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The Cauchy Problem and Stability of Solitary-Wave Solutions for RLW–KP-Type Equations[1]

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The Kadomtsev–Petviashvilli (KP) equation,

$$(u_t + u_x + uu_x + u_{xxx})_x + \varepsilon u_{yy} = 0, \qquad (*)$$

arises in various contexts where nonlinear dispersive waves propagate principally along the x-axis, but with weak dispersive effects being felt in the direction parallel to the y-axis perpendicular to the main direction of propagation. We propose and analyze here a class of evolution equations of the form

$$(u_t + u_x + u^p u_x + Lu_t)_x + \varepsilon u_{yy} = 0, \qquad (**)$$

which provides an alternative to Eq. (*) in the same way the regularized long-wave equation is related to the classical Korteweg–de Vries (KdV) equation. The operator L is a pseudo-differential operator in the x-variable, $p \geq 1$ is an integer and $\varepsilon = \pm 1$.

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After discussing the underlying motivation for the class $(**)$, a local well-posedness theory for the initial-value problem is developed. With assumptions on L and p that include conditions appertaining to models of interesting physical phenomenon, the solutions defined locally in time t are shown to be smoothly extendable to the entire time-axis. In the particularly interesting case where $L = -\partial_x^2$ and $\varepsilon = -1$, $(*)$ possesses travelling-wave solutions $u(x, y, t) = \phi_c(x - ct, y)$ provided $c > 1$ and $0 < p < 4$. It is shown here that these solitary waves are stable for $0 < p < \frac{4}{3}$ and $c > 1$ and for $\frac{4}{3} < p < 4$ if $c > (4p)/(4 + p)$. The paper concludes with commentary on extensions of the present theory to more than two space dimensions.

Key Words: regularized long-wave equation; anisotropic Sobolev spaces; nonlinear dispersive waves; Kadomtsev–Petviashvili equation; transverse propagation.

1. INTRODUCTION

Studied here are initial-value problems of the form

$$\begin{cases} (u_t + u_x + u^p u_x + Lu_t)_x + \varepsilon u_{yy} = 0, & (x, y) \in \mathbb{R}^2, \quad t > 0, \\ u(x, y, 0) = \phi(x, y), \end{cases} \tag{1.1}$$

that include natural generalizations of a regularized version of the Kadomtsev–Petviashvili equation for the propagation of surface water waves. In (1.1), $\varepsilon = \pm 1$, $p \geqslant 1$ is an integer, and L is an operator formally defined by

$$\widehat{Lf}(k, l) = m(k)\hat{f}(k, l). \tag{1.2}$$

Here a circumflex over a function denotes the function's Fourier transform and the symbol m of L will be assumed homogeneous, though this is not necessary for most of the results in view.

The goal of the present paper is to establish qualitative results for the initial-value problem (1.1). Thus a theory is put forward that asserts, under suitable assumptions about the symbol m of the operator L, the power p in the nonlinearity, and the initial data ϕ, problem (1.1) is globally well-posed. That is corresponding to a given ϕ, there is a unique solution u defined for $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}$ and, moreover, u depends continuously on ϕ. A more detailed aspect of the evolution equation is then considered for the special but important case where $L = -\partial_x^2$ and $\varepsilon = -1$. In this situation, the equation admits solitary-wave solutions (see [14–16, 41]). Building on the global well-posedness, a theory of nonlinear stability of certain of these travelling waves is developed. The paper also includes commentary about the extension of the present ideas to more than two space dimensions.

Before embarking on the mathematical development, it is worth putting the evolution equations featured in (1.1) into context. The Kadomtsev–

Petviashvili equations

$$\begin{cases} \eta_t + \eta_x + \frac{3}{2}\eta\eta_x + \frac{1}{6}\eta_{xxx} + \frac{1}{2}w_y = 0, \\ w_x - \eta_y = 0, \end{cases} \tag{1.3}$$

were first put forward as a model to describe wave propagation on the surface of water of constant depth h, say. Here, x and y are longitudinal and lateral coordinates in the horizontal plane and these variables, the wave height η, the transverse velocity w and time t have been rendered nondimensional with respect to h and the gravitational acceleration g. The underlying assumptions leading to (1.3) are that the motion is that of an ideal fluid, so viscosity is ignored, that the flow is irrotational, the wave amplitude small, the wavelength in the x-direction large, and the variations in the y-direction even more gradual. Moreover, it is assumed that the waves move primarily in the direction of increasing values of x. If the nondimensional variables in (1.3) are scaled so that η and its first few partial derivatives are all of order one, Eq. (1.3) takes the revealing form

$$\begin{cases} \eta_t + \eta_x + \delta\eta\eta_x + \delta\eta_{xxx} + \delta w_y = 0, \\ w_x - \eta_y = 0, \end{cases} \tag{1.4}$$

where δ represents the order of the ratio of wave amplitude a to the undisturbed height h. The parameter δ is assumed also to be the order of h^2/λ_x^2, where λ_x is a typical wavelength in the x-direction, and is also assumed to be of order of h/λ_y, where λ_y is a typical wavelength in the y-direction. (The constants $\frac{3}{2}$, $\frac{1}{6}$ and $\frac{1}{2}$ naturally appearing in the original nondimensionalization have been scaled out in (1.4).) In fact, the zeroes on the right-hand side of (1.4) appear from ignoring higher-order terms, and a more complete accounting would reveal terms of formal order δ^2 on the right-hand side of the first equation, and of order δ on the right-hand side of the second equation. At the lowest order, $\eta_t + \eta_x = O(\delta)$ as $\delta \to 0$, a reflection of the essential unidirectionality of the wave motion. If this relation is differentiated twice with respect to x, it appears formally $\eta_{xxx} = -\eta_{xxt} + O(\delta)$, as $\delta \to 0$. If this latter relation is used in the first equation in (1.4), there obtains the system

$$\begin{cases} \eta_t + \eta_x + \delta\eta\eta_x - \delta\eta_{xxt} + \delta w_y = 0, \\ w_x - \eta_y = 0, \end{cases} \tag{1.5}$$

which is formally equivalent to (1.4) in that the difference between the two equations lies at order δ^2, and such terms have all been systematically ignored.

Another way to draw the same conclusion is to consider the linearized KP-equation

$$\begin{cases} u_t + u_x + \frac{1}{6}u_{xxx} + \frac{1}{2}\, w_y = 0, \\ w_x - u_y = 0, \end{cases}$$

which can be written as the single equation

$$(u_t + u_x + \tfrac{1}{6}\, u_{xxx})_x + \tfrac{1}{2}\, u_{yy} = 0 \tag{1.6}$$

by cross-differentiating. If we search for a simple-wave solution of the form $e^{i(kx+ly-\omega t)}$, then the dispersion relation

$$\omega = \omega(k, l) = \frac{k^2 + \frac{1}{2}l^2 - \frac{1}{6}k^4}{k} \tag{1.7}$$

is determined.

Indeed, (1.7) results from truncating the linearized dispersion relation

$$\omega^2(k, l) = \kappa \tanh(\kappa), \tag{1.8}$$

where $\kappa^2 = k^2 + l^2$, for the full Euler equations under the scaling assumptions in force here (that $k^2 = O(\delta)$ and $l = O(\delta)$ as $\delta \to 0$). Since κ is small, we may write

$$\begin{aligned} \omega^2(k, l) &= \kappa^2 - \tfrac{1}{3}\, \kappa^4 + \text{higher-order terms} \\ &= k^2 + l^2 - \tfrac{1}{3}(k^4 + 2k^2l^2 + \cdots) + \text{ higher-order terms} \\ &= k^2\left(1 + \frac{l^2}{k^2} - \frac{1}{3}\, k^2\right) + \text{higher-order terms,} \end{aligned}$$

where the higher-order terms are all $O(\delta^{5/2})$, and so formally negligible compared to k^4 and l^2, both of which have order δ^2. Taking the positive square root of ω corresponding to waves moving to the right, there appears

$$\begin{aligned} \omega(k, l) &= k\left(1 + \frac{1}{2}\frac{l^2}{k^2} - \frac{1}{6}\, k^2\right) + \text{higher-order terms} \\ &= \frac{k^2 + \frac{1}{2}l^2 - \frac{1}{6}k^4}{k} + \text{higher-order terms,} \end{aligned} \tag{1.9}$$

just as in (1.7). With the same scaling assumptions, (1.8) may be approximated to the same order in δ by

$$\tilde{\omega}(k, l) = \frac{k^2 + \frac{1}{2}l^2}{k(1 + \frac{1}{6}k^2)} \tag{1.10}$$

(since $k^2 l^2$ is of higher order), and this is precisely the linearized dispersion relation for the regularized version

$$(\eta_t + \eta_x + \tfrac{3}{2}\eta\eta_x - \tfrac{1}{6}\eta_{xxt})_x + \tfrac{1}{2}\eta_{yy} = 0 \tag{1.11}$$

of (1.3).

Note that in case the wave motion does not vary at all with y, (1.3) and (1.11) reduce to the Korteweg–de Vries equation

$$\eta_t + \eta_x + \tfrac{3}{2}\eta\eta_x + \tfrac{1}{6}\eta_{xxx} = 0 \tag{1.12}$$

and the regularized long-wave equation or BBM-equation

$$\eta_t + \eta_x + \tfrac{3}{2}\eta\eta_x - \tfrac{1}{6}\eta_{xxt} = 0, \tag{1.13}$$

respectively, which govern to a good approximation the unidirectional propagation of small-amplitude long water waves in a channel where variation across the channel can be safely ignored (see [10, 18, 21, 22, 44]).

There are many other physical systems besides the surface of water under gravity that feature waves where a balance is struck between nonlinearity and dispersion. Sometimes, the lowest level description of such systems is (1.12) and (1.13), but not always. Nonlinearity occasionally enters at other than quadratic order while the linearized dispersion relation need not be a quadratic polynomial. This has led to a study of the generalized Korteweg–de Vries equations

$$u_t + u_x + u^p u_x - Lu_x = 0, \tag{1.14}$$

and their regularized counterparts

$$u_t + u_x + u^p u_x + Lu_t = 0, \tag{1.15}$$

(cf. [1, 3, 5, 11, 27, 42, 43]), where $p \geqslant 1$ is an integer and L is as in (1.2),

$$\widehat{Lv}(k) = m(k)\hat{v}(k),$$

but the Fourier transforms are taken with respect to a single spatial variable. The linearized dispersion relations corresponding to (1.14) and (1.15) are

$$\omega(k) = k(1 - m(k))$$

and

$$\omega(k) = \frac{k}{1 + m(k)},$$

respectively. These are typically close to one another for long waves (small values of k) for symbols m that arise in practice (see [3, 11]).

If weaker, but not vanishingly small variations along wave crests are contemplated, it is natural to start with a quadratic dependence on wavenumber in the y-direction, and this leads immediately to the augmented relations

$$\omega(k, l) = k\left(1 - m(k) + \frac{l^2}{k^2}\right) \tag{1.16}$$

or its regularized version

$$\omega(k, l) = \frac{k^2 + l^2}{k(1 + m(k))}, \tag{1.17}$$

analogous to (1.9) and (1.10), respectively. At this level of modelling, (1.14) and (1.15) become

$$(u_t + u_x + u^p u_x - Lu_x)_x \pm u_{yy} = 0 \tag{1.18}$$

and

$$(u_t + u_x + u^p u_x + Lu_t)_x \pm u_{yy} = 0, \tag{1.19}$$

where the sign depends on the particular system being modelled. Class (1.18) has been studied recently in [36, 39] (see also [19,20,24,26,34,35,40,45] for studies of subclasses of (1.18), and in particular for the original Kadomtser-Petviashvili equation). In this script, attention will be given to the collection of models depicted in (1.19).

The first goal with regard to (1.19) is to establish that the pure initial-value problem in $\mathbb{R}^2$ is globally well-posed. To obtain such a result, it is shown in Section 3 that under assumptions that include physically interesting models, the problem is locally well-posed. This is accomplished by means of the contraction-mapping principle in a suitably chosen space. Global existence, uniqueness and continuous dependence on initial data is proved in Section 4 making use of the special structure of the equations exposed in Section 2. Section 2 also features notation and some formal manipulations that motivate the rigorous theory to follow.

It is worth remarking that settling the issue of global existence is not just an idle exercise in making rigorous what is otherwise easily understood. Indeed, the Cauchy problem for a generalized Kadomtsev–Petviashvili equation of the form

$$(u_t + u_x + u^p u_x - D_x^\alpha u_x)_x + \varepsilon u_{yy} = 0, \tag{1.20}$$

where $D_x = (-\partial_x^2)^{1/2}$ is the positive square root of the Laplace operator, has been studied by various authors most especially in the case $\alpha = 2$. Bourgain [17] has proved that the pure initial-value problem for what is usually called the KPII equation ($\alpha = 2$, $\varepsilon = +1, p = 1$) is locally well-posed, and hence, in light of one of the conservation laws for the equation, globally well-posed for data in $L_2(\mathbb{R}^2)$. A compactness method that uses only the divergence form of the nonlinearity and the skew-adjointness of the linear dispersion operator was employed by Iorio and Nunes [23] to establish local well-posedness for data in $H^s(\mathbb{R}^2)$, $s > 2$ for the KPI equation ($\alpha = 2$, $\varepsilon = -1$, $p = 1$). The Iorio–Nunes approach applies equally well to KPII-type equations. It was shown that Eq. (1.20) has global solutions corresponding to large initial data for $\varepsilon = -1$ and $\alpha \geqslant 2$ provided that $p < (4\alpha)/(4 + \alpha)$ (see [39]). It has also been shown that certain solutions of (1.20) cannot remain in the Sobolev space $H^1(\mathbb{R}^2)$ for all time if $\varepsilon = -1$ and $p \geqslant 4$. Indeed, it is demonstrated that the $L_2(\mathbb{R}^2)$-norm of u_y blows up in finite time (see [33, 41]). This blow-up result has little to do with the dispersion in x and depends solely on the *transverse* dispersion. Indeed the same result can be shown to hold for the inviscid Burgers version of the KP equation in two space dimensions, viz.,

$$(u_t + u_x + u^p u_x)_x - u_{yy} = 0.$$

One of the important features of the one-dimensional equations (1.12) and (1.13) is their solitary-wave solutions (see [6, 9, 28, 41]). In many particular instances of these equations, the solitary travelling waves play a distinguished role in the longer-time asymptotics of solutions. In the special, but important case where $L = -\partial_x^2$, it has been shown that (1.1) possesses nontrivial travelling-wave solutions if and only if $\varepsilon = -1$ and $1 \leqslant p < 4$ [15, 41]. The symmetry of these travelling-wave solutions with respect to the transverse coordinate y and the fact that they decay to zero as $x^2 + y^2 \to +\infty$ was proven by de Bouard and Saut [15]. In [41], the set of ground-state solitary wave solutions was shown to be orbitally stable if $p < \frac{4}{3}$, while a proof of instability of the solitary waves corresponding to $p > 4$ was also offered. Several of the technical points in the proof of the instability in [41] were carried out in detail in [14]. When $\varepsilon = -1$ and $L = -\partial_x^2 + \partial_x^4$, Eq. (1.1) is the regularized version of a two-dimensional fifth-order KdV-type equation introduced by Abramyan and Stepanyants [2] and Karpman and Belashov [25]. In this case, for $\varepsilon = -1$ and for any integer p, (1.1) possesses nontrivial solitary-wave solutions (see [15]). These were observed numerically for $p = 1$ and $\varepsilon = -1$ in [25].

In Section 5, the prospect in view is the stability of the de Bouard–Saut travelling-wave solutions. Thus attention is given to the case $L = -\partial_x^2$ and conditions on the speed of propagation c and the power of the nonlinearity p are determined so that the associated solitary-waves are stable when considered as solutions of the full evolution equation. Unfortunately, our theory relies on the homogeneity of the operator L, and so it does not apply as it stands to the operator $L = -\partial_x^2 + \partial_x^4$, for example.

