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Asymptotic decomposition of nonlinear, dispersive wave equations with dissipation

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Abstract

Provided $\nu > 0$, solutions of the generalized regularized long wave-Burgers equation

$$u_t + u_x + P(u)_x - \nu u_{xx} - u_{xxt} = 0 \qquad (*)$$

that begin with finite energy decay to zero as t becomes unboundedly large. Consideration is given here to the case where P vanishes at least cubically at the origin. In this case, solutions of (*) may be decomposed exactly as the sum of a solution of the corresponding linear equation and a higher-order correction term. An explicit asymptotic form for the L_2-norm of the higher-order correction is presented here. The effect of the nonlinearity is felt only in the higher-order term. A similar decomposition is given for the generalized Korteweg–de Vries–Burgers equation

$$u_t + u_x + P(u)_x - \nu u_{xx} + u_{xxx} = 0. \qquad (**)$$

Keywords: Generalized Korteweg–de Vries–Burgers equation; Generalized regularized long wave-Burgers equation; Weak nonlinearity; Asymptotic decay

1. Introduction

When quantitative agreement of the predictions of mathematical models for nonlinear, dispersive wave propagation with laboratory experiments or field data is in question, dissipative effects cannot usually be ignored. In consequence, the detailed study of the long-term balances struck between nonlinearity, dispersion and dissipation in wave equations has come to the fore as an area worthy of extended study. It is the purpose of this script to add to the discussion of the large-time asymptotics of solutions of the pure initial-value problem for nonlinear, dispersive, dissipative wave equations.

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The context of the present discussion is solutions of the initial-value problems for the generalized Korteweg–de Vries–Burgers (GKdVB) equation

$$u_t + u_x + P(u)_x - \nu u_{xx} + u_{xxx} = 0, \quad x \in \mathbb{R}, \; t > 0, \tag{1.1}$$

and the generalized regularized long wave-Burgers (GRLW-B) equation

$$u_t + u_x + P(u)_x - \nu u_{xx} - u_{xxt} = 0, \quad x \in \mathbb{R}, \; t > 0. \tag{1.2}$$

In Eqs. (1.1) and (1.2), $u = u(x, t)$ is a real-valued function of the two real variables x and t, ν a positive number and the initial datum

$$u(x, 0) = f(x), \quad x \in \mathbb{R} \tag{1.3}$$

will presently be restricted in smoothness and evanescence as $x \to \pm\infty$. These nonlinear, dispersive, dissipative wave equations have appeared frequently as models of physical phenomena, particularly when P is quadratic (cf. [8,12,13]). Indeed, when $P(u) = u^2$ and $\nu = 0$, these are the classical Korteweg–de Vries equation [17] and the regularized long-wave equation or BBM equation [2,19], respectively. For theory concerning the equations, with $\nu = 0$, see [15,16,23].

Because the parameter ν is positive, dissipation acts continuously and so the 'energy' of solutions decreases. If the evolution equation is initiated with a wave profile of finite energy, then it seems obvious, and indeed it is a fact (see [1,3,6,7,9–11,14,18,20–22]) that the solution corresponding to a given initial disturbance will decrease to zero as t becomes unboundedly large. For example, if $p \geq 1$ and $P(u) = cu^{p+1}$, then the $L_2(\mathbb{R})$-norm of solutions tends to zero at the rate $t^{-1/4}$ as t tends to infinity (see [1,18] for $p = 1$ and [6,11] for $p \geq 2$). For generic initial data, this rate of decay is optimal. For special initial data whose Fourier transform vanishes at the origin at some algebraic rate, enhanced decay of the $L_2(\mathbb{R})$-norm occurs (see [7,11]).

There is a subtle difference between what occurs for the quadratic case $P(u) = \frac{1}{2}u^2$ and for the general, homogeneous nonlinearity $P(u) = cu^{p+1}$, $p \geq 2$. Dix [11] referred to the latter case as having asymptotically weak nonlinearity. The rate $t^{-1/4}$ for the decay of the $L_2(\mathbb{R})$-norm is exactly that occurring for the linearized initial-value problems obtained by dropping the term $P(u)_x$ in (1.1) or (1.2). However, for the quadratic case, the limit

$$\lim_{t \to +\infty} \frac{\|u(\cdot, t)\|_{L_2(\mathbb{R})}}{\|w(\cdot, t)\|_{L_2(\mathbb{R})}} \tag{1.4}$$

exists and is not equal to 1, in general. Here, u is the solution of (1.1) or (1.2) with $P(u) = \frac{1}{2}u^2$ emanating from an initial datum f, while w is the solution of the corresponding linear equation with the same starting point. Thus the effect of the nonlinearity is in evidence at the lowest order in the long-time asymptotics. (The quadratic case has been further illuminated in the recent paper by Karch [14].) It is otherwise when $P(u) = cu^{p+1}$ with $p \geq 2$. In this case, the limit in (1.4) is equal to 1, and the difference

$$\|u(\cdot, t) - w(\cdot, t)\|_{L_2(\mathbb{R})} \tag{1.5}$$

tends to zero at a rate higher than $t^{-1/4}$ (see [9–11]). In fact, the quantity in (1.5) also tends to zero at a higher rate if, instead of w, one forms the difference between u and the solution v of the heat equation with a simple convection term

$$v_t + v_x - \nu v_{xx} = 0, \quad x \in \mathbb{R}, \; t > 0,$$

$$v(x, 0) = f(x), \quad x \in \mathbb{R}$$

(see [9,10]). Thus, neither nonlinearity nor dispersion appears in the lowest-order, long-time asymptotics. To see these effects, one must look to higher-order aspects of the decay.

The present study concentrates on (1.1) and (1.2) in the case of an asymptotically weak nonlinearity. We decompose solutions of (1.1) as a sum of the solution of the corresponding linear equation plus a term that decays at higher order. A quantitative description of the decay of the higher-order term is then obtained, in which the effects of nonlinearity and dispersion appear clearly. These results are comparable to those obtained for KdV-type equations with more general classes of dissipation in [10]. However, where the results can be compared, the present theory requires far less of the initial data than the theory in [10] made by way of renormalization techniques. Moreover, we are able to remove an annoying slight lack of sharpness in the decay rates appearing in [10].

The paper is organized as follows. In Section 2, the notation is set, earlier theory is reviewed and the results derived herein stated. In Section 3, a detailed analysis of the GRLW-B equation (1.2) is presented. The final section contains a sketch of the proof for the GKdVB equation (1.1), emphasizing the points where the reasoning is different from that exposed in Section 3.

2. Notation and main results

The L_q-norm of a function f which is qth-power Lebesgue integrable on $\mathbb{R}$ is denoted by $|f|_q$ for $1 \le q < \infty$, and similarly $|f|_\infty = \|f\|_{L_\infty}$. If $m \ge 0$ is an integer, $H^m(\mathbb{R})$ will be the Sobolev space consisting of those $L_2(\mathbb{R})$-functions whose first m generalized derivatives lie in $L_2(\mathbb{R})$, equipped with the usual norm

$$\|f\|_{H^m(\mathbb{R})} = \|f\|_m = \sum_{k=0}^{m} |f^{(k)}|_2.$$

The Fourier transform $\hat{f}$ of a function f is defined to be $\hat{f}(k) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$.

In the theory to follow, assumptions will be imposed that imply existence of globally defined solutions to the initial-value problems (1.1)–(1.3) and (1.2)–(1.3). More precisely, conditions will be imposed that yield time-independent bounds on the $H^1(\mathbb{R})$-norm of solutions. For Eq. (1.2), no restriction except smoothness on P is needed for the veracity of the latter condition. For Eq. (1.1), the assumption that the H^1-norm is bounded involves either a limit on the growth of P or else initial data which is sufficiently small (see [4–6]).

The following theorem is a slight generalization of the theory developed in [6,7], where it was simply assumed that $P(u) = cu^{p+1}$ for some constant $c \neq 0$.

Theorem 2.1. *Let $\nu > 0$ be fixed. Suppose P is a C^∞-function for which there is a constant $c > 0$ such that $|P(w)| \le c|w|^{p+1}$ for values of w near 0 and some integer $p \ge 2$. Suppose initial data f to lie in $L_1(\mathbb{R}) \cap H^2(\mathbb{R})$. In the case of Eq.* (1.1), *suppose also that $\|f\|_1 < \gamma_P$, where γ_P is the ceiling mentioned in* [6, *Theorem* 3.1]. *(Initial data that respects the ceiling γ_P leads to a time-independent bound on $\|u(\cdot,t)\|_1$ and so to global solutions, whereas those that do not may exhibit the formation of singularities in finite time. Of course, $\gamma_P = +\infty$ if $p < 4$.)*

Then there is a unique global solution u of (1.1) *or* (1.2) *corresponding to the initial data f, and, moreover, u depends continuously in $C(0,T;H^s)$ on variations of f in H^s for $s \ge 2$ and any finite value of T. Additionally, there are constants C_j, $1 \le j \le 3$, depending only on $\|f\|_1$ and ν, such that*

$$|u(\cdot,t)|_2 \le C_1(1+t)^{-1/4}, \qquad |u(\cdot,t)|_\infty \le C_2(1+t)^{-1/2}, \qquad |u_x(\cdot,t)|_2 \le C_3(1+t)^{-3/4} \tag{2.1}$$

for all $t \ge 0$. Furthermore, one has

$$\lim_{t\to+\infty} t^{1/2}|u(\cdot,t)|_2^2 = \lim_{t\to+\infty} t^{1/2}|w(\cdot,t)|_2^2 = (8\nu\pi)^{-1/2}\left(\int_{-\infty}^{\infty} f(x)\,dx\right)^2, \tag{2.2}$$

where w is the solution of the linearized version of (1.1) *or* (1.2) *in which the nonlinear term is simply dropped. For* $l = 0, 1$, *it is also true that*

$$|\partial_x^l (u-w)(\cdot,t)|_2^2 \le \begin{cases} C_N t^{-((3/2)+l)} & \text{if } p > 2, \\ C_N t^{-((3/2)+l)}(\log(1+t))^2 & \text{if } p = 2. \end{cases} \tag{2.3}$$

A detailed proof of this result may be made following the general line of argument exposed already in [6,7]. The difference in development here is that the homogeneous upper bound $|P(w)| \le c|w|^{p+1}$ is only enforced for w near zero. The reason one can get by with such local assumptions on the nonlinearity is that, in the presence of the assumed time-independent H^1-bound, the calculations in the proof of Lemma 4.1 and Corollary 4.2 in [6] yield that

$$|u_x(\cdot,t)|_2 \to 0 \quad \text{as } t \to +\infty.$$

Because of this, it follows that $|u(\cdot,t)|_\infty \to 0$ as $t \to +\infty$. Hence, for $t \ge T$ for a suitable choice of T, the values $w = u(x,t)$ all lie close enough to zero that the inequality $|f(w)| \le c|w|^{p+1}$ applies. Hence for $t \ge T$, the further developments in [6, Sections 4, 5] go over without substantive change, thus leading to the advertised conclusions.

