Contemporary Mathematics
Volume 371, 2005

WAVE GENERATION BY A MOVING BOUNDARY

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ABSTRACT. Considered here is an initial-boundary-value problem for a Korteweg-de Vries-type equation. The particular problem put forward involves a moving boundary condition and is argued to serve as a model for the generation of water waves by a piston- or flap-type wavemaker in a channel. An interesting feature of the problem is the appearance of a forced nonlinear oscillator equation (an Emden-Fowler-type equation) relating the motion of the wavemaker to the wave amplitude at the boundary. Another point of interest is a pair of higher order consistency conditions between initial and boundary data derived to insure solutions are classical. These conditions, and their obvious lower order counterparts, are automatically satisfied in a practically interesting configuration.

1. INTRODUCTION

The present study is concerned with the generation of waves by a wavemaker mounted at one end of a long, uniform, horizontal channel. Of particular interest are wavemaker motions that generate small-amplitude long waves corresponding to regimes in which the Korteweg-de Vries equation (KdV-equation henceforth) or its relatives might apply. Thus an idealized situation is envisioned in which the fluid is inviscid, the flow irrotational and uniform across the channel, so sensibly two-dimensional. We are motivated especially by the laboratory experiments of Zabusky & Galvin (1971), Hammack (1973), Hammack & Segur (1974) and Bona,

1991 *Mathematics Subject Classification.* Primary: 35G30, 35B40; Secondary: 74J30.

Key words and phrases. Nonlinear waves, moving boundary condition, existence and uniqueness, generalized and classical solutions.

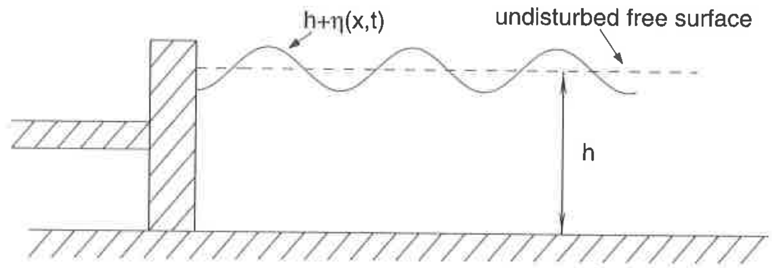

Fig. 1

Pritchard & Scott (1981). These experiments, which were designed explicitly to test the range of validity of KdV-type equations, all generally conform to the situation indicated in Figure 1.

Earlier analyses based on the just indicated physical situation have taken as a mathematical model the so-called quarter-plane problem, written here for the Korteweg-de Vries equation, namely

$$\begin{aligned} &\eta_t + \eta_x + \eta\eta_x + \eta_{xxx} = 0, \quad \text{for} \quad x, t > 0, \\ &\eta(x,0) = f(x), \quad \text{for} \quad x \geq 0, \\ &\eta(0,t) = g(t), \quad \text{for} \quad t \geq 0. \end{aligned} \tag{1.1}$$

This corresponds to a situation wherein measurements of the wave amplitude are made at a fixed station located down the channel from the wavemaker. The initial-boundary-value problem (1.1) and its relatives that include dissipation or are posed for the regularized long-wave or BBM-equation (Benjamin *et al.*, 1972)

$$u_t + u_x + uu_x - u_{xxt} = 0$$

(an alternative to the KdV-equation) have been analysed (see Bona & Bryant, 1973, Bona & Winther, 1983, 1989, and especially the extensive references in the recent paper of Bona, Sun & Zhang, 2002) and a satisfactory theory of global well-posedness in Hadamard's sense is available.

Our purpose here is to develop theory for this context based directly on the motion of the wavemaker rather than on an auxiliary measurement. Thus we presume to be given the motion of the wavemaker as represented by $\gamma(t)$ in Figure 1 and attempt to determine directly from this an approximation of the waves generated in the channel.

The plan of the paper is as follows. Section 2 is devoted to discussion of the modelling and the formal derivation of the differential equations governing the wave propagation and the boundary regime. The mathematical framework is specified in Section 3 which also deals with the issues of linear stability for the partial differential equation in question. An integral equation corresponding to the initial-boundary-value problem is analysed in Section 4 leading to a result of local well-posedness of the model. A priori bounds and global existence are developed in Section 5.

2. Modeling considerations, boundary behavior

We recall briefly some standard points arising in the derivation of model equations for small-amplitude long waves on the surface of water. The usual starting point for such studies is the full Euler equations for the two-dimensional motion of an ideal liquid in a uniform horizontal channel under the force of gravity. This classical system may be put in the form

$$\begin{aligned} &\beta\phi_{xx} + \phi_{yy} = 0, \quad \text{for} \quad (x,y) \in \Omega(t), \\ &\eta_t + \alpha\phi_x\eta_x - \frac{1}{\beta}\phi_y = 0, \qquad \text{at the free surface} \qquad y = 1 + \alpha\eta, \\ &\phi_t + \frac{1}{2}\left(\alpha\phi_x^2 + \frac{\alpha}{\beta}\phi_y^2\right) + g\eta = 0, \quad \text{at the free surface} \quad y = 1+\alpha\eta, \qquad (2.1) \\ &\phi_y = 0, \qquad \text{on the bottom} \quad y = 0. \end{aligned}$$