2. NOTATION AND PRELIMINARY DISCUSSION

In the present section, notation is introduced and some preliminary mathematical points are brought to the fore. The calculations in this section motivate the theory to follow in the later sections. Some are presented without formal justification, but this is easily provided as soon as the local well-posedness result is in hand.

The norm in $L_2(\mathbb{R}^2)$ will be written $\|\cdot\|_0$, while $\|\cdot\|_s$ will stand for the norm in the classical Sobolev spaces

$$H^s(\mathbb{R}^2) = \{f \in L_2(\mathbb{R}^2) \colon (1 + \xi^2 + \eta^2)^{s/2}\hat{f}(\xi, \eta) \in L_2(\mathbb{R}^2)\},$$

where the circumflex connotes the Fourier transform as before. For $1 \leq p \leq \infty$, the norm in $L_p(\mathbb{R}^2)$ will be written $|\cdot|_p$. For any $s \geqslant 1$, let

$$X_s = \{f \in H^s(\mathbb{R}^2) \colon \partial_x^{-1} f_y \in H^{s-1}(\mathbb{R}^2)\}$$

equipped with the norm

$$\|f\|_{X_s} = \|f\|_s + \|\partial_x^{-1} f_y\|_{s-1}.$$

Here and below, $\partial_x^{-1} f_y$ is defined via the Fourier transform as

$$\widehat{\partial_x^{-1} f_y} = \frac{\eta}{\xi} \hat{f}(\xi, \eta).$$

Note that if $s \geqslant 1$, then $\partial_x^{-1} f_y \in L_2(\mathbb{R}^2)$, so there is a $g \in L_2(\mathbb{R}^2)$ such that $f_y = g_x$ at least in the sense of distribution. On the other hand, since $f \in H^s(\mathbb{R}^2) \subseteq H^1(\mathbb{R}^2)$, so $f_y \in L_2(\mathbb{R}^2)$, whence $g_x \in L_2(\mathbb{R}^2)$. Thus g lies in the Hilbert space $H_x^1(\mathbb{R}^2)$, where

$$H_x^\lambda(\mathbb{R}^2) = \{f \in L_2(\mathbb{R}^2) \colon (1 + \xi^2)^{\lambda/2}\hat{f}(\xi, \eta) \in L_2(\mathbb{R}^2)\}.$$

In the same vein, let

$$H_y^\lambda(\mathbb{R}^2) = \{f \in L_2(\mathbb{R}^2)\colon\ (1+\eta^2)^{\lambda/2}\hat{f}(\xi,\eta) \in L_2(\mathbb{R}^2)\}$$

supplied with the obvious norm and, for $\beta \geqslant 0$,

$$V_\beta(\mathbb{R}^2) = \left\{ f \in L_2(\mathbb{R}^2)\colon\ |\xi|^\beta \hat{f}(\xi,\eta), \frac{\eta}{\xi}\hat{f}(\xi,\eta) \in L_2(\mathbb{R}^2) \right\}$$

with the norm

$$\|f\|_{V_\beta(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} \left(1 + |\xi|^{2\beta} + \frac{\eta^2}{\xi^2}\right) |\hat{f}(\xi,\eta)|^2 \, d\xi \, d\eta \right)^{1/2}.$$

If $\beta \geqslant 0$, let

$$W_\beta(\mathbb{R}^2) = \{f \in H_x^{\beta+1}(\mathbb{R}^2)\colon\ \partial_x^{-1} f_y \in H_x^\beta(\mathbb{R}^2)\}$$

with norm

$$\|f\|_{W_\beta(\mathbb{R}^2)} = \|f\|_{H_x^{\beta+1}(\mathbb{R}^2)} + \|\partial_x^{-1} f_y\|_{H_x^{\alpha/2}(\mathbb{R}^2)}$$

and let $\tilde{W}_\beta(\mathbb{R}^2)$ be the space

$$\tilde{W}_\beta(\mathbb{R}^2) = \{f \in H_x^\beta(\mathbb{R}^2)\colon\ \partial_x^{-1} f_y \in H_x^\beta(\mathbb{R}^2)\}$$

with the norm

$$\|f\|_{\tilde{W}_\beta(\mathbb{R}^2)} = \|f\|_{H_x^\beta(\mathbb{R}^2)} + \|\partial_x^{-1} f_y\|_{H_x^\beta(\mathbb{R}^2)}.$$

Note that if $\beta > \frac{1}{2}$, then for any $f \in W_\beta(\mathbb{R}^2)$,

$$\int_{-\infty}^{\infty} f_y(x,y)\, dx = 0, \tag{2.1}$$

where for almost every y, the left-hand side of (2.1) is an improper Riemann integral. As above when delineating $\partial_x^{-1} f_y$, define $\partial_x^{-2} f_{yy}$ via the relation

$$\widehat{\partial_x^{-2} f_{yy}} = \frac{\eta^2}{\xi^2}\hat{f}(\xi,\eta).$$

If this quantity lies in $L_2(\mathbb{R}^2)$, then $f_{yy} = h_{xx}$ for some $h \in L_2(\mathbb{R}^2)$, and thus as in (2.1), if $\partial_x^{-2} f_{yy} \in H_x^\beta(\mathbb{R}^2)$ for some $\beta > \frac{1}{2}$, then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{yy}(x_1, y)\, dx_1\, dx = 0, \tag{2.2}$$

where, again, the left-hand side may be interpreted as an iterated improper Riemann integral. The local existence theory for the initial-value problem (1.1) subsists in part on the fact that the initial data ϕ lies in $H_x^2(\mathbb{R}^2)$. This is implied if $\phi \in W_{\alpha/2}(\mathbb{R}^2)$ for $\alpha \geqslant 2$ and equally if $\phi \in \tilde{W}_{\alpha/2}(\mathbb{R}^2)$ for $\alpha \geqslant 4$. In the interest of sharpening the theory a little, we have chosen to work with the two different versions of these W-spaces.

We now enter into a set of formal calculations wherein it is presumed that the wave-profile $u(x, y, t)$ decays to 0 suitably rapidly as $x \to \pm\infty$, and that u is appropriately bounded in the y-variable. Integrating the evolution equation in (1.1) with respect to x and applying zero boundary conditions at $x = \pm\infty$ in concert with our decay-assumptions, it is determined that

$$\int_{-\infty}^{\infty} u_{yy}(x, y, t)\, dx = 0,$$

and thus that

$$\int_{-\infty}^{\infty} u(x, y, t)\, dx = c_1(t) + c_2(t)y, \tag{2.3}$$

where c_1 and c_2 are independent of y. The boundedness condition in the y-variable forces $c_2(t) \equiv 0$.

If, as will be assumed later, the initial data ϕ has the property that $\phi_y = \psi_x$ for some $L_2(\mathbb{R}^2)$-function ψ, then the initial-value problem (1.1) may be written as an equivalent system

$$\begin{cases} u_t + u_x + u^p u_x + Lu_t + \varepsilon v_y = 0, \\ v_x = u_y, \end{cases} \tag{2.4}$$

(see de Bouard and Saut [15]) with $u(x, y, 0) = \phi(x, y)$ and $v(x, y, 0) = \psi(x, y)$. In the original application to water waves where $p = 1$, $L = -\partial_x^2$ and $\varepsilon = +1$, v represents the horizontal velocity along the crest, so in the y-direction. Integrating the first equation in (2.4) with respect to x, and using the assumption that $L = N\partial_x$ where the symbol $n(\xi) = m(\xi)/i\xi$ of M is assumed to be bounded near $\xi = 0$, it transpires that

$$\partial_t \int_{-\infty}^{\infty} u(x, y, t)\, dx = -\varepsilon \int_{-\infty}^{\infty} v_y(x, y, t)\, dx = -\varepsilon\partial_y \int_{-\infty}^{\infty} v(x, y, t)\, dx. \tag{2.5}$$

The left-hand side of (2.5) is $c_1'(t)$, which is independent of y. Averaging (2.5) over the interval $\{y: -k \leqslant y \leqslant k\}$, where $k > 0$, leads to

$$c_1'(t) = \frac{\varepsilon}{2k}\left\{\int_{-\infty}^{\infty} v(x, k, t)\, dx - \int_{-\infty}^{\infty} v(x, -k, t)\, dx\right\}. \tag{2.6}$$

Imposing boundedness of the integrated transverse velocity,

$$\int_{-\infty}^{\infty} v(x, y, t)\, dx,$$

as a function of y and taking the limit as $k \to +\infty$ in (2.6) gives $c_1'(t) \equiv 0$, whence $c_1(t) \equiv c_1(0) = c_1$, say. Note this conclusion was drawn without asking that the solution u tend to 0 as $y \to \pm\infty$. If this latter condition is imposed then it follows formally that $c_1 = 0$, or that u satisfies the compatibility condition

$$\int_{-\infty}^{\infty} u(x, y, t)\, dx = 0, \tag{2.7}$$

for all $y \in \mathbb{R}$ and t for which the solution exists.

Now if (2.7) holds, let $w = u_t$. Then w formally satisfies the equation

$$w_t + w_x + (u^p w)_x + Lw_t + \varepsilon \partial_x^{-1} w_{yy} = 0,$$

where $L = N\partial_x$ as above. Integrating the above equation for w with respect to x over $\mathbb{R}$, it is adduced that

$$\frac{d}{dt} \int_{\mathbb{R}} w\, dx + \varepsilon \int_{\mathbb{R}} \partial_x^{-1} w_{yy} = 0.$$

Because $c_1'(t) \equiv 0$, as determined above, it follows that

$$\frac{d}{dt} \int_{\mathbb{R}} u(x, y, t)\, dx = \int_{\mathbb{R}} w\, dx = 0,$$

whence,

$$\int_{\mathbb{R}} \partial_x^{-1} w_{yy}\, dx = 0$$

and thus

$$\int_{\mathbb{R}} \partial_x^{-1} w_{yy}\, dx = \int_{\mathbb{R}} \partial_x^{-1} u_{yyt}\, dx = 0.$$

It is thereby inferred that

$$\int_{\mathbb{R}} \partial_x^{-1} u_{yy}(x, y, t)\, dx = \int_{\mathbb{R}} \partial_x^{-1} \phi_{yy}(x, y)\, dx$$

and, in particular, if

$$\int_{\mathbb{R}} \int_{-\infty}^{x} \phi_{yy}(x_1, y)\, dx_1\, dx = 0, \tag{2.8}$$

then

$$\int_{\mathbb{R}} \int_{-\infty}^{x} u_{yy}(x_1, y, t)\, dx_1\, dx = 0$$

for all $t > 0$.

If the initial data ϕ is such that $\phi_y = \psi_x$ for some $\psi \in L_2(\mathbb{R}^2)$, then it seems likely because of (2.7), and is in fact true as will appear later, that $\partial_x^{-1} u_y \in L_2(\mathbb{R}^2)$ for all $t \geqslant 0$. If the equation in (1.1) is multiplied by u and the result integrated over $\mathbb{R}^2$, then after appropriate integrations by parts, it is found that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} [u^2 + uLu]\, dx\, dy = -\int_{\mathbb{R}^2} uu_x\, dx\, dy - \int_{\mathbb{R}^2} u^{p+1} u_x\, dx\, dy$$
$$- \varepsilon \int_{\mathbb{R}^2} u\partial_x^{-1} u_{yy}\, dx\, dy = 0,$$

since

$$\int_{\mathbb{R}^2} u\partial_x^{-1} u_{yy}\, dx\, dy = -\int_{\mathbb{R}^2} u_y \partial_x^{-1} u_y\, dx\, dy = -\frac{1}{2}\int_{\mathbb{R}^2} (\partial_x^{-1} u_y)_x^2\, dx\, dy = 0.$$

Hence, the functional V defined by

$$V(u) = \frac{1}{2}\int_{\mathbb{R}^2} [u^2 + uLu]\, dx\, dy \tag{2.9}$$

is independent of t, being therefore determined by its value on the initial data ϕ. Under added restrictions on the initial data, the same is true of the functional

$$E(u) = -\int_{\mathbb{R}^2} \left[\frac{\varepsilon}{2}(\partial_x^{-1} u_y)^2 + \frac{u^2}{2} + \frac{u^{p+2}}{(p+1)(p+2)}\right] dx\, dy. \tag{2.10}$$

To see this, assume that the initial data ϕ is such that $\phi_{yy} = h_{xx}$ for some $h \in L_2(\mathbb{R}^2)$. Because of (2.8), we expect that $\partial_x^{-2} u_{yy}$ will lie in $L_2(\mathbb{R}^2)$ for $t > 0$. Introduce the function

$$w = -(1 + L)^{-1}\left[\varepsilon\partial_x^{-2} u_{yy} + u + \frac{u^{p+1}}{p+1}\right].$$

A simple calculation reveals that

$$w_x = -(1+L)^{-1}[\varepsilon\partial_x^{-1}u_{yy} + u_x + u^p u_x] = u_t,$$

and hence

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\left[\frac{\varepsilon}{2}(\partial_x^{-1}u_y)^2 + \frac{u^2}{2} + \frac{u^{p+2}}{(p+1)(p+2)}\right]dx\,dy \\
&= \varepsilon\int_{\mathbb{R}^2}\partial_x^{-1}u_y\partial_x^{-1}u_{yt}\,dx\,dy + \int_{\mathbb{R}^2}uu_t\,dx\,dy + \frac{1}{p+1}\int_{\mathbb{R}^2}u^{p+1}u_t\,dx\,dy \\
&= \int_{\mathbb{R}^2}\left[\varepsilon\partial_x^{-2}u_{yy} + u + \frac{u^{p+1}}{p+1}\right]u_t\,dx\,dy \\
&= -\int_{\mathbb{R}^2}(1+L)ww_x\,dx\,dy = 0.
\end{aligned}$$

In the last step, use has been made of the fact that the operator $(1+L)\partial_x$ is skew-symmetric. Because

$$\frac{d}{dt}V(u) = 0,$$

we have

$$\frac{d}{dt}\int_{\mathbb{R}^2}u^2\,dx\,dy = -\frac{d}{dt}\int_{\mathbb{R}^2}uLu\,dx\,dy,$$

and hence the functional $E(u)$ can also be written as

$$E(u) = -\int_{\mathbb{R}^2}\left[\frac{\varepsilon}{2}(\partial_x^{-1}u_y)^2 - \frac{1}{2}uLu + \frac{u^{p+2}}{(p+1)(p+2)}\right]dx\,dy.$$

Using the conserved quantities $V(u)$ and $E(u)$, it is possible to draw some preliminary conclusions. Suppose that L is homogeneous, say, $L = D_x^\alpha$ for some $\alpha > 0$, where D_x as defined earlier has the Fourier symbol $|\xi|$. Then the term

$$\int_{\mathbb{R}^2}uLu\,dx\,dy = \int_{\mathbb{R}^2}(D_x^{\alpha/2}u)^2\,dx\,dy,$$

and the invariance of V expressed in (2.9) implies that if the initial data $\phi \in H_x^{\alpha/2}(\mathbb{R}^2)$, then the corresponding solution u of (1.1) with $L = D_x^\alpha$ lies in $H_x^{\alpha/2}(\mathbb{R}^2)$ for all $t \geqslant 0$ for which it exists. To draw an inference based on the invariance of E, the following lemma is helpful. This lemma is closely related to the imbedding theorems for anisotropic Sobolev spaces studies in [7].