The purpose of the present script is to improve upon the foregoing theory. Establishing the following result is our primary goal.

Main result. Suppose that $\nu > 0$ and $P(z) = cz^{p+1}$ for some $c > 0$ and z near 0, where $p \ge 2$. Let initial data $f \in L_1(\mathbb{R}) \cap H^2(\mathbb{R})$ be given and let u be the solution of (1.1) or (1.2) corresponding to f. (In case (1.1), suppose f to satisfy the conditions in Theorem 2.1 that imply global $H^1(\mathbb{R})$-bounds.) Let w be the solution of the corresponding linear equation with initial data f. Then, we have that $u = w + R$, where the remainder R decays to zero as $t \to +\infty$ more rapidly that does w. In fact if $p > 2$, then

$$\lim_{t\to+\infty} t^{3/2}|R(\cdot,t)|_2^2 = \frac{c^2}{4\nu(8\nu\pi)^{1/2}}\left(\int_0^\infty\int_{-\infty}^\infty u^{p+1}(x,\tau)\,dx\,d\tau\right)^2. \tag{2.4}$$

For $p = 2$, suppose $P(u) = cu^3$ for some constant c and that $xf(x)$ lies in $L_2(\mathbb{R})$ for Eq. (1.1) or $xf(x)$ and $xf'(x)$ lie in $L_2(\mathbb{R})$ if Eq. (1.2) is considered. Then, we have the following conclusions relative to a solution $u = u(x,t)$.

1. If $\lambda(t) = |\int_0^t\int_{-\infty}^\infty u^3(x,\tau)\,dx\,d\tau| \to +\infty$ as $t \to +\infty$, then

$$\lim_{t\to+\infty} \frac{t^{3/2}}{\lambda^2(t)}|u(\cdot,t) - w(\cdot,t)|_2^2 = \frac{c^2}{4\nu(8\nu\pi)^{1/2}}. \tag{2.5}$$

2. If $\int_0^\infty|\int_{-\infty}^\infty u^3(x,\tau)\,dx|\,d\tau < +\infty$, then one obtains

$$\lim_{t\to+\infty} t^{3/2}|u(\cdot,t) - w(\cdot,t)|_2^2 = \frac{c^2}{4\nu(8\nu\pi)^{1/2}}\left(\int_0^{+\infty}\int_{-\infty}^\infty u^3(x,\tau)\,dx\,d\tau\right)^2. \tag{2.6}$$

3. If $\int_0^\infty\int_{-\infty}^\infty|u^3(x,\tau)|\,dx\,d\tau = +\infty$, then the higher-order term is bounded above as follows:

$$\limsup_{t\to+\infty} \frac{t^{3/2}}{(\log(1+t))^2}|u(\cdot,t) - w(\cdot,t)|_2^2 \le \frac{c^2}{384\sqrt{2\pi^5\nu^7}}\left|\int_{-\infty}^\infty f(x)\,dx\right|^6. \tag{2.7}$$

Remark. Analogous results hold if it is only presumed that P is smooth and

$$|P(z)| \le c|z|^{p+1} \tag{2.8}$$

for z near 0. The double integral appearing in (2.4) is replaced by

$$\int_0^\infty \int_{-\infty}^\infty P(u(x,\tau))\,dx\,d\tau, \tag{2.9}$$

and similarly in (2.5)–(2.7).

The proofs of the main results for Eqs. (1.1) and (1.2) are similar. We present the proof for Eq. (1.2) in detail and then content ourselves with a sketch of the proof for (1.1). The complete proof for solutions of (1.1) is in several respects easier than that for Eq. (1.2).

The plan of the remainder of the paper is simple. In Section 3, the just mentioned detailed proof of the decay results for solutions of (1.2) is presented. Section 4 is devoted to sketching the proof for Eq. (1.1) emphasizing the points where the analysis departs from that in Section 3.

3. The proof of the main theorem for the GRLW-B equation

We begin with some facts about the linear initial-value problem

$$w_t + w_x - \nu w_{xx} - w_{xxt} = 0, \tag{3.1a}$$

$$w(x,0) = f(x). \tag{3.1b}$$

These provide a context for the nonlinear theory to follow. Problem (3.1a)–(3.1b) can be solved by formally taking the Fourier transform of Eq. (3.1a) with respect to the spatial variable x, thereby deducing that

$$\hat{w}(y,t) = \exp\left(-\frac{\nu y^2 t + iyt}{1+y^2}\right)\hat{w}(y,0), \tag{3.2}$$

and so concluding

$$w(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \exp\left(\frac{-\nu y^2 t - iyt}{1+y^2} + iyx\right)\hat{f}(y)\,dy. \tag{3.3}$$

The integral on the right-hand side of (3.3) will be denoted by $S(t)f(x)$. The proof of the following lemma may be found in [1, Lemma 2.1] (see also [7]).

Lemma 3.1. *If* $f \in H^1(\mathbb{R}) \cap L_1(\mathbb{R})$, *then the solution* w *of* (3.1a) *and* (3.1b) *with initial datum* f *satisfies*:

(a) $$\lim_{t\to\infty} t^{1/2}\int_{-\infty}^\infty w^2(x,t)\,dx = \lim_{t\to\infty} t^{1/2}|S(t)f(x)|_2^2 = (8\nu\pi)^{-1/2}\left(\int_{-\infty}^\infty f(x)\,dx\right)^2,$$

and

(b) $$\lim_{t\to\infty} t^{3/2}\int_{-\infty}^\infty w_x^2(x,t)\,dx = (128\nu^3\pi)^{-1/2}\left(\int_{-\infty}^\infty f(x)\,dx\right)^2.$$

We now come to the heart of the matter, which is to determine asymptotically the difference between a solution of (1.2) and (1.3), and the corresponding solution of (3.1a) and (3.1b).

Lemma 3.2. *Let f be in $H^2(\mathbb{R}) \cap L_1(\mathbb{R})$ and let P be smooth and satisfy* (2.8) *for z near* 0, *where $p > 2$. Then the difference between the solution u of* (1.2) *and* (1.3) *and the solution w of* (3.1a) *and* (3.1b), *both with initial value f, has the property*

$$\lim_{t\to+\infty} t^{3/2}|u(\cdot,t) - w(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{1/2}}\left(\int_0^\infty\int_{-\infty}^\infty P(u)\,\mathrm{d}x\,\mathrm{d}t\right)^2. \tag{3.4}$$

Proof. It is convenient to make the change of variables $U(x,t) = u(x+t,t)$ and $W(x,t) = w(x+t,t)$. Then if $V = U - W$, V satisfies the initial-value problem

$$V_t - \nu V_{xx} + V_{xxx} - V_{xxt} + P(U)_x = 0, \tag{3.5a}$$

$$V(x,0) = 0. \tag{3.5b}$$

This traveling-coordinate change of variables does not affect the value of any of the norms being considered here.

Take the Fourier transform of (3.5a) with respect to the spatial variable x and solve the resulting ordinary differential equation to reach the integral equation

$$\hat{U}(y,t) - \hat{W}(y,t) = -\int_0^t \frac{\mathrm{i}y}{1+y^2}\,\mathrm{e}^{-(((\nu y^2+\mathrm{i}y^3)/(1+y^2))(t-\tau))}\widehat{P(U)}(y,\tau)\,\mathrm{d}\tau. \tag{3.6}$$

According to Theorem 2.1, the solution u of Eq. (1.2) respects the inequalities

$$|u(\cdot,t)|_2 \le C(1+t)^{-1/4}, \qquad |u_x(\cdot,t)|_2 \le C(1+t)^{-3/4} \tag{3.7}$$

for $t \ge 0$, where the constants C depend only on ν and the norm of f. Because of (3.7), we see that

$$|U^{p+1}(\cdot,t)|_1 = |u^{p+1}(\cdot,t)|_1 \le |u(\cdot,t)|_\infty^{p-1}|u(\cdot,t)|_2^2 \le C(1+t)^{-p/2}. \tag{3.8}$$

Since $|U(\cdot,t)| \to 0$ uniformly in x as $t\to\infty$, it follows that for $t \ge T$ for some sufficiently large T, the values of $U(x,t)$ lie in the range where (2.8) applies. In consequence of this and of (3.8), it follows that there is a $C > 0$ independent of $t \ge 0$, such that

$$|P(U(\cdot,t))|_1 \le C(1+t)^{-p/2}$$

for all $t \ge 0$. Thus, the right-hand side of (3.4) is finite because $p > 2$. Apply Parseval's theorem and the substitution $s = y\sqrt{t}$ to (3.6) to obtain

$$\begin{aligned}
\lim_{t\to+\infty} t^{3/2}|U(\cdot,t) - W(\cdot,t)|_2^2 &= \lim_{t\to+\infty} t^{3/2}|\hat{U}(\cdot,t) - \hat{W}(\cdot,t)|_2^2 \\
&= \lim_{t\to+\infty} t^{3/2}\left|\frac{\mathrm{i}y}{1+y^2}\int_0^t \mathrm{e}^{-((\nu y^2-\mathrm{i}y^3)/(1+y^2))(t-\tau)}\widehat{P(U)}(y,\tau)\,\mathrm{d}\tau\right|_2^2 \\
&= \lim_{t\to+\infty} t^{3/2}\int_{-\infty}^\infty \frac{y^2}{(1+y^2)^2}\left|\int_0^t \mathrm{e}^{-((\nu y^2-\mathrm{i}y^3)/(1+y^2))(t-\tau)}\widehat{P(U)}(y,\tau)\,\mathrm{d}\tau\right|^2\mathrm{d}y \\
&= \lim_{t\to+\infty}\int_{-\infty}^\infty \frac{s^2\,\mathrm{e}^{-(2\nu s^2/(1+s^2/t))}}{(1+s^2/t)^2}\left|\int_0^t \mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{P(U)}\left(\frac{s}{\sqrt{t}},\tau\right)\mathrm{d}\tau\right|^2\mathrm{d}s.
\end{aligned} \tag{3.9}$$

