\eta_{xxt} = 0, \\ \eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{\beta}{6}\eta_{xxx} = 0, \end{cases}$$

are singled out for study. The second one is the famous KdV-equation discussed in the introduction, first derived by Boussinesq in 1871 and later by Korteweg and deVries in 1895.

With these formalities in front of us, the historical perspective presented in Section 1 may be given more precision. The model put forward by Airy in 1845 corresponds to taking $\alpha$ small and $\beta = 0$ in the present notation. Thus, Airy put forward what we would now call shallow water theory as a model for what Scott Russel observed. In this model, small, but finite, amplitude effects are contemplated, but finite wavelength effects are ignored. It is a model that retains validity only for waves of extreme length. Indeed, it is straightforward to see that the evolution equation

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x = 0$$

does not possess a travelling-wave solution $\eta(x,t) = \phi(x - ct)$, $c > 0$ a positive constant, that has the form of a solitary wave of elevation. Stokes, on the other hand, viewed the regime in which Scott Russel made his experiemnts as corresponding to infinitesimal waves.

He ignored finite-amplitude effects by taking $\alpha = 0$. However, he kept the effect of finite wavelength on wave speed by taking $\beta$ small, but non-zero. He thus put forward the model

$$\eta_t + \eta_x + \frac{\beta}{6}\eta_{xxx} = 0,$$

in the present notation. Fourier analysis shows that this model also has no solution of the form $\eta(x,t) = \phi(x-ct)$ where $\phi$ is an even function decaying rapidly to zero at $\pm\infty$. There are periodic wavetrains travelling at constant velocity, but a heap of water would decompose into components travelling at different speeds, and so continously spreading. Described in terms of the Stokes or Ursell number

$$S = \frac{\alpha}{\beta^2},$$

Airy took this quantity to be infinite, Stokes took it to be zero, whereas the presumption that corresponds to Scott Rusell's observations is $S \sim 1$. In the latter regime, the nonlinear and dispersive effects come in at the same order, hence the equations in (2.36). In general, $S$ is a rough measure of the relative importance of nonlinear effects as compared to dispersive effects, with $S$ small corresponding to a linear system and $S$ large a much more nonlinear regime.

Once the equations in (2.36) have been obtained, the need for the small parameters disappears. For mathematical analysis, it is convenient to dispense with $\alpha$, $\beta$ and the coefficients $\frac{3}{2}$ and $\frac{1}{6}$. This may be accomplished by redefining the variables $\eta$, $x$ and $t$, *viz.* $\tilde{\eta}(\tilde{x},\tilde{x}) = \frac{3}{2}\alpha\eta(\sqrt{\frac{\beta}{6}}x, \sqrt{\frac{\beta}{6}}t)$. Dropping the tildes, the dimensionless equations

(2.37) $$\eta_t + \eta_x + \eta\eta_x - \eta_{xxt} = 0$$

and

(2.38) $$\eta_t + \eta_x + \eta\eta_x + \eta_{xxx} = 0$$

emerge. The small parameters are not really absent, however; they appear in the imposition of auxiliary conditions. For example, if it is supposed the waveform is known initially, then we are concerned with the pure initial-value problem with $\eta(x,0)$ given. In the variables appertaining to (2.36), $\eta(x,0)$ is of order one along with its derivatives, whereas in the variables appearing in (2.37)-(2.38) $\eta(x,0)$ has the form $\frac{3}{2}\alpha g(\sqrt{\frac{\beta}{6}}x)$ where $g$ and its first few derivatives are of order one.



# Chapter 3. Mathematical Theory for the Initial-value Problems

In this chapter, some details of the mathematical theory pertaining to the evolution equations put forward in Chapter 2 is presented. Included in the discussion are results pertaining to the comparison between the two principal models under discussion.

## *3.1 Theory for the BBM-RLW equation*

The discussion of rigorous theory begins with the initial-value problem

(3.1) $$\begin{cases} \eta_t + \eta_x + \eta\eta_x - \eta_{xxt} = 0 & \text{for } t \geq 0, x \in \mathbb{R}, \\ \eta(x,0) = g(x), & \text{for } x \in \mathbb{R}. \end{cases}$$

The following formal calculation gives an indication of some of the mathematics that follows. Suppose $\eta$ is a smooth solution of (3.1), that, with all its derivatives, decays to 0 at $\pm\infty$. Multiply the equation (3.1) by $\eta$ and integrate over $\mathbb{R}$ to obtain

$$0 = \int_{\mathbb{R}} (\eta\eta_t + \eta\eta_x + \eta^2\eta_x - \eta\eta_{xxt})\,dx = \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} (\eta^2 + \eta_x^2)\,dx$$

after integration by parts and imposition of zero boundary conditions at $\pm\infty$. This is equivalent to

(3.2) $$\int_{-\infty}^{\infty} \left[\eta(x,t)^2 + \eta_x(x,t)^2\right] dx = \|\eta(\cdot,t)\|_{H^1}^2 = \|g\|_{H^1}^2 = \int_{-\infty}^{\infty} \left[g(x)^2 + g_x(x)^2\right] dx.$$

Thus the $H^1$-norm of solutions is a conserved quantity; the law (3.2) corresponds to conservation of momentum in some physical systems. Similarly, conservation of mass is expressed in the form

(3.3) $$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} g(x)\,dx.$$

Rewrite (3.1) in the form

$$(1-\partial_x^2)u_t = -(u + \frac{1}{2}u^2)_x,$$

view it as an ordinary differential equation $v - v'' = f$, where $f = -(u+\frac{1}{2}u^2)_x$, and solve it explicitly by inverting the operator $1-\partial_x^2$ by variation of constants or by Fourier analysis, for example, to obtain the formula

$$u_t(x,t) = -\int_{-\infty}^{\infty} M(x-y)\left[u_y(y,t) + u(y,t)u_y(y,t)\right] dy,$$

where $M(x) = \frac{1}{2}e^{-|x|}$. Provided that $u$ is bounded (or at least not exponentially growing as $x \to \pm\infty$,) integration by parts gives the alternative

(3.4) $$u_t(x,t) = -\int_{-\infty}^{\infty} K(x-y)\big[u(y,t) + \frac{1}{2}u^2(y,t)\big]\,dy,$$

where

$$K(z) = \frac{1}{2}\mathrm{sgn}(z)e^{-|z|}.$$

*Remark:* To obtain (3.4), break the integral in the previous equation at $y = x$ and integrate these two by parts separately, *viz.*

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-|x-y|}f'(y)\,dy &= \int_{-\infty}^{x} e^{-(x-y)}f'(y)\,dy + \int_{x}^{\infty} e^{-(y-x)}f'(y)\,dy\\ &= e^{y-x}f(y)\Big|_{-\infty}^{x} - \int_{-\infty}^{x} e^{y-x}f(y)\,dy + e^{x-y}f(y)\Big|_{x}^{\infty} + \int_{x}^{\infty} e^{x-y}f(y)\,dy\\ &= f(x) - \int_{-\infty}^{x} e^{y-x}f(y)\,dy - f(x) + \int_{x}^{\infty} e^{x-y}f(y)\,dy\\ &= \int_{-\infty}^{\infty} \mathrm{sgn}(x-y)e^{-|x-y|}f(y)\,dy.\end{aligned}$$

A formal integration with respect to time $t$ in (3.4) and application of the fundamental theorem of calculus yields

$$u(x,t) - u(x,0) = \int_0^t\int_{-\infty}^{\infty} K(x-y)\left[u(y,\tau) + \frac{1}{2}u^2(y,\tau)\right]\,dy\,d\tau,$$

or, since $u(x,0)$ is known,

(3.5) $$u(x,t) = g(x) + \int_0^t\int_{-\infty}^{\infty} K(x-y)\left[u(y,\tau) + \frac{1}{2}u^2(y,\tau)\right]\,dy\,d\tau.$$

Write (3.5) in the form

(3.6) $$u = Au,$$

where $A$ is the integral operator defined by the right-hand side of (3.5); that is, if $v = v(x,t)$ is a bounded continuous function, say, then

(3.7) $$Av(x,t) = g(x) + \int_0^t\int_{-\infty}^{\infty} K(x-y)\left[v(y,\tau) + \frac{1}{2}v^2(y,\tau)\right]\,dy\,d\tau.$$

The initial-value problem (3.1) has thereby been converted into the issue of existence of a fixed point of the operator $A$.



At least over a small interval time, existence of a fixed point follows from the Contraction Mapping Principle as we now show. For $T > 0$, let $C_T$ be the Banach space

$$C_T = C(\mathbb{R}\times[0,T]) = \left\{v : \mathbb{R}\times[0,T]\to\mathbb{R} : v \text{ is continuous and } \sup_{x,t}|v| < +\infty\right\},$$

normed by $\|v\|_{C_T} = \sup_{x\in\mathbb{R},0\le t\le T}|v(x,t)|$. Let $B_R = \{v : \|v\|_{C_T} \le R\}$ in $C_T$, where the constants $R > 0$ and $T > 0$ remain to be chosen. For any $R > 0$, the set $B_R$ is a closed subset of the Banach space $C_T$, so it is certainly a complete metric space.

THEOREM 3.1. *Let $g$ be a bounded and continuous function, say, $\sup|g(x)| \le b$. Then there is $T = T(b) > 0$ such that the integral equation (3.5) has a solution in $C_T$.*

*Proof.* The idea is to show that $A$ is a contraction mapping of $B_R$ into itself for suitable choices of $R$ and $T$. The result then follows from the Contraction Mapping Principle.

Step 1. If $v \in C_T$, then $Av \in C_T$ since $g$ and $v$ are bounded and continuous and $K \in L_1(\mathbb{R})$. Indeed, we have

(3.8) $$\|Av\|_{C_T} \le \sup_{x\in\mathbb{R}}|g(x)| + T(\|v\|_{C_T} + \frac{1}{2}\|v\|_{C_T}^2) < \infty$$

since $\int|K(z)|\,dz = 1$.

Step 2. If $v, w \in C_T$, then

(3.9) $$\begin{aligned}\|Av - Aw\|_{C_T} &= \sup_{x,t}\left|\int_0^t\int_{\mathbb{R}} K(x-y)\Big[(v-w) + \frac{1}{2}(v^2-w^2)\Big]\,dy\,d\tau\right|\\ &\le T\|v-w\|_{C_T} + \frac{1}{2}\Big(\|v\|_{C_T} + \|w\|_{C_T}\Big)\|v-w\|_{C_T}\\ &\le T\Big(1 + \frac{1}{2}(\|v\|_{C_T} + \|w\|_{C_T})\Big)\|v-w\|_{C_T}.\end{aligned}$$

Step 3. Now suppose $v, w \in B_R$. Then (3.9) implies

$$\|Av - Aw\|_{C_T} \le T(1+R)\|v-w\|_{C_T}.$$

To apply the Contraction Mapping Principle, first demand that $T$ and $R$ are such that

$$\Theta = T(1+R) = \frac{1}{2},$$

say. Then choose $R = 2b$ where $b = \sup_{x\in\mathbb{R}}|g(x)|$ and notice this choice means that

$$\|Au\|_{C_T} \le \|Au - A0\|_{C_T} + \|A0\|_{C_T} \le \Theta\|u-0\|_{C_T} + b \le \frac{1}{2}\|u\|_{C_T} + \frac{1}{2}R \le R$$

if $u \in B_R$. Thus $A$ is a contraction of $B_R$ and the result follows. □

*Remark:* As the initial data gets larger, the interval of existence $T$ obtained by the above argument gets smaller.

To insure the fixed point $u$ of (3.5) is a solution of the initial-value problem (3.1), the regularity of $u$ is brought into focus.

PROPOSITION 3.2. *If $g \in C_b^2(\mathbb{R})$ and $u$ is a solution in $C_T$ of the integral equation (3.5), then $u$, $u_x$ and $u_{xx}$ are infinitely smooth functions of $t$, and $u$ solves (3.1) pointwise. More precisely, $\partial_t^m u \in C_T$, $\partial_t^m u_x \in C_T$, $\partial_t^m u_{xx} \in C_T$ for all $m \geq 0$, $\lim_{t\to 0} u(x,t) = g(x)$ in $C_b^2(\mathbb{R})$, and the continuous function $u_t + u_x + uu_x - u_{xxt}$ is identically equal to zero for $(x,t) \in \mathbb{R} \times [0,T]$.*

*Proof.* This is established by bootstrap-type arguments. Since

$$u(x,t) = g(x) + \int_0^t \int_{\mathbb{R}} K(x-y)(u + \frac{1}{2}u^2)\,dy\,d\tau$$

where $K(x) = \frac{1}{2}\mathrm{sgn}(x)e^{-|x|}$, then plainly $u$ is differentiable with respect to $t$ and

$$u_t = \int_{\mathbb{R}} K(x-y)(u + \frac{1}{2}u^2)\,dy \in C_T$$

is a bounded and continuous function. Elementary considerations then imply that $u_{tt}$ exists and

$$u_{tt} = \int_{\mathbb{R}} K(x-y)(u_t + uu_t)\,dy \in C_T$$

since $u, u_t \in C_T$. An inductive argument leads to the conclusion $\partial_t^m u \in C_T$ for all $m \geq 0$.

Now write the integral equation (3.5) as

$$u(x,t) = g(x) + \int_0^t \int_{-\infty}^x K(x-y)(u + \frac{1}{2}u^2)\,dy\,d\tau + \int_0^t \int_x^\infty K(x-y)(u + \frac{1}{2}u^2)\,dy\,d\tau$$

and use Leibnitz rule for the differentiation of integrals to obtain

$$u_x = g' + \int_0^t \left[u(x,\tau) + \frac{1}{2}u^2(x,\tau)\right] d\tau - \int_0^t \int_{-\infty}^\infty \frac{1}{2}e^{-|x-y|}(u + \frac{1}{2}u^2)\,dy\,d\tau.$$

This is plainly in $C_T$ since $u \in C_T$ and $g \in C_b^1(\mathbb{R})$. Note that $u_x$ is expressed in terms of $u$, so another inductive argument demonstrates that $\partial_t^m u_x \in C_T$ for $m \geq 0$. A similar argument shows $u_{xx}$ to exist and to be given by

$$(3.10)\quad u_{xx} = g''(x) + \int_0^t [u_x(x,\tau) + u(x,\tau)u_x(x,\tau)]\,d\tau + \int_0^t \int_{-\infty}^\infty K(x-y)(u + \frac{1}{2}u^2)\,dy\,d\tau.$$

The right-hand side clearly lies in $C_T$, and again, as $\partial_t^m u_x \in C_T$ for $m \geq 0$, so also $\partial_t^m u_{xx} \in C_T$. Using (3.5) in (3.10) gives

$$u_{xx} = g'' + \int_0^t (u_x + uu_x)\,d\tau + u - g(x).$$



Differentiating the last expression with respect to $t$, there appears

$$u_{xxt} = u_x + uu_x + u_t,$$

as hoped.