LEMMA 2.1. *Let $\alpha \geqslant 1$ be given and let $p \leqslant 4\alpha/(4-\alpha)$ for $1 \leqslant \alpha \leqslant 2$ or $p \leqslant 2\alpha$ for $\alpha \geqslant 2$. Then there is a constant c depending only on α and p such that for any $f \in V_{\alpha/2}(\mathbb{R}^2)$,*

$$|f|^{p+2}_{L_{p+2}(\mathbb{R}^2)} \leqslant c\|f\|_0^{2/q'} \|f\|^{(p/2)+(2/q)}_{H_x^{\alpha/2}(\mathbb{R}^2)} \|\partial_x^{-1} f_y\|_0^{p/2}, \tag{2.11}$$

where

$$q = \frac{2(\alpha+2)}{2p+(2-\alpha)(p+2)}, \qquad q' = \frac{2(\alpha+2)}{4\alpha-(4-\alpha)_p} \qquad \text{if } 1 \leqslant \alpha \leqslant 2,$$

and

$$q = \frac{4\alpha}{(4-\alpha)p}, \qquad q' = \frac{4\alpha}{4\alpha-(4-\alpha)p} \qquad \text{if } \alpha \geqslant 2.$$

As a consequence, it follows that there is a constant c such that for all $f \in V_{\alpha/2}(\mathbb{R}^2)$,

$$|f|_{L^{p+2}(\mathbb{R}^2)} \leqslant c\|f\|_{V_{\alpha/2}(\mathbb{R}^2)},$$

which is to say $V_{\alpha/2}(\mathbb{R}^2)$ is embedded in $L_{p+2}(\mathbb{R}^2)$.

Proof. The lemma is established for $C_0^\infty(\mathbb{R}^n)$-functions and then limits are taken to complete the proof. First, consider the case $\alpha \geqslant 2$. Because of Sobolev theory and interpolation, we have that if $g \in H^{\alpha/2}(\mathbb{R})$, then

$$\|g\|_{H^{p/[2(p+2)]}(\mathbb{R})} \leqslant c\|g\|^{p/[\alpha(p+2)]}_{H^{\alpha/2}(\mathbb{R})} \|g\|^{[p(\alpha-1)+2\alpha]/[\alpha(p+2)]}_{L_2(\mathbb{R})},$$

(see [29]). It follows that

$$\begin{aligned}
\int_{\mathbb{R}^2} |f|^{p+2}\,dx\,dy &\leqslant c\int_{\mathbb{R}} \|f(\cdot,y)\|^{p+2}_{H^{p/[2(p+2)]}(\mathbb{R})}\,dy \\
&\leqslant c\int_{\mathbb{R}} \|f(\cdot,y)\|^{p/\alpha}_{H^{\alpha/2}(\mathbb{R})} \|f(\cdot,y)\|^{[p(\alpha-1)+2\alpha]/\alpha}_{L_2(\mathbb{R})}\,dy \\
&\leqslant c\left(\int_{\mathbb{R}} \|f(\cdot,y)\|^2_{H^{\alpha/2}(\mathbb{R})}\,dy\right)^{p/(2\alpha)} \\
&\quad\times\left(\int_{\mathbb{R}} \|f(\cdot,y)\|^{(2[p(\alpha-1)+2\alpha])/(2\alpha-p)}_{L_2(\mathbb{R})}\,dy\right)^{(2\alpha-p)/(2\alpha)} \\
&\leqslant c\|f\|^{p/\alpha}_{H_x^{\alpha/2}(\mathbb{R}^2)} \|f\|_0^{(2\alpha-p)/\alpha} \sup_{y\in\mathbb{R}} \|f(\cdot,y)\|^{\gamma}_{L_2(\mathbb{R})},
\end{aligned} \tag{2.12}$$

where $\gamma = \frac{2[p(\alpha-1)+2\alpha]}{2\alpha-p}$. But, for each $y \in \mathbb{R}$,

$$\begin{aligned}
\|f(\cdot,y)\|^2_{L_2(\mathbb{R})} &= \int_{\mathbb{R}} f^2(x,y)\,dx \\
&= 2\int_{\mathbb{R}}\int_{-\infty}^{y} f(x,y_1)f_y(x,y_1)\,dy_1\,dx \\
&= -2\int_{-\infty}^{y}\int_{\mathbb{R}} f_x(x,y_1)\partial_x^{-1}f_y(x,y_1)\,dx\,dy_1 \\
&\leqslant c\int_{\mathbb{R}} \|f_x(\cdot,y)\|_{L_2(\mathbb{R})}\|\partial_x^{-1}f_y(\cdot,y)\|_{L_2(\mathbb{R})}\,dy \\
&\leqslant c\int_{\mathbb{R}} \|f(\cdot,y)\|_{H^1(\mathbb{R})}\|\partial_x^{-1}f_y(\cdot,y)\|_{L_2(\mathbb{R})}\,dy \\
&\leqslant c\left(\int_{\mathbb{R}} f(\cdot,y)\|^2_{H^1(\mathbb{R})}\,dy\right)^{1/2}\left(\int_{\mathbb{R}} \|\partial_x^{-1}f_y(\cdot,y)\|^2_{L_2(\mathbb{R})}\,dy\right)^{1/2} \\
&\leqslant c\left(\int_{\mathbb{R}} (\|f(\cdot,y)\|^{\frac{2}{\alpha}}_{H^{\frac{\alpha}{2}}(\mathbb{R})}\|f(\cdot,y)\|^{1-\frac{2}{\alpha}}_{L_2(\mathbb{R})})^2\,dy\right)^2 \|\partial_x^{-1}f_y\|_0.
\end{aligned}$$

It follows that for any $y \in \mathbb{R}$,

$$\begin{aligned}
&\|f(\cdot,y)\|^2_{L_2(\mathbb{R})} \\
&\leqslant c\left(\int_{\mathbb{R}} \|f(\cdot,y)\|^{\frac{4}{\alpha}}_{H^{\alpha/2}(\mathbb{R})}\|f(\cdot,y)\|^{2-\frac{4}{\alpha}}_{L_2(\mathbb{R})}\,dy\right)^{1/2}\|\partial_x^{-1}f_y\|_0 \\
&\leqslant \left(\left(\int_{\mathbb{R}} \|f(\cdot,y)\|^2_{H^{\alpha/2}(\mathbb{R})}\,dy\right)^{2/\alpha}\left(\int_{\mathbb{R}} \|f(\cdot,y)\|^2_{L_2(\mathbb{R})}\,dy\right)^{1-\frac{2}{\alpha}}\right)^{1/2}\|\partial_x^{-1}f_y\|_0 \\
&\leqslant c\left(\int_{\mathbb{R}} \|f(\cdot,y)\|^2_{H^{\alpha/2}(\mathbb{R})}\,dy\right)^{1/\alpha}\left(\int_{\mathbb{R}} \|f(\cdot,y)\|^2_{L_2(\mathbb{R})}\,dy\right)^{\frac{1}{2}(1-\frac{2}{\alpha})}\|\partial_x^{-1}f_y\|_0 \\
&\leqslant c\|f\|_0^{1-\frac{2}{\alpha}}\|f\|^{\frac{2}{\alpha}}_{H^{\alpha/2}(\mathbb{R}^2)}\|\partial_x^{-1}f_y\|_0,
\end{aligned}$$

and hence

$$\sup_{y\in\mathbb{R}} \|f(\cdot,y)\|^2_{L_2(\mathbb{R})} \leqslant c\|f\|_0^{1-\frac{2}{\alpha}}\|f\|^{\frac{2}{\alpha}}_{H^{\alpha/2}(\mathbb{R}^2)}\|\partial_x^{-1}f_y\|_0. \tag{2.13}$$

Combining the last inequality with (2.12) yields the result for $\alpha \geqslant 2$. Observe in particular that when $\alpha = 4$, $\frac{1}{q} = 0$ and $q' = 1$.

Now consider the case where $1 \leqslant \alpha \leqslant 2$. Again, interpolations gives

$$\int_{\mathbb{R}^2} |f|^{p+2}\,dx\,dy \leqslant c \int_{\mathbb{R}} \|f(\cdot,y)\|^{p+2}_{H^{\frac{p}{2(p+2)}}(\mathbb{R})}\,dy$$
$$\leqslant c \int_{\mathbb{R}} \|f(\cdot,y)\|^{\tilde{\theta}}_{H^r(\mathbb{R})} \|f(\cdot,y)\|^{\theta}_{H^{\alpha/2}(\mathbb{R})}\,dy, \tag{2.14}$$

where

$$\theta = \frac{p - 2r(p+2)}{\alpha - 2r} \qquad \text{and} \qquad \tilde{\theta} = \frac{\alpha(p+2) - p}{\alpha - 2r},$$

with r to be chosen presently. The next step is to gain control of the norm $\|f(\cdot,y)\|_{H^r(\mathbb{R})}$. In this regard, write

$$\|f\|^2_{H^r(\mathbb{R})} = \int_{\mathbb{R}} (\Lambda^r f)^2\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{y} \Lambda^r f(x,z) \Lambda^r \partial_z f(x,z)\,dz\,dx,$$

in which Λ^r is a standard Bessel potential given by

$$\widehat{\Lambda^r g} = (1+\xi^2)^{r/2} \hat{g}(\xi).$$

Suppose that $\partial_x^{-1} f_y \in L_2(\mathbb{R}^2)$, say $\partial_x^{-1} f_y = g$ and use Fubini's theorem to derive

$$\begin{aligned}
\frac{1}{2}\|f(\cdot,y)\|^2_{H^r(\mathbb{R})} &= \int_{-\infty}^{y}\int_{-\infty}^{\infty} \Lambda^r f(x,z) \partial_z \Lambda^r f(x,z)\,dx\,dz \\
&= \int_{-\infty}^{y}\int_{-\infty}^{\infty} \Lambda^r f(x,z) \partial_x \partial_x^{-1} \Lambda^r f(x,z)\,dx\,dz \\
&= -\int_{-\infty}^{y}\int_{-\infty}^{\infty} \partial_x \Lambda^r f(x,z) \Lambda^r g\,dx\,dz \\
&= -\int_{-\infty}^{y}\int_{-\infty}^{\infty} \partial_x \Lambda^{2r} f(x,z) g\,dz\,dx \\
&\leqslant c \int_{-\infty}^{\infty} \|f(\cdot,y)\|_{H^{1+2r}(\mathbb{R})} \|\partial_x^{-1} f_y\|_{L_2(\mathbb{R})}\,dy \\
&\leqslant c \|f\|_{H_x^{1+2r}(\mathbb{R}^2)} \|\partial_x^{-1} f_y\|_0.
\end{aligned}$$

Choose r so that $1 + 2r = \frac{\alpha}{2}$. (Notice that $r \leqslant 0$ since $\alpha \leqslant 2$.) With this choice of r, we have

$$\sup_{y\in\mathbb{R}} \|f(\cdot,y)\|^2_{H^r(\mathbb{R})} \leqslant c \|f\|_{H_x^{1+2r}(\mathbb{R}^2)} \|\partial_x^{-1} f_y\|_0. \tag{2.15}$$

Putting this inequality back into (2.14) gives a helpful relation. With the present choice of r, the quantities θ and $\tilde{\theta}$ in (2.14) take the values

$$\theta = \frac{2p + (2-\alpha)(p+2)}{\alpha+2} \qquad \text{and} \qquad \tilde{\theta} = \frac{4\alpha + 2p(\alpha-1)}{\alpha+2}.$$

To make further progress with this line of argument, we need $\theta \leqslant 2$. This amounts to the restriction

$$p \leqslant \frac{4\alpha}{4-\alpha}.$$

Supposing p respects this inequality, let $q \geqslant 1$ be such that $\theta q = 2$, and let q' be the conjugate index. Applying Young's inequality to the right-hand side of (2.13) yields

$$\int_{\mathbb{R}^2} |f|^{p+2}\, dx\, dy \leqslant \left(\int_{-\infty}^{\infty} \|f(\cdot, y)\|_{H^r(\mathbb{R})}^{\tilde{\theta}q'}\, dy \right)^{1/q'} \left(\int_{-\infty}^{\infty} \|f(\cdot, y)\|_{H^{\alpha/2}(\mathbb{R})}^{2}\, dy \right)^{1/q},$$

where

$$q = \frac{2}{\theta} = \frac{2(\alpha+2)}{2p + (2-\alpha)(p+2)}$$

and

$$q' = \frac{q}{q-1} = \frac{2(\alpha+2)}{2(\alpha+2) - [2p + (2-\alpha)(p+2)]} = \frac{2(\alpha+2)}{4\alpha - (4-\alpha)p}.$$

It thus transpires that

$$\tilde{\theta} q' = \frac{2[4\alpha + 2p(\alpha-2)]}{4\alpha - (4-\alpha)p}.$$

The right-hand side of the last integral inequality is

$$\left(\int_{-\infty}^{\infty} \|f(\cdot, y)\|_{H^r(\mathbb{R})}^{\tilde{\theta}q'}\, dy \right)^{1/q'} \|f\|_{H_x^{\alpha/2}(\mathbb{R}^2)}^{2/q}.$$

Considering what we have in hand, it is natural to bound the first term above by

$$\sup_{y\in\mathbb{R}} \|f(\cdot,y)\|_{H^r(\mathbb{R})}^{\frac{\tilde{\theta}q'-2}{q'}} \left(\int_{-\infty}^{\infty} \|f(\cdot,y)\|_{H^r(\mathbb{R})}^2 \, dy \right)^{1/q'}.$$

Using (2.15) and the fact that $r\leqslant 0$, the latter quantity is bounded above by

$$\|f\|_{H_x^{\alpha/2}(\mathbb{R}^2)}^{\frac{\tilde{\theta}q'-2}{2q'}} \|\partial_x^{-1} f_y\|_0^{\frac{\tilde{\theta}q'-2}{2q'}} \|f\|_0^{2/q'}.$$

Summarizing, we have derived the inequality

$$\int_{\mathbb{R}^2} |f|^{p+2} \, dx\, dy \leqslant \|f\|_{H_x^{\alpha/2}(\mathbb{R}^2)}^{\frac{\tilde{\theta}q'-2}{2q'}+\frac{2}{q}} \|\partial_x^{-1} f_y\|_0^{\frac{\tilde{\theta}q'-2}{2q'}} \|f\|_0^{\frac{2}{q'}}.$$

On the other hand,

$$\frac{\tilde{\theta}q'-2}{2q'} = \frac{\tilde{\theta}}{2} - \frac{1}{q'} = \frac{p}{2},$$

and thus the last inequality is the advertised result for the case $1\leqslant\alpha\leqslant 2$. ∎

Remark. When inequality (2.10) is used, we will need the exponent of $\|\partial_x^{-1} f_y\|_0$ to be less than or equal to 2, and this entails the restriction $p<4$.

Now suppose initial data ϕ in the class $V_{\alpha/2}(\mathbb{R}^2)$ is such that $\phi_{yy} = \psi_{xx}$ for some function $\psi \in L_2(\mathbb{R}^2)$. Because of the invariance of V, $u(\cdot, t)$ is bounded in $H_x^{\alpha/2}(\mathbb{R}^2)$ for all $t>0$ and any $p\geqslant 0$. The time-independence of $E(u)$ implies

$$\int_{\mathbb{R}^2} (\partial_x^{-1} u_y)^2 \, dx\, dy = \int_{\mathbb{R}^2} \left[(\partial_x^{-1}\phi_y)^2 + \frac{1}{\varepsilon}\phi^2 + \frac{2\phi^{p+2}}{\varepsilon(p+1)(p+2)} \right] dx\, dy - \frac{1}{\varepsilon}\|u\|_0^2 - \frac{2}{\varepsilon(p+1)(p+2)} \int_{\mathbb{R}^2} u^{p+2} \, dx\, dy.$$

Applying Lemma 2.1, there appears the relation

$$\begin{aligned}\|\partial_x^{-1}u_y(\cdot,t)\|_0^2 &\leqslant \|\partial_x^{-1}\phi_y\|_0^2 + \|\phi\|_0^2 + c\|\phi\|_0^{\frac{2}{q'}}\|\phi\|_{H_x^{\alpha/2}(\mathbb{R}^2)}^{\frac{p}{2}+\frac{2}{q}}\|\partial_x^{-1}\phi_y\|_0^{\frac{p}{2}} \\ &\quad + \|u(\cdot,t)\|_0^2 + c\|u(\cdot,t)\|_0^{\frac{2}{q'}}\|u(\cdot,t)\|_{H_x^{\alpha/2}(\mathbb{R}^2)}^{\frac{p}{2}+\frac{2}{q}}\|\partial_x^{-1}u_y(\cdot,t)\|_0^{\frac{p}{2}} \\ &\leqslant c(\|\phi\|_{V_{\alpha/2}(\mathbb{R}^2)}) + c\|\phi\|_{H_x^{\alpha/2}(\mathbb{R}^2)}^{\frac{p}{2}+2}\|\partial_x^{-1}u_y(\cdot,t)\|_0^{\frac{p}{2}},\end{aligned} \tag{2.16}$$

since $|\varepsilon| = 1$. It follows that if $p<4$, then

$$\sup_{t>0}\|\partial_x^{-1}u_y(\cdot,t)\|_0 \leqslant c(\|\phi\|_{V_{\alpha/2}(\mathbb{R}^2)}). \tag{2.17}$$

Note however, that for $1\leqslant\alpha<2$, inequality (2.17) will only be valid for $p\leqslant\frac{4\alpha}{4-\alpha}$. In particular, for the regularized Benjamin–Ono–KP model where $\alpha=1$, we require $p\leqslant\frac{4}{3}$. The following proposition emerges from this discussion.