The strategy is to apply the dominated-convergence theorem to evaluate the right-hand side of (3.9). This requires a number of relatively straightforward estimates which we include for the reader's convenience. First remark that the L_∞-norm of the inner integrand with respect to the variable s may be bounded as follows:

$$\left|\mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{P(U)}\left(\frac{s}{\sqrt{t}},\tau\right)\right| \le C\,\mathrm{e}^{\nu s^2\tau/(t+s^2)}|P(U(\cdot,\tau))|_1 \le C\,\mathrm{e}^{\nu s^2\tau/(t+s^2)}(1+\tau)^{-p/2}, \tag{3.10}$$

where the constants C do not depend on s, τ and t. Since $p > 2$, it follows that

$$\begin{aligned}
&\left|\int_0^t e^{(\nu s^2 - is^3/\sqrt{t})\tau/(t+s^2)}\widehat{P(U)}\left(\frac{s}{\sqrt{t}}, \tau\right) d\tau\right| \le C\left(\int_0^{t/2} + \int_{t/2}^t\right)\frac{e^{\nu s^2\tau/(t+s^2)}\, d\tau}{(1+\tau)^{p/2}} \\
&\le C\, e^{\nu s^2 t/2(t+s^2)}\int_0^{t/2}\frac{d\tau}{(1+\tau)^{p/2}} + \frac{C}{(1+t/2)^{p/2}}\int_{t/2}^t e^{\nu s^2\tau/(t+s^2)}\, d\tau \\
&\le \frac{C\, e^{\nu s^2 t/2(t+s^2)}}{1-(1+t/2)^{p/2-1}} + \frac{C(t+s^2)\, e^{\nu s^2 t/(t+s^2)}}{\nu s^2(1+t/2)^{p/2}}(1-e^{-(\nu s^2 t/2(t+s^2))}) \\
&\le C\left[e^{\nu s^2 t/2(t+s^2)} + \frac{(1+s^2/t)\, e^{\nu s^2 t/(t+s^2)}}{\nu s^2}(1-e^{-(\nu s^2 t/2(t+s^2))})\right].
\end{aligned}$$

With this inequality in hand, one shows that the integrand with respect to the variable s in (3.9) is bounded above in the following way:

$$\begin{aligned}
&\frac{s^2 e^{-2\nu s^2/(1+s^2/t)}}{(1+s^2/t)^2}\left|\int_0^t e^{(\nu s^2 - is^3/\sqrt{t})\tau/(t+s^2)}\widehat{P(U)}\left(\frac{s}{\sqrt{t}}, \tau\right) d\tau\right|^2 \\
&\le C\frac{s^2 e^{-2\nu s^2/(1+s^2/t)}}{(1+s^2/t)^2}\left[e^{\nu s^2 t/2(t+s^2)} + \frac{(1+s^2/t)\, e^{\nu s^2 t/(t+s^2)}}{\nu s^2}(1-e^{-\nu s^2 t/2(t+s^2)})\right]^2 \\
&\le \frac{Cs^2 e^{-\nu s^2/(1+s^2/t)}}{(1+s^2/t)^2} + \frac{C(1-e^{-\nu s^2/2(1+s^2/t)})^2}{\nu s^2} \\
&\le \frac{C}{1+s^2}\left\{\frac{(1+s^2)^2}{(1+s^2/t)^2}e^{-\nu(1+s^2)/(1+s^2/t)}\right\} e^{\nu/(1+s^2/t)} + \frac{C(1-e^{-\nu s^2/2(1+s^2/t)})^2}{\nu s^2} \\
&\le C\left[\frac{1}{1+s^2} + \frac{(1-e^{-\nu s^2/2})^2}{\nu s^2}\right],
\end{aligned} \tag{3.11}$$

where the constants C are again independent of s and t. Thus the integrand in the last integral in (3.9) is seen to be bounded above by a fixed L_1-function. It follows that we may pass the limit inside the integral with respect to s.

If $g(s, t, \tau)$ temporarily denotes the integrand in the inner integral on the right-hand side of (3.9), then

$$\int_0^t g(s, t, \tau)\, d\tau = \int_0^\infty g(s, t, \tau)\chi_{[0,t]}(\tau)\, d\tau = \int_0^\infty G(s, t, \tau)\, d\tau,$$

where $\chi_{[0,t]}(\tau) = 1$ if $\tau \le t$ and $\chi_{[0,t]}(\tau) = 0$ if $\tau > t$. Because of the uniform estimate (3.10) on G, one shows that for fixed s,

$$|G(s, t, \tau)| \le C\, e^{\nu s^2\tau/(t+s^2)}(1+\tau)^{-p/2} \le C\, e^{\nu s^2/(1+s^2/t)}(1+\tau)^{-p/2}.$$

Note that the function on the right-hand side of the above inequality is an L_1-function with respect to τ since $p > 2$. Using this information and the dominated-convergence theorem, it is inferred that

$$\lim_{t\to\infty}\int_0^\infty G(s, t, \tau)\, d\tau = \int_0^\infty \left(\lim_{t\to\infty} G(s, t, \tau)\right) d\tau,$$

provided the latter limit exists. But,

$$\exp\left(\frac{(\nu s^2 - is^3/\sqrt{t})\tau}{t+s^2}\right) \to 1$$

as $t \to +\infty$ for any fixed $s \in \mathbb{R}$ and $\tau \in \mathbb{R}^+$. Also, $U \in H^1$, so $|U(x,t)| \to 0$ as $|x| \to \infty$, for any $t \geq 0$. Hence for $|x| \geq M$ for suitably chosen M, which may depend on t, the value of $U(x,t)$ lies in the range where (2.8) applies. In consequence, we see that

$$|P(U)| \leq c|U^{p+1}|$$

for $x \geq M$, and then it is inferred that $P(U) \in L_1$. It follows from the Riemann–Lebesgue lemma that $\widehat{P(U)}$ is continuous, whence

$$\lim_{t\to\infty} \widehat{P(U)}\left(\frac{s}{\sqrt{t}}, \tau\right) = \widehat{P(U)}(0,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P(U(x,\tau))\,dx.$$

Combining these ruminations allows us to continue Eq. (3.9) as follows:

$$\begin{aligned}\lim_{t\to+\infty} t^{3/2}|U(\cdot,t) - W(\cdot,t)|_2^2 &= \int_{-\infty}^{\infty} s^2\,e^{-2\nu s^2}\,ds \left(\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\int_{-\infty}^{\infty} P(U(x,\tau))\,dx\,d\tau\right)^2 \\ &= \frac{1}{2\pi(2\nu)^{3/2}} \int_0^{\infty} s^{1/2}\,e^{-s}\,ds \left(\int_0^{\infty}\int_{-\infty}^{\infty} P(u(x,\tau))\,dx\,d\tau\right)^2 \\ &= \frac{1}{4\nu(8\nu\pi)^{1/2}} \left(\int_0^{\infty}\int_{-\infty}^{\infty} P(u(x,\tau))\,dx\,d\tau\right)^2.\end{aligned}$$

The lemma is established. □

If $p = 2$, then the higher-order decay of solutions of (1.2) takes a slightly different form from that just described in case $p > 2$. The next lemma is a step toward understanding the case $p = 2$. For simplicity, the nonlinearity is specialized to be a pure cubic.

Lemma 3.3. *Let f lie in $H^2(\mathbb{R}) \cap L_1(\mathbb{R})$. Moreover, suppose $xf(x)$ and $xf'(x)$ are members of $L_2(\mathbb{R})$. Let u be the solution of (1.2) and (1.3) with the nonlinearity $P(u) = cu^3$. If $|\int_0^t \int_{-\infty}^{\infty} u^3(x,\tau)\,dx\,d\tau| \to +\infty$ as $t \to +\infty$, then one has*

$$\lim_{t\to+\infty} \frac{\int_0^t e^{(\nu s^2 - is^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}(s/\sqrt{t},\tau)\,d\tau}{(1/\sqrt{2\pi})\int_0^t\int_{-\infty}^{\infty} U^3(x,\tau)\,dx\,d\tau} = \lim_{t\to+\infty} \frac{\int_0^t e^{(\nu s^2 - is^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}(s/\sqrt{t},\tau)\,d\tau}{\int_0^t \widehat{U^3}(0,\tau)\,d\tau} = 1 \tag{3.12}$$

for any fixed s, where $U(x,t) = u(x+t,t)$ as before.

Proof. First, by using (3.8) with $p = 2$, it is seen that for any $s \in \mathbb{R}$ and $t > 0$,

$$\left|\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\right| \leq \frac{1}{\sqrt{2\pi}}|U(\cdot,\tau)|_3^3 \leq \frac{C}{1+\tau}. \tag{3.13}$$

Hence for any fixed s, the use of (3.13) shows that

$$\begin{aligned}&\left|\int_{t/2}^t \left[e^{(\nu s^2 - is^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right) - \widehat{U^3}(0,\tau)\right] d\tau\right| \leq C\int_{t/2}^t e^{\nu s^2\tau/(t+s^2)}|U(\cdot,\tau)|_3^3\,d\tau \\ &\quad\leq \int_{t/2}^t \frac{C\,e^{\nu s^2\tau/(t+s^2)}}{1+\tau}\,d\tau \leq \frac{C}{1+t/2}\int_{t/2}^t e^{\nu s^2\tau/(t+s^2)}\,d\tau \leq \frac{C(t+s^2)\,e^{\nu s^2 t/(t+s^2)}}{s^2(1+t/2)}\left[1 - e^{-\nu s^2 t/2(t+s^2)}\right].\end{aligned} \tag{3.14}$$

By using (3.14) and the hypothesis that $|\int_0^t\int_{-\infty}^{\infty}u^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau| \to +\infty$ as $t \to +\infty$, one shows that

$$\lim_{t\to+\infty}\frac{\int_{t/2}^t[\mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}(s/\sqrt{t},\tau)-\widehat{U^3}(0,\tau)]\,\mathrm{d}\tau}{\int_0^t\widehat{U^3}(0,\tau)\,\mathrm{d}\tau}=0. \tag{3.15}$$

From (3.15) it is concluded that to establish (3.12), one only need to prove

$$\lim_{t\to+\infty}\frac{\int_0^{t/2}\mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}(s/\sqrt{t},\tau)\,\mathrm{d}\tau}{\int_0^t\widehat{U^3}(0,\tau)\,\mathrm{d}\tau}=1. \tag{3.16}$$