The fact that $u(\cdot,t)$ converges to $g$ as $t \downarrow 0$ is obvious and the proposition is established.

□

*Remark:* In fact, $u$ is an analytic function of $t$; i.e. $u(x,t)$ can be expanded as $\sum_{m=0}^\infty u_m(x)t^m$ for suitable functions $\{u_m\}_{m\geq 0}$, and the series has a positive radius of convergence.

Note that a solution cannot acquire more spatial regularity than that of the initial data. Suppose, for example, that $g \in C_b^k$ but $g \notin C_b^{k+1}$, and suppose that for some $t > 0$, $u(x,t) \in C_b^{k+1}(\mathbb{R})$. At this value of $t$,

$$g(x) = u(x,t) - \int_0^t \int_{\mathbb{R}} K(x-y)(u + \frac{1}{2}u^2)(y,s)\,dy\,ds.$$

At time $t$, $u \in C_b^{k+1}(\mathbb{R})$, and since $u \in C_b^k(\mathbb{R})$ for all $t$, so is $u + \frac{1}{2}u^2$. Hence after convolution with $K$, there obtains a function in $C_b^{k+1}$ in the spatial variable. The integration with respect to $t$ does not change the spatial regularity, and consequently it is adduced that $g \in C_b^{k+1}(\mathbb{R})$, contrary to assumption.

The issue in front of us now is how to extend the local existence theory to arbitrary time intervals. The following result will be helpful in pursuit of this goal.

LEMMA 3.3. *Let $k \geq 0$. Corresponding to given initial data $g \in C_b^k(\mathbb{R})$ and some $T > 0$, let $u$ be the solution in $C_T$ constructed via the Contraction Mapping Principle of the integral equation (3.5). Suppose additionally that for some $p \leq k$,*

$$(3.11)\qquad g, g', \cdots, g^{(p)} \to 0 \qquad \text{as } x \to \pm\infty.$$

*Then for any $t \in [0,T]$,*

$$(3.12)\qquad \partial_x^l \partial_t^m u(x,t) \to 0 \qquad \text{as } x \to \pm\infty, \text{ for } 0 \leq l \leq p \text{ and any } m \geq 0.$$

*Proof.* This will follow in three steps.

Step 1. First, it is demonstrated that the collection $C_T^0$ of functions in $C_T$ that tend to zero at infinity is a closed subspace. Clearly it is a linear subspace. If $v_n \in C_T^0$ and $v_n \to v$ in $C_T$, then $v \to 0$ at $\pm\infty$. To see this, fix $t \in [0,T]$ and write

$$|v(x,t)| \leq |v(x,t) - v_n(x,t)| + |v_n(x,t)|.$$

Let $\epsilon > 0$ be given and choose a corresponding $n_0$ so large that

$$\sup_{x,t} |v(x,t) - v_{n_0}(x,t)| \leq \frac{\epsilon}{2}.$$

Since $v_{n_0}$ is known to be null at $\pm\infty$, there exists $M$ such that $|v_{n_0}(x,t)| \leq \frac{\epsilon}{2}$ for $|x| \geq M$. Therfore if $|x| \geq M$, $|v(x,t)| \leq \epsilon$, which is to say $v \to 0$ at $\pm\infty$.

Step 2. Note that if $v \in C_T^0$, then so is $\int_{\mathbb{R}} e^{-|x-y|} v(y,t)\,dy$ and $\int_{\mathbb{R}} K(x-y)v(y,t)\,dy$. Let $\epsilon > 0$ be given. For $x \geq \xi$,

$$\begin{aligned} \left|\int_{\mathbb{R}} e^{-|x-y|} v(y,t)\,dy\right| &\leq e^{-x} \int_{-\infty}^{\xi} e^{y} |v(y,t)|\,dy + 2\sup_{y\geq\xi} |v(y,t)| \\ &\leq e^{-x+\xi} \sup_{y,t} |v(y,t)| + 2\sup_{y\geq\xi} |v(y,t)|. \end{aligned}$$

Since $v \to 0$ at $\pm\infty$, there exists $\xi$ such that $|v(y,t)| \leq \frac{\epsilon}{4}$ for $y \geq \xi$. Once $\xi$ is fixed, then $x$ can be chosen large enough that the first term is made smaller than $\frac{\epsilon}{2}$, hence the sum is smaller than $\epsilon$. A similar argument applies as $x \to -\infty$.

Step 3. Let $u_1(x,t) = g(x)$, for $0 \leq t \leq T$. By assumption, $u_1$ is null at $\pm\infty$. By Step 2, then $u_2 = Au_1$ is null at $\pm\infty$ and inductively, $u_n = Au_{n-1}$ is null at $\pm\infty$. Hence by Step 1, $u$ is asymptotically null, since $u_n \to u$ in $C_T$.

Now

$$u_t = \int K(x-y)(u + \frac{1}{2}u^2)\,dy$$

is asymptotically null by step 2, and by induction, so too are higher temporal derivatives. As discovered already,

$$u_x = g' + \int_0^t \int_{\mathbb{R}} e^{-|x-y|}(u + \frac{1}{2}u^2)\,dy\,d\tau + \int_0^t \left[u + \frac{1}{2}u^2\right] d\tau.$$

Since $u + \frac{1}{2}u^2$ is bounded and asymptotically null, so is $\int_0^t \left[u + \frac{1}{2}u^2\right] d\tau$ by the Dominated Convergence Theorem. Thus $u_x$ is asymptotically null by Step 2. Then

$$u_{xt} = g'' + u_x + uu_x + u - g$$

is asymptotically null and so on. A double induction finishes the proof. □

LEMMA 3.4. *Suppose $g \in C_b^k(\mathbb{R})$, $k \geq 2$ and $g \in H^1(\mathbb{R})$. Then there exists $T > 0$ such that the solution $u$ of the initial-value problem*

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} = 0, \\ u(x,0) = g(x), \end{cases}$$



*lies in $H^1(\mathbb{R})$ for all $t \in [0,T]$, and*

$$\|u(\cdot,t)\|_{H^1} = \|g(\cdot)\|_{H^1}.$$

*Proof.* As we already know, $u$ is a classical solution of the partial differential equation. Multiply (3.1) by $u$, then integrate with respect to $x$ over $[-M,M]$ and integrate by parts as in the calculations leading to (3.2). Then integrate with respect to time over $[0,t]$. Since elements $f \in H^1(\mathbb{R})$ which also lie in $C_b^2(\mathbb{R})$ have $f$ and $f'$ asymptotically null, it follows from Lemma 3.3 that $u$ and $u_x$ are null at infinity. Hence upon taking the limit as $M \to \infty$, there obtains

$$\int_{\mathbb{R}} [u^2(x,t) + u_x^2(x,t)]\,dx = \int_{\mathbb{R}} [u^2(x,0) + u_x^2(x,0)]\,dx,$$

for all $t$ for which the solution exists. That is to say,

$$\|u(\cdot,t)\|_{H^1(\mathbb{R})} = \|g\|_{H^1(\mathbb{R})}. \qquad \square$$

for all $t$ for which the solution exists.

This last point suffices to establish a global existence theorem.

THEOREM 3.6. *Suppose $g \in H^1(\mathbb{R}) \cap C_b^2(\mathbb{R})$, then there exists a unique global solution $u$ of the initial-value problem (3.1) such that the solution $u$ satisfies $\partial_t^m \partial_x^k u \in C_T$ for all $T > 0$, $0 \leq k \leq 2$, and $m \geq 0$. Moreover, for all $t \geq 0$ and the same range of $k$ and $m$, $\partial_t^m \partial_x^k u(x,t)$ tends to 0 at $\infty$ .*

*Proof.* As remarked already, if $g \in C_b^2 \cap H^1$ then $g, g' \to 0$ at $\pm\infty$. Hence there exists a local solution of the desired type at least on a small interval $[0,T]$, where $T$ depends only on $\sup_x |g(x)| = b$.

To extend this local solution, repeat the contraction-mapping argument using $u(x,T)$ as new data. Because of uniqueness, this has the effect of extending the range of the solution. A straightforward induction would get us out to $T = \infty$ if $\sup_x |u(x,t)|$ is bounded on bounded time intervals. In fact, instead of controlling $\sup_x |u(x,t)|$ directly, we control $\|u(\cdot,t)\|_{H^1}$, then use the property $\sup |f| \leq \|f\|_{H^1}$. The latter inequality follows because for any $f \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} f^2(x) &= 2\int_{-\infty}^{x} f(y)f'(y)\,dy \leq \int_{-\infty}^{x} (f^2 + f'^2)\,dy \\ &\leq \int_{-\infty}^{\infty} (f^2 + f'^2) = \|f\|_{H^1}^2. \end{aligned} \tag{3.13}$$

Any $f \in H^1(\mathbb{R})$ is a limit of elements in $C_c^\infty(\mathbb{R})$, so let $\{f_n\}_{n=1}^\infty \subset C_c(\mathbb{R})$ be such that $f_n \to f$ in $H^1$; then

$$\|f_n - f_m\|_\infty \leq \|f_n - f_m\|_{H^1} \to 0,$$

so $f_n$ is Cauchy in $C_b(\mathbb{R})$, say $f_n \to g$ in $C_b$; but $g = f$ pointwise a.e. since $f_n \to f$ in $H^1$, which means $f_n \to f$ *a.e.* Formula (3.13) then follows for any $f \in H^1(\mathbb{R})$ and $x \in \mathbb{R}$. It thus transpires that the solution is uniformly bounded on $\mathbb{R}$, independently of $t$. It may therefore be extended to a globally defined solution.

*Remark*: What were the crucial ingredients that went into the contraction-mapping argument? A moment's reflection reveals that the same argument would work in any Banach space $X = X(\mathbb{R})$ that is a Banach algebra, so that if $f, g \in X$, then $fg \in X$ and there is a universal constant $c_1$ for which $\|fg\|_X \le c_1 \|f\|_X \|g\|_X$ and for which convolution with the kernel $K$ is a bounded linear operator on $X$. In ordinary Sobolev spaces, $X$ is a Banach algebra if and only if the elements of $X$ are $L_\infty(\mathbb{R})$-functions and there is another universal constant $c_2$ such that if $f \in X$, then $\|f\|_\infty \le c_2 \|f\|_X$. In this situation, if $Y = C(0, T; X)$, then

$$\begin{aligned}\|u\|_Y &\le \|g\|_X + T\|K\|_{\mathcal{B}(X,X)} \|u + \frac{1}{2}u^2\|_Y \\ &\le \|g\|_X + T\|K\|_{\mathcal{B}(X,X)} \left(\|u\|_Y + \frac{c_1}{2}\|u\|_Y^2\right)\end{aligned}$$

and

$$\begin{aligned}\|Av - Aw\|_Y &\le T\|K\|_{\mathcal{B}(X,X)} \left(\|v - w\|_Y + \frac{1}{2}(\|v\|_Y + \|w\|_Y)\|v - w\|_Y\right) \\ & T\|K\|_{\mathcal{B}(X,X)} \left(1 + \frac{c_1}{2}(\|v\|_Y + \|w\|_Y)\right) \|v - w\|_Y.\end{aligned}$$

Let $M = \{u \in Y : \|u\| \le R\}$. To make the contraction-mapping argument work, it suffices to choose $R$ and $T$ so that

$$\begin{aligned}&\|g\|_X + T\|K\|_{\mathcal{B}(X,X)}(R + \frac{c_1}{2}R^2) \le R \quad \text{and} \\ &T\|K\|_{\mathcal{B}(X,X)}(1 + \frac{c_1}{2}R) = \theta < 1.\end{aligned}$$

These two conditions hold if $R$ is chosen large enough and $T$ is taken small.

Uniqueness follows straightforwardly because the solution is locally presented as the fixed point of a contraction mapping. We pass over the details, which are straightforward.

*Remark*: Uniqueness may also be established via a Gronwall-type argument.

COROLLARY 3.7. *Let $g \in H^k(\mathbb{R})$, for some $k \ge 1$. There is a $T > 0$ and a unique solution of the integral equation (3.5)*

$$u(x,t) = g(x) + \int_0^t \int_{\infty}^{\infty} K(x - y)\left(u(y,\tau) + \frac{1}{2}u^2(y,\tau)\right) dy\, d\tau$$

*in $C(0, T; H^k(\mathbb{R}))$, where $K(z) = \frac{1}{2}sgn(z)e^{-|z|}$. Moreover, $\partial_t^m u$ lies in $C(0, T; H^{k+1}(\mathbb{R}))$ for all $m > 0$.*



*Proof.* $H^k(\mathbb{R})$ is a Banach algebra embedded in $C_b(\mathbb{R})$, provided only that $k > \frac{1}{2}$, so the preceding remarks suffice to produce a local well-posedness result in the space $C(0, T; H^{k+1}(\mathbb{R}))$ for suitably small values of $T$. Differentiating the integral equation with respect to $t$ yields

$$u_t = \int_{\mathbb{R}} K(x - y)(u + \frac{1}{2}u^2)\, dy,$$

where $u + \frac{1}{2}u^2 \in C(0, T; H^k)$. It follows straightforwardly that $K*(u + \frac{1}{2}u^2) \in C(0, T; H^{k+1})$. Thus if $g \in H^1(\mathbb{R})$, then $u \in C(0, T; H^1)$ and $u_t \in C(0, T; H^2)$. The differential equation is satisfied pointwise almost everywhere. The result concerning higher-order derivatives follows easily by induction.