PROPOSITION 2.2. *If $p\leqslant\frac{4\alpha}{4-\alpha}$ for $1\leqslant\alpha<2$ or $p<4$ for $\alpha\geqslant2$, then a solution u that starts in $V_{\alpha/2}(\mathbb{R}^2)$ will remain in this space throughout its period of existence, regardless of the sign of ε. In case $\alpha\geqslant2$, the same conclusion holds if $p=4$ and the initial data ϕ is not too large in $H_x^{\alpha/2}(\mathbb{R}^2)$.*

3. LOCAL EXISTENCE AND UNIQUENESS OF SOLUTIONS

Hereafter, it will be assumed that the dispersion operator L has the homogeneous form D_x^α for some $\alpha>0$. Thus (1.1) takes the form

$$\begin{cases}(u_t+u_x+u^pu_x+D_x^\alpha u_t)_x+\varepsilon u_{yy}=0, & (x,y)\in\mathbb{R}^2, \quad t\geqslant0,\\ u(x,y,0)=\phi(x,y).\end{cases} \tag{3.1}$$

The first step in the analysis of the initial-value problem (3.1) is to establish existence and uniqueness of solutions over a small time interval. This is accomplished by writing the solution of the initial-value problem (3.1) formally as

$$u(x,y,t) = K_t\phi - \int_0^t K_{t-\tau}Q\left(\frac{u^{p+1}(\cdot,\tau)}{p+1}\right)d\tau, \tag{3.2}$$

where the operators K_t and Q are defined via their Fourier transforms, viz.,

$$\widehat{K_t f}(\xi,\eta) = e^{\frac{-i(\xi^2+\varepsilon\eta^2)t}{\xi(1+|\xi|^\alpha)}}\hat{f}(\xi,\eta),$$

$$\widehat{Qf}(\xi,\eta) = \frac{i\xi}{1+|\xi|^\alpha}\hat{f}(\xi,\eta).$$

The first result records the way in which convolution with K_t and Q, and thus the composition K_tQ, maps various function spaces.

PROPOSITION 3.1. *The operator K_t is a unitary operator on all the spaces $L_2(\mathbb{R}^2)$, $H^s(\mathbb{R}^2)$, X_s, $H_x^\lambda(\mathbb{R}^2)$, $V_\alpha(\mathbb{R}^2)$, $W_\alpha(\mathbb{R}^2)$ and $\tilde{W}_\alpha(\mathbb{R}^2)$. If $\alpha \geqslant 1$, then Q is a bounded linear operator from $L_2(\mathbb{R}^2)$ into $H_x^{\alpha-1}(\mathbb{R}^2)$, from $H_y^1(\mathbb{R}^2)$ into $V_{\alpha-1}(\mathbb{R}^2)$, from $H^1(\mathbb{R}^2)$ into $W_{\alpha-1}(\mathbb{R}^2)$, from $H_y^1(\mathbb{R}^2)$ into $\tilde{W}_{\alpha-1}(\mathbb{R}^2)$, and from $H^s(\mathbb{R}^2)$ into X_s, for any $s \geqslant 1$.*

Proof. The operator K_t is unitary operator on the indicated spaces because its symbol has modulus 1. The facts about Q are also straightforward. For example, if $g \in H_y^1(\mathbb{R}^2)$, then

$$\begin{aligned}
\|Qg\|^2_{V_{\alpha-1}(\mathbb{R}^2)} &= \int_{\mathbb{R}^2}\left(1+|\xi|^{\alpha-1}+\frac{\eta^2}{\xi^2}\right)\left|\frac{i\xi\hat{g}}{1+|\xi|\alpha}\right|^2 d\xi\, d\eta \\
&= \int_{\mathbb{R}^2}\left(1+|\xi|^{\alpha-1}+\frac{\eta^2}{\xi^2}\right)\frac{\xi^2}{(1+|\xi|^\alpha)^2}|\hat{g}(\xi,\eta)|^2\, d\xi\, d\eta \\
&\leqslant \left[1+\sup_\xi \frac{\xi^{\alpha+1}}{(1+|\xi|^\alpha)^2}\right]\int_{\mathbb{R}^2}(1+\eta^2)|\hat{g}(\xi,\eta)|^2\, d\xi\, d\eta \\
&\leqslant 2\|g\|^2_{H_y^1(\mathbb{R}^2)}
\end{aligned}$$

since $\alpha \geqslant 1$. ∎

COROLLARY 3.2. (1) *The composite operator K_tQ is a bounded linear operator from $L_2(\mathbb{R}^2)$ into $H_x^{\alpha-1}(\mathbb{R}^2)$* for all $\alpha \geqslant 1$.

(2) *K_tQ is bounded from $H_y^1(\mathbb{R}^2)$ into $V_{\alpha-1}(\mathbb{R}^2)$ for all $\alpha \geqslant 1$.*

(3) *K_tQ maps $H^1(\mathbb{R}^2)$ and $H_y^1(\mathbb{R}^2)$ into $W_{\alpha-1}(\mathbb{R}^2)$ and $\tilde{W}_{\alpha-1}(\mathbb{R}^2)$, respectively.*

(4) *K_tQ is bounded from $H^s(\mathbb{R}^2)$ into X_s for all $s \geqslant 1$.*

The following embedding result will be helpful in the proof of the local well-posedness of the initial-value problem.

LEMMA 3.3. *For $\beta > \frac{1}{2}$, $W_\beta(\mathbb{R}^2)$ is continuously embedded in $C_b(\mathbb{R}^2)$, the bounded continuous functions defined on $\mathbb{R}^2$.*

Proof. The norm of a function f in $W_\beta(\mathbb{R}^2)$ is equivalent to

$$\left(\int_{\mathbb{R}^2}\left[1+|\xi|^{2\beta+2}+\frac{\eta^2}{\xi^2}(1+|\xi|^{2\beta})\right]|\hat{f}(\xi,\eta)|^2\,d\xi\,d\eta\right)^{1/2}.$$

Note that if $g \in L_2(\mathbb{R}^n; w(x)\,dx)$ where $w > 0$ and $1/w \in L_1(\mathbb{R}^n)$, then $g \in L_1(\mathbb{R}^n)$. Thus if $f \in W_\beta(\mathbb{R}^2)$ and

$$\frac{1}{1+|\xi|^{2\beta+2}+\frac{\eta^2}{\xi^2}(1+|\xi|^{2\beta})} \quad \text{lies in } L_1(\mathbb{R}^2),$$

then $\hat{f} \in L_1(\mathbb{R}^2)$ so $f \in C_b(\mathbb{R}^2)$ and $f \to 0$ at ∞ by the Riemann–Lebesgue Lemma. To prove the lemma at hand, it therefore suffices to show that if $\beta > \frac{1}{2}$, then

$$\frac{1}{1+|\xi|^{2\beta+2}+\frac{\eta^2}{|\xi|^2}(1+|\xi|^{2\beta})} \in L_1(\mathbb{R}^2).$$

Let $z = \frac{(1+|\xi|^{2\beta})^{1/2}}{|\xi|}\eta$. The following calculation is decisive in proving the above claim:

$$\begin{aligned}
&\int_{\mathbb{R}^2}\frac{d\xi\,d\eta}{1+|\xi|^{2\beta+2}+\frac{\eta^2}{|\xi|^2}(1+|\xi|^{2\beta})}\\
&=\int_{\mathbb{R}^2}\frac{|\xi|\,d\xi\,dz}{(1+|\xi|^{2\beta})^{1/2}(1+|\xi|^{2\beta+2}+z^2)}\\
&=\int_{\mathbb{R}}\frac{|\xi|}{1+|\eta|^{2\beta})^{1/2}}\left(\int_{\mathbb{R}}\frac{dz}{1+|\xi|^{2\beta+2}+z^2}\right)d\xi\\
&=\int_{\mathbb{R}}\frac{|\xi|}{(1+|\xi|^{2\beta})^{1/2}(1+|\xi|^{2\beta+2})^{1/2}}\tan^{-1}\left(\frac{z}{(1+|\xi|^{2\beta+2})^{1/2}}\right)\Bigg|_{-\infty}^{\infty}d\xi\\
&=\pi\int_{\mathbb{R}}\frac{|\xi|}{[(1+|\xi|^{2\beta})(1+|\xi|^{2\beta+2})]^{1/2}}\,d\xi\\
&\leqslant c\pi\int_{\mathbb{R}}\frac{|\xi|\,d\xi}{1+|\xi|^{2\beta+1}}\\
&<+\infty
\end{aligned}$$

if $\beta > \frac{1}{2}$, and hence the weight w is in L_1. This implies the desired result as indicated above. ∎

Here is a local existence theory under very weak hypotheses.

THEOREM 3.4. *If $\alpha \geqslant 1$ and if the initial data ϕ is such that $\hat{\phi} \in L_1(\mathbb{R}^2)$, then there exists a $T > 0$ and unique weak solution u of the integral equation* (3.2) *such that $\hat{u} \in C(0, T; L_1(\mathbb{R}^2))$. The correspondence $\hat{\phi} \mapsto \hat{u}$ is continuous from $\widehat{L_1(\mathbb{R}^2)}$ into $C(0, T; \widehat{L_1(\mathbb{R}^2)})$.*

Proof. In the Fourier-transformed variables, the integral equation has the form

$$\hat{u}(\xi, \eta, t) = \widehat{K_t \phi}(\xi, \eta) + \int_0^t \widehat{K_{t-s}} \hat{Q}\left(\frac{1}{p+1} \widehat{u^{p+1}}\right) ds \equiv A(\hat{u}) = A_\phi(\hat{u}). \tag{3.3}$$

If $B_R(0)$ is the closed ball of radius R about zero in $C(0, T; \widehat{L_1(\mathbb{R}^2)})$ and $\hat{w}$ and $\hat{v}$ are both in $B_R(0)$, then

$$\begin{aligned} A(\hat{w}) - A(\hat{v}) &= \int_0^t \hat{K_{t-s}} \hat{Q} \frac{1}{p+1} (\widehat{w^{p+1}} - \widehat{v^{p+1}})\, ds \\ &= \int_0^t \hat{K_{t-s}} \hat{Q} \frac{1}{p+1} (\hat{w} - \hat{v}) * [\widehat{w^p} + \widehat{w^{p-1} v} + \cdots + \widehat{v^p}]\, ds. \end{aligned}$$

Since $\alpha \geqslant 1$, $\frac{\xi}{1+|\xi|^\alpha}$ is bounded and therefore there is a constant c_1 such that for $0 \leqslant t \leqslant T$,

$$|A(\hat{w}) - A(\hat{v})|_1 \leqslant T |\hat{w} - \hat{v}|_1 c_1 R^p.$$

With this estimate, we see that if $R = 2|\hat{\phi}|_1$ and then T is chosen so that $cTR^p = \frac{1}{2}$ then A is both contractive and maps $B_R(0)$ into itself. From the contraction mapping principle, it is inferred that there is unique solution u of the integral equation (3.2) such that $u \in C(0, T; \widehat{L_1(\mathbb{R}^2)})$. Since $\hat{u} \in L_1$, it is implied that $u \in C_b(\mathbb{R}^2)$, and that $u \to 0$ at ∞. The continuous dependence of the solution upon the initial data follows immediately since $A_\phi(u) - A_\psi(u) = K_t(\phi - \psi)$. Indeed, if ϕ, ψ are both initial data whose Fourier transform lies in $L_1(\mathbb{R}^2)$, and $R = 2\max\{|\hat{\phi}|_1, |\hat{\psi}|_1\}$, then if T is chosen so that $c_1 TR^p = \frac{1}{2}$, it follows that

$$\begin{aligned} \sup_{0 \leqslant t \leqslant T} |\hat{u}(\cdot, t) - \hat{v}(\cdot, t)|_1 &= \sup_{0 \leqslant t \leqslant T} |A_\phi(\hat{u})(\cdot, t) - A_\psi(\hat{v})(\cdot, t)|_1 \\ &\leqslant \sup_{0 \leqslant t \leqslant T} |A_\phi(\hat{u})(\cdot, t) - A_\psi(\hat{u})(\cdot, t)|_1 + \sup_{0 \leqslant t \leqslant T} |A_\psi(\hat{u})(\cdot, t) - A_\psi(v)(\cdot, t)|_1 \\ &\leqslant \frac{1}{2} \sup_{0 \leqslant t \leqslant T} |\hat{u}(\cdot, t) - \hat{v}(\cdot, t)|_1 + |\hat{\phi} - \hat{\psi}|_1, \end{aligned}$$

where u and v are the solutions in $C(0, T; \widehat{L_1})$ of (3.2) corresponding to initial data ϕ and ψ, respectively. It transpires that

$$\|u - v\|_{C(0,T;\hat{L}_1)} \leqslant 2|\hat{\phi} - \hat{\psi}|_1.$$

Thus the correspondence $\phi \mapsto u$ of data with the associated solution is in fact locally Lipschitz. ∎

Attention is now turned to a well-posedness theory in function spaces small enough to yield classical solutions, and in which the stability theory of Section 4 can be cast.

THEOREM 3.5. *Suppose $\alpha \geqslant 1$ and let $\phi \in W_\beta(\mathbb{R}^2)$ (respectively $\tilde{W}_\beta(\mathbb{R}^2)$) where $\beta \geqslant 1$. Then there exists a maximal time T_0 depending only on ϕ such that for each $T < T_0$, the initial-value problem* (3.1) *has a unique solution u that lies in $C(0, T; W_\beta(\mathbb{R}^2))$ (respectively $C(0, T; \tilde{W}_\beta(\mathbb{R}^2))$. The maximal time interval has a lower bound that is related inversely to $\|\phi\|_{W_1(\mathbb{R}^2)}$ and approaches $+\infty$ as $\|\phi\|_{W_1(\mathbb{R}^2)}$ approaches zero.*

For given $R > 0$, there is a $T(R) > 0$ such that the mapping that associates to the initial data $\phi \in W_\beta(\mathbb{R}^2)$ (respectively $\tilde{W}_\beta(\mathbb{R}^2)$) the solution u of (3.1) *is continuous from the ball of radius R about zero in $W_\beta(\mathbb{R}^2)$ (respectively $\tilde{W}_\beta(\mathbb{R}^2)$) into $C(0, T; W_\beta(\mathbb{R}^2))$ (respectively $C(0, T; \tilde{W}_\beta(\mathbb{R}^2))$).*

Proof. The proof is made for the case $\alpha = 2$ and $\beta = 1$ just to simplify notation. The proofs for the general case follow the same line of argument.