For any fixed s, (3.13) and the fact that $\sin(x) \le x$ for $x \ge 0$ implies

$$\begin{aligned}&\left|\int_0^{t/2}\mathrm{e}^{\nu s^2\tau/(t+s^2)}\sin\left(\frac{s^3\tau}{(t+s^2)\sqrt{t}}\right)\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\,\mathrm{d}\tau\right|\\&\quad\le C\,\mathrm{e}^{\nu s^2t/2(t+s^2)}\int_0^{t/2}\frac{|s|^3\tau}{t^{3/2}(1+s^2/t)(1+\tau)}\,\mathrm{d}\tau\le\frac{C|s|^3\,\mathrm{e}^{\nu s^2t/2(t+s^2)}}{t^{1/2}(1+s^2/t)},\end{aligned} \tag{3.17}$$

where the constant C is independent of s and t. Thus, for any fixed s, one certainly infers

$$\lim_{t\to+\infty}\frac{\int_0^{t/2}\mathrm{e}^{\nu s^2\tau/(t+s^2)}\sin(s^3\tau/(t+s^2)\sqrt{t})\widehat{U^3}(s/\sqrt{t},\tau)\,\mathrm{d}\tau}{\int_0^t\int_{-\infty}^{\infty}U^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau}=0. \tag{3.18}$$

Since $p = 2$, the function U satisfies the equation

$$U_t-\nu U_{xx}+U_{xxx}-U_{xxt}+(cU^3)_x=0. \tag{3.19}$$

As in Lemma 4.1 in [6], one straightforwardly shows that $xU(x,t)$ and $xU_x(x,t)$ lie in $L_2(\mathbb{R})$ for any $t \ge 0$. Because of this, the following calculation makes sense. Multiply (3.19) by $2x^2U$ and then integrate the result by parts over $\mathbb{R}\times[0,t]$ to obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}x^2U^2(x,t)\,\mathrm{d}x+\int_{-\infty}^{\infty}x^2U_x^2(x,t)\,\mathrm{d}x+2\nu\int_0^t\int_{-\infty}^{\infty}x^2U_x^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\&\quad=2|U(\cdot,t)|_2^2-2|f|_2^2+\int_{-\infty}^{\infty}x^2[f^2(x)+(f'(x))^2]\,\mathrm{d}x+2\nu\int_0^t\int_{-\infty}^{\infty}U^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\&\qquad+\int_0^t\int_{-\infty}^{\infty}[4xU_x(x,\tau)U_t(x,\tau)-6xU_x^2(x,\tau)+3cxU^4(x,\tau)]\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{3.20}$$

The first two terms in the second double integral on the right-hand side of (3.20) can be estimated via Young's inequality as

$$\begin{aligned}&\left|\int_0^t\int_{-\infty}^{\infty}[4xU_x(x,\tau)U_t(x,\tau)-6xU_x^2(x,\tau)]\,\mathrm{d}x\,\mathrm{d}\tau\right|\le\int_0^t\left[\frac{36}{\nu}|U_x(\cdot,\tau)|_2^2+\frac{16}{\nu}|U_t(\cdot,\tau)|_2^2\right]\mathrm{d}\tau\\&\quad+\frac{\nu}{2}\int_0^t\int_{-\infty}^{\infty}x^2U_x^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{3.21}$$

As remarked earlier, $|U_x(\cdot,t)|_2=|u_x(\cdot,t)|_2$. Since $U_t=u_x+u_t$, it follows that $|U_t(\cdot,t)|_2\le|u_x(\cdot,t)|_2+|u_t(\cdot,t)|_2$. If we invert the operator $1-\partial_x^2$ in Eq. (1.2), subject to zero boundary conditions at $\pm\infty$, there appears a formula for u_t, namely

$$u_t=-(1-\partial_x^2)^{-1}(u_x+c(u^3)_x-\nu u_{xx})=-K(u_x+c(u^3)_x)+\nu Mu_x, \tag{3.22}$$

where $K(z) = \frac{1}{2}\,\mathrm{e}^{-|z|}$ and $M(z) = -\frac{1}{2}\,\mathrm{sgn}(z)\,\mathrm{e}^{-|z|}$. The fundamental energy-type relationships (3.7) assures that $|u(\cdot,t)|_2$, $|u_x(\cdot,t)|_2$ and $|u(\cdot,t)|_\infty$ are bounded, independently of $t \geq 0$, in terms of $\|f\|_1$, and that $|u_x(\cdot,t)|_2 \in L_2(0,\infty)$ with a bound that also depends only on ν and $\|f\|_1$. Since both the convolution kernels K and M lie in $L_1(\mathbb{R})$, it thus follows from (3.22) that

$$|u_t(\cdot,t)|_2^2 \leq C(\nu, \|f\|_1)|u_x(\cdot,t)|_2^2.$$

Hence, the first integral on the right-hand side of (3.21) is bounded by a constant depending only on ν and $\|f\|_1$. On the other hand

$$xU^2(x,t) = \int_{-\infty}^{x} [2zU(z,t)U_z(z,t) + U^2(z,t)]\,\mathrm{d}z,$$

so, by the Cauchy–Schwartz inequality, one has

$$\sup_{-\infty<x<\infty} |xU^2(x,t)| \leq 2|U(\cdot,t)|_2 \left(\int_{-\infty}^{\infty} x^2U_x^2(x,\tau)\,\mathrm{d}x\right)^{1/2} + |U(\cdot,t)|_2^2. \tag{3.23}$$

By using (3.23) and Young's inequality, the last term in the second double integral on the right-hand side of (3.20) is bounded above, viz.

$$\begin{aligned}\left|\int_0^t\int_{-\infty}^{\infty} xU^4(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\right| &\leq \int_0^t |xU^2(x,\tau)|_\infty |U(\cdot,\tau)|_2^2\,\mathrm{d}\tau \\ &\leq C\int_0^t |U(\cdot,\tau)|_2^4\,\mathrm{d}\tau + \frac{\nu}{2}\int_0^t\int_{-\infty}^{\infty} x^2U_x^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau,\end{aligned} \tag{3.24}$$

where $C = C(\nu, \|f\|_1)$. Using (3.21) and (3.24) in (3.20), making further estimates and applying (3.7) again, one obtains

$$\begin{aligned}&\int_{-\infty}^{\infty} x^2U^2(x,t)\,\mathrm{d}x + \int_{-\infty}^{\infty} x^2U_x^2(x,t)\,\mathrm{d}x + \nu\int_0^t\int_{-\infty}^{\infty} x^2U_x^2(x,t)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad\leq C + C\int_{-\infty}^{\infty} x^2[f^2(x) + (f'(x))^2]\,\mathrm{d}x + C\int_0^t\int_{-\infty}^{\infty} u^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \leq C(1+t)^{1/2}.\end{aligned} \tag{3.25}$$

Suppose that $t \geq \tau \geq 1$, say. Then note that (3.25) implies

$$\int_{|x|\geq t^{1/2}} U^2(x,\tau)\,\mathrm{d}x \leq \frac{1}{t}\int_{|x|\geq t^{1/2}} x^2U^2(x,\tau)\,\mathrm{d}x \leq \frac{1}{t}\int_{-\infty}^{\infty} x^2U^2(x,\tau)\,\mathrm{d}x \leq C\frac{\tau^{1/2}}{t} \leq \frac{C}{t^{1/2}} \tag{3.26}$$

for some constant C depending on ν and $\|f\|_1$ as before. The inequality (3.25) can be combined with (3.7) and the Cauchy–Schwartz inequality to adduce

$$\begin{aligned}\int_{-\infty}^{\infty} |x||U(x,\tau)|^3\,\mathrm{d}x &\leq |U(\cdot,\tau)|_4^2\left(\int_{-\infty}^{\infty} x^2U^2(x,\tau)\,\mathrm{d}x\right)^{1/2} \leq |U(\cdot,\tau)|_\infty|U(\cdot,\tau)|_2\left(\int_{-\infty}^{\infty} x^2U^2(x,\tau)\,\mathrm{d}x\right)^{1/2} \\ &\leq C(1+\tau)^{-1/2-1/4+1/4} \leq C(1+\tau)^{-1/2}.\end{aligned} \tag{3.27}$$

With (3.26) and (3.27) in hand, one shows that

$$\begin{aligned}
&\left|\int_0^{t/2}\left[e^{\nu s^2\tau/(t+s^2)}\cos\left(\frac{s^3\tau}{(t+s^2)\sqrt{t}}\right)\int_{-\infty}^{\infty}\sin\left(\frac{sx}{\sqrt{t}}\right)U^3(x,\tau)\,dx\right]d\tau\right| \\
&\quad=\left|\int_0^{t/2} e^{\nu s^2\tau/(t+s^2)}\cos\left(\frac{s^3\tau}{(t+s^2)\sqrt{t}}\right)\left\{\int_{|x|>t^{1/2}}+\int_{|x|\le t^{1/2}}\right\}\sin\left(\frac{sx}{\sqrt{t}}\right)U^3(x,\tau)\,dx\,d\tau\right| \\
&\quad\le C\,e^{\nu s^2 t/2(t+s^2)}\int_0^{t/2}\left[|U(\cdot,\tau)|_\infty\int_{|x|>t^{1/2}}|U(x,\tau)|^2\,dx+\int_{|x|\le t^{1/2}}\frac{|s|}{t^{1/2}}|xU^3(x,\tau)|\,dx\right]d\tau \\
&\quad\le C\,e^{\nu s^2 t/2(t+s^2)}\int_0^{t/2}\frac{1+|s|}{t^{1/2}(1+\tau)^{1/2}}\,d\tau\le C(1+|s|)\,e^{\nu s^2 t/2(t+s^2)},
\end{aligned} \tag{3.28}$$

where the constant C is independent of s and t. It follows from (3.28) that for any fixed value of s,

$$\lim_{t\to+\infty}\frac{\int_0^{t/2}[e^{\nu s^2\tau/(t+s^2)}\cos(s^3\tau/(t+s^2)\sqrt{t})\int_{-\infty}^{\infty}\sin(sx/\sqrt{t})U^3(x,\tau)\,dx]\,d\tau}{\int_0^t\int_{-\infty}^{\infty}U^3(x,\tau)\,dx\,d\tau}=0, \tag{3.29}$$

because of the hypothesis about $\int_0^t\int_{-\infty}^{\infty}u^3(x,\tau)\,dx\,d\tau$. It also follows from (3.28) that