Here, x is the horizontal coordinate along the channel, y is the vertical coordinate, ϕ is the velocity potential, $1 + \alpha\eta(x,t)$ is the total water depth at the point (x,t) in space-time and $\Omega(t) = \{(x,y)|x \in I$ and $0 < y < 1 + \alpha\eta(x,t)\}$ is the flow domain at time t. The channel bottom is located at $y = 0$ and is flat and horizontal. The system (2.1) has already been non-dimensionalized and scaled, so that

$$\bar{x} = lx, \quad \bar{y} = h(y-1), \quad \bar{t} = \frac{lt}{c_0}, \quad \bar{\eta} = a\eta, \quad \bar{\phi} = g\frac{la}{c_0}\phi,$$

where l is a typical wavelength of the disturbance, h is the undisturbed depth, $c_0 = \sqrt{gh}$ is the kinematic velocity (the velocity of waves of extreme length), a is the maximum wave amplitude, g is the gravity constant and the bars adorn the original dimensional variables. Reflecting Boussinesq and KdV-type assumptions about the wave motion, the two nondimensional parameters

$$\alpha = \frac{a}{h} \qquad \text{and} \qquad \beta = \frac{h^2}{l^2}$$

are both assumed to be small compared to 1 but of the same order, so that the Stokes number

$$S = \frac{\alpha}{\beta} \tag{2.2}$$

is of order 1. One approach to simplifying the Euler system (2.1) (see Whitham, 1974) is to expand the velocity potential in the vertical coordinate, *viz.*

$$\phi(x, y, t) = \sum_{m=0}^{\infty} \phi_m(x, t) y^m. \tag{2.3}$$

Demanding that ϕ satisfy Laplace's equation in the flow domain and the no flow condition at the bottom boundary implies the formal expansion (2.3) to have the more specific form wherein $\phi_m = 0$ if m is odd and

$$\phi_{2k}(x, t) = \frac{(-1)^k \beta^k}{(2k)!} \partial_x^{2k} \phi_0(x, t) \tag{2.4}$$

for $k = 1, 2, \ldots$. Putting the representation (2.3)-(2.4) into the two free-surface boundary conditions in (2.1) and keeping only the terms of order 0 and 1 in powers of α and β leads to a coupled system of the form first put forward by Boussinesq (1872). In the particular representation (2.3)-(2.4), one comes to the system

$$\eta_t + [(1 + \alpha\eta)q]_x - \frac{\beta}{6} q_{xxx} = 0,$$

$$q_t + \eta_x + \alpha q q_x - \frac{\beta}{2} q_{xxt} = 0$$

where $q(x, t) = \phi_x(x, 0, t) = \partial_x \phi_0(x, t)$ is the horizontal velocity at the bottom and η is as before. By expressing the motion in terms of the horizontal velocity w at the height θh above the bottom, where $0 \le \theta \le 1$, and taking advantage of the lowest order relations

$$\eta_t + w_x = order(\alpha, \beta),$$

$$w_t + \eta_x = order(\alpha, \beta) \tag{2.5}$$

one may derive instead the three-parameter system

$$\eta_t + w_x + \alpha(w\eta)_x + \beta(a w_{xxx} - b\eta_{xxt}) = 0,$$

$$w_t + \eta_x + \alpha w w_x + \beta(c\eta_{xxx} - d w_{xxt}) = 0, \tag{2.6}$$

where

$$a = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda, \; b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1 - \lambda),$$

$$c = \frac{1}{2}\left(1-\theta^2\right)\mu, \; d = \frac{1}{2}\left(1-\theta^2\right)(1-\mu),$$

$0 \le \theta \le 1$, and λ, μ are real numbers (see Bona & Smith, 1976, Bona, Chen & Saut, 2002). The systems depicted in (2.6) are all formally equivalent descriptions of the two-dimensional motion of water waves in which dissipation, surface tension effects, compressibility and rotational aspects are ignored. A detailed mathematical analysis of this entire class appears in Bona *et al.* (2002, 2004).

A further simplification may be effected by restricting attention to waves travelling only in one direction, say in the direction of increasing values of x. Such an assumption is certainly warranted in attempting to describe the aforementioned experiments. To insure unidirectional propagation, a relation between the velocity and surface elevation is enforced. At the crudest level of approximation, when one ignores even terms of order α and β, the model simplifies to the one-dimensional linear wave equation written in factored form

$$\eta_t + w_x = 0,$$
$$w_t + \eta_x = 0$$

as in (2.5). Propagation in the direction of increasing values of x is implied exactly by the restriction $\eta = w$, in which case the model simplifies to basic hydraulics, namely

$$\eta_t + \eta_x = 0.$$

Reinterpreted in physical variables, this simply expresses the contention that very small, very long disturbances propagate at velocity $c_0 = \sqrt{gh}$, where recall that h is the undisturbed depth and g is the gravity constant. As is well known (see e.g. Benjamin (1974), Bona, Chen & Saut (2002), Whitham (1974)), the relation $\eta = w$ must be corrected at order α and β. The usual route from bidirectional systems to single equations governing approximately unidirectional waves is to derive a relationship between η and w which renders the system of two equations consistent and which reduces to $\eta = w$ when $\alpha = \beta = 0$. In the present variables, this amounts to the relation

$$w = \eta - \frac{\alpha}{4}\eta^2 - \beta\left(\frac{1}{3} - \frac{1}{2}\theta^2\right)\eta_{xt} + \text{terms quadratic in } \alpha,\ \beta. \tag{2.7}$$

Because of this, the lowest order relations (2.5) collapse as above to simply

$$\eta_t + \eta_x = order(\alpha, \beta).$$

Thus, the relation (2.7) could be given the equivalent form

$$w = \eta - \frac{\alpha}{4}\eta^2 + \beta\left(\frac{1}{3} - \frac{1}{2}\theta^2\right)\eta_{xx} + \text{terms quadratic in } \alpha,\ \beta, \tag{2.8}$$

or, indeed, any convex combination of these two relations. When one of these formally equivalent relations is enforced, the Boussinesq systems in (2.6) condense to a single equation for either η or w which has the form

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{\beta}{6}\left(\delta\eta_{xxx} - (1-\delta)\eta_{xxt}\right) = 0 \tag{2.9}$$

where δ is any real parameter.