THEOREM 3.8. *(continuous dependence on the initial data). The mapping $g \mapsto u$ is continuous from $H^k \to C(0, T; H^k)$. It is also continuous from $C_b^k(\mathbb{R})$ into the associated solution space.*

*Proof.* This follows directly since continuity is a local property and the solution is given locally in time as the fixed point of a contraction mapping.

### *3.2. Bore Propagation*

In this subsection, consideration is given to an alternative type of initial disturbance, namely that associated with a bore. A bore is a surge of water, often generated by high tides in confined bays that have a river emptying into them. There are two general classes of such motions, a strong bore and the so-called weak or undular bore. Strong bores feature breaking and they can be quite steep. Undular bores are more gradual and do not have breaking as part of their makeup. Undular bores can fall into the regime where Boussinesq-KdV-type and BBM-type equations can serve as approximate models. If the initial-value problem is posited, then interest will focus on

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} = 0, & \text{for } x \in \mathbb{R}, \quad t \ge 0, \\ u(x,0) = g(x), \end{cases} \tag{3.12}$$

where $g \in C_B^2(\mathbb{R})$, $g' \in L_2(\mathbb{R})$, and $g(x) \to a$ as $x \to +\infty$, $g(x) \to b$ as $x \to -\infty$.

As in Section 3.1, the local existence, regularity, and global existence of solutions for (3.12) will be discussed. Actually, the local existence and regularity follow from the results developed in Section 3.1. Change the dependent variable to $v = u - g$ in (3.12). In terms of $v$, (3.12) becomes

$$\begin{cases} v_t - v_{xxt} + (g + v + \frac{1}{2}g^2 + gv + \frac{1}{2}v^2)_x = 0. \\ v(x,0) = 0. \end{cases} \tag{3.13}$$

The initial-value problem (3.13) is equivalent to the integral equation

$$v = \int_0^t M * (g + v + \frac{1}{2}g^2 + gv + \frac{1}{2}v^2)_x \, d\tau, \tag{3.14}$$

or, after integration with respect to $t$,

$$v = -\int_0^t M * (v_y + vv_y + gv_y + g'v)\, d\tau - \int_0^t M * (g' + \dot{g}g')\, d\tau$$

where $M(z) = \frac{1}{2}e^{-|z|}$. Since $g' \to 0$ at $\pm\infty$, $\int_0^t M * (g' + gg') \to 0$ at $\pm\infty$. Consequently, we can essentially argue as in Lemma 3.4 to conclude that $v, v_x, v_{xt} \to 0$ at $\pm\infty$.

With local existence settled, a search is initiated for *a priori* bounds that will allow the local solution to be extended globally. To this end, multiply (3.13) by $v$ and integrate over $[-R, R] \times [0, t]$. After suitable integrations by parts and then passing to the limit as $R \to \infty$, (which may be justified since $v \to 0$ at infinity), there follows the relation

$$\begin{aligned} \frac{1}{2}\int_{-\infty}^{\infty} [v^2 + v_x^2]\, dx &= \int_{-\infty}^{\infty} [v^2(x,0) + v_x^2(x,0)]\, dx + \int_0^t \int_{-\infty}^{\infty} [(1+g)g'v - gvv_x]\, dx \\ &\le (1 + \|g\|_\infty) \int_0^t \|g'\| \|v\|\, d\tau + \|g\|_\infty \int_0^t \|v\| \|v_x\|\, d\tau \\ &\le \frac{C}{2} \int_0^t \int_{\mathbb{R}} [v^2 + v_x^2]\, dx + \frac{C_1}{2}, \end{aligned} \tag{3.15}$$

where $C$ and $C_1$ are constants only dependent on $\|g'\|_{L_2}$ and $\|g\|_{L_\infty}$. Gronwall's lemma yields

$$\int_{-\infty}^{\infty} [v^2(x,t) + v_x^2(x,t)]\, dx \le \frac{C_1}{C}(e^{ct} - 1),$$

which is enough to extend the local solution to $T = +\infty$ by iteration of the contraction-mapping argument.

### *3.3. Theory for the Korteweg-de Vries equation.*

Without the small parameters, the Boussinesq-KdV-equation can be written in the form

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{for } x \in \mathbb{R}, \quad t \ge 0, \\ u(x,0) = g(x). \end{cases} \tag{3.16}$$

A shift to travelling coordinates by the change of variables $\tilde{u}(x,t) = u(x+t,t)$, turns (3.16) into the slightly simpler equation

$$\begin{cases} \tilde{u}_t + \tilde{u}\tilde{u}_x + \tilde{u}_{xxx} = 0, & \text{for } x \in \mathbb{R}, \quad t \ge 0, \\ \tilde{u}(x,0) = g(x). \end{cases} \tag{3.17}$$



Dropping the tildes and regularizing the differential equation, there emerges the initial-value problem

$$\begin{cases} u_t + uu_x + u_{xxx} - \epsilon u_{xxt} = 0 & \text{for } x \in \mathbb{R}, \quad t \ge 0 \\ u(x,0) = g(x), \end{cases} \tag{3.18}$$

where $\epsilon > 0$ is fixed for the time being.

*Remark*: The differential equation in (3.18) looks a little peculiar. A more standard regularization would be $u_t + u_x + uu_x + u_{xxt} - \epsilon u_{xxxx} = 0$, so making the equation parabolic. The present regularization is motivated by the physics - in particular by the dispersion relation. Moreover, a certain interesting question falls out easily if the regularization appearing above is contemplated.

Consider the change of variables $v(x,t) = \epsilon u(\epsilon^{\frac{1}{2}}(x-t), \epsilon^{\frac{3}{2}}t)$. In terms of $v$, (3.18) becomes

$$\begin{cases} v_t + v_x + vv_x - v_{xxt} = 0, \\ v(x,0) = \epsilon g(\epsilon^{\frac{1}{2}}x). \end{cases} \tag{3.19}$$

For fixed positive $\epsilon$, (3.19) is the problem which was dealt with in Section 3.1. Hence attention turns to the limit $\epsilon \to 0$, where one hopes to recover a solution of the Boussinesq-KdV initial-value problem. The crux of the matter is $\epsilon$-independent bounds on the solutions of (3.19).

*A priori bounds*

Because of the continuous dependence result from the last section, we might as well work with initial data $g \in H^\infty(\mathbb{R})$. For such initial data, the corresponding solution $u$ is $C^\infty$ in $x$ and $t$, and $u(\cdot,t) \in L^2(\mathbb{R})$ along with its partial derivatives of all orders. Moreover, everything in sight goes to 0 at $\pm\infty$.

Multiply (3.18) by $u$, integrate with respect to $x$ over $\mathbb{R}$ to obtain

$$\int_{-\infty}^{\infty} (uu_t + u^2u_x + uu_{xxx} - \epsilon uu_{xxt})\, dx = 0,$$

or after integration by parts,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (u^2 + \epsilon u_x^2)\, dx = 0.$$

Hence, we have

$$\int_{-\infty}^{\infty} (u^2 + \epsilon u_x^2)\, dx = \int_{-\infty}^{\infty} (g^2 + \epsilon g'^2)\, dx \le \|g\|_{H^1}^2.$$

where the restriction $\epsilon \leq 1$ has been imposed. Independently of $\epsilon \in (0,1]$,

$$\|u(\cdot,t)\|_{L^2} \leq \|g\|_{H^1}. \tag{3.20}$$

Rewrite (3.18) as

$$u_t + (\frac{1}{2}u^2 + u_{xx} - \epsilon u_{xt})_x = 0,$$

multiply it by $\frac{1}{2}u^2 + u_{xx} - \epsilon u_{xt}$ and integrate over $\mathbb{R}$ to obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty}\left(\frac{1}{6}u^3 - u_x^2\right) dx = 0,$$

which is to say

$$\int_{-\infty}^{\infty}(u_x^2 - \frac{1}{3}u^3)\,dx = \text{constant} = \int_{-\infty}^{\infty}(g'^2 - \frac{1}{3}g^3)\,dx.$$

In consequence of this conservation law, which is independent of $\epsilon$, there follows the inequality

$$\begin{aligned}\int_{\mathbb{R}} u_x^2 &\leq |u|_\infty \int_{\mathbb{R}} u^2\,dx + \int_{\mathbb{R}} g_x^2\,dx + \frac{1}{3}|g|_\infty \int_{\mathbb{R}} g^2\,dx \\ &\leq \|g\|_{H^1}\|u\|_{L^2}{}^{\frac{1}{2}}\|u_x\|_{L^2}{}^{\frac{1}{2}} + +\|g\|_{H^1}^2 + \frac{1}{3}\|g\|_{H^1}^3 \\ &\leq \|g\|_{H^1}^{\frac{3}{2}}\|u_x\|_{L^2}^{\frac{1}{2}} + \|g\|_{H^1}^2 + \frac{1}{3}\|g\|_{H^1}^3.\end{aligned} \tag{3.21}$$

If $A = A(t) = \|u_x(\cdot,t)\|$, then (3.24) is rewritten as:

$$A^2 \leq CA^{\frac{1}{2}} + D \leq \frac{1}{2}A^2 + \frac{1}{4}C^2 + D \tag{3.22}$$

where $C = \|g\|_{H^1}^{\frac{3}{2}}$ and $D = \|g\|_{H^1}^2 + \frac{1}{3}\|g\|_{H^1}^3$ are only dependent on the initial data $g$. It follows that

$$\|u_x\|^2 = A^2 \leq 2(C_1 + D) = D_1 = D_1(\|g\|_{H^1}).$$

Thus in summary, for all $t \geq 0$,

$$\|u(\cdot,t)\|_{H^1} \leq a(\|g\|_{H^1}). \tag{3.23.}$$

The next step is to obtain $\epsilon$-independent bounds on the $H^2$-norm of solutions. This turns out to somewhat tedious and we content ourselves here with a statement of the relevant result. A proof may be found in the paper of Bona and Smith (1975).



LEMMA 3.8. *Let $u$ be a smooth solution of regularized equation (3.18) corresponding to $g \in H^\infty(\mathbb{R})$. Then there exists $\epsilon_0 = \epsilon_0(T, \|g\|_{H^3})$ such that if $0 < \epsilon \leq \epsilon_0$, then*

$$\|u\|_{C(0,T;H^2)} \leq a(\|g\|_{H^3}),$$

*where $a$ is independent of $T$.*

With an $H^2$-bound in hand, further progress is much easier.

LEMMA 3.9. *Let $m > 2$ and suppose that $g \in H^m$, and that $\|u\|_{C(0,T;H^{m-1})}$ is bounded, independently of $\epsilon \leq \epsilon_0$, with a bound dependent only on $T$ and $\epsilon_0$. It is then adduced that $u$ is bounded in $C(0,T;H^m)$ with a bound dependent only on $T$, $\epsilon_0$, $\|g\|_{H^m}$ and $\epsilon^{\frac{1}{2}}\|g\|_{H^{m+1}}$.*

*Proof.* . Multiply (3.18) by $u_{(2m)} = \partial_x^{2m}u$, integrate over $\mathbb{R}$ and integrate by parts; there appears

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}(u_{(m)}^2 + \epsilon u_{(m+1)}^2)\,dx &= -\int_{\mathbb{R}}(u^2)_{(m+1)}u_{(m)} \\ &= \epsilon_0\int_{\mathbb{R}} uu_{(m+1)}u_{(m)}\,dx \\ &+ c_1\int_{-\infty}^{\infty}\left(u_xu_{(m)}^2 + u_{(m)}\sum_{r=2}^{m-2}c_ru_{(r)}u_{(m+1-r)} + u_{(m-1)}^2u_{(m)}\right)dx \\ &\leq |u_x|_\infty\int_{\mathbb{R}} u_{(m)}^2 + c\|u_{(m)}\|_{L^2} \leq c_1\|u_{(m)}\|^2 + c_2\|u_{(m)}\|.\end{aligned} \tag{3.32}$$

The result in view now follows from a Gronwall-type estimate. □

With these $\epsilon$-independent bounds in hand, the limit $\epsilon \downarrow 0$ is investigated. There are two or three ways to handle the passage to the limit. The way chosen here leads to sharper regularity results, but an easier method could be adopted, using weak compactness arguments in $L^\infty(0,T;H^m)$, for example.

Fix initial data $g \in H^s$, where $s \geq 3$, say, and consider the regularized equation

$$(P_\epsilon)\qquad \begin{cases} u_t + uu_x + u_{xxx} - \epsilon u_{xxt} = 0, \\ u(x,0) = g_\epsilon(x), \end{cases}$$

where $\hat{g}_\epsilon(\xi) = \phi(\epsilon^{\frac{1}{6}}\xi)\hat{g}(\xi)$. The function $\phi$ is a $C^\infty$-function, with $0 \leq \phi \leq 1$ everywhere, $\phi(0) = 1$, $\phi \to 0$ exponentially rapidly at $\pm\infty$, and such that

$$\psi(\xi) = 1 - \phi(\xi)$$

has a zero of infinite order at $\xi = 0$. That is, $\phi$ is very flat at 0 !

The following lemma detailing how various Sobolev norms of $g_\epsilon$ behave as a function of $\epsilon$ will be useful.