$$\begin{aligned}
&\left|\int_0^{t/2}\left[e^{\nu s^2\tau/(t+s^2)}\cos\left(\frac{s^3\tau}{(t+s^2)\sqrt{t}}\right)\int_{-\infty}^{\infty}\left[\cos\left(\frac{sx}{\sqrt{t}}\right)-1\right]U^3(x,\tau)\,dx\right]d\tau\right| \\
&\quad\le C\,e^{\nu s^2 t/2(t+s^2)}\int_0^{t/2}\left[|U(\cdot,\tau)|_\infty\int_{|x|>t^{1/2}}|U(x,\tau)|^2\,dx+\int_{|x|\le t^{1/2}}\left|G\left(\frac{sx}{\sqrt{t}}\right)xU^3(x,\tau)\right|\frac{|s|}{t^{1/2}}\,dx\right]d\tau \\
&\quad\le C\,e^{\nu s^2 t/2(t+s^2)}\int_0^{t/2}\frac{1+|s|}{t^{1/2}(1+\tau)^{1/2}}\,d\tau\le C(1+|s|)\,e^{\nu s^2 t/2(t+s^2)},
\end{aligned} \tag{3.30}$$

where $G(z)=(\cos(z)-1)/z$ is a bounded function for $z\in\mathbb{R}$. Finally, note that

$$\begin{aligned}
&\left|\int_0^{t/2}\left[\left[e^{\nu s^2\tau/(t+s^2)}\cos\left(\frac{s^3\tau}{(t+s^2)\sqrt{t}}\right)-1\right]\int_{-\infty}^{\infty}\cos\left(\frac{sx}{\sqrt{t}}\right)U^3(x,\tau)\,dx\right]d\tau\right| \\
&\quad\le\left|\int_0^{t/2}\left[e^{\nu s^2\tau/(t+s^2)}\left[\cos\left(\frac{s^3\tau}{(t+s^2)\sqrt{t}}\right)-1\right]+[e^{\nu s^2\tau/(t+s^2)}-1]\right]|U(\cdot,\tau)|_3^3\,d\tau\right| \\
&\quad\le C\,e^{\nu s^2 t/2(t+s^2)}\int_0^{t/2}\left[\frac{|s|^3\tau}{t^{3/2}(1+s^2/t)}\left|H\left(\frac{\tau}{t},s\right)\right|+\frac{\nu s^2\tau}{t+s^2}\right]\frac{C}{1+\tau}\,d\tau \\
&\quad\le\frac{C\,e^{\nu s^2 t/2(t+s^2)}}{1+s^2/t}\left(\frac{|s|^3}{t^{3/2}}+\frac{s^2}{t}\right)\int_0^{t/2}\frac{\tau}{1+\tau}\,d\tau\le\frac{C\,e^{\nu s^2 t/2(t+s^2)}}{1+s^2/t}\left(\frac{|s|^3}{\sqrt{t}}+s^2\right),
\end{aligned} \tag{3.31}$$

where $H(y,s)$ is a bounded function for $y\in[0,1]$ and any fixed s, and the elementary inequality $e^x-1\le x\,e^x$

for $x > 0$ has been used. From (3.16), one concludes that

$$\lim_{t\to+\infty}\left[\frac{\int_0^{t/2}[\mathrm{e}^{\nu s^2\tau/(t+s^2)}\cos(s^3\tau/(t+s^2)\sqrt{t})\int_{-\infty}^{\infty}\cos(sx/\sqrt{t})U^3(x,\tau)\,\mathrm{d}x]\,\mathrm{d}\tau}{\int_0^t\int_{-\infty}^{\infty}U^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau}-1\right]$$
$$=\lim_{t\to+\infty}\frac{\int_0^{t/2}[[\mathrm{e}^{\nu s^2\tau/(t+s^2)}\cos(s^3\tau/(t+s^2)\sqrt{t})-1]\int_{-\infty}^{\infty}\cos(sx/\sqrt{t})U^3(x,\tau)\,\mathrm{d}x]\,\mathrm{d}\tau}{\int_0^t\int_{-\infty}^{\infty}U^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau}$$
$$+\lim_{t\to+\infty}\frac{\int_0^{t/2}[\mathrm{e}^{\nu s^2\tau/(t+s^2)}\cos(s^3\tau/(t+s^2)\sqrt{t})\int_{-\infty}^{\infty}[\cos(sx/\sqrt{t})-1]U^3(x,\tau)\,\mathrm{d}x]\,\mathrm{d}\tau}{\int_0^t\int_{-\infty}^{\infty}U^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau}=0. \tag{3.32}$$

Combining the limits (3.16), (3.18), (3.29) and (3.32), it is determined that

$$\lim_{t\to+\infty}\frac{\int_0^t \mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}(s/\sqrt{t},\tau)\,\mathrm{d}\tau}{\int_0^t\widehat{U^3}(0,\tau)\,\mathrm{d}\tau}$$
$$=\lim_{t\to+\infty}\int_0^t[\mathrm{e}^{\nu s^2\tau/(t+s^2)}[\cos(s^3\tau/(t+s^2)\sqrt{t})-\mathrm{i}\sin(s^3\tau/(t+s^2)\sqrt{t})]$$
$$\times\int_{-\infty}^{\infty}[\cos(sx/\sqrt{t})+\mathrm{i}\sin(sx/\sqrt{t})]U^3(x,\tau)\,\mathrm{d}x]\,\mathrm{d}\tau\left(\int_0^t\int_{-\infty}^{\infty}U^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\right)^{-1}$$
$$=\lim_{t\to+\infty}\frac{\int_0^t[\mathrm{e}^{\nu s^2\tau/(t+s^2)}\cos(s^3\tau/(t+s^2)\sqrt{t})\int_{-\infty}^{\infty}\cos(sx/\sqrt{t})U^3(x,\tau)\,\mathrm{d}x]\,\mathrm{d}\tau}{\int_0^t\int_{-\infty}^{\infty}U^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau}=1.$$

The lemma is thereby proved. □

When $p=2$, it is not clear whether $\int_0^{\infty}\int_{-\infty}^{\infty}u^3\,\mathrm{d}x\,\mathrm{d}\tau$ is a finite or not for general initial data even if xf and xf' lie in $L_2(\mathbb{R})$. But, the quantity $|\int_0^t\int_{-\infty}^{\infty}u^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau|$ grows no faster than $\log(1+t)$ as t becomes large. This is because

$$\left|\int_{-\infty}^{\infty}u^3(x,\tau)\,\mathrm{d}x\right|\le|u(\cdot,\tau)|_\infty|u(\cdot,\tau)|_2^2\le C(1+\tau)^{-1}, \tag{3.33}$$

from which follows

$$\left|\int_0^t\int_{-\infty}^{\infty}u^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\right|\le C\log(1+t).$$

In the remainder of this section, decay estimates for $p=2$ are established. In some cases, optimal decay rates are obtained.

Lemma 3.4. *Let f satisfy the conditions in Lemma* (3.3). *Let u be the solution of Eq.* (1.2) *with $P(u)=cu^3$ and w be the solution of Eq.* (3.1a) *and* (3.1b), *both with initial value f.*

1. *If $\lambda(t)=|\int_0^t\int_{-\infty}^{\infty}u^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau|\to+\infty$ as $t\to+\infty$, then*

$$\lim_{t\to+\infty}\frac{t^{3/2}}{\lambda^2(t)}|u(\cdot,t)-w(\cdot,t)|_2^2=\frac{c^2}{4\nu(8\nu\pi)^{1/2}}. \tag{3.34}$$

2. *If $\int_0^{\infty}|\int_{-\infty}^{\infty}u^3(x,\tau)\,\mathrm{d}x|\,\mathrm{d}\tau<\infty$, then*

$$\lim_{t\to+\infty}t^{3/2}|u(\cdot,t)-w(\cdot,t)|_2^2=\frac{c^2}{4\nu(8\nu\pi)^{1/2}}\left(\int_0^{+\infty}\int_{-\infty}^{\infty}u^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\right)^2. \tag{3.35}$$

3. *If* $\int_0^\infty\int_{-\infty}^\infty |u^3(x,\tau)|\,dx\,d\tau = +\infty$, *then*

$$\limsup_{t\to+\infty}\frac{t^{3/2}}{(\log(1+t))^2}|u(\cdot,t)-w(\cdot,t)|_2^2 \le \frac{c^2}{384\sqrt{2\pi^5\nu^7}}\left|\int_{-\infty}^\infty f(x)\,dx\right|^6. \tag{3.36}$$

Remark 3.5. Notice that if $\lambda(t)$ as defined above is not bounded as $t\to\infty$, whether or not it actually converges to $+\infty$ there, then the hypothesis 3 holds. Notice also that if the quantity appearing in the hypothesis 2 is not bounded, then the hypothesis 3 holds.

Proof. (1) It is supposed that $xf(x)$ and $xf'(x)$ lie in $L_2(\mathbb{R})$. Suppose also that the corresponding solution u of (1.2) and (1.3) with $P(u) = cu^3$ has the property

$$\left|\int_0^t\int_{-\infty}^\infty u^3(x,\tau)\,dx\,d\tau\right| \to +\infty$$

as $t\to+\infty$. By applying Parseval's theorem and Lemma 3.3, one shows that

$$\begin{aligned}
&\lim_{t\to+\infty} t^{3/2}\frac{|u(\cdot,t)-w(\cdot,t)|_2^2}{|\int_0^t\int_{-\infty}^\infty u^3(x,\tau)\,dx\,d\tau|^2}\\
&\quad=\lim_{t\to+\infty} t^{3/2}\frac{|U(\cdot,t)-W(\cdot,t)|_2^2}{|\int_0^t\int_{-\infty}^\infty U^3(x,\tau)\,dx\,d\tau|^2}\\
&\quad=\lim_{t\to+\infty} t^{3/2}\left|\frac{\int_0^t cyi/(1+y^2)\,e^{-((\nu y^2-iy^3)/(1+y^2))(t-\tau)}\widehat{U^3}(y,\tau)\,d\tau}{\sqrt{2\pi}\int_0^t\widehat{U^3}(0,\tau)\,d\tau}\right|_2^2\\
&\quad=\lim_{t\to+\infty} t^{3/2}\int_{-\infty}^\infty\frac{c^2y^2|\int_0^t e^{-((\nu y^2-iy^3)/(1+y^2))(t-\tau)}\widehat{U^3}(y,\tau)\,d\tau|^2}{2\pi(1+y^2)^2|\int_0^t\widehat{U^3}(0,\tau)\,d\tau|^2}\,dy\\
&\quad=\lim_{t\to+\infty}\int_{-\infty}^\infty\frac{c^2s^2\,e^{-2\nu s^2/(1+s^2/t)}|\int_0^t e^{(\nu s^2-is^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}(s/\sqrt{t},\tau)\,d\tau|^2}{2\pi(1+s^2/t)^2|\int_0^t\widehat{U^3}(0,\tau)\,d\tau|^2}\,ds.
\end{aligned}\tag{3.37}$$