The strategy is to first show that corresponding to given $\phi \in W_1(\mathbb{R}^2)$, the integral equation (3.1) has a solution in $C(0, T; W_1(\mathbb{R}^2))$ for suitable values of $T > 0$. This will be accomplished via the contraction-mapping principle as in Theorem 3.4. For given ϕ in $W_1(\mathbb{R}^2)$ and any $v \in C(0, T; W_1(\mathbb{R}^2))$, define the action of the operator $A = A_\phi$ on v to be

$$Av(x, y, t) = K_t\phi(x, y) - \int_0^t K_{t-\tau} Q\left(\frac{v^{p+1}(\cdot, \tau)}{p+1}\right)(x, y)\, d\tau \tag{3.4}$$

for $(x, y, t) \in \mathbb{R}^2 \times [0, T]$. The aim is just as above, to show that the operator A is a contraction of the closed ball $B_R(0)$ of radius R about the zero function in $C(0, T; W_1(\mathbb{R}^2))$ provided R and T are well chosen. The crux of the matter is to understand the temporal integral in the definition of A.

Let $u, v \in C(0, T; W_1(\mathbb{R}^2))$ be given and consider the difference $Au - Av$. The norm of this difference in $W_1(\mathbb{R}^2)$ is bounded thusly: for fixed $t \in [0, T]$,

$$\left\| \int_0^t K_{t-\tau} Q\left(\frac{1}{p+1} u^{p+1}\right) d\tau - \int_0^t K_{t-\tau} Q\left(\frac{1}{p+1} v^{p+1}\right) d\tau \right\|_{W_1(\mathbb{R}^2)}$$

$$\leqslant T \sup_{0 \leqslant t \leqslant T} \left\| K_{t-\tau} Q\left(\frac{1}{p+1} (u^{p+1} - v^{p+1})\right) \right\|_{W_1(\mathbb{R}^2)}$$

$$\leqslant cT \sup_{0 \leqslant t \leqslant T} \left\| (u-v)\left(\frac{1}{p+1} [u^p + u^{p-1}v + \cdots + v^p]\right) \right\|_{H^1(\mathbb{R}^2)}, \tag{3.5}$$

where c is a universal constant coming from the use of Corollary 3.2. Of course, $H^1(\mathbb{R}^2)$ is not an algebra, but on account of Lemma 3.3, both u and v are bounded in terms of their $W_1(\mathbb{R}^2)$-norms. In consequence, Leibniz's rule comes to our aid in the following calculation:

$$\left\| (u-v)\left(\frac{1}{p+1} [u^p + u^{p-1}v + \cdots + v^p]\right) \right\|_{H^1(\mathbb{R}^2)}$$

$$\leqslant \frac{1}{p+1} \|u-v\|_{H^1(\mathbb{R}^2)} |u^p + u^{p-1}v + \cdots + v^p|_\infty$$

$$+ \frac{1}{p+1} |u-v|_\infty (\|\partial_x(u^p + u^{p-1}v + \cdots + v^p)\|_0$$

$$+ \|\partial_y(u^p + u^{p-1}v + \cdots + v^p)\|_0)$$

$$\leqslant \frac{1}{p+1} \|u-v\|_{W_1(\mathbb{R}^2)} \left(\|u\|^p_{W_1(\mathbb{R}^2)} + \|u\|^{p-1}_{W_1(\mathbb{R}^2)} \|v\|_{W_1(\mathbb{R}^2)} + \cdots + \|v\|^p_{W_1(\mathbb{R}^2)}\right)$$

$$+ \frac{p}{p+1} \|u-v\|_{W_1(\mathbb{R}^2)} \left(\|u\|^p_{W_1(\mathbb{R}^2)} + \|u\|^{p-1}_{W_1(\mathbb{R}^2)} \|v\|_{W_1(\mathbb{R}^2)} + \cdots + \|v\|^p_{W_1(\mathbb{R}^2)}\right)$$

$$\leqslant c\|u-v\|_{W_1(\mathbb{R}^2)} \left(\|u\|^p_{W_1(\mathbb{R}^2)} + \|v\|^p_{W_1(\mathbb{R}^2)}\right),$$

where the constant c depends only upon p. Hence, if u and v are both in $B_R(0)$, then

$$\|Au - Av\|_{C(0,T;W_1(\mathbb{R}^2))} \leqslant cTR^p \|u-v\|_{W_1(\mathbb{R}^2)}. \tag{3.6}$$

Just as in Theorem 3.4, if we define R to be $2\|\phi\|_{W_1(\mathbb{R}^2)}$ and if T is fixed so that $cTR^p = \frac{1}{2}$, then A is a contractive map of $B_R(0)$ in $C(0, T; W_1(\mathbb{R}^2))$ into itself.

It follows from the contraction-mapping principle that with these choices of R and T, the mapping A has a unique fixed point u in $B_R(0)$. That is, there exists a unique u in $B_R(0)$ such that

$$u(x, y, t) = Au = K_t\phi - \int_0^t K_{t-\tau}Q\left(\frac{u^{p+1}(\cdot, \tau)}{p+1}\right) d\tau. \tag{3.7}$$

Moreover, if (3.6) is differentiated with respect to t, there appears the relation

$$u_t = K_t'\phi - Q\left(\frac{u^{p+1}}{p+1}\right) - \int_0^t K_{t-\tau}'Q\left(\frac{u^{p+1}(\cdot, \tau)}{p+1}\right) d\tau, \tag{3.8}$$

where

$$\begin{aligned}\widehat{K_t'f}(\xi, \eta) &= -\frac{i(\xi^2 + \varepsilon\eta^2)}{\xi(1+\xi^2)} e^{-\frac{i(\xi^2+\varepsilon\eta^2)t}{\xi(1+\xi^2)}} \hat{f}(\xi, \eta) \\ &= -\frac{i(\xi^2 + \varepsilon\eta^2)}{\xi(1+\xi^2)} \widehat{K_t f}(\xi, \eta).\end{aligned} \tag{3.9}$$

From relation (3.8), it follows that

$$K_t' = -(\partial_x + \varepsilon\partial_x^{-1}\partial_y^2)(1 + \partial_x^2)^{-1}K_t, \tag{3.10}$$

and since $Q = (1 + \partial_x^2)^{-1}\partial_x$, (3.7) becomes

$$\begin{aligned}(1 + \partial_x^2)u_t &= -u^p u_x \\ &\quad - (\partial_x + \varepsilon\partial_x^{-1}\partial_y^2)\left[K_t\phi - \int_0^t K_{t-\tau}Q\left(\frac{u^{p+1}(\cdot, \tau)}{p+1}\right) d\tau\right] \\ &= -u^p u_x - u_x - \varepsilon\partial_x^{-1}u_{yy},\end{aligned} \tag{3.11}$$

at least as an equation relating distributions in $W_1(\mathbb{R}^2)$. Furthermore, it is clear that

$$\lim_{t\to 0} u(\cdot, t) = \phi,$$

in $W_1(\mathbb{R}^2)$ since

$$\widehat{K_t\phi}(\xi, \eta) = e^{i(x\xi + y\eta)}e^{-\frac{i(\xi^2+\varepsilon\eta^2)t}{\xi(1+\xi^2)}} \hat{\phi}(\xi, \eta).$$

Following the arguments exposed in [3, 6, 12], it is determined that the solution u obtained by use of the contraction-mapping principle is automatically unique in the large, not just on $[0, T]$ and in the ball $B_R(0)$.

Iterating the contraction-mapping argument leads to an increasing sequence $\{T_k\}_{k=1}^{\infty}$ such that a solution of (3.1) exists on the time interval $[0, T_k]$ for all $k = 1, 2, \ldots 2, \ldots$. By its construction, the solution u on each interval $[T_k, T_{k+1}]$, $k = 1, 2, \ldots$ is given as the fixed point of an integral equation like (3.7). Two possibilities can occur here, either

$$\lim_{k\to\infty} T_k = T_\infty < +\infty \text{ or the sequence } \{T_k\}_{k=1}^{\infty} \text{ is unbounded.}$$

If $T_\infty = +\infty$, then the solution of (3.1) is global, while if $T_\infty < +\infty$, then it must be case that

$$\limsup_{t\to T_\infty} \|u(\cdot, t)\|_{W_1(\mathbb{R}^2)} = +\infty. \tag{3.12}$$

Otherwise, if M is an upper bound for $\|u(\cdot, t)\|_{W_1(\mathbb{R}^2)}$ for $t \in [0, T_\infty)$, then the local existence obtained via the contraction-mapping principle can be applied with initial data $u(\cdot, t_0)$, where $t_0 \in [0, T_\infty)$ is close to T_∞, to extend the solution by at least $\varepsilon_0 = \frac{1}{2}(1 + (2M)^P)^{-1}c^{-1}$, where c is the constant appearing on the right-hand side of (3.5). As a consequence of this lower bound, the solution is certainly extended to the temporal interval $[0, T_\infty + \frac{1}{2}\varepsilon_0]$, say. This would contradict the definition of T_∞. It follows from these arguments that

$$T_\infty = \sup\{T\colon \text{ there exists a solution } u \in C(0, T; W_1(\mathbb{R}^2)) \text{ of (3.1) with } u(\cdot, 0) = \phi\}.$$

Moreover, the solution u can be extended over any time interval $[0, T]$ for which one has an a priori estimate on the norm of u in $W_1(\mathbb{R}^2)$.

The argument for continuous dependence follows the lines given in the proof of Theorem 3.4. Thus let u be a solution obtained at least locally in time by iterating the operator A on any function in $B_R(0)$. More precisely, let $R > 0$ be given and let T_1 be determined by the relation $cT_1(1 + R^P) = \frac{1}{2}$. Let ϕ_1 and ϕ_2 be in $W_1(\mathbb{R}^2)$ and suppose $\|\phi_1\|_{W_1(\mathbb{R}^2)}, \|\phi_2\|_{W_1(\mathbb{R}^2)} \leqslant \frac{1}{2}R$. Define A_1 and A_2 to be the operators given by the right-hand side of (3.2) with ϕ replaced by ϕ_1 and ϕ_2, respectively. Then, as above, we see that

$$\|u_1 - u_2\|_{C(0,T;W_1(\mathbb{R}^2))} \leqslant 2\|\phi_1 - \phi_2\|_{W_1(\mathbb{R}^2)}.$$

Thus the solution depends continuously on the data at least on $[0, T_1]$. In particular

$$\|u_1(\cdot, T_1) - u_2(\cdot, T_1)\|_{W_1(\mathbb{R}^2)} \leqslant 2\|\phi_1 - \phi_2\|_{W_1(\mathbb{R}^2)}.$$

Using the same argument starting with the data $u_1(\cdot, T_1)$ and $u_2(\cdot, T_1)$ rather than ϕ_1 and ϕ_2 leads to the conclusion that the solution depends continuously on the data on the interval $[0, T_2]$. Continuing in this manner leads to the full conclusion about continuous dependence.

The statement concerning the maximal interval of existence approaching $+\infty$ in case ϕ approaches zero in $W_1(\mathbb{R}^2)$ is now verified. To this end, consider the equation (3.11) in the form

$$u_t = -(1+\partial_x^2)^{-1}\partial_x\left[u + \frac{u^{p+1}}{p+1}\right] - \varepsilon(1+\partial_x^2)^{-1}\partial_x^{-1}u_{yy}.$$

It is not hard to see that for sufficiently smooth solutions,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}[u^2+u_x^2+u_{xx}^2+(\partial_x^{-1}u_y)^2+u_y^2]\,dx\,dy \\
&= -\int_{\mathbb{R}^2} u(1+\partial_x^2)^{-1}\partial_x\left(\frac{u^{p+1}}{p+1}\right)dx\,dy \\
&\quad - \int_{\mathbb{R}^2} u_x(1+\partial_x^2)^{-1}\partial_x(u^p u_x)\,dx\,dy \\
&\quad - \int_{\mathbb{R}^2} u_x(1+\partial_x^2)^{-1}\partial_x(u^p u_x)\,dx\,dy \\
&\quad - \int_{\mathbb{R}^2} u_{xx}(1+\partial_x^2)^{-1}\partial_x(u^p u_x)_x\,dx\,dy \\
&\quad - \int_{\mathbb{R}^2} (\partial_x^{-1}u_y)(1+\partial_x^2)^{-1}(u^p u_y)\,dx\,dy \\
&\quad - \int_{\mathbb{R}^2} u_y(1+\partial_x^2)^{-1}\partial_x(u^p u_y)\,dx\,dy \\
&\leqslant C\|u\|_0\|u^{p+1}\|_0 + C\|u_{xx}\|_0\|u^{p+1}\|_0 + C\|u_{xx}\|_0\|u^p u_x\|_0 \\
&\quad + C\|\partial_x^{-1}u_y\|_0\|u^p u_y\|_0 + C\|u_y\|_0\|u^p u_y\|_0,
\end{aligned} \tag{3.13}$$

where use has been made of the fact that $(1+\partial_x^2)^{-1}$, $(1+\partial_x^2)^{-1}\partial_x$ and $(1+\partial_x^2)^{-1}\partial_x^2$ are all bounded linear operators on $L_2(\mathbb{R}^2)$ since we are working on the case $\alpha = 2$. (For other values of α, the energy functional

$$\int_{\mathbb{R}^2}[u^2+u_x^2+(D_x^\alpha u)^2+(\partial_x^{-1}u_y)^2+u_y^2]\,dx\,dy$$

comes naturally to the fore.) It then follows from (3.13) that

$$\frac{d}{dt}\|u(\cdot,t)\|^2_{W_1(\mathbb{R}^2)} \leqslant C\|u(\cdot,t)\|^{p+2}_{W_1(\mathbb{R}^2)}.$$

Integrating the differential equation obtained by demanding equality in the last inequality leads to the upper bound

$$\|u(\cdot, t)\|^2_{W_1(\mathbb{R}^2)} \leqslant \frac{\|\phi\|^2_{W_1(\mathbb{R}^2)}}{[1 - Cp\|\phi\|^p_{W_1(\mathbb{R}^2)} t]^{2/p}}. \tag{3.14}$$

Inequality (3.14) was obtained assuming the solution is smooth. As we will see in the next theorem, smooth data leads to smooth solutions, and the time interval of existence depends only on the $W_1(\mathbb{R}^2)$-norm of the initial data. Hence the continuous-dependence result just established allows one to infer that (3.13) continues to hold for $W_1(\mathbb{R}^2)$-solutions. On the other hand, bound (3.14) implies that

$$T_\infty \geqslant \frac{1}{Cp\|\phi\|^p_{W_1(\mathbb{R}^2)}}.$$

By combining this with the result of the last paragraph, the stated conclusion on the maximal time of existence is obtained. ∎

THEOREM 3.5. *Let $\alpha \geqslant 1$ and let $\phi \in X_s$ with $s > \frac{3}{2}$. Then there exists $T > 0$ such that the initial-value problem* (3.1) *has a unique solution*

$$u \in C(0, T; X_s) \cap C^1(0, T; H^{s-2}(\mathbb{R}^2)).$$

In particular, the solution u also satisfies

$$u \in C(0; T; H^s(\mathbb{R}^2)) \quad \text{and} \quad \partial_x^{-1} u_y \in C(0, T; H^{s-1}(\mathbb{R}^2)).$$

The solution depends continuously in these function classes on variations of ϕ in X_s.