LEMMA 3.10. *Let $g \in H^s$, $s \geq 3$ and let $g_\epsilon$ be as above. Then $g_\epsilon \in H^\infty$ and*

$$\|g_\epsilon\|_{H^{s+j}} = O(\epsilon^{-\frac{j}{6}}), \qquad j = 1, 2, \cdots$$

$$\|g - g_\epsilon\|_{H^{s-j}} = o(\epsilon^{\frac{j}{6}}), \qquad j = 0, 1, 2, \cdots$$

*as $\epsilon \downarrow 0$. The first bounds hold uniformly on bounded sets and the second set of bounds hold uniformly on compact sets. (If $o$ is replaced by $O$, then the second bound holds uniformly on bounded sets).*

*Proof.* This is an easy calculation in the Fourier transformed variables. □

COROLLARY 3.11. *Let $u_\epsilon$ be the solution of problem $P_\epsilon$, where $\epsilon$ is in $(0, 1]$, Then for each $T > 0$ and $m = 1, 2, \cdots$,*

*i) $u_\epsilon$ is bounded in $C(0, T; H^s)$ independently of $\epsilon$ sufficiently small, and*

*ii) $\epsilon^{\frac{m}{6}} u_\epsilon$ is bounded in $C(0, T; H^{s+m})$ independently of $\epsilon$ sufficiently small.*

*Proof.* We know from Lemma 3.8 that, for $T > 0$ given,

$$\|u_\epsilon\|_{C(0,T;H^s)} \leq C(T, \epsilon_0, \|g_\epsilon\|_{H^s}, \epsilon^{\frac{1}{2}}\|g_\epsilon\|_{H^{s+1}}).$$

Hence part $(i)$ is seen to be valid since both $\|g_\epsilon\|_{H^s}$ and $\epsilon^{\frac{1}{2}}\|g_\epsilon\|_{H^{s+1}}$ are bounded. For $m > 0$, a careful assessment of the energy-type estimates appearing earlier shows that

$$\|\epsilon^{\frac{m}{6}} u_\epsilon\|_{H^{s+m}} \leq C(T, \epsilon_0, \|g\|_{H^s}). \tag{3.35}$$

The result follows from this. □

*Remark:* The inequality (3.35) just put forward is not completely trivial to derive. Here is one way it may be ascertained. To make things concrete, consider the case $s = 3$. It is known already in this case that, independently of $\epsilon$ sufficiently small,

(a) $\|u_{(\cdot}, t)\epsilon\|_{H^3} \leq C(T, \epsilon_0, \|g_\epsilon\|_{H^3}, \epsilon^{\frac{1}{2}}\|g_\epsilon\|_{H^4}) \leq C(T, \epsilon_0, \|g\|_{H^3})$ for $0 \leq t \leq T$. As before, by multiplying by $u_{\epsilon(8)}$, and interating by parts, we obtain

(b)

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} (u^2_{\epsilon(4)} + \epsilon u^2_{\epsilon(5)})\,dx &= \int_{\mathbb{R}} (u^2_\epsilon)_{(5)} u_{\epsilon(4)}\,dx \\
&= \int_{\mathbb{R}} u_{\epsilon x} u^2_{\epsilon(4)}\,dx + \int_{\mathbb{R}} u_{xx} u_{\epsilon(3)} u_{\epsilon(4)}\,dx \\
&\leq c(\|g\|_{H^3}) \int_{\mathbb{R}} u^2_{\epsilon(4)}\,dx + c(\|g\|_{H^3}) \sqrt{\int_{\mathbb{R}} u^2_{\epsilon(4)}\,dx} \\
&\leq c_1(\|g\|_{H^3}) \int_{\mathbb{R}} u^2_{\epsilon(4)}\,dx + c_2(\|g\|_{H^3}).
\end{aligned}$$



Multiply (b) by $\epsilon^{\frac{1}{3}}$ and apply Gronwall's inequality to reach

$$\int_{\mathbb{R}} \left(\epsilon^{\frac{1}{3}} u^2_{\epsilon(4)} + \epsilon^{1+\frac{1}{3}} u^2_{\epsilon(5)}\right) dx \leq e^{c_1 t} \int_{\mathbb{R}} \left(\epsilon^{\frac{1}{3}} g^2_{(4)} + \epsilon^{1+\frac{1}{3}} g^2_{(5)}\right) dx + \frac{c_2 \epsilon^{\frac{1}{3}}}{c_1}(e^{c_1 t} - 1).$$

Because of Lemma 3.9, the right-hand side is bounded as in (3.35). Continue by multiplying the differential equation by $u_{\epsilon(10)}$ and integrating to come to

(c)

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} \left(u^2_{\epsilon(5)} + \epsilon u^2_{\epsilon(6)}\right) dx &= \int_{\mathbb{R}} (u^2_\epsilon)_{(6)} u_{\epsilon(5)}\,dx \\
&= \int_{\mathbb{R}} u_{\epsilon x} u^2_{\epsilon(5)}\,dx + \int_{\mathbb{R}} u_{\epsilon xx} u_{\epsilon(4)} u_{\epsilon(5)}\,dx + \int_{\mathbb{R}} u^2_{\epsilon xxx} u_{\epsilon(4)} u_{\epsilon(5)}\,dx \\
&\leq c \int_{\mathbb{R}} u^2_{\epsilon(5)}\,dx + c\|u_{\epsilon(4)}\|\|u_{\epsilon(5)}\| + c\|u_{\epsilon(4)}\|u_{\epsilon(5)}\|,
\end{aligned}$$

whence

$$\frac{d}{dt}\int_{\mathbb{R}} \left(\epsilon^{\frac{2}{3}} u^2_{\epsilon(5)} + \epsilon^{1+\frac{2}{3}} u^2_{\epsilon(6)}\right) dx \leq \int_{\mathbb{R}} \epsilon^{\frac{2}{3}} u^2_{\epsilon(5)}\,dx + c\epsilon^{\frac{1}{3}}\|u_{\epsilon(4)}\|\epsilon^{\frac{1}{3}}\|u_{\epsilon(5)}\|.$$

Gronwall's Lemma still gives (3.35) in light of Lemma 3.9. In general, write

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} \left(u^2_{\epsilon(k)} + \epsilon u_{\epsilon(k+1)}\right) dx &= \int_{\mathbb{R}} u^2_{\epsilon(k+1)} u_{\epsilon(k)}\,dx \\
&\leq C(\|g\|_{H^2}) \int_{-\infty}^{\infty} u^2_{\epsilon(k)}\,dx + D(\|g\|_{H^2})\|u_{\epsilon(k-1)}\|\|u_{\epsilon(k)}\| + E(\|g\|_{H^3} \\
&\leq C(\|g\|)_{H^3} \int_{-\infty}^{\infty} u^2{}_{(k)}\,dx + D(\|g\|_{H^3})\|u_\epsilon\|^2_{(k)}.
\end{aligned}$$

Letting $k = m + s$ and multiplying by $\epsilon^{\frac{m}{3}}$ yields

$$\frac{d}{dt}\int_{\mathbb{R}} \left(\epsilon^{\frac{m}{3}} u^2_{(m+s)} + \epsilon^{1+\frac{3}{m}} u^2_{(m+s+1)}\right) dx \leq C\int_{\mathbb{R}} \epsilon^{\frac{m}{3}} u^2_{(m+s)}\,dx + D\epsilon^{\frac{m}{3}}\|u\|^2_{(m+s+1)}. \tag{3.37}$$

An inductive argument shows the term connoted $D$ is bounded and even tends to 0 as $\epsilon \downarrow 0$. Hence, Gronwall's inequality gives the advertised result.

COROLLARY 3.12. *$\partial_t u_\epsilon$ is bounded in $C(0, T; H^{s-3})$ and $\epsilon^{\frac{m}{6}}\partial_x^{s+m-3}\partial_t u_\epsilon$ is bounded in $C(0, T; H^1)$, independently of $\epsilon$ sufficiently small for $m = 1, 2, 3, 4, 5$.*

*Proof.* Invert the operator $(1 - \epsilon\partial_x^2)$ to reach

$$u_{\epsilon t} = (1 - \epsilon\partial_x^2)^{-1}(-u_\epsilon u_{\epsilon x} - u_{\epsilon xxx}),$$

from which it follows that, independently of $\epsilon > 0$,

$$\begin{aligned}\|u_{\epsilon t}\|_{s-3} &\le \|u_\epsilon\|_{s-3}\|u_\epsilon\|_{s-2} + \|u_\epsilon\|_s \\ &\le \|u_\epsilon\|_s^2 + \|u_\epsilon\|_s \le C,\end{aligned}$$

and similarly,

$$\begin{aligned}\epsilon^{\frac{m}{6}}\|\partial_x^{s+m-3}\partial_t u_\epsilon\| &\le \epsilon^{\frac{m}{6}}\|\partial_t u_\epsilon\|_{m+s-3} \\ &\le \epsilon^{\frac{m}{6}}\left(\|u_\epsilon\|_{s+m-3}\|u_\epsilon\|_{s+m-2} + \|u_\epsilon\|_{s+m}\right) \\ &\le \epsilon^{\frac{m}{6}}\|u_\epsilon\|_{s+m} + \left(\epsilon^{\frac{m-3}{6}}\|u_\epsilon\|_{s+m-3}\|u_\epsilon\|\epsilon^{\frac{m-2}{6}}\right)\epsilon^{\frac{m-(m-3+m-2)}{6}} \\ &\le C + C\epsilon^{\frac{5-m}{6}} \le C\end{aligned}$$

if $m \le 5$. □

PROPOSITION 3.13. *Let $\{u_\epsilon\}$ be the solutions of $(P_\epsilon)$. Then $\{u_\epsilon\}$ is Cauchy in $C(0,T;H^s)$, for $g \in H^s$.*

*Proof.* Let $u = u_\epsilon$ and $v = u_\delta$, where $\delta \le \epsilon$, say. It is enough to show that $\|u - v\|_{H^s}$ can be made arbitrarily small, independently of $t \in [0,T]$, by choosing $\epsilon$ small enough. If $w = u - v$, then

$$w_t + (uw + \frac{1}{2}w^2)_x + w_{xxx} - \delta w_{xxt} = (\epsilon - \delta)u_{xxt} \tag{3.38}$$

with $w(x,0) = g_\epsilon(x) - g_\delta(x) = h(x)$, say. Multiply (3.38) by $w_{(2j)}$ and integrate over $\mathbb{R}$ with respect to $x$, to get

$$\begin{aligned}\int_{\mathbb{R}}(w_{(j)}^2 + \delta w_{(j+1)}^2)\,dx &= \int_{\mathbb{R}}(h_{(j)}^2 + \delta h_{(j+1)}^2)\,dx \\ &\quad -2\int_0^t\int_{\mathbb{R}}\left((uw + \frac{1}{2}w^2)_{(j+1)} - (\epsilon - \delta)u_{t,(j+2)}\right)w_{(j)}\,dx\,d\tau.\end{aligned} \tag{3.39}$$

Denote $V_j^2(t) = \int_{\mathbb{R}}(w_{(j)}^2 + \delta w_{(j+1)}^2)\,dx$. The details are developed for $s = 3$. First for $j = 0$, the master relation (3.39) looks like

(3.40)

$$\begin{aligned}\int_{\mathbb{R}}(w^2 + \delta w_x^2)\,dx &= \int_{\mathbb{R}}(h^2 + \delta h_x^2)\,dx - 2\int_0^t\int_{\mathbb{R}}((uw)_x + ww_x - (\epsilon - \delta)u_{xxt})\,w\,dx d\tau \\ &\le \int_{\mathbb{R}}(h^2 + \delta h_x^2)\,dx + 2\int_0^t\left[|u_x|_\infty\int_{\mathbb{R}}w^2\,dx + c\epsilon^{\frac{2}{3}}\sqrt{\int w^2}\right]d\tau.\end{aligned}$$

In consequence, we have

$$V_0^2(t) \le V_0^2(0) + 2\int_0^t[c_1V_0^2(\tau) + c_2\epsilon^{\frac{2}{3}}V_0(\tau)]\,d\tau.$$



Applying Gronwall's Lemma gives

$$\|w(\cdot,t)\| \le V_0(t) \le V_0(0)e^{c_1T} + \frac{\epsilon^{\frac{2}{3}}c_2}{c_1}(e^{c_1T} - 1),$$

where

$$\begin{aligned}V_0(0) &= \left[\int_{\mathbb{R}}(g_\delta - g_\epsilon)^2 + \delta(g_\delta' - g_\epsilon')^2\right]^{\frac{1}{2}} \\ &\le \|g - g_\delta\|_{H^1} + \|g - g_\epsilon\|_{H^1} \\ &\le C\epsilon^{\frac{1}{3}} \quad \text{for} \quad \epsilon \le 1.\end{aligned}$$

Hence $\{u_\epsilon\}$ is indeed Cauchy in $C(0,T;L^2)$ and we have the estimate

$$\|u_\epsilon - u_\delta\|_{L^2} \le C\epsilon^{\frac{1}{3}}$$

for $\delta \le \epsilon$ and $\epsilon$ sufficiently small. Next for $j = 1$, (3.39) gives

$$V_1^2(t) = V_1^2(0) - 2\int_0^t\int_{\mathbb{R}}\left(\frac{1}{2}w_x^2 + \frac{3}{2}u_x\right]w_x^2\,dx - 2\int_0^t\int_{\mathbb{R}}[u_{xx}w - (\epsilon - \delta)u_{xxt}]\,w_x\,dx,$$

where $|w_x|_\infty$, $|u_x|_\infty$, $|u_{xx}|_\infty$ and $\epsilon^{\frac{1}{2}}\|u_{xxxt}\|$ are all bounded, independently of $\epsilon$ sufficiently small. Hence,

$$V_1^2(t) \le V_1^2(0) + 2\int_0^t c_1V_1^2(\tau)\,d\tau + 2\int_0^t\left[c_2\|w\|V_1(\tau) + c_3\epsilon^{\frac{1}{2}}V_1(\tau)\right]d\tau,$$

or, by use of Gronwall's lemma again,

$$V_1(t) \le V_1(0)e^{c_1T} + C\frac{\epsilon^{\frac{1}{3}} + \|w\|_{C(0,T;L^2)}}{c_1}(e^{c_1T} - 1)$$

where

$$V_1(0) \le \|g - g_\epsilon\|_1 + \|g - g_\delta\|_1 + \delta^{\frac{1}{2}}\|g - g_\epsilon\|_2 + \delta^{\frac{1}{2}}\|g - g_\delta\|_2 \le C\epsilon^{\frac{1}{3}}.$$