Let

$$\theta(s,t,\tau) = \frac{\int_0^t e^{(\nu s^2-is^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}(s/\sqrt{t},\tau)\,d\tau}{\int_0^t\widehat{U^3}(0,\tau)\,d\tau}. \tag{3.38}$$

The use of the estimates (3.14), (3.17), (3.28), (3.30) and (3.31) in Lemma 3.3 shows that

$$\begin{aligned}
|\theta(s,t,\tau)-1| \le C\Bigg[&\frac{(t+s^2)\,e^{\nu s^2t/(t+s^2)}}{s^2(1+t/2)}[1-e^{-\nu s^2t/2(t+s^2)}] + \frac{e^{\nu s^2t/2(t+s^2)}}{1+s^2/t}\left(\frac{|s|^3}{\sqrt{t}}+s^2\right)\\
&+(1+|s|)\,e^{\nu s^2t/2(t+s^2)}\Bigg].
\end{aligned}\tag{3.39}$$

By using (3.39), one deduces that

$$\begin{aligned}
&\frac{s^2\,e^{-2\nu s^2 t/(t+s^2)}}{(1+s^2/t)^2}|\theta(s,t,\tau)|^2 \le 2\frac{s^2\,e^{-2\nu s^2 t/(t+s^2)}}{(1+s^2/t)^2}[1+|\theta(s,t,\tau)-1|^2]\\
&\le C\frac{s^2\,e^{-2\nu s^2 t/(t+s^2)}}{(1+s^2/t)^2}\left[1+\frac{(1+s^2/t)\,e^{\nu s^2 t/(t+s^2)}}{s^2}[1-e^{-\nu s^2 t/2(t+s^2)}]\right.\\
&\quad\left.+\left[\frac{|s|^3/\sqrt{t}+s^2}{1+s^2/t}+(1+|s|)\right]e^{\nu s^2 t/2(t+s^2)}\right]^2\\
&\le \frac{C}{1+s^2}\frac{(1+s^2)^2}{(1+s^2/t)^2}\,e^{-2\nu(1+s^2)/(1+s^2/t)}\,e^{2\nu/(1+s^2/t)}+\frac{C(1-e^{-\nu s^2 t/2(t+s^2)})^2}{s^2}\\
&\quad+\frac{C\,e^{-\nu s^2/(1+s^2/t)}}{(1+s^2/t)^4}\left(\frac{s^8}{t}+s^6\right)+\frac{Cs^2(1+|s|)^2\,e^{-\nu s^2/(1+s^2/t)}}{(1+s^2/t)^2}\\
&\le C\left[\frac{1}{1+s^2}+\frac{(1-e^{-\nu s^2/2(1+s^2/t)})^2}{s^2}+\left(\frac{s^6}{(1+s^2/t)^3}+\frac{s^4+s^2}{(1+s^2/t)^2}\right)e^{-\nu s^2/(1+s^2/t)}\right].
\end{aligned}\tag{3.40}$$

For $s^2 < t$, the last function in the inequality (3.40) is an L_1-function with respect to s. In fact, one has

$$\begin{aligned}
&\frac{1}{1+s^2}+\frac{(1-e^{-\nu s^2/2(1+s^2/t)})^2}{s^2}+\left(\frac{s^6}{(1+s^2/t)^3}+\frac{s^4+s^2}{(1+s^2/t)^2}\right)e^{-\nu s^2/(1+s^2/t)}\\
&\le \frac{1}{1+s^2}+\frac{\left(1-e^{-\nu s^2/2}\right)^2}{s^2}+(s^6+s^4+s^2)\,e^{-(\nu s^2/2)}.
\end{aligned}\tag{3.41}$$

On the other hand, for $s^2 \ge t$, it is easily seen that

$$\begin{aligned}
&\frac{s^2\,e^{-2\nu s^2/(1+s^2/t)}|\int_0^t e^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}(s/\sqrt{t},\tau)\,d\tau|^2}{(1+s^2/t)^2}\\
&\le \frac{s^2\,e^{-2\nu s^2/(1+s^2/t)}[\int_0^{t/2}+\int_{t/2}^t e^{\nu s^2\tau/(t+s^2)}|U^3(\cdot,\tau)|_1\,d\tau]^2}{(1+s^2/t)^2}\\
&\le \frac{s^2}{(1+s^2/t)^2}\,e^{-2\nu s^2 t/(t+s^2)}\left[e^{\nu s^2 t/2(t+s^2)}\int_0^{t/2}\frac{C}{1+\tau}\,d\tau+\frac{C}{1+(t/2)}\int_{t/2}^t e^{\nu s^2\tau/(t+s^2)}\,d\tau\right]^2\\
&\le C\left[\frac{s^2\log^2(1+t)}{(1+s^2/t)^2}\,e^{-\nu s^2 t/(t+s^2)}+\frac{(1-e^{-\nu s^2 t/2(t+s^2)})^2}{s^2}\right]\\
&\le C\left[\frac{\log^2(1+s^2)}{1+s^2}\frac{(1+s^2)^2}{(1+s^2/t)^2}\,e^{-\nu(1+s^2)/(1+s^2/t)}\,e^{\nu/(1+s^2/t)}+\frac{(1-e^{-\nu s^2/2(1+s^2/t)})^2}{s^2}\right]\\
&\le C\left[\frac{\log^2(1+s^2)}{1+s^2}+\frac{(1-e^{-\nu s^2/2})^2}{s^2}\right].
\end{aligned}\tag{3.42}$$

The last function in the inequality (3.42) is also an L_1-function. By the dominated-convergence theorem, it follows from (3.37) and Lemma 3.3 that

$$\lim_{t\to+\infty} t^{3/2}\frac{|u(\cdot,t)-w(\cdot,t)|_2^2}{|\int_0^t\int_{-\infty}^{\infty}u^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau|^2}$$
$$=\frac{c^2}{2\pi}\int_{-\infty}^{\infty}\left[s^2\,\mathrm{e}^{-2\nu s^2}\lim_{t\to+\infty}\left(\frac{|\int_0^t \mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}(s/\sqrt{t},\tau)\,\mathrm{d}\tau|^2}{|\int_0^t\widehat{U^3}(0,\tau)\,\mathrm{d}\tau|^2}\right)\right]\mathrm{d}s$$
$$=\frac{c^2}{2\pi}\int_{-\infty}^{\infty}s^2\,\mathrm{e}^{-2\nu s^2}\,\mathrm{d}s=\frac{c^2}{2\pi(2\nu)^{3/2}}\int_0^{\infty}s^{1/2}\,\mathrm{e}^{-s}\,\mathrm{d}s=\frac{c^2}{4\nu(8\nu\pi)^{1/2}}.$$

(2) Assume now that $\int_0^{+\infty}|\int_{-\infty}^{\infty}u^3(x,\tau)\,\mathrm{d}x|\,\mathrm{d}\tau$ is finite, whence $\int_0^{+\infty}\int_{-\infty}^{\infty}U^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau$ is likewise finite. Note that for any fixed τ, the function $\widehat{U^3}(y,\tau)$ is a continuous function in y since U^3 is in $L_1(\mathbb{R})$ with respect to the spatial variable. Hence for $s\le t^{1/3}$ and t large enough, one deduces that for any fixed τ,

$$\left|\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)-\widehat{U^3}(0,\tau)\right|\le\frac{1}{2}|\widehat{U^3}(0,\tau)|. \tag{3.43}$$

Thus for $s\le t^{1/3}$ and t large enough, the use of (3.43) shows that

$$\int_0^{t/2}\left|\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\right|\mathrm{d}\tau\le 2\int_0^{t/2}|\widehat{U^3}(0,\tau)|\,\mathrm{d}\tau\le C\int_0^{\infty}\left|\int_{-\infty}^{\infty}U^3(x,\tau)\,\mathrm{d}x\right|\mathrm{d}\tau. \tag{3.44}$$

On the other hand, for $s>t^{1/3}$ inequality (3.33) yields

$$\int_0^{t/2}\left|\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\right|\mathrm{d}\tau\le\int_0^{t/2}\frac{C}{1+\tau}\,\mathrm{d}\tau\le C\log(1+t)\le C\log(1+s^3). \tag{3.45}$$

Combining the estimates (3.44) and (3.45), one obtains that for any s,

$$\begin{aligned}\int_0^{t/2}\left|\mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\right|\mathrm{d}\tau&\le\int_0^{t/2}\mathrm{e}^{\nu s^2\tau/(t+s^2)}\left|\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\right|\mathrm{d}\tau\\ &\le C\,\mathrm{e}^{\nu s^2/2(1+s^2/t)}\left[\log(1+s^3)+\int_0^{\infty}\left|\int_{-\infty}^{\infty}U^3(x,\tau)\,\mathrm{d}x\right|\mathrm{d}\tau\right].\end{aligned} \tag{3.46}$$

Applying (3.33) again yields the upper bound

$$\begin{aligned}\int_{t/2}^{t}\left|\mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\right|\mathrm{d}\tau&\le\int_{t/2}^{t}\mathrm{e}^{\nu s^2\tau/(t+s^2)}|U^3(\cdot,\tau)|_1\,\mathrm{d}\tau\\ &\le\int_{t/2}^{t}\frac{C\,\mathrm{e}^{\nu s^2\tau/(t+s^2)}}{1+\tau}\,\mathrm{d}\tau\le\frac{C}{1+t/2}\int_{t/2}^{t}\mathrm{e}^{\nu s^2\tau/(t+s^2)}\,\mathrm{d}\tau\le\frac{C(t+s^2)}{1+t/2}\frac{\mathrm{e}^{\nu s^2t/(t+s^2)}-\mathrm{e}^{\nu s^2t/2(t+s^2)}}{\nu s^2}\\ &\le C\left(1+\frac{s^2}{t}\right)\mathrm{e}^{\nu s^2/(1+s^2/t)}\frac{1-\mathrm{e}^{-(\nu s^2/2)}}{s^2}\end{aligned} \tag{3.47}$$

for some constant C. The inequalities in (3.46) and (3.47) allow us to further conclude that

$$\begin{aligned}\left|\int_0^{t}\mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\mathrm{d}\tau\right|&\le\int_0^{t/2}+\int_{t/2}^{t}\mathrm{e}^{\nu s^2\tau/(t+s^2)}\left|\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\right|\mathrm{d}\tau\\ &\le C\,\mathrm{e}^{\nu s^2/2(1+s^2/t)}\left[\log(1+s^3)+\int_0^{\infty}\left|\int_{-\infty}^{\infty}U^3(x,\tau)\,\mathrm{d}x\right|\mathrm{d}\tau\right]+C(1+s^2/t)\,\mathrm{e}^{\nu s^2/(1+s^2/t)}\frac{1-\mathrm{e}^{-\nu s^2/2}}{s^2}.\end{aligned} \tag{3.48}$$