It is reasonable to assume that the horizontal speed of the moving boundary coincides with the speed of the particles of fluid at the boundary (see Mei and Ünlüata, 1972), which is to say that

$$\eta - \frac{\alpha}{4}\eta^2 - \beta\left(\frac{1}{3} - \frac{1}{2}\theta^2\right)\eta_{xt}\Bigg|_{x=\gamma(t)} = \gamma'(t) \tag{2.10}$$

where $\gamma(t)$ is the position of the wavemaker at time t. Two important variables are the quantities g and h defined to be

$$g(t) = \eta(\gamma(t), t) \quad\text{and}\quad h(t) = \eta_x(\gamma(t), t). \tag{2.11}$$

These are the elevation and slope of the free surface at the wavemaker. It will be important in our analysis to determine a relation between $g(t)$ and $\gamma(t)$. For any dependent variable $f(x,t)$, denote by $\tilde{f}(t)$ the restriction $f(\gamma(t), t)$. Thus, equation (2.11) may be expressed as $g(t) = \tilde{\eta}(t)$ and $h(t) = \tilde{\eta}_x(t)$.

Attention is now given to determining approximate relationships between g and γ. Assuming that all the relevant functions are suitably differentiable, a straightforward calculation reveals that

$$g'(t) = \frac{d}{dt}\tilde{\eta}(t) = \eta_t(\gamma(t),t) + \gamma'(t)\eta_x(\gamma(t),t) = [\gamma'(t) - 1]\tilde{\eta}_x + h.o.t. \tag{2.12}$$

where $h.o.t.$ connotes terms of higher order in α and β. In consequence, it transpires that

$$\frac{d^2\tilde{\eta}(t)}{dt^2} = \tilde{\eta}_{tt} + 2\gamma'(t)\tilde{\eta}_{xt} + (\gamma'(t))^2\tilde{\eta}_{xx} + \gamma''(t)\tilde{\eta}_x$$

$$= -[1 - \gamma'(t)]^2\tilde{\eta}_{xt} + \frac{\gamma''(t)}{\gamma'(t) - 1}g'(t) + h.o.t. = g''(t).$$

Assuming that $|\gamma'(t)| < 1$, there obtains the relation

$$\widetilde{\eta}_{xt} = -\frac{\gamma''(t)g'(t)}{[1-\gamma'(t)]^3} - \frac{g''(t)}{[1-\gamma'(t)]^2} + h.o.t.$$

Substituting this relation into (2.10) and using (2.11), there emerges the second-order differential equation

(2.13)

$$\beta\left[g''(t) + \frac{\gamma''(t)}{1-\gamma'(t)}g'(t)\right] + \frac{[1-\gamma'(t)]^2}{1/3-\theta^2/2}\left[g(t) - \frac{\alpha}{4}g^2(t)\right] = \frac{\gamma'(t)[1-\gamma'(t)]^2}{1/3-\theta^2/2}$$

for g. Here, and henceforth, it is presumed that θ is chosen so that $\theta^2 \neq 2/3$. The latter case requires different treatment which will not concern us here. We seek a formal asymptotic solution of this equation in the form of an expansion in integer powers of the small parameters α and β, *viz.*

$$g(t) = \gamma'(t) + \alpha A(t) + \beta B(t) + O(\alpha^2 + \beta^2)$$

as α, $\beta \to 0$. Substituting the last formula into (2.13) and comparing coefficients of equal powers of α and β yields

$$g(t) = \gamma'(t) + \frac{\alpha}{4}(\gamma'(t))^2 - \beta\left(\frac{1}{3} - \frac{1}{2}\theta^2\right)\left\{\frac{\gamma'''}{[1-\gamma'(t)]^2} + \frac{(\gamma''(t))^2}{[1-\gamma'(t)]^3}\right\}$$
$$+O(\alpha^2+\beta^2).$$

Once g known, it is then straightforward to determine from (2.12) that

$$h(t) = \tilde{\eta}_x(t) = \frac{g'(t)}{\gamma'(t)-1} \tag{2.14}$$

$$= \frac{1}{\gamma'(t)-1}\frac{d}{dt}\left\{\gamma'(t) + \frac{\alpha}{4}(\gamma'(t))^2 - \beta\left(\frac{1}{3} - \frac{1}{2}\theta^2\right)\left[\frac{\gamma'''}{[1-\gamma'(t)]^2} + \frac{(\gamma''(t))^2}{[1-\gamma'(t)]^3}\right]\right\}$$
$$+O(\alpha^2+\beta^2).$$

The evolution equation (2.9) together with the relationship between γ and g will allow us to formulate a mathematical theory. Once (2.9) is derived, we may dispense with the small parameters α and β. The rescaling

$$u(x,t) = \frac{3}{2}\alpha\eta\left(\sqrt{\frac{\beta}{6}}x, \sqrt{\frac{\beta}{6}}t\right)$$

defines the function u satisfying the equation

$$u_t + u_x + uu_x + \delta u_{xxx} - (1-\delta)u_{xxt} = 0. \tag{2.15}$$

While the choice of $\delta \in \mathbf{R}$ is at our disposal, it will prove helpful to choose $\delta = -1$, thus arriving at the model

$$u_t + u_x - u_{xxx} - 2u_{xxt} + uu_x = 0.$$

This equation was listed in the family of third-order dispersive equations of the form

$$u_t + c_0 u_x + c_1 u_{xxx} - c_2^2 u_{xxt} = \left(c_3 u^2 + c_4 u_x^2 + c_5 u u_{xx}\right)_x \tag{2.16}$$

where c_i, $i = 0, \cdots, 5$, are real constants, studied by Degasperis & Procesi (1999) from the point of view of asymptotic inegrability. Only three equations in the class (2.16) were found to satisfy an asymptotic integrability condition within this family, namely the KdV equation, the Camassa-Holm equation (for which $c_3 = -3c_5/\left(2c_2^2\right)$, $c_4 = c_5/2$) and the equation obtained from (2.16) by setting $c_3 = -2c_5/c_2^2$, $c_4 = c_5$.