Proof. The proof is similar to the one given for Theorem 3.4 except that in showing that A maps $C(0, T; X_s)$ into itself, one first recalls from Corollary 3.2 that $K_{t-\tau}Q$ maps $H^s(\mathbb{R}^2)$ into X_s, then uses that fact that $H^s(\mathbb{R}^2)$ is a Banach-algebra since $s > \frac{3}{2}$. Once a solution of the equation is at hand, the equation implies

$$\begin{aligned} u_t &= -Q\left[u + \frac{u^{p+1}}{p+1} + \varepsilon \partial_x^{-2} u_{yy}\right] \\ &= -Q\left[u + \frac{u^{p+1}}{p+1}\right] - \varepsilon (1 + D_x^\alpha)^{-1} \partial_x^{-1} u_{yy}. \end{aligned}$$

Since both Q and $(1 + D_x^\alpha)^{-1}$ and are bounded on $H^s(\mathbb{R}^2)$, it follows because $\partial_x^{-1} u_y \in H^{s-1}(\mathbb{R}^2)$ and $s > \frac{3}{2}$ that

$$u_t \in C(0, T; H^{s-2}(\mathbb{R}^2)). \quad ∎$$

4. GLOBAL EXISTENCE

Having established local well-posedness for the initial-value problem under study, attention is given to whether the locally defined solution can be extended to the entire time axis.

THEOREM 4.1. (1) *If* $2 \leqslant \alpha < 2\sqrt{2}$ *and* $\phi \in W_{\alpha/2}(\mathbb{R}^2)$ *is such that* $\phi_{yy} = \psi_{xx}$ *for some function* $\psi \in L_2(\mathbb{R}^2)$, *then there exists a unique solution* u *to the initial-value problem* (3.1) *which for any* $T > 0$, *lies in* $C(0, T; W_{\alpha/2}(\mathbb{R}^2))$ *provided* $p \leqslant \frac{16(\alpha-1)}{4+4\alpha-\alpha^2}$.

(2) *If* $2\sqrt{2} \leqslant \alpha < 4$ *and* $\phi \in W_{\alpha/2}(\mathbb{R}^2)$, *then the conclusion in* (1) *continues to hold provided only that* $p < 4$.

(3) *If* $\alpha \geqslant 4$ *and* $\phi \in \tilde{W}_{\alpha/2}(\mathbb{R}^2)$, *then for any* $T > 0$, *the initial-value problem* (3.1) *possesses a unique solution* u *in the class* $C(0, T; \tilde{W}_{\alpha/2}(\mathbb{R}^2))$ *if* $p < 4$.

(4) *The conclusions in* (2) *and* (3) *are still valid if* $p = 4$ *and the initial data is small in* $H_x^{\alpha/2}(\mathbb{R}^2)$.

Proof. Recall that the condition that $\phi_{yy} = \psi_{xx}$ for some $L_2(\mathbb{R}^2)$-function ψ is needed to show the quantity $E(u)$ is conserved. The invariance of $E(u(\cdot, t))$ allowed one to infer that if $\phi \in V_{\alpha/2}(\mathbb{R}^2)$, then the corresponding solution remains bounded in $V_{\alpha/2}(\mathbb{R}^2)$ throughout its time of existence if $p<4$. As noted earlier in the proof of Theorem 3.4, the local solution u can be globally continued if it can be shown that u is bounded in $W_{\alpha/2}(\mathbb{R}^2)$ (respectively $\tilde{W}_{\alpha/2}(\mathbb{R}^2)$) on bounded time intervals.

If $2 \leqslant \alpha < 4$, and $\phi \in W_{\alpha/2}(\mathbb{R}^2)$, then ϕ, ϕ_x, $\partial_x^{-1}\phi_y \in H_x^{\alpha/2}(\mathbb{R}^2)$. It follows from the discussion in Section 2 that if $p<4$, the solution $u(\cdot, t) \in H_x^{\alpha/2}(\mathbb{R}^2)$ and $\partial_x^{-1}u_y(\cdot, t) \in L_2(\mathbb{R}^2)$ independently of time t. Thus, to show $u \in W_{\alpha/2}(\mathbb{R}^2)$ for any $T > 0$, it suffices to obtain bounds for the terms u_x and $\partial_x^{-1}u_y$ in $H_x^{\alpha/2}(\mathbb{R}^2)$.

First, differentiate Eq. (3.1) with respect to x. Multiply the resulting expression by u_x and integrate over $\mathbb{R}^2$ to obtain the relation

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} [u_x^2 + (D_x^{\alpha/2}u_x)^2] &= -\int_{\mathbb{R}^2} (u^p u_x)_x u_x \\ &= \int_{\mathbb{R}^2} u^p u_x u_{xx} \\ &\leqslant |u|_\infty^p \|u_x\|_0 \|u_{xx}\|_0 \\ &\leqslant c\|u_x\|_0^{\frac{p}{4}+1} \|\partial_x^{-1}u_y\|_0^{\frac{p}{4}} \|u_y\|_0^{\frac{p}{4}} \|u_{xx}\|_0^{\frac{p}{4}+1} \\ &\leqslant c\|u_y\|_0^{\frac{p}{4}} \|u_{xx}\|_0^{\frac{p}{4}+1}, \end{aligned} \tag{4.1}$$

where use has been made of the fact that u_x and $\partial_x^{-1}u_y$ are both a priori bounded in $L_2(\mathbb{R}^2)$ independently of t. This relation obtains formally for sufficiently smooth initial data. The extra smoothness needed for the derivation may be dispensed with upon resorting to the continuous dependence result.

Next, apply the operator $\partial_x^{-1}\partial_y$ to both sides of (3.1), multiply the resulting expression by $\partial_x^{-1}\partial_y u$ and integrate over $\mathbb{R}^2$ to find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}[(\partial_x^{-1}u_y)^2+(D_x^{\alpha/2}\partial_x^{-1}u_y)^2] &= -\int_{\mathbb{R}^2}u^p u_y\partial_x^{-1}u_y \\ &\leqslant |u|_\infty^p\|u_y\|_0\|\partial_x^{-1}u_y\|_0 \\ &\leqslant c\|u_x\|_0^{\frac{p}{4}}\|\partial_x^{-1}u_y\|_0^{\frac{p}{4}+1}\|u_y\|_0^{\frac{p}{4}+1}\|u_{xx}\|_0^{\frac{p}{4}} \\ &\leqslant c\|u_y\|_0^{\frac{p}{4}+1}\|u_{xx}\|_0^{\frac{p}{4}}. \end{aligned} \tag{4.2}$$

By adding relations (4.1) and (4.2), the inequality

$$\frac{d}{dt}\int_{\mathbb{R}^2}[u_x^2+(\partial_x^{-1}u_y)^2+(D_x^{\alpha/2}u_x)^2+(D_x^{\alpha/2}\partial_x^{-1}u_y)^2] \leqslant c[\|u_y\|_0^{\frac{p}{4}}\|u_{xx}\|_0^{\frac{p}{4}+1}+\|u_y\|_0^{\frac{p}{4}+1}\|u_{xx}\|_0^{\frac{p}{4}}] \tag{4.3}$$

emerges. The above inequality is then integrated with respect to t from 0 to T to yield

$$\|u(\cdot,T)\|_{W_{\alpha/2}(\mathbb{R}^2)}^2 \leqslant c\|\phi\|_{W_{\alpha/2}(\mathbb{R}^2)}^2 + c\int_0^T [\|u_y\|_0^{\frac{p}{4}}\|u_{xx}\|_0^{\frac{p}{4}+1}+\|u_y\|_0^{\frac{p}{4}+1}\|u_{xx}\|_0^{\frac{p}{4}}]\,dt. \tag{4.4}$$

Interpolation is applied as follows to control the terms under the integrand in (4.4): first since $\alpha<4$,

$$\|u_{xx}\|_0 \leqslant \|u\|_{H_x^2(\mathbb{R}^2)} \leqslant c\|u\|_{H_x^{1+\frac{\alpha}{2}}(\mathbb{R}^2)}^{2-\frac{\alpha}{2}}\|u\|_{H_x^{\alpha/2}(\mathbb{R}^2)}^{\frac{\alpha}{2}-1}, \tag{4.5}$$

and secondly, for any $\varepsilon_0 \geqslant 0$,

$$\|u_y\|_0 \leqslant c\|u_y\|_{H_y^{\varepsilon_0}(\mathbb{R}^2)}^{\frac{1}{1+\varepsilon_0}}\|\partial_x^{-1}u_y\|_0^{\frac{\varepsilon_0}{1+\varepsilon_0}}.$$

Choosing $\varepsilon_0 = (\alpha/2) - 1$ which is nonnegative since $\alpha \geqslant 2$, the last inequality is specialized to

$$\|u_y\|_0 \leqslant c\|u_y\|^{\frac{2}{\alpha}}_{H_x^{\frac{\alpha}{2}-1}(\mathbb{R}^2)} \|\partial_x^{-1} u_y\|_0^{1-\frac{2}{\alpha}}. \tag{4.6}$$

Relations (4.5) and (4.6) are then inserted into (4.4) to obtain

$$\|u(\cdot, T)\|^2_{W_{\alpha/2}(\mathbb{R}^2)} \leqslant \|\phi\|^2_{W_{\alpha/2}(\mathbb{R}^2)} + c\int_0^T \Bigg\{ \|u_y\|^{\frac{p}{2\alpha}}_{H_x^{\frac{\alpha}{2}-1}(\mathbb{R}^2)} \|u\|^{(\frac{p}{4}+1)(2-\frac{\alpha}{2})}_{H_x^{1+\frac{\alpha}{2}}(\mathbb{R}^2)}$$

$$+ \|u_y\|^{\frac{2}{\alpha}(\frac{p}{4}+1)}_{H_x^{\frac{\alpha}{2}-1}(\mathbb{R}^2)} \|u\|^{\frac{p}{4}(2-\frac{2}{\alpha})}_{H_x^{1+\frac{\alpha}{2}}(\mathbb{R}^2)} \Bigg\}\, dt,$$

where as previously noted, $\|u\|_{H_x^{\alpha/2}(\mathbb{R}^2)}$ and $\|\partial_x^{-1} u_y\|_0$ are known to be bounded independently of t. In consequence,

$$\|u(\cdot, T)\|^2_{W_{\alpha/2}(\mathbb{R}^2)} \leqslant \|\phi\|^2_{W_{\alpha/2}(\mathbb{R}^2)} + c\int_0^T \Bigg\{ \|u\|^{\frac{p}{2\alpha}+\frac{p}{2}-\frac{p\alpha}{8}+2-\frac{\alpha}{2}}_{W_{\alpha/2}(\mathbb{R}^2)} + \|u\|^{\frac{p}{2\alpha}+\frac{2}{\alpha}+\frac{p}{2}-\frac{p\alpha}{8}}_{W_{\alpha/2}(\mathbb{R}^2)} \Bigg\}\, dt, \tag{4.7}$$

and so a Gronwall-type argument may be applied to conclude that

$$\sup_{0\leqslant t\leqslant T} \|u(\cdot, t)\|^2_{W_{\alpha/2}(\mathbb{R}^2)} \leqslant \|\phi\|^2_{W_{\alpha/2}(\mathbb{R}^2)} \exp(cT), \tag{4.8}$$

provided

$$\frac{p}{2\alpha} + \frac{p}{2} - \frac{p\alpha}{8} + 2 - \frac{\alpha}{2} \leqslant 2,$$

and

$$\frac{p}{2\alpha} + \frac{2}{\alpha} + \frac{p}{2} - \frac{p\alpha}{8} \leqslant 2.$$

It then follows that there is a uniform bound on the solution u in the space $W_{\alpha/2}(\mathbb{R}^2)$ on the interval $[0, T)$, and since $T > 0$ was arbitrary, global well-posedness then follows. Since $2 \leqslant \alpha < 4$, a calculation reveals

$$\frac{p}{2\alpha} + \frac{p}{2} - \frac{p\alpha}{8} + 2 - \frac{\alpha}{2} \leqslant \frac{p}{2\alpha} + \frac{p}{2} - \frac{p\alpha}{8} + \frac{2}{\alpha},$$

and therefore

$$p \leqslant \frac{16(\alpha-1)}{4+4\alpha-\alpha^2}.$$

As $p \leqslant 4$ in any event, this inequality is only binding when $2 \leqslant \alpha < 2\sqrt{2}$. For $\alpha \geqslant 2\sqrt{2}$, the restriction $p<4$ (or $p=4$ and $\|\phi\|_{H_x^{\alpha/2}(\mathbb{R}^2)}$ small) is the stronger restriction. Thus the situation for $\alpha<4$ is seen to be as advertised in the statement of the theorem.

For $\alpha \geqslant 4$, let $\phi \in \tilde{W}_{\alpha/2}(\mathbb{R}^2)$. Since $\|u\|_{H^{\alpha/2}(\mathbb{R}^2)}$ and $\|\partial_x^{-1} u_y\|_0$ are a priori bounded independently of t if $p<4$, $u_{xx} \in L_2(\mathbb{R}^2)$ because $\alpha \geqslant 4$. Hence, to derive a uniform bound on the solution u in the space $\tilde{W}_{\alpha/2}(\mathbb{R}^2)$, it is enough to estimate $\partial_x^{-1} u_y$ in the space $H_x^{\alpha/2}(\mathbb{R}^2)$. From relation (4.2), it follows that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} &[(\partial_x^{-1}u_y)^2 + (D_x^{\alpha/2}\partial_x^{-1}u_y)^2] \\
&= -\int_{\mathbb{R}^2} u^p u_y \partial_x^{-1} u_y \\
&\leqslant |u|_\infty^p \|u_y\|_0 \|\partial_x^{-1}u_y\|_0 \\
&\leqslant c\|u_x\|_0^{p/4} \|\partial_x^{-1}u_y\|_0^{(p/4)+1} \|u_y\|_0^{(p/4)+1} \|u_{xx}\|_0^{p/4} \\
&\leqslant c\|u_y\|_0^{(p/4)+1}.
\end{aligned} \tag{4.9}$$

On the other hand,

$$\begin{aligned}
\|u_y\|_0^2 &= \int_{\mathbb{R}^2} u_y u_y \\
&= -\int_{\mathbb{R}^2} u_{xy}\partial_x^{-1}u_y \\
&\leqslant c\|\partial_x^{-1}u_y\|_0 \|u_{xy}\|_0 \\
&\leqslant c\|\partial_x^{-1}u_y\|_0 \|u_y\|_{H_x^{(\alpha/2)-1}(\mathbb{R}^2)}^{2/(\alpha-2)} \|u_y\|_0^{(\alpha-4)/(\alpha-2)},
\end{aligned}$$

and so

$$\|u_y\|_0^{\alpha/(\alpha-2)} \leqslant c\|\partial_x^{-1}u_y\|_0 \|u_y\|_{H_x^{(\alpha/2)-1}(\mathbb{R}^2)}^{2/(\alpha-2)},$$

or, what is the same,

$$\|u_y\|_0 \leqslant c\|\partial_x^{-1}u_y\|_0^{1-(2/\alpha)} \|u_y\|_{H_x^{(\alpha/2)-1}(\mathbb{R}^2)}^{2/\alpha}. \tag{4.10}$$

Hence, (4.9) may be extended to

$$\frac{d}{dt}\|\partial_x^{-1}u_y\|^2_{H_x^{\alpha/2}(\mathbb{R}^2)} \leqslant c\|u_y\|^{(2/\alpha)((p/4)+1)}_{H_x^{(\alpha/2)-1}(\mathbb{R}^2)}, \tag{4.11}$$

where c depends on $\|\phi\|_{V_{\alpha/2}(\mathbb{R}^2)}$. Integrating relation (4.11) with respect to t and noting that for $\alpha \geqslant 4$ and $p \leqslant 4$

$$\frac{2}{\alpha}\left(\frac{p}{4}+1\right) = \beta \leqslant 1.$$

Gronwall's lemma may be applied to obtain

$$\sup_{t\in[0,T]} \|\partial_x^{-1}u_y(\cdot, t)\|^2_{H_x^{\alpha/2}(\mathbb{R}^2)} \leqslant \|\partial_x^{-1}\phi_y\|^2_{H_x^{\alpha/2}(\mathbb{R}^2)} e^{cT}.$$