As a consequence, we have

$$\|w_x(\cdot,t)\| \le V_1(t) \le C\epsilon^{\frac{1}{3}}. \tag{3.41}$$

The relation (3.39) for $j = 2$ allows us to infer

$$\begin{aligned}&\int_{\mathbb{R}}(w_{xx}^2 + \delta w_{xxx}^2)\,dx \\ &= \int_{\mathbb{R}}(h_{xx}^2 + \delta h_{xxx}^2)\,dx - 2\int_0^t\left((uw + \frac{1}{2}w^2)_{xxx}w_{xx} - (\epsilon - \delta)u_{xxxxt}w_{xx}\right)dx\,d\tau \\ &= 2\int_0^t\int_{\mathbb{R}}\left(-\frac{5}{2}(u_x + w_x)w_{xx}^2 - 3u_{xx}w_xw_{xx} - u_{xxx}ww_{xx}\right)dx\,d\tau \\ &\quad - (\epsilon - \delta)\int_0^t\int_{\mathbb{R}}u_{xxxxt}w_{xx}\,dx\,d\tau,\end{aligned}$$

in which we know that, for $0 \leq t \leq T$, $|u_x||w_x| \leq C$, $|u_{xx}| \leq C$, $\|u_{xxx}\| \leq C$, $\|u_{xxxt}\| \leq C\epsilon^{-\frac{2}{3}}$, $\|w_x\| \leq C\epsilon^{\frac{1}{3}}$, $|w| \leq C\epsilon^{\frac{1}{3}}$, where the $C's$ are constants dependent only on $T$ and on $\|g\|_{H^3}$. Thus it transpires that

$$\begin{aligned}
\int_{\mathbb{R}} (w_{xx}^2 + \epsilon w_{xxx}^2)\, dx &\leq \int_{\mathbb{R}} (h_{xx}^2 + \delta h_{xxx}^2)\, dx + 2\int_0^t \int_{\mathbb{R}} (Cw_{xx}^2 + C\epsilon^{\frac{1}{3}}|w_{xx}|)\, dx \\
&\quad + 2\epsilon \int_0^t \|w_{xx}\|\|u_{xxxxt}\|\, dx \\
&\leq \int_{\mathbb{R}} (h_{xx}^2 + \delta h_{xxx}^2)\, dx + 2\int_0^t (C\|w_{xx}\|^2 + C\epsilon^{\frac{1}{3}}\|w_{xx}\|)\, dx,
\end{aligned}$$

where

$$V_2^2(0) = \int_0^t (h_{xx}^2 + \delta h_{xxx}^2)\, dx \leq C\epsilon^{\frac{1}{3}};$$

in consequence,

$$V_2^2(t) \leq C\epsilon^{\frac{1}{3}} + 2C\int_0^t [V_2^2(\tau) + \epsilon^{\frac{1}{3}} V_2(\tau)]\, d\tau,$$

from which it follows that

$$\|w_{xx}(\cdot,t)\| \leq V_2(t) \leq C\epsilon^{\frac{1}{6}} \qquad \text{for} \quad 0 \leq t \leq T. \tag{3.42}$$

To finish, take $j = 3$ in the master relation (3.39) and derive

$$V_3^2(t) = V_3^2(0) + 2\int_0^t \int_{\mathbb{R}} \left( (uw + \frac{1}{2}w^2)_{xxxx} w_{xxx} - (\epsilon - \delta) u_{xxxxxt} w_{xxx} \right) dx\, d\tau.$$

It is known that

$$\|u_{xxxxxt}\| \leq \frac{C}{\epsilon^{\frac{5}{6}}},$$

and so

$$\begin{aligned}
2(\epsilon - \delta) \int_0^t \int_{\mathbb{R}} u_{xxxxxt} w_{xxx}\, dx &\leq 2\epsilon C \epsilon^{-\frac{5}{6}} \int_0^t \|w_{xxx}\|\, d\tau \\
&\leq 2C\epsilon^{\frac{1}{6}} \int_0^t V_3(\tau)\, d\tau.
\end{aligned}$$

The other term under the integral is estimated as follows:

$$\begin{aligned}
&2\int_0^t \int_{\mathbb{R}} [(uw + \frac{1}{2}w^2)_{xxxx} w_{xxx}]\, dx\, d\tau \\
&= 2\int_0^t \int_{\mathbb{R}} \left( \frac{7}{2}(u_x + w_x) w_{xxx}^2 - 4w_x u_{xxx} w_{xxx} - 6u_{xx} w_{xx} w_{xxx} - u_{xxxx} w w_{xxx} \right) dx\, d\tau \\
&\leq 2C \int_0^t \left( \|w_{xxx}\|^2 + \|w_{xxx}\|(\epsilon^{\frac{1}{6}} + \epsilon^{\frac{1}{6}} + \epsilon^{-\frac{1}{6}}\epsilon^{\frac{1}{3}}) \right) d\tau \\
&\leq 2C \int_0^t \left( V_3^2(\tau) + \epsilon^{\frac{1}{6}} V_3(\tau) \right) d\tau.
\end{aligned}$$



Since we know already that

$$V_3^2(0) = \int_{-\infty}^{\infty} (h_{xxx}^2 + \delta h_{xxxx}^2)\, dx = o(1),$$

as $\epsilon \to 0$ it follows that

$$V_3^2(t) \leq o(1) + C\int_0^t (V_3^2(\tau) + \epsilon^{\frac{1}{6}} V_3(\tau)\, d\tau.$$

Applying Gronwall's lemma gives

$$\|w_{xxx}(\cdot,t)\| \leq V_3(t) \leq o(1) \qquad \text{as } \epsilon \downarrow 0.$$

Summing the preceding estimates on $\|w\|$, $\|w_x\|$, $\|w_{xx}\|$ and $\|w_{xxxx}\|$ leads to

$$\|w(\cdot,t)\|_{H^3} = o(1) \qquad \text{as } \epsilon \downarrow 0,$$

uniformly on $0 \leq t \leq T$.

Corollary 3.14. *$\{\partial_t u_\epsilon\}_{\epsilon>0}$ is Cauchy in $C(0,T;H^{s-3})$.*

Theorem 3.15. *Let $g \in H^s$, $s \geq 3$. Then there exists unique solution $u$ which lies in $C(0,T;H^s)$ for all $T > 0$, to the KdV-equation posed with initial data $g$.*

*Proof.* Uniqueness is a simple Gronwall estimate. Existence is likewise easy. Let $\{u_\epsilon\}$ be associated solutions of problem $\{P_\epsilon\}$. Then, because of Proposition 3.1 and Corollary 3.14, there exists $u$ that, for each $T > 0$, lies in $C(0,T;H^s)$ and is such that

$$\begin{aligned}
u_\epsilon &\to u && \text{in} \quad C(0,T;H^s), \\
\partial_t u_\epsilon &\to v && \text{in} \quad C(0,T;H^{s-3}), \\
\partial_x(u_\epsilon^2) &\to \partial_x(u^2) && \text{in} \quad C(0,T;H^{s-1}), \\
\partial_{xxx} u_\epsilon &\to \partial_{xxx} u && \text{in} \quad C(0,T;H^{s-3}), \\
\epsilon^{\frac{1}{3}} \partial_x^2 \partial_t u_\epsilon & && \text{is bounded in} \quad C(0,T;H^{s-3}), \\
\text{so,} \quad \epsilon \partial_x^2 \partial_t &\to 0 && \text{in} \quad C(0,T;H^{s-3}).
\end{aligned}$$

We'd like to know that $v = u_t$. If $\phi \in C_0^\infty(\mathbb{R} \times [0,T])$, then

$$\int_0^T \int_{\mathbb{R}} u_\epsilon \phi_t\, dx\, d\tau \to \int_0^T \int_{\mathbb{R}} u\phi_t\, dx\, d\tau \qquad \text{as } \epsilon \downarrow 0.$$

On the other hand,

$$\int_0^T \int_{\mathbb{R}} u_\epsilon \phi_t\, dx\, d\tau = -\int_{\mathbb{R}} \int_0^T (\partial_t u_\epsilon)\phi\, dx\, d\tau \to \int_0^T \int_{\mathbb{R}} v\phi\, dx\, d\tau \quad \text{as } \epsilon \downarrow 0,$$

showing that $u$ is weakly differentiable and that $u_t = v$. Since $v \in C(0,T; H^{s-3})$, it follows that $u$ is strongly differentiable and that $u \in C^1(0,T; H^{s-3})$. This establishes the existence and uniqueness. The continuous dependence of the solution on initial data follows using the fact that the bounds leading to the conclusion that the net $\{u_\epsilon\}$ is Cauchy are in fact uniform on compact subsets of $H^s$. □

The method of proof used for existence of smooth solutions of KdV has an interesting implication. Consider again the regime that is of physical interest – namely small waves with long wavelength.

In dimensionless, but unscaled variables, where $u$ and its derivatives are all of order 1, the initial-value problems considered heretofore have the form

$$\begin{aligned} &\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{1}{6}\beta\eta_{xxt} = O(\alpha^2, \alpha\beta, \beta^2), \\ &\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} = O(\alpha^2, \alpha\beta, \beta^2), \\ &\eta(x,0) = g(x), \end{aligned} \tag{3.43}$$

where the Stokes number $S = \frac{\alpha}{\beta} = \frac{al^2}{h^3}$. Inherent in keeping both the $\alpha$ and $\beta$ terms on an equal footing is that $S \backsim 1$.

Now, over what time scales is it expected that the model will be valid? Consider

$$\eta_t + \eta_x = \epsilon, \quad \eta(x,0) = g(x).$$

The solution of this equation is

$$\eta(x,t) = g(x-t) + \epsilon t.$$

Thus the long-term effect of a small perturbation is, in general, to grow linearly in time. Hence presuming $S \backsim 1$, the effect of the small nonlinear term and small dispersive term is to grow, over a time-scale of order $\frac{1}{\alpha} \backsim \frac{1}{\beta}$, to order 1. Thus these terms can have a significant effect on the shape of the wave profile on a time-scale of order $\frac{1}{\alpha}$. Equally, the neglected terms of order $\alpha^2 \backsim \beta^2$, can make an order-one contribution on a time scale of order $\frac{1}{\alpha^2} \backsim \frac{1}{\beta^2}$. Thus we have the following situation:

when $t \backsim \frac{1}{\alpha}$, nonlinear and dispersive effects can affect the basic wave profile;

when $t \backsim \frac{1}{\alpha^2}$, neglected effects can have accumulated to the order of the basic wave, so the model may no longer be reliable.

Now let's consider the following pair of problems: take $\alpha = \beta$, $S = 1$, and consider the pair of initial-value problems

$$\begin{aligned} &\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{1}{6}\beta\eta_{xxt} = 0, \\ &\xi_t + \xi_x + \frac{3}{2}\alpha\xi\xi_x + \frac{1}{6}\beta\xi_{xxx} = 0, \\ &\eta(x,0) = \xi(x,0) = g(x), \qquad \text{an order-one initial profile.} \end{aligned} \tag{3.44}$$



Over what time scales are $\eta$ and $\xi$ close together? By "close together", we shall mean that

$$|\eta(x,t) - \xi(x,t)| \le C\alpha.$$

Order $\alpha$ is the resolution of either $\eta$ or $\xi$, so this result would mean that practically we couldn't tell the two apart.

Let $\tilde{u}(x,t) = \frac{3}{2}\alpha\eta(\sqrt{\frac{\alpha}{6}}x, \sqrt{\frac{\alpha}{6}}t)$, and $\tilde{v}(x,t) = \frac{3}{2}\alpha\xi(\sqrt{\frac{\alpha}{6}}x, \sqrt{\frac{\alpha}{6}}t)$. Note that the initial data becomes

$$g_\alpha(x) = \frac{3}{2}\alpha g(\sqrt{\frac{\alpha}{6}}\, x).$$

The new variables $\tilde{u}$ and $\tilde{v}$, satisfy

$$\begin{aligned} &\tilde{u}_t + \tilde{u}_x + \tilde{u}\tilde{u}_x - \tilde{u}_{xxt} = 0, \\ &\tilde{v}_t + \tilde{v}_x + \tilde{v}\tilde{v}_x + \tilde{v}_{xxx} = 0, \\ &\tilde{u}(x,0) = \tilde{v}(x,0) = g_\alpha(x), \end{aligned} \tag{3.45}$$

and we want $|\tilde{u} - \tilde{v}| \le C\alpha^2$ as $\alpha \downarrow 0$. In this formulation, the small parameter is hidden in the initial data. It is easier to follow the evolution if we magnify the solution and move with the wave. To this end, let

$$u(x,t) = \alpha^{-1}\tilde{u}(\alpha^{-\frac{1}{2}}x + \alpha^{-\frac{3}{2}}t, \alpha^{-\frac{3}{2}}t),$$

and

$$v(x,t) = \alpha^{-1}\tilde{v}(\alpha^{-\frac{1}{2}}x + \alpha^{-\frac{3}{2}}t, \alpha^{-\frac{3}{2}}t).$$

It is easy to verify that

$$\begin{cases} u_t + u_x + u_{xxx} - \alpha u_{xxt} = 0, \\ v_t + v_x + v_{xxx} = 0, \\ u(x,0) = v(x,0) = g(x). \end{cases}$$

This is the problem just investigated. It is known that, as $\alpha \downarrow 0$, $u \to v$, but more precise information than that is needed. Setting $w = u - v$ as before, so that $u = w + v$, then

$$\begin{cases} w_t + ww_x + w_{xxx} - \epsilon w_{xxt} = \alpha v_{xxt} - (vw)_x \\ w(x,0) = 0. \end{cases}$$

The next step is to estimate $\|\partial_x^j w\|_{L^2}$ for $j = 0, 1, 2, \cdots$. Sharp bounds are obtained via detailed aspects of the KdV-equation to be explained now.

Let $u$ be an $H^\infty$-solution of the KdV-equation. Such solutions satisfy an infinite collection of polynomial conservation laws. These polynomial invariants lead to bounds on the $H^s$-norms of solutions that are independent of time, depending only on the $H^s$-norm of the initial data. These bounds are the key to the following result.