Equation (2.13) is conveniently rewritten to take account of the relation (2.2) between the parameters α and β and the Stokes number S. The new form is

$$\alpha\left[g''(t) + \kappa(t)g'(t)\right] + b(t)\left[g(t) - \frac{\alpha}{4}g^2(t)\right] = \gamma'(t)b(t), \tag{2.17}$$

where

$$\kappa(t) = \frac{\gamma''(t)}{1-\gamma'(t)}, \quad b(t) = \frac{S[1-\gamma'(t)]^2}{1/3 - \theta^2/2},$$

or, alternatively,

$$g''(t) + \kappa(t)g'(t) - b(t)\left(g(t) - \frac{2}{\alpha}\right)^2 = \frac{1}{\alpha}\left(\gamma'(t) - \frac{1}{\alpha}\right)b(t).$$

Notice that by introducing the new dependent variable $z(t) = g(t) - 2/\alpha$, equation (2.17) may be rewritten as a forced Emden-Fowler-type equation, namely

$$z''(t) + \kappa(t)z'(t) - \frac{b(t)}{4}z^2(t) = G(t) \tag{2.18}$$

where

$$G(t) = \frac{1}{\alpha}\left(\gamma'(t) - \frac{1}{\alpha}\right)b(t).$$

Since our interest is focused upon long waves of small amplitude, only small solutions of the equation (2.17) are of physical interest. A typical example of the wavemaker movement is described by the function $\gamma(t) = \varepsilon\sin(\varepsilon^{1/2}t)$. Thus we are led to pose the Cauchy problem

$$\varepsilon\left[g''(t) + \kappa(t,\varepsilon)g'(t)\right] + b(t,\varepsilon)\left[g(t) - \frac{\varepsilon}{4}g^2(t)\right] = \varepsilon^{3/2}\cos(\varepsilon^{1/2}t)b(t,\varepsilon), \ t > 0,$$

where

$$\kappa(t,\varepsilon) = -\frac{\varepsilon^2 \sin(\varepsilon^{1/2}t)}{1-\varepsilon^{3/2}\cos(\varepsilon^{1/2}t)}, \quad b(t,\varepsilon) = \frac{S\left[1-\varepsilon^{3/2}\cos(\varepsilon^{1/2}t)\right]^2}{1/3-\theta^2/2}$$

and

$$g(0) = \varepsilon^{3/2}, \ g'(0) = 0.$$

We are interested in small perturbations of the stationary solution $g(t) = 0$. Of course, there is another, realtively large stationary solution $g(t) = 4/\varepsilon$ which is not of interest in the present context. Introducing a new variable $\zeta = \varepsilon^{1/2}t$, the equation may be rewritten as

$$\varepsilon^2\frac{d^2g}{d\zeta^2} - \varepsilon^{7/2}\frac{\sin\zeta}{1-\varepsilon^{3/2}\cos\zeta}\frac{dg}{d\zeta} + b(\zeta,\varepsilon)\left(g - \frac{\varepsilon}{4}g^2\right) = \varepsilon^{3/2}b(\zeta,\varepsilon)\cos\zeta$$

where

$$b(\zeta,\varepsilon) = \frac{S(1-\varepsilon^{3/2}\cos\zeta)^2}{1/3-\theta^2/2}$$

and

$$g(0) = \varepsilon^{3/2}, \ \frac{dg}{d\zeta}(0) = 0.$$

Looking for a solution in the form

$$g(\zeta,\varepsilon) = g_0(\zeta) + \varepsilon^{1/2}g_1(\zeta) + \varepsilon g_2(\zeta) + \varepsilon^{3/2}g_3(\zeta) + \varepsilon^2 g_4(\zeta) + \cdots,$$

we find that

$$g_0(\zeta) = g_1(\zeta) = g_2(\zeta) = g_4(\zeta) = 0, \ g_3(\zeta) = \cos\zeta.$$

Returning to the original variables, there thus appears

$$g(t,\varepsilon) = \varepsilon^{3/2}\cos(\varepsilon^{1/2}t) + O(\varepsilon^2).$$

In any event, whether we accept the first non-zero term in the asymptotic solution just derived as adequate or we go ahead and integrate (2.17) with the given forcing $\gamma(t)$, the point is that the surface elevation $g(t)$ at the wavemaker is determined via separate considerations and may thus be viewed as independent auxuliary data that helps to determine the appropriate solution of the partial differential equation. As already pointed out, once g is determined, the second boundary function $h(t) = \eta_x(\gamma(t),t)$ is then specified via (2.14).

3. Preliminaries

In this section, further notation is introduced and some mathematical preliminaries are set forth.

Notation

The function spaces to be described now are all taken to be real-valued. For $T > 0$ the space $C(0,T)$ is the usual collection of continuous functions h defined on $[0,T]$ with the norm

$$\|h\|_{C(0,T)} = \sup_{0\le s\le T} |h(s)|\,.$$

The subspace $C^1(0,T)$ is the collection of continuously differentiable functions g defined on $[0,T]$ with

$$\|g\|_{C^1(0,T)} = \|g\|_{C(0,T)} + \|g'\|_{C(0,T)}\,,$$

and similarly for $C^k(0,T)$, $k = 2,3,\cdots$. The space $C_b^k(\mathbf{R}^+)$ is the space of bounded continuous functions f defined on $\mathbf{R}^+$ whose first k derivatives are likewise bounded and continuous. It is a Banach space with the norm

$$\|f\|_{C_b^n(\mathbf{R}^+)} = \sum_{k=0}^{n} \sup_{x\in\mathbf{R}^+} \left|f^{(k)}(x)\right|.$$

The standard symbol $H^k(\mathbf{R}^+)$ connotes the Sobolev space of measurable square integrable functions defined on $\mathbf{R}^+$ whose generalized derivatives to order k are also square integrable over $\mathbf{R}^+$. This is a Hilbert space with the usual inner product

$$\langle f,g\rangle_k = \sum_{j=0}^{k} \int_0^\infty f^{(j)}(x)g^{(j)}(x)dx.$$

The norm on $H^k(\mathbf{R}^+)$ is denoted by

$$\|f\|_{H^k(\mathbf{R}^+)} = \langle f,f\rangle_k^{1/2}.$$

The space $H^0(\mathbf{R}^+)$ is simply $L_2(\mathbf{R}^+)$.