This completes the proof of the theorem. ∎

Attention is now turned to specifying conditions under which the local solution obtained in Theorem 3.5 can be extended to a global one. If $\phi \in X_2 \cap W_{\alpha/2}(\mathbb{R}^2)$ (respectively $\tilde{W}_{\alpha/2}(\mathbb{R}^2)$) is such that $\phi_{yy} = \psi_{xx}$ for some function $\psi \in L_2(\mathbb{R}^2)$, then by Theorem 4.1, the solution u lies in $C(0, T; W_{\alpha/2}(\mathbb{R}^2))$ (respectively $C(0, T; \tilde{W}_{\alpha/2}(\mathbb{R}^2))$ for any $T > 0$ when p and α satisfy the conditions of Theorem 4.1. Thus, to show that the solution is global in X_2, say, it is sufficient to estimate $\partial_x^{-1}u_{yy}$ and u_{yy} in $L_2(\mathbb{R}^2)$. To this end, the integral equation

$$u = K_t\phi - \int_0^t K_{t-\tau}Q\left(\frac{u^{p+1}}{p+1}\right)d\tau \tag{3.2$'$}$$

comes to the fore. It follows from this relation that

$$\|\partial_x^{-1}u_{yy}\|_0 \leqslant \|\partial_x^{-1}\phi_{yy}\|_0 + \int_0^t \left\| K_{t-\tau}Q\partial_x^{-1}\left(\frac{u^{p+1}}{p+1}\right)_{yy}\right\|_0 d\tau.$$

Observing that $K_{t-\tau}Q\partial_x^{-1} = K_{t-\tau}(1+D_x^\alpha)^{-1}$ is a bounded operator on $L_2(\mathbb{R}^2)$, it follows that

$$\|\partial_x^{-1}u_{yy}\|_0 \leqslant \|\partial_x^{-1}\phi_{yy}\|_0 + c\int_0^t \|(u^{p+1})_{yy}\|_0\, d\tau.$$

Similarly,

$$\|u_{yy}\|_0 \leqslant \|\phi_{yy}\|_0 + \int_0^t \|(u^{p+1})_{yy}\|_0\, d\tau,$$

and therefore

$$\begin{aligned}
\|\partial_x^{-1} u_{yy}\|_0 + \|u_{yy}\|_0 &\leqslant \|\partial_x^{-1}\phi_{yy}\|_0 + \|\phi_{yy}\|_0 + c\int_0^T \|(u^{p+1})_{yy}\|_0\, d\tau \\
&\leqslant \|\partial_x^{-1}\phi_{yy}\|_0 + \|\phi_{yy}\|_0 + c\int_0^T (\{\|pu^{p-1}u_y^2\|_0 + \|u^p u_{yy}\|_0\})\, d\tau \\
&\leqslant \|\partial_x^{-1}\phi_{yy}\|_0 + \|\phi_{yy}\|_0 + c\int_0^T \{|u|_\infty^{p-1}|u_y|_4^2 + |u|_\infty^p\|u_{yy}\|_0\}\, d\tau \\
&\leqslant \|\partial_x^{-1}\phi_{yy}\|_0 + \|\phi_{yy}\|_0 \\
&\quad + c\int_0^T \{|u|_\infty^{p-1}\|u_y\|_0^{1/2}\|u_{xx}\|_0^{1/2}\|u_{yy}\|_0^{1/2}\|\partial_x^{-1}u_{yy}\|_0^{1/2} \\
&\quad + |u|_\infty^p\|u_{yy}\|_0\}\, d\tau \\
&\leqslant \|\partial_x^{-1}\phi_{yy}\|_0 + \|\phi_{yy}\|_0 \\
&\quad + c\int_0^T \{\|\partial_x^{-1}u_{yy}\|_0^{1/2}\|u_{yy}\|_0^{1/2} + \|u_{yy}\|_0\}\, d\tau.
\end{aligned}$$

In these inequalities, use has been made of the fact that $|u|_\infty$ is bounded on $[0, T]$ by virtue of the condition $u \in C(0, T; W_{\alpha/2}(\mathbb{R}^2))$ (respectively $C(0, T; \tilde{W}_{\alpha/2}(\mathbb{R}^2))$) for any $T > 0$ and that

$$|u_y|_4^2 \leqslant c\|u_y\|_0^{1/2}\|u_{xx}\|_0^{1/2}\|u_{yy}\|_0^{1/2}\|\partial_x^{-1}u_{yy}\|_0^{1/2}.$$

It follows that if $X_2 \cap W_{\alpha/2}(\mathbb{R}^2)$ is normed by

$$\|\phi\|_{X_2 \cap W_{\alpha/2}(\mathbb{R}^2)} = \|\phi\|_{X_2} + \|\phi\|_{W_{\alpha/2}(\mathbb{R}^2)},$$

then an application of Gronwall's Lemma yields

$$\|\partial_x^{-1}u_{yy}\|_0 + \|u_{yy}\|_0 \leqslant \|\phi\|_{X_2 \cap W_{\alpha/2}(\mathbb{R}^2)} \exp(cT)$$

for $0 \leqslant t \leqslant T$ or equivalently,

$$\sup_{t \in [0, T]} \|u(\cdot, t)\|_{X_2} \leqslant \|\phi\|_{X_2 \cap W_{\alpha/2}(\mathbb{R}^2)} \exp(cT). \tag{4.12}$$

A similar estimate obtains for $\alpha \geqslant 4$, but the right-hand side of (4.12) features the norm in the space $X_2 \cap \tilde{W}_{\alpha/2}(\mathbb{R}^2)$.

For $s > 2$, the integral equation (3.2)–(3.2′) may be used again to derive the inequality

$$\|u\|_{X_s} \leqslant \|\phi\|_{X_s} + \int_0^t \|u^{p+1}\|_s \, d\tau$$
$$\leqslant \|\phi\|_{X_s} + \int_0^t a(\|u\|_{s-1})\|u\|_s \, d\tau,$$

where a is a monotone increasing function (see [26]). Thus

$$\|u\|_{X_s} \leqslant \|\phi\|_{X_s} + \int_0^t a(M_{s-1})\|u(\cdot\,,\tau)\|_{X_s} \, d\tau, \tag{4.13}$$

where for any real number q,

$$M_q = M_q(T) = \sup_{0 \leqslant t \leqslant T} \|u(\cdot, t)\|_{X_q}.$$

From the previous paragraph, it is known that

$$M_2(T) = \sup_{0 \leqslant t \leqslant T} \|u(\cdot, t)\|_{X_2} \leqslant \|\phi\|_{X_2 \cap W_{\alpha/2}(\mathbb{R}^2)} e^{cT}$$

for any $T > 0$. By applying Gronwall's Lemma to (4.13) and arguing inductively from the bound on $M_2(T)$, it is ascertained that $M_s(T)$ is uniformly bounded on $[0, T]$ for any s such that $\phi \in X_s \cap W_{\alpha/2}(\mathbb{R}^2)$ (respectively $X_s \cap \tilde{W}_{\alpha/2}(\mathbb{R}^2)$ when $\alpha \geqslant 4$). Hence, $\|u(\cdot, t)\|_{X_s}$ is uniformly bounded on $[0, T]$ and in consequence, we have the following global existence and uniqueness result.

THEOREM 4.2. *Let $\alpha \geqslant 2$ and $\phi \in X_s \cap W_{\alpha/2}(\mathbb{R}^2)$ (respectively $\phi \in X_s \cap \tilde{W}_{\alpha/2}(\mathbb{R}^2)$ if $\alpha \geqslant 4$) be such that $\partial_x^{-2}\phi_{yy} \in L_2(\mathbb{R}^2)$. Assume p and α satisfy the conditions of Theorem* 4.1. *Then for any $T > 0$, the initial-value problem* (3.1) *has a unique solution*

$$u \in C(0, T; X_s) \cap C^1(0, T; H^{s-2}(\mathbb{R}^2)).$$

In particular, for any $T > 0$, the solution u of (3.1) *has the properties*

$$u \in C(0, T; H^s(\mathbb{R}^2)), \qquad \partial_x^{-1} u_y \in C(0, T; H^{s-1}(\mathbb{R}^2)).$$

For any $T > 0$, the solution map $\phi \mapsto u$ is locally Lipschitz-continuous from $X_s \cap W_{\alpha/2}(\mathbb{R}^2)$ (respectively, $X_s \cap \tilde{W}_{\alpha/2}(\mathbb{R}^2)$) into these spaces.

Remark 4.3. (1) For the generalized RLW–KP equations (1.5) and (1.11) with a power nonlinearity or (1.18) with $L = -\partial_x^2$, global unique solution exists for large data in X_2 if $p \leqslant 2$ regardless of the sign of ε (even though the solution belongs to $V_1(\mathbb{R}^2)$ independently of t if $p < 4$), while for the generalized KPI equation ($\varepsilon = -1$), global solutions persist for large data in $V_1(\mathbb{R}^2)$ if $p < \frac{4}{3}$ [36].

(2) The regularized version of the fifth-order KP-equation has a unique global solution in $X_2 \cap \tilde{W}_2(\mathbb{R}^2)$ if $p<4$. On the other hand, its counterpart, Eq. (1.12) with $\alpha = 4$ possesses global solution in $V_2(\mathbb{R}^2)$ if $p<2$.

5. STABILITY OF SOLITARY-WAVE SOLUTIONS

In this section, attention is restricted to the case $\alpha = 2$ and $\varepsilon = -1$. The case $\alpha = 2$ arose in the original derivation of the KP-equations as a model for surface water waves, and it continues to hold a distinguished place in the pantheon of KP-like models, including those considered here. The mathematical advantage to the case $\alpha = 2$, and the principal reason why we restrict to this case in the present development, is that the dispersion operator $L = -\partial_x^2$ is local. Thus the evolution equation takes the special form

$$(u_t + u_x + u^p u_x - u_{xxt})_x = u_{yy}. \tag{5.1}$$

Here, as before, the parameter $p \geqslant 1$ is a positive integer.

The focus of the development in this section is the solitary-wave, or lump solutions of (5.1). Localized, travelling-wave solutions of nonlinear, dispersive wave equations are known in many circumstances to play a distinguished role in the long-time evolution of an initial disturbance. In consequence, and because the issues are interesting in their own right, the orbital and asymptotic stability of these special solutions has been a central theme of development for more than three decades (cf. [4, 8, 30, 31, 41, 42], etc.).

In the context of (5.1), if $c > 1$ is a specified speed of propagation, a solitary-wave solution $u(x, y, t) = \phi_c(x - ct, y)$ satisfies the time-independent partial differential equation

$$-c\phi_{cxx} + (c-1)\phi_c + \partial_x^{-2}\phi_{cyy} - \frac{1}{p+1}\phi_c^{p+1} = 0. \tag{5.2}$$

A central role will be played by the functionals $I = I_c$ and K defined for $u \in V_1(\mathbb{R}^2)$ by

$$I_c(u) = \int_{\mathbb{R}^2} [(c-1)u^2 + cu_x^2 + (\partial_x^{-1}u_y)^2]\, dx\, dy$$

and

$$K(u) = \frac{1}{p+1} \int_{\mathbb{R}^2} u^{p+2}\, dx\, dy,$$

respectively. The functional K is well-defined on $V_1(\mathbb{R}^2)$ by virtue of Lemma 2.1. Equation (5.2) is the Euler–Lagrange equation of the functional

$$L_c(u) = \frac{1}{2} I_c(u) - \frac{1}{p+2} K(u).$$

Thus it is interesting with regard to the analysis of travelling-wave solutions to consider the minimization problem

$$M_c = \inf\{I_c\colon u \in V_1(\mathbb{R}^2) \text{ and } K(u) = 1\}. \tag{5.3}$$

If $\psi_c \in V_1(\mathbb{R}^2)$ is a minimizer of problem (5.3), so that $M_c = I_c(\psi_c)$ and $K(u) = 1$, then ψ_c is necessarily a solution to the equation

$$-c\partial_x^2\psi_c + (c-1)\psi_c + \partial_x^{-2}\partial_y^2\psi_c = \lambda\psi_c^{p+1}, \tag{5.4}$$

for some Lagrange-multiplier λ. As $\psi \neq 0$, it is easy to see that $\lambda > 0$ and so

$$\phi_c = \lambda^{1/p}\psi_c$$

is a solution of (5.2), often referred to as a *ground state* of (5.1). It is clear that

$$I_c(\phi_c) = K(\phi_c) = M_c^{(2+p)/p} = \lambda^{2/p} I_c(\psi_c).$$

The following result is proved in [15, 30, 31, 41].

THEOREM 5.1. *Suppose $c > 1$ and $0 < p < 4$. Let $\{\psi_k\}_{k=1}^{\infty}$ be a minimizing sequence for problem (5.3). Then there exists a subsequence $\{\psi_n = \psi_{k_n}\}_{n=1}^{\infty}$ of $\{\psi_k\}_{k=1}^{\infty}$, a sequence $\{y_j\}_{j=1}^{\infty}$ of real numbers and an element $\psi_c \in V_1(\mathbb{R}^2)$ such that $\psi_{k_j}(\cdot - y_j) \to \psi_c$ in $V_1(\mathbb{R}^2)$. The function ψ_c is a minimizer of (5.3) subject to the constraint $K(\psi_c) = 1$ and is therefore a solution of the Euler–Lagrange equation (5.4). It then follows that $\phi_c = \lambda^{1/p}\psi_c$ is a ground state of (5.1), where $\lambda > 0$ is the Euler–Lagrange multiplier associated to the solution ψ_c of (5.3).*

Remark 5.2. In [15], the problem that was actually posed and solved by de Bouard and Saut was to minimize

$$J(u) = \int_{\mathbb{R}^2} [u^2 + u_x^2 + (\partial_x^{-1}u_y)^2]\, dx\, dy,$$

with $K(u) = \lambda > 0$, say. Solutions of the de Bouard–Saut minimization problem are related to ground-state solutions as defined above by a simple change of variables; if

$$\phi_c(x, y) = (c-1)^{1/p} w\left(\sqrt{\frac{c-1}{c}}\, x, \frac{c-1}{\sqrt{c}}\, y\right),$$

where ϕ_c is a ground state, then w is a ground state of

$$-w_{xx} + w - \frac{1}{p+1}\, w^{p+1} + \partial_x^{-2} w_{yy} = 0$$

and

$$I_c(\phi_c) = (c-1)^{(2+p)/p} J(w).$$

For a given $c > 1$, the set of all associated ground states S_c may be characterized as

$$S_c = \{\phi_c \in V_1(\mathbb{R}^2)\colon K(\phi_c) = I_c(\phi_c) = M_c^{(p+2)/p}\}.$$

By Theorem 5.1, S_c is not empty. Moreover, as both I_c and K are invariant under translations in the spatial variable x, y, S_c is also invariant under such translations.

Definition. Let X be a Banach space of real valued functions whose domain is $\mathbb{R}^2$. A set $S \subset X$ is called X-stable for the RLW–KP equation (5.1) if for any $\varepsilon_0 > 0$, there exists $\delta > 0$ such that for $u_0 \in X \cap W_1(\mathbb{R}^2)$ with

$$\inf_{v \in S} \|u_0 - v\|_X < \delta,$$

the solution u of Eq. (5.1) with initial value $u(\cdot, 0) = u_0(\cdot)$ can be extended to a global solution in $C(0, \infty; X \cap W_1(\mathbb{R}^2))$ and

$$\sup_{0 \leqslant t < \infty} \inf_{v \in S} \|u(t) - v\|_X < \varepsilon_0.$$

Otherwise, S is called *X-unstable* or just *unstable* if X is understood.