THEOREM 3.16. *Let $g \in H^{k+5}$, where $k \geq 0$. Let $\alpha > 0$ and let $\eta^\alpha$ and $\xi^\alpha$ be the unique solutions of*

$$\eta_t + \eta_x + \alpha\eta\eta_x - \alpha\eta_{xxt} = 0$$

*and*

$$\xi_t + \xi_x + \alpha\xi\xi_x + \alpha\xi_{xxx} = 0$$

*with initial value $\eta(x,0) = \xi(x,0) = g(x)$, respectively. Then there are order-one constants $C_j$ and $D_j, j = 0, 1, 2\cdots$ such that*

$$\|\eta^\alpha_{(j)} - \xi^\alpha_{(j)}\|_{L^2} \leq C_j\alpha^{\frac{7}{4}+\frac{j}{2}}(\alpha^{\frac{3}{2}}t),$$

*and*

$$\|\eta^\alpha_{(j)} - \xi^\alpha_{(j)}\|_{L^\infty} \leq D_j\alpha^{2+\frac{j}{2}}(\alpha^{\frac{3}{2}}t)$$

*at least for $0 \leq t \leq \alpha^{-\frac{3}{2}}$. The constants $C_j$ and $D_j$ are not dependent on $\alpha$ for $\alpha$ in a bounded domain $[0, \alpha_0]$.*

*Conjecture.* These bounds hold over time-scales of order $t \backsim \alpha^{-\frac{3}{2}}$. Moreover, we expect the bounds are sharp, in which case the two models will generically have diverged from one another by times of order $\alpha^{-\frac{3}{2}}$. Numerical evidence supports this conjecture. But one must be careful in discounting the models beyond $t \backsim \alpha^{-\frac{3}{2}}$, for reasons that will appear later.

### *3.4 The Quarter-Plane Problem*

The pure initial-value problems that have been treated so far, are not always as useful as a model problem, as initial-boundary-value problems to be treated presently. The fact is that the pure initial-value problem is often not well-suited to comparison with or prediction of real data.

An often encountered experimental set-up is as follows. In a channel, a disturbance is created at one end, which subsequently propagates down the channel. At a certain point $x_0$ in the channel the disturbance is recorded as it passes. One hopes to predict what the wave will look like further downstream from this data. Similar situations arise in other physcial systems where KdV-type equations arise as models. Such an experimental or practical configuration leads naturally to initial- and boundary-value problems for the evolution equations studied heretofore, and so to problems like

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{1}{6}\beta\eta_{xxt} = 0,$$
$$\xi_t + \xi_x + \frac{3}{2}\alpha\xi\xi_x + \frac{1}{6}\beta\xi_{xxx} = 0,$$



with

$$\eta(x, t_0) = \xi(x, t_0) = f(x), \qquad \text{for} \quad x \geq x_0,$$
$$\xi(x_0, t) = \eta(x_0, t) = g(t), \qquad \text{for} \quad t \geq t_0.$$

(Henceforth, and without loss of generality, take $x_0 = t_0 = 0$, and scale out the $\frac{1}{6}\beta$ and the $\frac{3}{2}\alpha$.) Inquiry is made as to whether or not either of these comprises a well-posed mathematical problem. It is not quite obvious that this is the case since the equations are, respectively, second- and third-order in the space variable and so in principle require not just one, but two or three spatial boundary conditions to single out a unique solution.

We begin the detailed discussion with the initial-boundary-value problem

$$\begin{aligned} &u_t + u_x + uu_x - u_{xxt} = 0, \\ &u(x,0) = f(x), \qquad u(0,t) = g(t), \qquad \text{for} \quad x, t \geq 0, \end{aligned} \tag{3.46}$$

for the BBM-RLW equation, where the consistency condition $f(0) = g(0)$ is imposed at the outset. This is an initial-boundary-value problem posed in a quarter plane. It will be shown to be well posed by methods very similar to those used to study the pure initial-value problem in Section 3.1. Since the idea is the same and the details not so very different, we content outselves with a sketch of the development. Start by writing the differential equation as

$$(1 - \partial_x^2)u_t = -\partial_x(u + \frac{1}{2}u^2)$$

and regard this as an ordinary differential equation in the spatial variable $x$. Solving for $u_t$, there obtains

$$u_t(x,t) = -\frac{1}{2}\int_0^\infty e^{-|x-\xi|}\partial_\xi(u + \frac{1}{2}u^2)\,d\xi + \frac{1}{2}\int_0^\infty e^{-(x+\xi)}\partial_\xi(u + \frac{1}{2}u^2)\,d\xi + g'(t)e^{-x}, \tag{3.47}$$

where use has been made of the fact that $u_t(0,t) = g'(t)$. A formal integration by parts followed by integration over $[0, t]$ yields

$$u(x,t) = f(x) + (g(t) - g(0))e^{-x} + \int_0^t\int_0^\infty K(x,\xi)[u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau)]\,d\xi\,d\tau. \tag{3.48}$$

with $K(x,\xi) = \frac{1}{2}\,\mathrm{sgn}(x-\xi)e^{-|x-\xi|} + \frac{1}{2}e^{-(x+\xi)}$.

LEMMA 3.17. *Let $f \in C_b(\mathbb{R}^+)$ and $g \in C(0,T)$. Then there exists $S$ with $0 < S \leq T$ depending only on $\|f\|_{C_b}$ and $\|g\|_{C(0,T)}$ and a unique solution $u \in C_b(\mathbb{R}^+ \times [0,T])$ of (3.48) corresponding to $f$ and $g$. Moreover, $u$ depends continuously in $C_b(\mathbb{R}^+ \times [0,T])$ on variations of $f$ and $g$ within their function classes.*

*Proof.* Write (3.48) as $u = Au = g(x) + e^{-x}(g(t) - g(0)) + B(u)$ say. View this as a mapping of the space $C(0, S; C_b(\mathbb{R}^+))$ into itself. We argue that, by taking $R$ large and $S$

small, $A$ is seen to be a contraction mapping of the closed ball $\mathcal{B}_R(0) \subset C(0,S;C_b(\mathbb{R}^+))$ into itself. The crucial estimate is

$$\|Au - Av\|_{C(0,S;C_b)} = \|Bu - Bv\|_{C(0,S;C_b)}.$$
$$\leq S\left(1 + \frac{1}{2}\left[\|u\|_{C(0,S;C_b)} + \|v\|_{C(0,S;C_b)}\right]\right)\|u - v\|_{C(C_b)}$$

which is easily deduced using the fact that $\sup_{x\geq 0}\int_0^\infty |K(x,\xi)|\,d\xi = 1$. Thus we have that

$$\|Au\|_{C(0,S;C_b)} \leq \|f\|_{C_b} + 2\|g\|_{C(0,T)} + S\|u\|_{C(0,S;C_b)}(1 + \frac{1}{2}\|u\|_{C(0,S;C_b)}),$$

since

$$\|Au\|_{C(0,S;C_b)} \leq \|Au - A(0)\|_{C(0,S;C_b)} + \|A(0)\|_{C(0,S;C_b)}.$$

We may now proceed as before to let $a = \|f\|_{C_b} + 2\|g\|_{C(0,T)}$, choose $R = 2a$ and $S = \frac{1}{2(1+R)}$. It then follows that $A : \mathcal{B}_R(0) \to \mathcal{B}_R(0)$ and that

$$\|Au - Av\|_{C(0,S;C_b)} \leq \frac{1}{2}\|u - v\|_{C(0,S;C_b)}.$$

for $u, v \in \mathcal{B}_R(0)$. The lemma is proved. □

PROPOSITION 3.18. *Suppose $f \in C_b^2(\mathbb{R}^+)$ and $g \in C^1(0,T)$, then any solution of the integral equation (3.48) has*

$$\partial_t^i \partial_x^j u \in C(0,T;C_b) \qquad \text{for } 0 \leq i \leq 1, 0 \leq j \leq 2.$$

*Moreover, $u$ is a classical solution of the problem (3.46).*

*Proof.* This follows as in the proof of Proposition 3.2. □

COROLLARY 3.19. *If $f \in C_b^l(\mathbb{R}^+)$ and $g \in C^k(0,T)$, $k \geq 1, l \geq 2$, then any solution $u$ of the integral equation in $C(0,T;C_b)$ has*

$$\partial_t^i \partial_x^j u \in C(0,T;C_b(\mathbb{R}^+)) \qquad \text{for } 0 \leq i \leq k, 0 \leq j \leq l.$$

LEMMA 3.20. *Let $f \in C_b^l$ and $g \in C^k(0,T)$, $l \geq 2$, $k \geq 1$ and suppose $f, f', \ldots f^{(p)}$ are null at $+\infty$ for some $p$ with $0 \leq p \leq l$. If $u \in C(0,T;C_b)$ is the solution of the integral equation coresopnding to $f$ and $g$, then $\partial_t^i \partial_x^j u$ is null at $+\infty$, for $0 \leq i \leq k, 0 \leq j \leq p$, uniformly for $0 \leq t \leq T$.*

*Proof.* This follows just as the analogous result did for the pure initial-value problem. □



LEMMA 3.21. *If $f \in C_b^2(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$ and $g \in C^1(0,T)$, then the classical solution of the initial-value problem that exists on $[0,S]$ for $S$ small enough satisfies the inequality*

$$\int_0^\infty [u^2(x,t) + u_x^2(x,t)]\,dx + \frac{1}{4}\int_0^t u_{xt}^2(0,\tau)\,d\tau \leq c\int_0^\infty [f^2(x) + f_x^2(x)]\,dx + C(t),$$

*where $c$ is a constant and $C(t)$ depends only on $g$ and $g'$ on $[0,T]$.*

*Proof.* We calculate formally as if the solution was infinitely smooth and vanished along with all its derivatives at $+\infty$. These machinations are easily justified because of the regularity theory above and the continuous dependence result in Lemma 3.17. Multiply (3.46) by $u$ and integrate over $\mathbb{R}^+$ with respect to the spatial variable $x$ to reach, after some manipulations,

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty u^2\,dx - \frac{1}{2}g^2(t) - \frac{1}{3}g^3(t) + g(t)u_{xt}(0,t) + \frac{1}{2}\frac{d}{dt}\int_0^\infty u_x^2\,dx = 0,$$

or

(3.49) $$\frac{d}{dt}\int_0^\infty (u^2 + u_x^2)\,dx = g^2(t) + g^3(t) - g(t)u_{xt}(0,t).$$

The term $u_{xt}(0,t)$ is troublesome and the next calculations are aimed at getting control of it. Multiply (3.46) by $u^2$ and integrate over $\mathbb{R}^+$ to reach

(3.50) $$\frac{1}{3}\int_0^\infty u^3\,dx - \frac{1}{3}g^3(t) - \frac{1}{4}g^4(t) + g^2(t)u_{xt}(0,t) + 2\int_0^\infty uu_xu_{xt}\,dx = 0,$$

a relation that does not look especially useful. Multiply (3.46) by $u_{xt}$ and integrate over $\mathbb{R}^+$ to obtain

(3.51) $$\frac{d}{dt}\int_0^\infty u_x^2\,dx = -2\int_0^\infty uu_xu_{xt}\,dx + g'(t)^2 - u_{xt}^2(0,t).$$

Form the combination (3.49) − (3.50) + (3.51) to come to

(3.52) $$\begin{aligned}\frac{d}{dt}\int_0^\infty \left(u^2 + 2u_x^2 - \frac{1}{3}u^3\right)dx + u_{xt}^2(0,t) &= g'(t)^2 - \frac{1}{3}g^3(t)\\ &- \frac{1}{4}g^4(t) + g^2(t)u_{xt}(0,t) + g^2(t) + \frac{2}{3}g^3(t) - g(t)u_{xt}(0,t).\end{aligned}$$

Young's inequality allows us to write an inequality based (3.52) of the form

(3.53) $$\frac{d}{dt}\int_0^\infty \left(u^2 + 2u_x^2 - \frac{1}{3}u^3\right)dx + \frac{1}{2}u_{xt}^2(0,t) = B(t),$$

where $B$ connotes a polynomial in $g$ and $g'$. Elementary inequalities imply that

$$\int_0^\infty u^3\,dx \le \|u(\cdot,t)\|_{H^1}^3.$$

Integrate (3.48) with respect to $t$ over $[0,t]$ and use the result to continue the last inequality. After a few elementary manipulations, we arrive at

$$\begin{aligned}\int_0^\infty u^3\,dx &\le \left(\|u(\cdot,0)\|_{H^1}^2 + \left\{\int_0^t u_{xt}(0,s)^2\,ds\int_0^t g(s)^2\,ds\right\}^{\frac12} - \int_0^t R(s)\,ds\right)^{\frac32} \\ &\le \|f\|_{H^1}^3 + \frac14\int_0^t u_{xt}(0,s)^2\,ds + C\|g\|_{L_2(0,T)}^2 - \int_0^t R(s)\,ds\end{aligned} \tag{3.54}$$

for an absolute constant $C$, where $R = g^2 + g^3$ and $u(\cdot,0) = f$. If (3.53) is integrated over $[0,t]$, there appears

$$\begin{aligned}\int_0^\infty (u^2+2u_x^2)\,dx + \frac12\int_0^t u_{xt}(0,s)^2\,ds &= \frac13\int_0^\infty u^3\,dx + \int_0^\infty (f^2+2f_x^2-\frac13 f^3)\,dx \\ +\int_0^t B(s)\,ds &\le V + \frac13\int_0^\infty u^3\,dx + C_1\|g\|_{H^1(0,T)}^2 + C_2\|g\|_{H^1(0,T)}^4,\end{aligned}$$

where $C_1$ and $C_2$ are constants and $V = \int_0^\infty f^2 + 2f_x^2 - \frac13 f^3$ is bounded since $f$ starts life in $H^1$. If (3.54) is used in the last inequality, the desired result emerges directly. □

This $H^1(\mathbb{R}^+)$-bound is enough to pass to a global solution by way of iterating the contraction mapping argument. The results of continuous dependence may now be derived much as before, as well as further regularity results. The details are ommitted.

As far as the same initial-boundary-value problem for the KdV-equation is concerned, similar results may be derived, but the details are considerably more complicated and will only be hinted at here. First note that, formally, the solution is uniquely specified by the initial data $f$ and boundary data $g$. For if $u$ and $v$ are two solutions and $w = u - v$, then

$$\begin{cases} w_t + w_x + \frac12[(u+v)w]_x + w_{xxx} = 0, \\ w(x,0) = 0, \quad w(0,t) = 0. \end{cases} \tag{3.55}$$

Multiply (3.53) by $w$ and integrate over $\mathbb{R}^+$ to obtain

$$\begin{aligned}-\frac12\frac{d}{dt}\int_{\mathbb{R}^+} w^2\,dx &= \frac12(u+v)w^2|_0^\infty + ww_{xx}|_0^\infty - \int_{\mathbb{R}^+} w_x w_{xx}\,dx - \frac12\int_{\mathbb{R}^+}(u+v)ww_x\,dx \\ &= -\frac14(u+v)w^2|_0^\infty + \frac14\int_{\mathbb{R}^+}(u_x+v_x)w^2\,dx - \frac12 w_x^2|_0^\infty,\end{aligned}$$



or

$$\frac{d}{dt}\int_{\mathbb{R}^+} w^2(x,t)\,dx + w_x^2(0,t) \le |u_x + v_x|_\infty \int_{\mathbb{R}^+} w^2.$$

Assuming $u_x$ and $v_x$ are bounded, then Gronwall's lemma implies $w = 0$, whence $u = v$.