On the other hand, (3.48) together with some simple calculations shows that

$$\begin{aligned}
&\frac{s^2}{(1+s^2/t)^2}\mathrm{e}^{-2\nu s^2/(1+s^2/t)}\left|\int_0^t \mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\mathrm{d}\tau\right|^2 \\
&\le \frac{s^2\,\mathrm{e}^{-2\nu s^2/(1+s^2/t)}}{(1+s^2/t)^2}\left[C\,\mathrm{e}^{\nu s^2/2(1+s^2/t)}\left(\log(1+s^3)+\int_0^\infty\left|\int_{-\infty}^\infty U^3(x,\tau)\,\mathrm{d}x\right|\mathrm{d}\tau\right)\right. \\
&\left. +C(1+s^2/t)\,\mathrm{e}^{\nu s^2/(1+s^2/t)}\frac{1-\mathrm{e}^{-\nu s^2/2}}{s^2}\right]^2 \\
&\le \frac{C(1+s^2)\,\mathrm{e}^{-\nu(1+s^2)/(1+s^2/t)}\,\mathrm{e}^{-\nu/(1+s^2/t)}}{(1+s^2/t)^2}\left(\log(1+s^3)+\int_0^\infty\left|\int_{-\infty}^\infty U^3(x,\tau)\,\mathrm{d}x\right|\mathrm{d}\tau\right)^2 \\
&+C\frac{(1-\mathrm{e}^{-\nu s^2/2})^2}{s^2}\le\frac{C}{1+s^2}\left(\log(1+s^3)+\int_0^\infty\left|\int_{-\infty}^\infty U^3(x,\tau)\,\mathrm{d}x\right|\mathrm{d}\tau\right)^2+C\frac{(1-\mathrm{e}^{-\nu s^2/2})^2}{s^2}.
\end{aligned}\tag{3.49}$$

Hence the function obtained in the last inequality in (3.49) may be taken as a dominating function. The use of the representation (3.6) and the dominated-convergence theorem implies that

$$\begin{aligned}
&\lim_{t\to+\infty} t^{3/2}|u(\cdot,t)-w(\cdot,t)|_2^2=\lim_{t\to+\infty}t^{3/2}|\hat{U}(\cdot,t)-\hat{W}(\cdot,t)|_2^2 \\
&=\lim_{t\to+\infty}t^{3/2}\int_{-\infty}^\infty\frac{c^2y^2}{(1+y^2)^2}\left|\int_0^t \mathrm{e}^{-((\nu y^2-\mathrm{i}y^3)/(1+y^2))(t-\tau)}\widehat{U^3}(y,\tau)\,\mathrm{d}\tau\right|^2\mathrm{d}y \\
&=\lim_{t\to+\infty}\int_{-\infty}^\infty\frac{c^2s^2\,\mathrm{e}^{-2\nu s^2/(1+s^2/t)}}{(1+s^2/t)^2}\left|\int_0^t \mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\mathrm{d}\tau\right|^2\mathrm{d}s \\
&=\int_{-\infty}^\infty c^2s^2\,\mathrm{e}^{-2\nu s^2}\lim_{t\to+\infty}\left|\int_0^t \mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\mathrm{d}\tau\right|^2\mathrm{d}s.
\end{aligned}\tag{3.50}$$

For any fixed s and t large enough, one infers that

$$\left|\mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\right|\le 2\,\mathrm{e}^{\nu s^2\tau/(t+s^2)}|\widehat{U^3}(0,\tau)|\le C\,\mathrm{e}^{\nu s^2}\left|\int_{-\infty}^\infty U^3(x,\tau)\,\mathrm{d}x\right|,\tag{3.51}$$

since $\widehat{U^{p+1}}(y,\tau)$ is continuous in y by the Riemann–Lebesgue lemma. Let

$$g(s,t,\tau)=\mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}(s/\sqrt{t},\tau).$$

Because $\int_{-\infty}^\infty U^3(x,\tau)\,\mathrm{d}x$ is an L_1-function in τ, it follows from (3.51) and the dominated-convergence theorem that

$$\lim_{t\to+\infty}\int_0^t g(s,t,\tau)\,\mathrm{d}\tau=\lim_{t\to+\infty}\int_0^\infty g(s,t,\tau)\chi_{[0,t]}(\tau)\,\mathrm{d}\tau=\int_0^\infty\lim_{t\to+\infty}g(s,t,\tau)\chi_{[0,t]}(\tau)\,\mathrm{d}\tau.\tag{3.52}$$

Note also that

$$\mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\to 1$$

as $t \to +\infty$ for any fixed $s \in \mathbb{R}$ and $\tau \in [0, t)$. It follows from (3.50) and (3.52), and then the Riemann–Lebesgue lemma that

$$\lim_{t\to+\infty} t^{3/2}|u(\cdot,t) - w(\cdot,t)|_2^2 = \lim_{t\to+\infty} t^{3/2}|\hat{U}(\cdot,t) - \hat{W}(\cdot,t)|_2^2$$

$$= c^2\int_{-\infty}^{\infty} s^2\,\mathrm{e}^{-2\nu s^2}\left(\int_{-\infty}^{\infty}\lim_{t\to+\infty} g(s,t,\tau)\chi_{[0,t]}(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s = c^2\int_{-\infty}^{\infty} s^2\,\mathrm{e}^{-2\nu s^2}\,\mathrm{d}s\left(\int_{-\infty}^{\infty}\hat{U}^3(0,\tau)\,\mathrm{d}\tau\right)^2$$

$$= c^2\int_{-\infty}^{\infty} s^2\,\mathrm{e}^{-2\nu s^2}\,\mathrm{d}s\left(\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\int_{-\infty}^{\infty} U^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\right)^2 = \frac{c^2}{4\nu(8\nu\pi)^{1/2}}\left(\int_0^{+\infty}\int_{-\infty}^{\infty} u^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\right)^2.$$

(3) By using the estimates (3.14), (3.17), (3.28), (3.30) and (3.31) and then following the derivation appearing in (3.40)–(3.42), it is straightforward to adduce that

$$\phi(s,t) = \frac{s^2\,\mathrm{e}^{-2\nu s^2 t/(t+s^2)}}{(1+s^2/t)^2}\left|\int_0^t \mathrm{e}^{(\nu s^2 - \mathrm{i}s|s|)\tau/t}\,\widehat{U^3}\left(\frac{s}{\sqrt{t}},\tau\right)\mathrm{d}\tau - \int_0^t \widehat{U^3}(0,\tau)\,\mathrm{d}\tau\right|^2$$

$$\le C\frac{s^2\,\mathrm{e}^{-2\nu s^2 t/(t+s^2)}}{(1+s^2/t)^2}\left[\frac{(t+s^2)\,\mathrm{e}^{\nu s^2 t/(t+s^2)}}{s^2(1+t/2)}[1-\mathrm{e}^{-\nu s^2 t/2(t+s^2)}]\right.$$

$$\left. + \frac{|s|^3\,\mathrm{e}^{\nu s^2 t/2(t+s^2)}}{\sqrt{t}(1+s^2/t)} + (1+|s|)\,\mathrm{e}^{\nu s^2 t/2(t+s^2)}\right]^2$$

$$\le C\left[\frac{1}{1+s^2} + \frac{(1-\mathrm{e}^{-\nu s^2/2(1+s^2/t)})^2}{s^2} + \left(\frac{s^6}{(1+s^2/t)^3} + \frac{s^4+s^2}{(1+s^2/t)^2}\right)\mathrm{e}^{-\nu s^2/(1+s^2/t)}\right]$$

$$\le \begin{cases} \left[\dfrac{1}{1+s^2} + \dfrac{(1-\mathrm{e}^{-\nu s^2/2})^2}{s^2} + (s^6+s^4+s^2)\,\mathrm{e}^{-\nu s^2/2}\right], & \text{when } s^2 < t, \\ C\left[\dfrac{\log^2(1+s^2)}{1+s^2} + \dfrac{(1-\mathrm{e}^{-\nu s^2/2})^2}{s^2}\right], & \text{when } s^2 \ge t \end{cases}$$

$$\le C\left[\frac{1}{(1+s^2)^{2/3}} + \frac{(1-\mathrm{e}^{-\nu s^2/2})^2}{s^2} + (1+s^6)\,\mathrm{e}^{-\nu s^2/2}\right] = \Phi(s) \tag{3.53}$$

for some suitable constant C. Note that Φ is an L_1-function. By hypothesis, as $t \to +\infty$, $\int_0^t\int_{-\infty}^{\infty}|u|^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \to +\infty$. The use of (3.2) with (2.1) and (2.2), and l'Hôpital's rule shows that

$$\lim_{t\to+\infty}\frac{\int_0^t\int_{-\infty}^{\infty}|u|^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau}{\log(1+t)} = \lim_{t\to+\infty}(1+t)\int_{-\infty}^{\infty}|u|^3(x,t)\,\mathrm{d}x = \lim_{t\to+\infty}(1+t)\int_{-\infty}^{\infty}|w|^3(x,t)\,\mathrm{d}x$$

$$= \frac{1}{4\sqrt{3}\nu\pi}\left|\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x\right|^3. \tag{3.54}$$

Hence, by using (3.53) and (3.54), it appears that for t fixed, but large enough

$$
\begin{aligned}
&\frac{s^2\,\mathrm{e}^{-2\nu s^2 t/(t+s^2)}}{(1+s^2/t)^2}\left|\frac{\int_0^t \mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}(s/\sqrt{t},\tau)\,\mathrm{d}\tau}{\log(1+t)}\right|^2\\
&\le \frac{\phi(s,t)+(s^2\,\mathrm{e}^{-2\nu s^2 t/(t+s^2)})/(1+s^2/t)^2|\int_0^t\widehat{U^3}(0,\tau)\,\mathrm{d}\tau|^2}{(\log(1+t))^2}\\
&\le \Phi(s)+(1/(1+s^2))((1+s^2)^2/(1+s^2/t)^2)\,\mathrm{e}^{-2\nu(1+s^2)/(1+s^2/t)}\,\mathrm{e}^{2\nu/(1+s^2/t)}((1/\sqrt{2\pi})\\
&\quad\times\int_0^t\int_{-\infty}^{\infty}|u|^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau)^2(\log(1+t))^{-2}\\
&\le \Phi(s)+\frac{C}{1+s^2}.
\end{aligned}\tag{3.55}
$$