Define the function d of the wavespeed c to be

$$d(c) = E(\phi_c) + cV(\phi_c)$$

for $\phi_c \in S_c$, where as in (2.10) with $\varepsilon = -1$,

$$E(u) = \int_{\mathbb{R}^2} \left[\frac{1}{2}\,(\partial_x^{-1} u_y)^2 - \frac{1}{2}\, u^2 - \frac{u^{p+2}}{(p+1)(p+2)}\right] dx\, dy$$

and

$$V(u) = \frac{1}{2} \int_{\mathbb{R}^2} (u^2 + u_x^2)\, dx\, dy.$$

It is easy to see that $d(c)$ depends only on c and not on the element $\phi_c \in S_c$. In fact, from the definition of d and the characterization of S_c, it follows that

$$d(c) = \frac{1}{2} I_c(\phi_c) - \frac{1}{p+2} K(\phi_c) = \frac{p}{2(p+2)} M_c^{(p+2)/p}.$$

THEOREM 5.3 (Nonlinear Stability). *The travelling wave S_c is $V_1(\mathbb{R}^2)$-stable if either* (1) $0<p<\frac{4}{3}$ *and* $c>1$, *or* (2) $\frac{4}{3}<p<4$ *and* $c>(4p)/(4+p)$.

Remark 5.4. Actually, it will turn out the nonlinear stability just asserted is determined by the sign of $d''(c)$. In the present circumstances, it is easy to determine the sign of $d''(c)$. Just note that

$$d(c) = \frac{p}{2(p+2)} (c-1)^{(4-p)/2p} cK(w),$$

where

$$K(w) = I(w) = \int_{\mathbb{R}^2} [w^2 + w_x^2 + (\partial_x^{-1} w_y)^2]\, dx\, dy > 0,$$

and w is a solution of

$$-w_{xx} + w + \partial^{-2} w_{yy} - \frac{1}{p+1} w^{p+1} = 0 \qquad (x, y) \in \mathbb{R}^2.$$

Of course, w is independent of c. Simple calculations show that

$$d'(c) = V(\phi_c) = (c-1)^{(4-3p)/2p} \left(\frac{4+p}{2p} c - 1 \right) \frac{p}{2(p+2)} K(w) > 0$$

for $0<p<4$, and that

$$d''(c) = \frac{1}{4p^2} (4-p)((4+p)c - 4p)(c-1)^{(4-5p)/2p} \frac{p}{s(p+2)} K(w).$$

In consequence, we have

$$d''(c) > 0 \tag{5.6}$$

if and only if (1) or (2) holds. Condition (5.6) turns out to imply stability.

In dealing with the stability issues just raised, the following two lemmas will be helpful.

LEMMA 5.5. *Let $c>1$ and suppose $d''(c)>0$. There exists a $\delta>0$ such that if $|c_1-c|<\delta$ then*

$$d(c_1)>d(c)+d'(c)(c_1-c)+\tfrac{1}{4}d''(c)(c_1-c)^2.$$

Proof. The functionals E and V are C^∞-mappings of $V_1(\mathbb{R}^2)$ into $\mathbb{R}$. Similarly, the mapping $c\mapsto I_c$ is smooth from $\mathbb{R}$ into $C^\infty(V_1(\mathbb{R}^2);\mathbb{R})$, the class of smooth maps from $V_1(\mathbb{R}^2)$ to $\mathbb{R}$. Hence the value M_c varies smoothly with $c\in(1,\infty)$, and consequently d is a smooth function of $c>1$. Taylor's Theorem applied to d implies the advertised result. ∎

For $\varepsilon_0>0$ and $c>0$, define an ε_0-neighborhood of the set of solitary-waves of speed c to be

$$U_{c,\varepsilon_0}=\left\{u\in V_1(\mathbb{R}^2)\colon \inf_{\phi_c\in S_c}\|u-\phi_c\|_{V_1(\mathbb{R}^2)}<\varepsilon_0\right\}.$$

Suppose $0<p<4$ so that for $c>1$, $d'(c)>0$. The Implicit-Function Theorem implies that for each $c>1$ there corresponds an $\varepsilon_0>0$ and C^1-mapping $C: U_{c,\varepsilon_0}\to\mathbb{R}^+$ such that $C(\phi_c)=c$ and,

$$C(u)=d^{-1}\left(\frac{p}{2(p+2)}K(u)\right) \tag{5.7}$$

for all $u\in U_{c,\varepsilon_0}$. For any compact subset $[m,M]$ of $(1,\infty)$, the value of ε_0 may be chosen so that (5.7) is valid uniformly for $c\in[m,M]$ (cf. [13]).

LEMMA 5.6. *Suppose $d''(c)>0$ for some $c>1$. Then there exists $\varepsilon_0>0$ such that for any $u\in U_{c,\varepsilon_0}$ and $\phi_c\in S_c$*

$$E(u)-E(\phi_c)+C(u)(V(u)-V(\phi_c))\geqslant\tfrac{1}{4}d''(c)|C(u)-c|^2,$$

where $C(u)$ is defined in (5.7) *above.*

Proof. Because

$$K(u)=\frac{2(p+2)}{p}d(C(u))=\frac{2(p+2)}{p}[E(\phi_{C(u)})+C(u)V(\phi_{C(u)})].$$

and, for any u and c,

$$E(u)+cV(u)=\frac{1}{2}I_C(u)-\frac{1}{p+2}K(u)$$

It follows since $I_c(\phi_c) = K(\phi_c)$ that

$$K(u) = K(\phi_{C(u)}). \tag{5.8}$$

This implies that

$$I_{C(u)}(u) \geqslant I_{C(u)}(\phi_{C(u)})$$

because $\phi_{C(u)}$ is a minimizer of $I_{C(u)}$ subject to the constraint $K(u) = K(\phi_{C(u)})$. Lemma 5.5 implies

$$\begin{aligned} E(u) + C(u)V(u) &\geqslant \frac{1}{2} I_{C(u)}(\phi_{C(u)}) - \frac{1}{p+2} K(\phi_{C(u)}) = d(C(u)) \\ &\geqslant d(c) + d'(c)(C(u) - c) + \tfrac{1}{4} d''(c)(C(u) - c)^2 \\ &= E(\phi_c) + C(u)V(\phi_c) + \tfrac{1}{4} d''(c)(C(u) - c)^2, \end{aligned}$$

as advertised. ∎

Proof of Theorem 5.4. Arguing by contradiction, assume that S_c is $V_1(\mathbb{R}^2)$-unstable. This means there exists $\delta > 0$ and initial data $u_k(0) \in U_{c,(1/k)}$ and times $t_k > 0$, $k = 1, 2, \ldots$ such that

$$\inf_{\phi \in S_c} \|u_k(\cdot, t_k) - \phi\|_{V_1(\mathbb{R}^2)} = \delta. \tag{5.9}$$

Because the functionals E and V are continuous on $V_1(\mathbb{R}^2)$ and are conserved quantities for the RLW–KP flow, there are elements $\phi_k \in S_c$, $k = 1, 2, \ldots$, such that

$$|E(u_k(\cdot, t_k)) - E(\phi_k)| = |E(u_k(\cdot, 0)) - E(\phi_k)| \to 0 \tag{5.10}$$

as $k \to \infty$ and

$$|V(u_k(\cdot, t_k)) - V(\phi_k)| = |V(u_k(\cdot, 0)) - V(\phi_k)| \to 0 \tag{5.11}$$

as $k \to \infty$. Choose δ small enough so that Lemma 5.6 applies, which is to say

$$\begin{aligned} &E(u_k(\cdot, t_k)) - E(\phi_k) + C(u_k(\cdot, t_k))(V(u_k(\cdot, t_k)) - V(\phi_k)) \\ &\geqslant \tfrac{1}{4} d''(c)(C(u_k(\cdot, t_k)) - c)^2 \end{aligned} \tag{5.12}$$

for all $k = 1, 2, \ldots$. Observe that for any $k \geqslant 1$,

$$\begin{aligned}\|u_k(\cdot, t_k)\|_{V_1(\mathbb{R}^2)} &\leqslant \|\phi_k\|_{V_1(\mathbb{R}^2)} + 2\delta \\ &\leqslant \left(1 + \frac{1}{c} + \frac{1}{c-1}\right) I_c(\phi_c) + 2\delta \\ &= \left(1 + \frac{1}{c} + \frac{1}{c-1}\right) M_c^{(p+2)/p} + 2\delta < +\infty.\end{aligned}$$

Thus the sequence $\{u_k(\cdot, t_k)\}_{k=1}^{\infty}$ is uniformly bounded in $V_1(\mathbb{R}^2)$ It follows immediately that the collection $\{K(u_k(\cdot, t_k)\}_{k=1}^{\infty}$ is bounded and hence so are the values $\{C(u_k(\cdot, t_k))\}_{k=1}^{\infty}$ since d^{-1} is continuous. Combining this with (5.10)–(5.12) yields that

$$C(u_k(\cdot, t_k) \to c. \tag{5.13}$$

Relation (5.13) implies in turn that

$$\begin{aligned}\lim_{k\to\infty} K(u_k(\cdot, t_k)) &= \lim_{k\to\infty} \frac{2(p+2)}{p} d(C(u_k(\cdot, t_k))) \\ &= \frac{2(p+2)}{p} d(c).\end{aligned}$$

From the foregoing facts, it follows reading that

$$\begin{aligned}I_c(u_k(\cdot, t_k)) &= 2\Big(E(u_k) + cV(u_k)\Big) + \frac{2}{p+2} K(u_k(\cdot, t_k)) \to 2d(c) + \frac{4}{p} d(c) \\ &= \frac{2(p+2)}{p} d(c) \qquad \text{as } k \to \infty,\end{aligned}$$

which is to say

$$I_c(u_k(\cdot, t_k)) \to M_c^{(p+2)/p} = I_c(\phi_c).$$

Define w_k by

$$w_k = K(u_k(\cdot, t_k))^{-[1/(p+2)]} u_k(\cdot, t_k).$$

Then $K(w_k) = 1$ and

$$\begin{aligned}I_c(w_k) &= (K(u_k(\cdot, t_k)))^{-[2/(p+2)]} I_c(u_k(\cdot, t_k)) \\ &\to \frac{M_c^{(p+2)/p}}{(\frac{2(p+2)}{p} d(c))^{2/(p+2)}} = M_c^{(p+2)/p} M_c^{-(2/p)} = M_c.\end{aligned}$$

Thus, the sequence $\{w_k\}_{k=1}^{\infty}$ minimizes E subject to the constraint $V = 1$. In consequence of Theorem 5.1, there is a subsequence, still denoted $\{w_k\}_{k=1}^{\infty}$, and a corresponding sequence $\{\psi\}_{k=1}^{\infty} \subset V_1(\mathbb{R})$ such that

$$\lim_{k\to\infty} \|w_k - \psi_k\|_{V_1(\mathbb{R}^2)} = 0,$$

where $K(\psi_k) = 1$ for all k. This in turn implies that

$$\lim_{k\to\infty} \|u_k(\cdot, t_k) - \phi_k\|_{V_1(\mathbb{R}^2)} = 0$$

where $\phi_k = M_c^{1/p}\psi_k \in S_c$, which contradicts (5.9). ∎

6. EXTENSION TO SEVERAL SPACE DIMENSIONS

In this section, we offer commentary on an extension of the foregoing theory to several space dimensions. The extension we have in mind is the Cauchy problem

$$\begin{cases} (u_t + u_x + u^p u_x + D_x^\alpha u_t)_x + \varepsilon\Delta_{n-1} u = 0, \\ u(x, y_1, \dots, y_{n-1}, 0) = \phi(x, y_1, \dots, y_{n-1}), \end{cases} \tag{6.1}$$

where $\Delta_{n-1} = \partial_{y_1}^2 + \cdots + \partial_{y_{n-1}}^2$ and $n \geqslant 2$. Two conserved quantities in this case are

$$V(u) = \int_{\mathbb{R}^n} [u^2 + (D_x^{\alpha/2} u)^2]\, dx\, dy_1 \dots dy_{n-1}$$

and

$$E(u) = \int_{\mathbb{R}^n} \left[\frac{\varepsilon}{2}(\partial_x^{-1} u_{y_1})^2 + \cdots + \frac{\varepsilon}{2}(\partial_x^{-1} u_{y_{n-1}})^2 + \frac{u^2}{2} + \frac{u^{p+2}}{(p+1)(p+2)}\right].$$

Define $V_{\alpha/2}(\mathbb{R}^n)$ to be the space of functions f such that $f \in H_x^{\alpha/2}(\mathbb{R}^n)$ and $\partial_x^{-1}\nabla_y f \in L_2(\mathbb{R}^n)$. The associated semi-norm is

$$\begin{aligned} \|\partial_x^{-1}\nabla_y u\|_{L_2(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} |\partial_x^{-1}\nabla_y u|^2\, dx\, dy_1 \dots dy_{n-1} \\ &= \left(\int_{\mathbb{R}^n} [(\partial_x^{-1} u_{y_1})^2 + \cdots + (\partial_x^{-1} u_{y_{n-1}})^2]\right) dx\, dy_1 \dots dy_{n-1}. \end{aligned}$$

The basic inequality

$$\|u\|_{L_{p+2}(\mathbb{R}^n)}^{p+2} \leqslant c\|u\|_{L_2(\mathbb{R}^n)}^{2-(pn/\alpha)-(p/2)(n-3)} \|u\|_{H_x^{\alpha/2}(\mathbb{R}^n)}^{pn/\alpha} \|\partial_x^{-1} u_y\|_{L_2(\mathbb{R}^n)}^{(p/2)(n-1)} \tag{6.2}$$

, valid for $n \geq 2$, $\alpha \geq 2$ and $0 \leq p \leq \frac{4\alpha}{2(n-1)+(n-2)\alpha}$ (see [37–39]). As before, a necessary condition for a function f to be in the class $V_{\alpha/2}(\mathbb{R}^n)$ is that $f_{y_i} = g_{ix}$ for some functions $g_i \in L_2(\mathbb{R}^n)$ for $i = 1, \ldots, n-1$. It therefore follows that if initial data $\phi \in V_{\alpha/2}(\mathbb{R}^n)$ is imposed, then the conserved quantities $V(u)$ and $E(u)$ and inequality (6.2) imply that the associated solution u defined at least for $0 \leq t \leq T$ will lie in the space $L_\infty(0, T; V_{\alpha/2}(\mathbb{R}^n))$ if $p < [4/(n-1)]$. If $p = [4/(n-1)]$, the same conclusion holds if $\|\phi\|_{H^{\alpha/2}(\mathbb{R}^n)}$ is small.

This remark should be compared with a similar one for the initial-value problem

$$\begin{cases} (u_t + u^p u_x - D_x^\alpha u_x)_x - \Delta_{n-1} u = 0, \\ u(x, y_1, \ldots, y_{n-1}) = \phi(x, y_1, \ldots, y_{n-1}), \end{cases} \tag{6.3}$$

for the Kadomtsev–Petviashvili equation in several space dimensions. It was recently shown that if $\phi \in V_{\alpha/2}(\mathbb{R}^n)$, $n \geq 1$, then all solutions u belong to the class $L_\infty(\mathbb{R}^+; V_{\alpha/2}(\mathbb{R}^n))$ provided $p < [4\alpha/(2n + (n-1)\alpha)]$. (In the case $n = 1$, this result reduces to the result of Saut [32] for the existence of global solutions for generalizations of the Korteweg–de Vries equation of the form

$$u_t + u^p u_x - D_x^\alpha u_x = 0$$

with $p < 2\alpha$.) Since $4\alpha/[2n + (n-1)\alpha] < 4/(n-1)$ for $n \geq 2$, there seems to be an improvement in the range of values of p for the existence of global solutions for the RLW–KP-type equations over the KDV–KP type. Indeed, this difference is real and not just an artifact of the proof. It is known (see [33, 38] that smooth initial data ϕ leads to a local solution u of (6.3), but if $p \geq 4/(n-1)$ it can happen that the $H^1(\mathbb{R}^n)$-norm $\|u(\cdot, t)\|_{H^1(\mathbb{R}^n)} \to +\infty$ as $t \to t_* < +\infty$.

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