We shall also need spaces of functions of two variables analogous to the one-dimensional spaces introduced above. For any Banach space X of functions of one variable x, say, $C(I;X)$ is the class of functions $u(x,t)$ of two variables such that the mapping $t \longrightarrow u(\cdot,t)$ is continuous from the closed interval I to X. This space carries the induced norm

$$\|u\|_{C(I;X)} = \sup_{t\in I} \|u(\cdot,t)\|_X\,.$$

In particular, for $T > 0$, we will systematically abbreviate $C([0,T]; C_b(\mathbf{R})) = C(0,T; C_b(\mathbf{R}))$ by C_T.

As a guide to what might be true for the nonlinear problem, we undertake an analysis of an associated linear initial-boundary-value problem, namely

$$\begin{aligned} u_t + u_x - u_{xxx} - 2u_{xxt} &= 0, \ x > 0, \ t > 0, \\ u(x,0) &= f(x), \ x > 0, \\ \lim_{x\to\infty} u(x,t) &= 0 \end{aligned} \tag{3.1}$$

with boundary conditions posed at the fixed point $x = 0$. This problem is analogous to the problems considered by Bona, Sun and Zhang (2002) and the references contained therein for the pure KdV-equation and in Bona and Tzvetkov (2004) and Bona, Chen, Sun and Zhang (2004) and the references mentioned there for the BBM-equation. Interest is focussed on what boundary conditions are needed at $x = 0$ to determine a solution. Following Bona, Sun and Zhang (2002), this issue is approached by reducing (3.1) to an ordinary differential equation in the variable x. Applying the Laplace transform

$$\widehat{u}(x,p) = \int_0^\infty e^{-pt} u(x,t)dt$$

in the variable t leads to

$$\widehat{u}_{xxx} + 2p\widehat{u}_{xx} - \widehat{u}_x - p\widehat{u} = f''(x) - f(x).$$

Seeking solutions of the corresponding homogeneous equation in the form $e^{\lambda x}$, there obtains the characteristic equation

$$\lambda^3 + 2p\lambda^2 - \lambda - p = 0. \tag{3.2}$$

Descarte's Theorem implies that, for all $p > 0$, there exists one positive root $\lambda_1(p)$ of this equation. Changing λ to $-\lambda$ yields

$$-\lambda^3 + 2p\lambda^2 + \lambda - p = 0.$$

This equation can have either zero or two positive roots. Therefore (3.2) has either two or no negative roots. Making the change of variable $\lambda = z - 2p/3$ reduces (3.2) to the canonical form

$$z^3 + 3Pz + Q = 0 \tag{3.3}$$

where

$$P = -\frac{1}{3}\left(\frac{4p^2}{3} + 1\right) \text{ and } Q = \frac{p}{3}\left(\frac{16p^2}{9} - 1\right).$$

The discriminant $D = 4P^3 + Q^2$ is calculated to be

$$D = -\frac{32p^4 + 19p^2 + 12}{27} < 0,$$

and this means that the cubic equation (3.3) always has three real roots for $p > 0$. Consequently, equation (3.2) also has three real roots, at least one of which is postive. As the sum of the roots of (3.2) is $-2p < 0$, it follows that they cannot all three be positive and therefore there are two negative roots $\lambda_2(p)$ and $\lambda_3(p)$. In any event, a solution of the homogeneous equation may be written in the form

$$\widehat{u}(x,p) = C_1(p)\exp\left(-\lambda_1(p)x\right) + C_2(p)\exp\left(\lambda_2(p)x\right) + C_3(p)\exp\left(\lambda_3(p)x\right).$$

Since it is demanded that u be bounded, and $\lambda_1(p) > 0$, we must require that $C_1(p) \equiv 0$. This leaves $C_2(p)$ and $C_3(p)$ to be determined and since $\lambda_2(p) < 0$ and $\lambda_3(p) < 0$, these functions are conveniently specified by boundary conditions at $x = 0$. Thus we see that two boundary conditions are required to specify the solution of the linear, fixed-domain problem. This calculation is helpful in understanding what might be expected when considering the full nonlinear problem with conditions imposed at a moving boundary.

4. Statement of the problem

Our purpose now is to consider the initial-boundary-value problem

$$u_t + u_x - u_{xxx} - 2u_{xxt} + uu_x = 0, \qquad x > \gamma(t),\ t > 0,$$

$$u(\gamma(t),t) = g(t),\ u_x(\gamma(t),t) = h(t), \qquad t > 0, \tag{4.1}$$

$$u(x,0) = f(x), \qquad x > \gamma(0),$$

$$\lim_{x\to\infty} u(x,t) = 0,$$

along with the consistency conditions

$$f(\gamma(0)) = g(0), \qquad f'(\gamma(0)) = h(0)$$

(see Figure 2). Without loss of generality, we suppose $\gamma(0) = 0$ henceforth. It is assumed throughout that the function $\gamma(t)$ describing the movement of the boundary satisfies the assumptions: $\gamma(t) \in C^1(0,\infty)$, $\gamma(0) = 0$ and $-\gamma_0 \le \gamma'(t) \le \gamma_1 < 1/2$ with $\gamma_0,\ \gamma_1 > 0$ and $\gamma_0 < \infty$. At this point, we take it that g and h are known functions.