The function on the right-hand side of (3.55) is in L_1. Another application of Lebesgue's dominated-convergence theorem and (3.54) shows that

$$
\begin{aligned}
&\limsup_{t\to+\infty}\frac{t^{3/2}}{(\log(1+t))^2}|u(\cdot,t)-w(\cdot,t)|_2^2=\limsup_{t\to+\infty}\frac{t^{3/2}}{(\log(1+t))^2}|\hat{U}(\cdot,t)-\hat{W}(\cdot,t)|_2^2\\
&=\limsup_{t\to+\infty}\frac{t^{3/2}}{(\log(1+t))^2}\int_{-\infty}^{\infty}\frac{c^2y^2}{(1+y^2)^2}\left|\int_0^t \mathrm{e}^{(-\nu y^2+\mathrm{i}y^3)(t-\tau)/(1+y^2)}\widehat{U^3}(y,\tau)\,\mathrm{d}\tau\right|^2\mathrm{d}y\\
&=c^2\limsup_{t\to+\infty}\int_{-\infty}^{\infty}\frac{s^2\,\mathrm{e}^{-2\nu s^2}}{(1+s^2/t)^2}\left|\frac{\int_0^t \mathrm{e}^{(\nu s^2-\mathrm{i}s^3/\sqrt{t})\tau/(t+s^2)}\widehat{U^3}(s/\sqrt{t},\tau)\,\mathrm{d}\tau}{\log(1+t)}\right|^2\mathrm{d}s\\
&\le c^2\int_{-\infty}^{\infty}s^2\,\mathrm{e}^{-2\nu s^2}\limsup_{t\to+\infty}\left|\frac{\int_0^t\widehat{U^3}(0,\tau)\,\mathrm{d}\tau}{\log(1+t)}\right|^2\mathrm{d}s\\
&\le c^2\limsup_{t\to+\infty}\left|\frac{(1/\sqrt{2\pi})\int_0^t\int_{-\infty}^{\infty}|u|^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau}{\log(1+t)}\right|^2\int_{-\infty}^{\infty}s^2\,\mathrm{e}^{-2\nu s^2}\,\mathrm{d}s\\
&=\frac{c^2}{2\pi(4\sqrt{3}\nu\pi)^2}\left|\int_{-\infty}^{\infty}f(x)\,\mathrm{d}x\right|^6\int_{-\infty}^{\infty}s^2\,\mathrm{e}^{-2\nu s^2}\,\mathrm{d}s=\frac{c^2}{384\sqrt{2\pi^5\nu^7}}\left|\int_{-\infty}^{\infty}f(x)\,\mathrm{d}x\right|^6.\quad\square
\end{aligned}\tag{3.56}
$$

4. Results for GKdVB equation

The proof of the theorem stated in Section 2 for the GKdVB equation (1.1) is quite similar to that just presented in Section 3 for the GRLW-B equation (1.2). In consequence, we only provide sketches of the proofs for the GKdVB equation, emphasizing the few points that are different from those appearing in Section 3.

The linearized KdV-Burgers equation is

$$w_t+w_x-\nu w_{xx}+w_{xxx}=0.\tag{4.1}$$

Let u be the solution of Eq. (1.1) and let w be the solution of the corresponding linear equation (4.1), both with the same initial value f. After replacing $u(x+t,t)$ by $U(x,t)$ and $w(x+t,t)$ by $W(x,t)$, the variable $V=U-W$ satisfies the equation

$$V_t-\nu V_{xx}+V_{xxx}+P(U)_x=0,\tag{4.2a}$$

with initial condition

$$V(x,0)=0. \tag{4.2b}$$

As before, the L_2-norm of u, w and $u-w$ are unaffected by the change to traveling coordinates.

Apply the Fourier transform with respect to the spatial variable x to Eq. (4.2a) and solve the resulting ordinary differential equation to reach the integral equation

$$\hat{U}(y,t)-\hat{W}(y,t)=-\int_0^t \mathrm{i}y\,\mathrm{e}^{(-\nu y^2+\mathrm{i}y^3)(t-\tau)}\hat{P}(U)(y,\tau)\,\mathrm{d}\tau, \tag{4.3}$$

which is (3.6) except the kernel $(y/(y^2+1))\,\mathrm{e}^{(-\nu y^2+\mathrm{i}y^3)/(1+y^2)t}$ is replaced by $y\,\mathrm{e}^{(-\nu y^2+\mathrm{i}y^3)t}$. It is straightforward to infer the analog of Lemma 3.3 for the GKdVB equation (1.1) by following the line of argument put forward in Section 3. Indeed, much of the development is easier on account of the exponential rather than algebraic decay of the kernel. However, in obtaining the analog of inequality (3.20), it is only required that the initial data f be such that $xf(x)$ lies in $L_2(\mathbb{R})$. This is because when the equation

$$U_t-\nu U_{xx}+U_{xxx}+c(U^3)_x=0 \tag{4.4}$$

is multiplied by $2x^2U$ and integrated over $\mathbb{R}\times[0,t]$, one obtains

$$\begin{aligned}&\int_{-\infty}^{\infty}x^2U^2(x,t)\,\mathrm{d}x+2\nu\int_0^t\int_{-\infty}^{\infty}x^2U_x^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\&=\int_{-\infty}^{\infty}x^2f^2(x)\,\mathrm{d}x+2\nu\int_0^t|U(\cdot,\tau)|_2^2\,\mathrm{d}\tau-\int_0^t\int_{-\infty}^{\infty}[6xU_x^2(x,\tau)-3cxU^4(x,\tau)]\,\mathrm{d}x\,\mathrm{d}\tau\end{aligned} \tag{4.5}$$

after integrations by parts. With Lemma 3.3 in hand, the case when $p=2$ where $|\int_0^t\int_{-\infty}^{\infty}U^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau|\to+\infty$ as $t\to+\infty$ and the other cases are easily established. The dominating functions that emerge for the kernel $y\,\mathrm{e}^{(-\nu y^2+\mathrm{i}y^3)t}$ for the cases $p>2$ and $p=2$ are

$$s^2\,\mathrm{e}^{-\nu s^2}+\frac{(1-\mathrm{e}^{-\nu s^2/2})^2}{s^2}$$

and

$$(1+s^8)\,\mathrm{e}^{-\nu s^2}+\frac{(1-\mathrm{e}^{-\nu s^2/2})^2}{s^2},$$

respectively. The results for the GKdVB equation are summarized in the following theorem.

Theorem 4.1. *Suppose that the nonlinearity P is smooth and satisfies $|P(u)|\le c|u|^{p+1}$ at least for small values of u, where $p\ge 2$. Suppose that the initial data f lies in $H^1(\mathbb{R})\cap L_1(\mathbb{R})$. Let u be the solution of Eq.* (1.1) *and w the solution of the corresponding linear equation* (4.2a), *both with the same initial value f. Then the difference between u and w has the property*

$$\lim_{t\to+\infty}t^{3/2}|u(\cdot,t)-w(\cdot,t)|_2^2=\frac{1}{4\nu(8\nu\pi)^{1/2}}\left(\int_0^{\infty}\int_{-\infty}^{\infty}P(u)\,\mathrm{d}x\,\mathrm{d}\tau\right)^2 \tag{4.6}$$

for $p > 2$. *If* $P(u) = cu^3$ *and* $xf(x)$ *lies in* $L_2(\mathbb{R})$, *the following decay results obtain*:

1. *If* $\lambda(t) = |\int_0^t \int_{-\infty}^{\infty} u^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau| \to +\infty$ *as* $t \to +\infty$, *then*

$$\lim_{t\to+\infty} \frac{t^{3/2}}{\lambda^2(t)} |u(\cdot,t) - w(\cdot,t)|_2^2 = \frac{c^2}{4\nu(8\nu\pi)^{1/2}}. \tag{4.7}$$

2. *If* $\int_0^{\infty} |\int_{-\infty}^{\infty} u^3(x,\tau)\,\mathrm{d}x|\,\mathrm{d}\tau < \infty$, *then*

$$\lim_{t\to+\infty} t^{3/2} |u(\cdot,t) - w(\cdot,t)|_2^2 = \frac{c^2}{4\nu(8\nu\pi)^{1/2}} \left(\int_0^{+\infty} \int_{-\infty}^{\infty} u^3(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \right)^2. \tag{4.8}$$

3. *If* $\int_0^{\infty} \int_{-\infty}^{\infty} |u^3(x,\tau)|\,\mathrm{d}x\,\mathrm{d}\tau = +\infty$, *then the higher-order term is bounded by*

$$\lim_{t\to+\infty} \frac{t^{3/2}}{(\log(1+t))^2} |u(\cdot,t) - w(\cdot,t)|_2^2 \le \frac{c^2}{384\sqrt{2\pi^5\nu^7}} \left| \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \right|^6. \tag{4.9}$$

5. Conclusion

Detailed aspects of the final decay of solutions of equations featuring nonlinear, dispersive and dissipative effects have been studied. Attention was given to the case of weak nonlinearities where $|P(u)| \le c|u|^{p+1}$ at least for small values of u and $p \ge 2$. In case $p > 2$, it is demonstrated that at the lowest order, u decays like the solution w of the corresponding linear equation in the sense that the L_2-norm of the difference $u - w$ decays at a higher rate than the rate $t^{-1/4}$ corresponding to either quantity $|u(\cdot,t)|_2$ or $|w(\cdot,t)|_2$ separately. Moreover, we compute exactly the coefficient $\lim_{t\to\infty} t^{3/4}|u - w|_2$ corresponding to the higher temporal rate of decay. This limit depends on the quantity

$$\int_0^{\infty} \int_{-\infty}^{\infty} P(u)(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.$$

When $p = 2$, although the long-time behavior of solutions of (1.1) and (1.2) is the same as the solutions of their corresponding linear equations, the decay rate in L_2-norm of the difference between solutions of the linear and nonlinear equations may possibly differ by a logarithmic term from the $t^{-3/4}$ rate that obtains when $p > 2$.

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