*Remark:* Notice that the sign of the term $u_{xxx}$ was important in the last calculation. This aspect can be more completely understood by consideration of the linearized problem

$$\begin{cases} u_t + u_x - u_{xxx} = 0, \\ u(x,0) = f(x), \quad u(0,t) = g(t), \end{cases} \tag{3.56}$$

which features a changed sign in front of the dispersive term $u_{xxx}$. As will become apparent presently, uniqueness of solutions is no longer true, even if the solution is assumed to be smooth and rapidly decaying to 0 at $+\infty$. To see this, let $u$ be a solution of (3.56) with zero initial and zero boundary conditions. Let $v = \mathcal{L}u$ denote the Laplace transform of $u$ with respect to $t$. Then $v(x,s)$ satisfies the one-parameter family of boundary-value problems

$$sv + v_x - v_{xxx} = 0, \quad v(0,s) = \mathcal{L}u(0,\cdot) = 0,$$

where $s \ge 0$. To solve this constant-coefficient problem, compute the characteristic equation

$$r^3 - r - s = 0,$$

and let $r, r_1, r_2$ be its roots. Then the general solution is

$$a(s)e^{rx} + b(s)e^{r_1x} + c(s)e^{r_2x}.$$

Inspection of the cubic characteristic equation reveals that one root is positive, say $r \ge 0$, and the other two $r_1$ and $r_2$ have negative real parts.

Imposing the condition that $u$ and its derivatives tend to zero at $+\infty$ amounts to asking that $a(s)e^{rx}+b(s)e^{r_1x}+c(s)e^{r_2x}$ decay at $x \to +\infty$. Since $r > 0$ and $\mathrm{Re}\{r_j\} < 0$ for $j = 1,2$, this simply means that $a(s) = 0$. The condition $v(0,s) = 0$ implies that $b(s) + c(s) = 0$, or $b(s) = -c(s)$. Hence a non-zero solution is obtained as $v(x,s) = b(s)(e^{r_1x} - e^{r_2x})$ for suitable choices of $b$. For example, one could choose $b(s)$ with compact support in $[0, s_0]$, say, and then have a smooth non-trivial solution $u = \mathcal{L}^{-1}v$ of (3.56).

Recent work on the quarter-plane problem for the KdV-equation has brought this theory into the range of the developments for the pure initial-value problem. These will be mentioned in the lectures, but details are beyond the scope of these presentations.

## Chapter 4. Stabilty of Solitary Waves

Scott Russell's original experiments showed the solitary wave to be a very stable aspect of surface wave motion. Similar observations of other physical systems in more recent times have confirmed that when such waves exist, they appear to be very persistent. Moreover, the inverse-scattering theory for the KdV-equation and certain other model equations shows that the solitary waves play a distinguished role in the long-time evolution of general disturbances. Namely, it can be shown on the basis of this theory that initial disturbances break up into a sequence of solitary waves that propagate ahead and leave behind a dispersive tail in their wake.

T. B. Benjamin in 1972 was the first to address the theoretical issue of the stability of solitary waves. His original theory, which was worked out for the KdV-equation and the BBM-equation, has since been refined and generalized in various ways by many authors. A set of results in this domain will be sketched here. The setting will be a general class of one-dimensional waves equations of KdV-BBM type, namely

$$u_t + u_x + f(u)_x - Lu_x = 0, \tag{4.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a super-linear $C^\infty(\mathbb{R})$-function, typically a polynomial, with $f(0) = 0$ and $L$ is the dispersion operator defined as a Fourier-multiplier, *viz.*

$$\mathcal{F}\{Lu(\xi)\} = \alpha(\xi)\mathcal{F}\{u(\xi)\}$$

where $\mathcal{F}$ connotes the Fourier transform with respect to the spatial variable. The symbol $\alpha$ of $L$ is typically real-valued and even. Suppose $\phi = \phi_C = \phi(x-(C+1)t)$ is a solitary-wave solution of (4.1). Then it satisfies the ordinary differential equation

$$-C\phi' + (f(\phi))' - (L\phi)' = 0,$$

or

$$(C + L)\phi = f(\phi),$$

where $C > 0$ and $C+1$ is the velocity of wave propagation. A strong form of zero boundary conditions at infinity has been applied to evaluate the constant of integration inherent in getting to the last formula.

We start by explaining what we mean by the term "stability". If initial data $\psi$ for (4.1) is close to a solitary wave $\phi$, then it might be expected that the solution $u$ of (4.1) corresponding to $\psi$ is close to $\phi$ for all time, i.e. for any small $\epsilon > 0$ given, there is a $\delta > 0$ such that if $\|\psi - \phi\| < \delta$ then $\|u(\cdot,t) - \phi_C(\cdot - Ct)\| \le \epsilon$ for all time. A result of this strength is generally not true because the principal speeds of propagation of $\phi_C$ and $u$ may be different. More precisely, suppose we let $\psi = \phi_D$ where $D \neq C$. If $D \to C$,



then $\phi_D \to \phi_C$ in any of our favorite norms. On the other hand, in this case $u$ is known explicitly to be

$$u(x,t) = \phi_D(x - (D+1)),$$

and hence

$$\|u - \phi_C\| = \|\phi_D - \phi_C\|$$

converges to a positive constant as $t \to \infty$, no matter how close $D$ is to $C$. Indeed, if $D \neq C$, then

$$\lim_{t\to\infty} \|\phi_C(x-(C+1)t) - \phi_D(x-(D+1)t)\| = \|\phi_C\| + \|\phi_D\|.$$

Thus, the strong form of stability just put forward is too much to hope for in the present context. Indeed, the fundamental reason the result fails arises frequently when stability of motion is contemplated – small differences in velocity can eventually move two neighboring states very far apart.

One way around this difficulty is to give up knowing where the solution is in exchange for knowing its shape very well. This idea leads to a new measure of distance. Consider a Banach space $X = X(\mathbb{R})$ of functions defined on $\mathbb{R}$ endowed with a translation-invariant norm $\|\cdot\|_X$. Let $f$ and $g$ be elements in $X = X(\mathbb{R})$ and define

$$d(f,g) = \inf_{y\in\mathbb{R}} \|f(\cdot) - g(\cdot + y)\|_X.$$

This "distance" is the closest approach of $f$ and $g$ under the translation group in $\mathbb{R}$.

Lemma 4.1. *The mapping d defined above is a pseudo-metric if the norm in the Banach space $X = X(\mathbb{R})$ is invariant under translation; d is a metric on the quotient space $X/\tau$.*

In terms of the pseudo-metric, another notion of stability of solitary waves may be defined.

Definition 4.2. *A solitary-wave solution $\phi_C$ of (4.1) is said to be stable if for any given $\epsilon > 0$, there is a $\delta > 0$ such that if $\|\psi - \phi_C\| < \delta$, then $\inf_{y\in\mathbb{R}} \|u(\cdot,t) - \phi_C(\cdot + y)\| < \epsilon$ for all $t > 0$.*

If we move to a travelling frame of reference in (4.1) by letting $\tilde{u}(x,t) = u(x+t,t)$, then (4.1) with its initial condition may be written as

$$\begin{cases} u_t + f(u)_x - Lu_x = 0, \\ u(x,0) = \psi(x), \end{cases} \tag{4.2}$$

where the tilde was immediately dropped for clarity.

LEMMA 4.3. *Let $u$ be a suitably smooth solution of (4.2) that is null at $\infty$. Then the functionals $\int_{-\infty}^{\infty} u(x,t)\,dx$, $V(u) = \int_{-\infty}^{\infty} u^2(x,t)\,dx$ and $M(u) = \int_{-\infty}^{\infty}[\frac{1}{2}uLu(x,t) - F(u)(x)]\,dx$ are time independent, where $F'(x) = f(x)$ and $F(0) = 0$.*

PROOF. *This is easily established by multiplying (4.1) by 1, $u$ and $f(u) - Lu$, respectively, and then integrating with respect to $x$ over $\mathbb{R}$. Use is made of the fact that the dispersion operator $L$ is self-adjoint since its symbol $\alpha$ is real and even.*

*Intermediate hypothesis:* For the time being, it is assumed that $\phi$ is a fixed solitary-wave solution of (4.1) and that $\psi$ is slightly perturbed initial data. It is assumed additionally that $V(\phi) = V(\psi)$. This restriction will be removed later.

Define $\Lambda(u) = M(u) + CV(u)$, $h(x,t) = u(x+a,t) - \phi(x - Ct)$, where $C$ is the phase speed of the solitary wave $\phi = \phi_C$ we are interested in. Then $\phi$ satisfies equation

$$(C + L)\phi = f(\phi). \tag{4.3}$$

Let $T(u) = \Lambda(u) - \Lambda(\phi)$. Because of the time-invariance of $M$ and $V$, we may think of $T$ as a real-valued map of the initial data $\psi$. Using the Taylor expansion, we may express $T$ as

$$T(\psi) = \Lambda(u) - \Lambda(\phi) = \Lambda(\phi + h) - \Lambda(\phi) = \Lambda'(\phi)h + \frac{1}{2}\left(\Lambda''(\phi)h, h\right) + O(\|h\|^3),$$

or, in more concrete terms,

$$\begin{aligned} T(\psi) = &\int \frac{1}{2}(\phi + h)L(\phi + h) - F(\phi + h) - \int \frac{1}{2}\phi L\phi + F(\phi) + \frac{C}{2}\int(\phi + h)^2 - \frac{C}{2}\int \phi^2 \\ = &\int hL\phi + \int hLh - \int f(\phi)h + \frac{1}{2}f'(\phi)h^2 + C\int \phi h + \frac{C}{2}\int h^2 + O(\|h\|^3). \end{aligned}$$

as $\|h\| \to 0$. Equation (4.3) together with the self-adjointness of $L$ implies $\Lambda'(\phi) = 0$, and consequently

$$m\|h\|^2 + a\|h\|^3 \geq \Lambda(u) - \Lambda(\phi) = T(\psi) - \frac{1}{2}\left(\Lambda''(\phi)h, h\right) + O(\|h\|^3).$$

If $\Lambda''$ is positive definite, then

$$m\|h\|^2 + a\|h\|^3 \geq \Lambda(u) - \Lambda(\phi) = T(\psi) \geq \lambda\|h\|^2 - b\|h\|^3,$$

for all $t$. As $T(\psi)$ does not depend upon $t$, this means that if at $t = 0$, $\|h\|_{H^1}$ is small, then $\Lambda$ will remain small for all $t \geq 0$.

Define $\mathcal{L} = \mathcal{L}_C = \Lambda''(\phi) = L + C - f'(\phi)$. Thus the self-adjoint, unbounded operator on $L_2(\mathbb{R})$. Notice that $\mathcal{L}(\phi') = 0$ since $(L + C)\phi' - f'(\phi)\phi' = 0$ from (4.3). Hence $\mathcal{L}$ has



zero as an eigenvalue with $\phi'$ as a corresponding eigenfunction. Therefore $\mathcal{L}$ is not strictly positive definite. Moreover, by considering $(\mathcal{L}(\lambda u), \lambda u)$ and observing the behavior of this quantity as $\lambda$ varies, it is often easy to see that $\mathcal{L} = \Lambda''(\phi)$ is not bounded below, and hence is not positive definite.

Denote by $\dot{\phi} = \frac{d\phi}{dC}$. Differentiate (4.3) with respect to $C$ to derive

$$(L + C)\dot{\phi} - f'(\phi)\dot{\phi} + \phi = 0,$$

which is to say,

$$\mathcal{L}(\dot{\phi}) = -\phi.$$

In consequence,

$$\left(\mathcal{L}(\dot{\phi}), \dot{\phi}\right) = -(\phi, \dot{\phi}) = -\frac{1}{2}\frac{d}{dC}(\phi, \phi).$$

It is convenient at this point to posit a set of assumptions about the operator $\mathcal{L}$ that allow a stability-theory to go forward. It then becomes an interesting objective to find conditions on $f$ and $L$ that imply these assumptions to be valid.

*Assumptions on $\mathcal{L}$*

(i) $\frac{d}{dC}(\phi, \phi) > 0$.

(ii) The spectrum of $\mathcal{L}$ is composed of a single negative eigenvalue $-\nu$, say, and 0 together with the interval $[C, +\infty)$.

(iii) 0 is a simple eigenvalue.

(iv) $-\nu$ is a simple eigenvalue.

(v) $\alpha(\xi) \geq \mu|\xi|$ at least for $|\xi|$ large for some $\mu > 0$.

Let $X$ be the Hilbert space

$$X = \left\{ f \in L_2 : \int (1 + \alpha(\xi))|\hat{f}(\xi)|^2\, d\xi < \infty \right\}.$$

Then the dispersion operator $L$ may be viewed as a bounded linear opeator from $X$ to $X^*$.