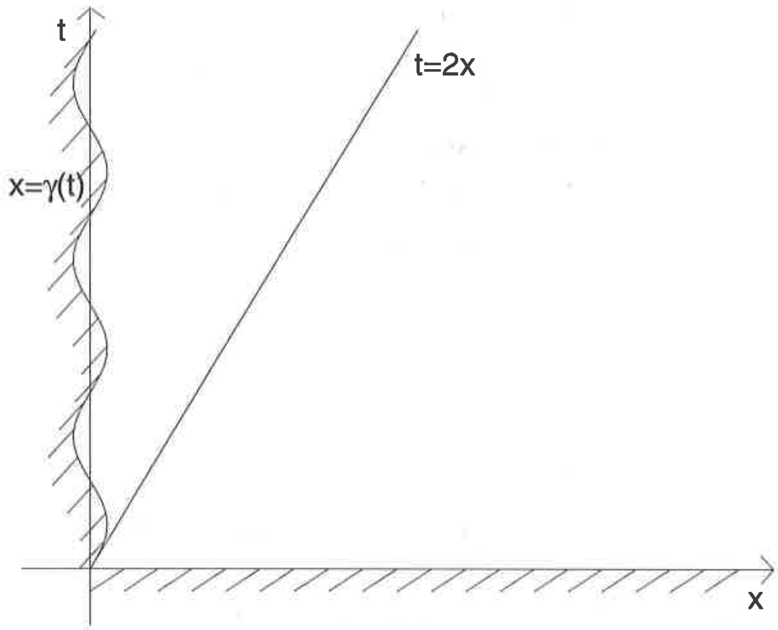

Fig. 2

Definition. A function $u(x,t)$ will be termed a *classical solution* of (4.1) if u and its partial derivatives included in the equation are continuous in the domain $\Omega_T = \{(x,t) : x > \gamma(t),\ 0 < t < T\}$, for some $T > 0$, and u satisfies the initial and boundary conditions and the partial differential equation pointwise in Ω_T.

4.1 Local existence

First, by formal operations, problem (4.1) is reduced to an associated integral equation. It is propitious first to make the change of variable

$$u(x,t) = v\left(x - \gamma(t), t/2 - \gamma(t)\right) \tag{4.2}$$

and use the notation

$$\tau = \frac{t}{2} - \gamma(t). \tag{4.3}$$

Since by assumption

$$\tau'(t) = \frac{1}{2} - \gamma'(t) > 0,$$

there exists a unique solution $t = \varphi(\tau)$ of equation (4.3). Also, $\tau(0) = 0$ and $\tau(t) \to +\infty$ as $t \to +\infty$. Thus, this is a legitimate change of variables and the function v, which is defined on $\mathbf{R}^+ \times [0,T]$ for some

$T > 0$, is seen to satisfy the equation

$$\begin{gathered}
\left(1-2\partial_\xi^2\right)\left(v_\tau+v_\xi\right)=-\frac{1}{2a(\tau)}\partial_\xi\left(v+v^2\right), \quad \text{for} \quad \xi>0, \tau>0, \\
v(0,\tau)=g(\varphi(\tau)),\ v_\xi(0,\tau)=h(\varphi(\tau)), \quad \text{for} \quad \tau>0, \\
v(\xi,0)=f(\xi), \quad \text{for} \quad \xi>0, \\
\lim_{\xi\to+\infty} v(\xi,\tau)=0
\end{gathered} \tag{4.4}$$

where

$$a(\tau)=\tau'(t)\bigg|_{t=\varphi(\tau)}=\frac{1}{2}-\gamma'(t)\bigg|_{t=\varphi(\tau)}.$$

Introducing the function $\widetilde{V}=v_\tau+v_\xi$, (4.4) becomes

$$\begin{gathered}
\left(1-2\partial_\xi^2\right)\widetilde{V}=-\frac{1}{2a(\tau)}\partial_\xi\left(v+v^2\right), \quad \text{for} \quad \xi>0, \tau>0, \\
\widetilde{V}(0,\tau)=\frac{d}{d\tau}g(\varphi(\tau))+h(\varphi(\tau))=F(\tau), \quad \text{for} \quad \tau>0, \\
\lim_{\xi\to+\infty}\widetilde{V}(\xi,\tau)=0.
\end{gathered}$$

Inverting $1-2\partial_\xi^2$ subject to the specified boundary conditions at $\xi=0$ and $\xi=+\infty$ leads to the integro-differential equation

$$\begin{aligned}
&\widetilde{V}(y,\tau)=\exp(-\xi/\sqrt{2})F(\tau) \\
&-\frac{\sqrt{2}}{4a(\tau)}\int_0^\infty\left[\exp(-|\xi-y|/\sqrt{2})-\exp(-(\xi+y)/\sqrt{2})\right]\partial_y(v+v^2)(y,\tau)dy.
\end{aligned} \tag{4.5}$$

Integrating by parts in (4.5) and recalling the definition of $\widetilde{V}$, there obtains

$$\begin{gathered}
v_\tau+v_\xi=\exp(-\xi/\sqrt{2})F(\tau) \\
+\frac{1}{4a(\tau)}\int_0^\infty K(\xi,y)\,(v+v^2)(y,\tau)dy, \quad \text{for} \quad \xi>0, \tau>0, \\
v(0,\tau)=g(\varphi(\tau)),\ v_\xi(0,\tau)=h(\varphi(\tau)), \quad \text{for} \quad \tau>0, \\
v(\xi,0)=f(\xi),\ \xi>0, \\
\lim_{\xi\to+\infty}v(\xi,\tau)=0,
\end{gathered} \tag{4.6}$$

where

$$K(\xi,y)=\exp(-|\xi-y|/\sqrt{2})\mathrm{sgn}(\xi-y)-\exp(-(\xi+y)/\sqrt{2}).$$

Introducing the characteristic variables $\chi = \xi - \tau,\ \tau = \tau$ maps the quarter plane $\{(\xi, t) : \xi > 0,\ t > 0\}$ into the domain $\{-\tau < \chi < +\infty,\ \tau > 0\}$. In (χ, τ), the differential operator on the left-hand side of the equation in (4.6) is a full derivative and thus (4.6) becomes

$$\begin{aligned} &\partial_\tau v(\chi+\tau,\tau) = \exp(-(\chi+\tau)/\sqrt{2})F(\tau) \\ (4.7)\quad &+\frac{1}{4a(\tau)}\int_0^\infty K(\chi+\tau,y)(v+v^2)(y,\tau)dy, \quad \text{for} \quad \chi > -\tau,\ \tau > 0, \\ &v(\chi+\tau,\tau)\Big|_{\tau=0} = f(\chi), \quad \text{for} \quad \chi \geq 0, \\ &v(\chi+\tau,\tau)\Big|_{\tau=-\chi} = g(\varphi(-\chi)), \quad \text{for} \quad \chi < 0. \end{aligned}$$

It is natural at this point to integrate (4.7) with respect to τ. The contour of integration is the segment of the line parallel to the τ-axis and connecting the points $(\chi, 0)$ and (χ, τ) when $\chi \geq 0$ and the points $(-\tau, \tau)$ and (χ, τ) when $\chi < 0$. Performing this integration and reverting to the variables (ξ, τ) leads to the formula

$$\begin{aligned} &v(\xi,\tau) = f(\xi-\tau) + e^{-\xi/\sqrt{2}}\int_0^\tau e^{(\tau-s)/\sqrt{2}}F(s)ds \\ (4.8)\quad &+\frac{1}{4}\int_0^\tau \frac{ds}{a(s)}\int_0^\infty M(\xi,\tau,y,s)(v+v^2)(y,s)dy, \quad \text{for} \quad \xi \geq \tau > 0, \end{aligned}$$

and

$$\begin{aligned} &v(\xi,\tau) = g(\varphi(\tau-\xi)) + e^{-\xi/\sqrt{2}}\int_{\tau-\xi}^\tau e^{(\tau-s)/\sqrt{2}}F(s)ds \\ (4.9)\quad &+\frac{1}{4}\int_{\tau-\xi}^\tau \frac{ds}{a(s)}\int_0^\infty M(\xi,\tau,y,s)(v+v^2)(y,s)dy, \quad \text{for} \quad 0 < \xi < \tau \end{aligned}$$

where

$$\begin{aligned} M(\xi,\tau,y,s) = &\exp\left(-\left|\xi-y-(\tau-s)\right|/\sqrt{2}\right)\operatorname{sgn}\left(\xi-y-(\tau-s)\right) \\ &+\exp\left(-\left(\xi+y-(\tau-s)\right)/\sqrt{2}\right) \end{aligned}$$

and, as above,

$$F(\tau) = \frac{d}{d\tau}g(\varphi(\tau)) + h(\varphi(\tau)).$$

This pair of relations is written in shorthand notation as

(4.10)

$$v = Av = A(f,g,h,\gamma)v = \begin{cases} f(\xi-\tau) + F_1(\xi,\tau) + B_1 v, & 0 < \tau \le \xi, \\ g(\varphi(\tau-\xi)) + F_2(\xi,\tau) + B_2 v, & 0 < \xi < \tau, \end{cases}$$

where F_1 and F_2 denote the second terms and $B_1 v$ and $B_2 v$ denote the third terms on the right-hand side of (4.8) and (4.9), respectively.

Lemma 1. *Let $T > 0$ be given and suppose that γ, $g \in C^1(0,T)$, $h \in C(0,T)$ and $f \in C_b(\mathbf{R}^+)$. Suppose also that $-\gamma_0 \le \gamma'(t) \le \gamma_1 < 1/2$ in the interval $0 \le t \le T$ for some constants γ_0 and γ_1. Then there is an S with $0 \le S \le T$, depending on the norms of γ, g, h, f, γ_0 and γ_1, such that the integral equation (4.8)-(4.9)-(4.10) has a unique solution in C_S. Moreover, the mapping that assigns to initial- and boundary data (f,g,h,γ) the solution u in C_S is locally Lipschitz continuous.*

Proof. View A as a mapping of the space C_S, where $S \le T$ will be specified presently. Note that F is continuous since φ is continuous. Let u, $v \in C_S$. If $(\xi,\tau) \in \mathbf{R}^+ \times [0,S]$ is such that $\xi \ge \tau$, then (4.8) applies and it is straightforwardly deduced that

$$(4.11)\quad |Au(\xi,\tau) - Av(\xi,\tau)| \le C_a(S)||u-v||_{C_S}(1 + ||u||_{C_S} + ||v||_{C_S}).$$

In deriving (4.11), we have used the inequality

$$\sup_{\xi-\tau\ge 0} \int_0^\infty |M(\xi,\tau,y,s)|\,dy \le 4\sqrt{2}$$

and the notation

$$C_a(S) = \sqrt{2}\int_0^S \frac{ds}{a(s)}$$

where $a(\tau) > 0$. Note that $C_a(S)$ is an increasing function of S and that there are constants C_a^0 and C_a^1 for which $SC_a^0 \le C_a(S) \le SC_a^1$.

If, instead, $\tau > \xi \ge 0$, then (4.9) applies and exactly the same inequality as in (4.11) is seen to hold. Taking the supremum of (4.11) and its counterpart when $\tau > \xi$, for $\xi \in \mathbf{R}^+$ and $\tau \in [0,S]$ yields

$$(4.12)\qquad ||Au - Av||_{C_S} \le C_a(S)||u-v||_{C_S}(1 + ||u||_{C_S} + ||v||_{C_S}).$$

Let $\theta(\xi,\tau)$ be the zero function in C_S. After an integration by parts in the temporal variable, (4.8) and (4.9) may be rewritten as

$$v(\xi,\tau) = f(\xi-\tau) + e^{-\xi/\sqrt{2}}g(\varphi(\tau)) - e^{-(\tau-\xi)/\sqrt{2}}g(0)$$

$$(4.13) \qquad +e^{-(\xi-\tau)/\sqrt{2}} \int_0^\tau e^{-s/\sqrt{2}} \mathcal{F}(s) ds + B_1 v \quad \text{for} \quad \xi \geq \tau > 0$$

and
(4.14)

$$v(\xi,\tau) = e^{-\xi/\sqrt{2}} g(\varphi(\tau)) + e^{-(\xi-\tau)/\sqrt{2}} \int_{\tau-\xi}^{\tau} e^{-s/\sqrt{2}} \mathcal{F}(s) ds + B_2 v \quad \text{for} \quad 0 < \xi < \tau$$

where

$$\mathcal{F}(s) = \frac{g(\varphi(s))}{\sqrt{2}} + h(\varphi(s)).$$

In consequence of these formulas, it transpires that

$$(4.15) \qquad ||A\theta||_{C_S} \leq \|f\|_{C_b(\mathbf{R}^+)} + 3\,\|g\|_{C(0,S)} + \|h\|_{C(0,S)} = r(S),$$