LEMMA 4.4. *Let $V$ be the subspace of $X$ of those functions such that $0 = (\psi, \phi') = (\psi, \phi)$ and let*

$$\eta = \inf\{< \mathcal{L}\psi, \psi >: \psi \in V, \|\psi\| = 1\}.$$

*Then, $\eta > 0$.*

*Proof.* Step 1. Certainly $\eta$ is not $-\infty$ since the spectrum of $\mathcal{L}$ is bounded below. If $\eta \leq 0$, then there exists a sequence $\{q_n\} \subset X$ such that for $n \geq 1$, $(q_n, \phi) = (q_n, \phi') =$

0, $(q_n, q_n) = 1$ and $< \mathcal{L}q_n, q_n > \to \eta \le 0$ as $n \to \infty$. Without loss of generality, it may be supposed that the $q_n$ are infinitely smooth. Remark that

$$0 \le (Lq_n, q_n) = \int_{-\infty}^{\infty} \alpha(\xi)\hat{q}_n(\xi)\hat{q}_n^*(\xi)\, d\xi = (\mathcal{L}q_n, q_n) - C(q_n, q_n) + \int_{-\infty}^{\infty} f'(\phi)q_n^2(x)\, dx.$$

Thus it is seen that the sequence $\{q_n\}$ is bounded in $X$, and so there exists a subsequence $\{q_{n_k}\}$, still called $\{q_n\}$, such that,

$$q_n \to q_* \text{ weakly in } X.$$

Because of assumption (v), $\alpha(\xi) > \mu|\xi|$ for some $\mu > 0$, so $X \subset H^{\frac{1}{2}}(\mathbb{R}) \subset L_p$, for all finite values of $p \ge 1$. Thus, $\{q_n\}$ is bounded in $H^{\frac{1}{2}}(-M, M)$ for any finite number $M$, and so by a Cantor diagolization, there is a further subsequence of $\{q_n\}$ that converges strongly in $L_2(-M, M)$ and $L_4(-M, M)$. for any finite $M$, and, by a second Cantor diagonalization, pointwise almost everywhere. Thus we may suppose that

$$q_n \to q_* \text{ weakly in } X \text{ and } q_n \to G \text{ in } L_{2,\mathrm{loc}}(\mathbb{R}) \cap L_{4,\mathrm{loc}}(\mathbb{R}) \text{ and pointwise a.e. in } \mathbb{R}.$$

It is asserted that $q_* = G$. To see this, let $\rho \in \mathcal{D}(\mathbb{R}) = C_0^\infty(\mathbb{R})$ and compute as follows:

$$\int_{-\infty}^{\infty} [q_n(x) - G(x)]\rho(x)\, dx \le \|\rho\|_{L_2}\|q_n - G\|_{L_2(\mathrm{supp}(\rho))}.$$

It follows that $q_n \to G$ in the sense of distributions $\mathcal{D}'$. But weak convergence in $X$ implies convergence in $\mathcal{D}'$. Therefore $G = q_*$. Moreover, it now follows that $q_n^2 \to q_*^2$ in $\mathcal{D}'$, in $L_{1,\mathrm{loc}}$ and weakly in $L_p(\mathbb{R})$ for all finite $p \ge 1$. For example, if $\rho \in \mathcal{D}(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} [q_n(x)^2 - q_*(x)^2]\rho(x)\, dx \le \|\rho\|_{L_\infty}\|q_n - q_*\|_{L_2(\mathrm{supp}(\rho))}\|q_n + q_*\|_{L_2(\mathrm{supp}(\rho))}.$$

With this information in hand, we return to the primary issue under consideration and remark that because of the weak convergence in $X$,

$$0 = \lim_{n\to\infty}(\phi, q_n) = (\phi, q_*) \quad \text{and} \quad 0 = \lim_{n\to\infty}(\phi', q_n) = (\phi', q_*).$$

The lower semi-continuity of semi-norms with respect to weak convergence implies that

$$(Lq_*, q_*) \le \liminf_{n\to\infty}(Lq_n, q_n) \text{ and}$$

$$\|q_*\|^2 = (q_*, q_*) \le \liminf_{n\to\infty}(q_n, q_n) = 1.$$



We also need that $\lim_{n\to\infty}(f'(\phi), q_n^2) = (f'(\phi), q_*^2)$. This follows, for example, because as mentioned already, the sequence $\{q_n^2\}$ can be assumed to converge weakly in $L_2(\mathbb{R})$ to $q_*^2$. Since $f'(\phi)$ lies in $L_2(\mathbb{R})$, the convergence result follows. An outcome of the foregoing musings is that

$$\begin{aligned} 0 \ge \eta = \lim_{n\to\infty}(\mathcal{L}q_n, q_n) &= (Lq_n, q_n) + C(q_n, q_n) - (f'(\phi), q_n^2) \\ &\ge (Lq_*, q_*) + C - \int f'(\phi)q_*^2\, dx \\ &\ge C - \int f'(\phi)q_*^2\, dx, \end{aligned}$$

because $(Lq_*, q_*) \ge 0$. Since $C > 0$, the last inequality implies that $q_* \ne 0$. Let $f_* = \frac{q_*}{\|q_*\|}$ be a normalized version of $q_*$ so that

$$(\mathcal{L}f_*, f_*) = \frac{1}{\|q_*\|^2}(\mathcal{L}q_*, q_*) \le 0.$$

Making use of assumption (i), we know from our ealier computations that

$$(\mathcal{L}\dot{\phi}, \dot{\phi}) < 0.$$

Suppose the simple negative eigenvalue $\nu$ of $\mathcal{L}$ has a corresponding eigenfunction $\mathcal{X}$; under assumption (iii) $\phi'$ spans the eigenspace associated to the eigenvalue 0. Decompose $\dot{\phi}$ as $\dot{\phi} = a\mathcal{X} + b\phi' + P_0$, and substitute this representation into the inequality $(\mathcal{L}\dot{\phi}, \dot{\phi}) < 0$ to obtain

$$\begin{aligned} 0 &> (\mathcal{L}(a\mathcal{X} + b\phi' + P_0), a\mathcal{X} + b\phi' + P_0) \\ &= a^2(\mathcal{L}\mathcal{X}, \mathcal{X}) + b^2(\mathcal{L}\phi', \phi') + (\mathcal{L}P_0, P_0) \\ &= -\nu a^2\|\mathcal{X}\|^2 + (\mathcal{L}P_0, P_0), \end{aligned}$$

whence

$$(\mathcal{L}P_0, P_0) < \nu a^2.$$

Remember $(\mathcal{L}f_*, f_*) \le 0$ and $f_*$ is orthogonal to $\phi'$ and $\phi$. Therefore, $f_*$ has a representation in the form $f_* = c\mathcal{X} + P$, which means

$$0 = (\phi, f_*) = -(\mathcal{L}\dot{\phi}, f_*) = -ac\nu + (\mathcal{L}P_0, P).$$

In consequence of this equation, it is seen that

$$\begin{aligned} (\mathcal{L}f_*, f_*) &= -c^2\nu + (\mathcal{L}P, P) \\ &\ge -c^2\nu + \frac{(\mathcal{L}P, P_0)^2}{(\mathcal{L}P_0, P_0)} \\ &> -c^2\nu + \frac{(ac\nu)^2}{a^2\nu} = 0. \end{aligned}$$

This contradiction proves the lemma. □

COROLLARY 4.5. *There is a positive constant $\eta$ such that $(\mathcal{L}y, y) \geq \eta\|y\|^2$ for any $y \in X$ that is orthogonal to both $\phi$ and $\phi_x$.*

The issue of stability in view here is orbital stability. As the orbit of a solitary-wave solution $\phi(x - Ct)$ of $u_t + f(u)_x - Lu_x = 0$ is just the set of all translates of the initial profile. So to prove the solitary wave is stable, it is necessary to show that given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|\phi - \psi\|_X < \delta \implies u \in \mathcal{U}_\epsilon = \{z : d(z, \phi) < \epsilon\}.$$

LEMMA 4.6. *There exist $\epsilon > 0$, $C > 0$ and a unique $C^1$-mapping $\alpha : U_\epsilon \to \mathbb{R}$ such that for $u \in U_\epsilon$,*

*(i) $(u(\cdot + \alpha), \phi_x) = 0$,*

*(ii) $\alpha(u(\cdot + r)) = \alpha(\tau_r u) = \alpha(u) - r$,*

*(iii) $\alpha'(u) = \dfrac{\phi_x(\cdot - \alpha(u))}{\int_{-\infty}^{\infty} u(x)\phi_{xx}(x - \alpha(u))\,dx}$.*

*Proof.* Consider the functional

$$G : L_2 \times \mathbb{R} \to \mathbb{R} \text{ defined by}$$

$$G(u, \alpha) = \int_{-\infty}^{\infty} u(x + \alpha)\phi_x(x)\,dx.$$

Simple considerations indicate that

$$G(u, \alpha)|_{(u=\phi, \alpha=0)} = 0$$

and

$$\frac{\partial G}{\partial \alpha}|_{(u=\phi,\alpha=0)} = \int_{-\infty}^{\infty} u_x(x + \alpha)\phi_x(x)\,dx|_{(u=\phi,\alpha=0)} = \int_{-\infty}^{\infty} \phi_x(x)^2\,dx \neq 0.$$

The Implicit-Function Theorem implies there is a neighbourhood $B_\epsilon(\phi)$ and a unique $C^1$ functional $\alpha : B_\epsilon(\phi) \to \mathbb{R}$ satisfying

i) $(u(\cdot + \alpha), \phi_x) = 0$;

ii) by translation invariance, $u(\cdot + \alpha(u)) = u(\cdot + r + (\alpha(u) - r)) = \tau_r u(\cdot + \alpha(u) - r)$, and by uniqueness, $\alpha(u) - r = \alpha(\tau_r(u))$.

By a change of variables, property (i) can be rewritten as

$$0 = \int_{-\infty}^{\infty} u(x)\phi_x(x - \alpha(u))\,dx.$$



Differentiating this latter relation with respect to $u$ leads to

$$0 = \phi_x(\cdot - \alpha(u)) - \alpha'(u)\int_{-\infty}^{\infty} u(x)\partial_x^2\phi(x - \alpha(u))\,dx,$$

and therefore

$$\alpha'(u) = \frac{\phi_x(\cdot - \alpha(u))}{\int_{-\infty}^{\infty} u(x)\phi_{xx}(x - \alpha(u))\,dx}.$$

The proof is complete. □

LEMMA 4.7. *Let $\phi = \phi_C$ be a fixed solitary-wave solution of (4.1). Then there exist constants $M > 0$ and $\epsilon > 0$ such that*

$$\Lambda(u) - \Lambda(\phi) \geq M\|u(\cdot + \alpha(u), t) - \phi\|^2$$

*for all $u \in U_\epsilon$ such that $\|u\|_{L_2} = \|\phi\|_{L_2}$.*

*Proof.* Let $h(x,t) = u(x + \alpha(u(x,t)), t) - \phi(x)$. Because of the properties of the mapping $\alpha$, $h$ is orthogonal to $\phi_x$. The solution $u$ can be written in form $u = (1 + a)\phi + y$ where $a$ is determined uniquely by the requirement that $y$ is orthogonal to both $\phi_x$ and $\phi$. By translation invariance and Taylor's Theorem,

$$V(\phi) = V(u) = V(u(\cdot + \alpha(u))) = V(\phi) + 2(\phi, h) + \|h\|^2,$$

from which there is adduced

$$a(\phi, \phi) = \frac{1}{2}\|h\|^2,$$

and so $a = O(\|h\|^2)$ as $h \to 0$. Applying Taylor's Theorem to $\Lambda$, we derive that

$$\begin{aligned}
\Lambda(u) &= \Lambda(u(\cdot + \alpha(u))) \\
&= \Lambda(\phi) + \frac{1}{2}(\mathcal{L}h, h) + o(\|h\|^2) \\
&= \Lambda(\phi) + \frac{1}{2}(\mathcal{L}(a\phi + y), a\phi + y) + o(\|h\|^2) \\
&\geq \Lambda(\phi) + O(a^2) + O(a\|y\|) + \eta\|y\|^2 + o(\|h\|^2).
\end{aligned}$$

On the other hand,

$$\|y\| = \|h - a\phi\| \geq \|h\| - a\|\phi\| = \|h\| - O(\|h\|^2) \geq \frac{1}{2}\|h\|$$

for $h$ small enough. In consequence,

$$\begin{aligned}
\Lambda(u) &\geq \Lambda(\phi) + \frac{\eta}{2}\|h\|^2 + o(\|h\|^2) \\
&\geq \Lambda(\phi) + \frac{\eta}{4}\|h\|^2
\end{aligned}$$

for $\|h\|$ small, and thus if we choose $M = \eta/4$, then

$$\Lambda(u) - \Lambda(\phi) \geq M\|h\|^2.$$

This is the advertised result. □

Since we are laboring under the presumption that $V(u) = V(\phi)$, this last result means that

$$E(u) - E(\phi) \geq C\|h\|^2$$

for $\|h\|$ small.

THEOREM 4.8. *The solitary wave $\phi = \phi_C$ is stable if and only if $m''(C) > 0$, where*

$$m(C) = E(\phi_C) + CV(\phi_C).$$

*Proof.* We only establish the sufficiency of the condition $m''(C) > 0$ for stability. The argument for the other direction is a little more involved.

Let $\{\psi_n\}$ be any sequence such that

$$\inf_{s\in\mathbb{R}} \|\psi_n - \phi(\cdot + s)\| \to 0 \quad \text{as } n \to \infty.$$

Let $u_n$ be the solution of (4.1) with initial value $\psi_n$, $n = 1, 2, \cdots$. Suppose the solitary wave in question not to be stable. Then there is an $\epsilon > 0$ and a sequence of times $\{t_n\}$ such that $u_n(\cdot, t_n) \in \partial U_\epsilon$ for $n = 1, 2, \cdots$. As both $E$ and $V$ are continuous and translation invariant,

$$E(u_n(\cdot, t_n)) = E(\psi_n) \to E(\phi) \quad \text{and} \quad V(u_n(\cdot, t_n)) = V(\psi_n) \to V(\phi).$$

Next, choose $w_n \in U_{2\epsilon}$ so that $V(w_n) = V(\phi)$ and $\|w_n - u_n(\cdot, t_n)\| \to 0$. This is easily arranged by letting $w_n = \lambda_n u_n(\cdot - t_n)$ and choosing $\lambda_n$ so that $V(\lambda_n \psi_n) = V(\phi)$. Because of Lemma 4.7, it is seen that

$$M\|w_n - \phi(\cdot - \alpha(w_n))\|^2 = M\|w_n(\cdot + \alpha(w_n)) - \phi\|^2 \leq E(w_n) - E(\phi) \to 0$$

as $n \to \infty$. It follows that

$$\|u_n(\cdot, t_n) - \phi(\cdot - \alpha(w_n))\| \to 0$$

as $n \to \infty$. This contradicts the fact that $u_n(\cdot, t_n)$ lies at distance at least $\epsilon$ from $\phi$, even when translations are taken account. The result follows. □



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