say. Hence if $u,\, v \in C_S$ and $||u||_{C_S},\ ||v||_{C_S} \leq R$, then

$$\|Av\|_{C_S} \leq \|Av - A\theta\|_{C_S} + \|A\theta\|_{C_S}$$

$$\leq C_a(S)\,\|v\|_{C_S}\left(1 + \|v\|_{C_S}\right) + r(S)$$

$$\leq C_a(S) R(1+R) + r(S)$$

and

$$\|Au - Av\|_{C_S} \leq C_a(S)(1+2R)\,\|u - v\|_{C_S}.$$

Thus, if R is chosen to be $2r(S)$, then since $C_a(S) \to 0$ as $S \to 0$ and $r(S)$ is decreasing with S, there are positive values of S such that

$$(4.16) \qquad C_a(S)\,(1 + 4r(S)) \leq 1/2.$$

For such a determination of R and S, the operator A is a contraction mapping of the ball B_R of radius R about θ in C_S. Thus A has a fixed point in C_S.

The local Lipschitz continuity of the solution mapping $\mathrm{Q} : C_b(\mathbf{R}^+) \times C^1(0,T) \times C(0,T) \times C^1(0,T) \to C_S$ defined by

$$\mathrm{Q}(f,g,h,\gamma) = u$$

follows since u is obtained via the contraction mapping principle. In a little more detail, let (f,g,h,γ) and $\left(\widetilde{f},\widetilde{g},\widetilde{h},\widetilde{\gamma}\right)$ be given and let $A(f,g,h,\gamma)$ and $\widetilde{A} = A\left(\widetilde{f},\widetilde{g},\widetilde{h},\widetilde{\gamma}\right)$ be the corresponding mappings as defined in (4.8)-(4.9)-(4.10). Let $S_0 \leq \min\left\{S, \widetilde{S}\right\}$, where S and $\widetilde{S}$ are determined by (4.16) for their respective quantities C_a and r. The difference $u - \widetilde{u}$ can be estimated as follows:

$$\|u-\widetilde{u}\|_{C_{S_0}} = \left\|Au - \widetilde{A}\widetilde{u}\right\|_{C_{S_0}} \leq \|Au - A\widetilde{u}\|_{C_{S_0}} + \left\|A\widetilde{u} - \widetilde{A}\widetilde{u}\right\|_{C_{S_0}}$$

$$\leq \frac{1}{2}\|u-\widetilde{u}\|_{C_{S_0}} + \left\|f - \widetilde{f}\right\|_{C_b(\mathbf{R}^+)} + 3\|g-\widetilde{g}\|_{C(0,S)} + \left\|h - \widetilde{h}\right\|_{C(0,S)}.$$

This completes the proof of Lipschitz continuity and of Lemma 1. □

A direct consequence of the contraction mapping principle is the following corollary.

Corollary. *Let γ, f, g and h be as in Lemma 1, and let*

$$v_0(\xi,\tau) = A\theta(\xi,\tau) = \begin{cases} f(\xi-\tau) + F_1(\xi,\tau), & \text{if} \quad 0 < \tau \leq \xi, \\ g(\varphi(\tau-\xi)) + F_2(\xi,\tau), & \text{if} \quad 0 < \xi < \tau. \end{cases}$$

Then the Picard sequence $v_n(\xi,\tau)$ defined by the formula

$$v_n(\xi,\tau) = Av_{n-1}(\xi,\tau) = v_0(\xi,\tau) + \begin{cases} B_1 v_{n-1}(\xi,\tau), & \text{for} \quad \xi \geq \tau > 0, \\ B_2 v_{n-1}(\xi,\tau), & \text{for} \quad 0 < \xi < \tau. \end{cases}$$

converges in C_S to the unique solution v of (4.8)-(4.9) in the ball $\|v\|_{C_S} \leq R$ (the operators B_1 and B_2 are defined in (4.10)).

Lemma 2. *Let $\gamma,\ g,\ h \in C^2([0,T])$, $f \in C^3(\mathbf{R}^+) \cap C_b^3(\mathbf{R}^+)$, $-\gamma_0 \leq \gamma'(t) \leq \gamma_1 < 1/2$, $\gamma(0) = 0$, and let $t = \varphi(\tau)$, where $\varphi(0) = 0$, be the unique solution of equation (4.3). Moreover, assume that the following consistency conditions hold:*

$$f(0) = g(0),\ f'(0) = h(0), \tag{4.17}$$

$$f''(0) = \frac{1}{\sqrt{2}}\left[g'(0)\varphi'(0) + h(0)\right] + h'(0)\varphi'(0) + \frac{g(0)\left[1+g(0)\right]}{4\varphi'(0)}, \tag{4.18}$$

$$f'''(0) = \frac{1}{\sqrt{2}}\left[g''(0)\left(\varphi'(0)\right)^2 + g'(0)\varphi''(0) + h'(0)\varphi'(0)\right]$$
$$+\frac{1}{2}\left[g'(0)\varphi'(0) + h(0)\right] + h''(0)\left(\varphi'(0)\right)^2 + h'(0)\varphi''(0)$$
$$-\frac{\varphi''(0)g(0)\left[1+g(0)\right]}{4\left(\varphi'(0)\right)^2} + \left(\frac{h(0)}{4\varphi'(0)} + \frac{g'(0)}{4}\right)\left[1+2g(0)\right] \tag{4.19}$$

where

$$\varphi'(0) = \frac{1}{1/2 - \gamma'(0)}, \qquad \varphi''(0) = \frac{\gamma''(0)}{\left[1/2-\gamma'(0)\right]^3}.$$

Then any solution $v \in C_S$ of (4.10) is a classical solution of the initial-boundary-value problem (4.4).

Remark. The first two conditions (4.17) are transparent, but (4.18) and (4.19) appear complicated. In fact, they are automatically satisfied for the practically interesting case where the liquid is quiescent at $t = 0$ and the wavemaker actually starts to move smoothly at a later time $t_0 > 0$ (see Zabusky & Galvin (1971), Hammack (1973), Hammack & Segur (1974), Bona, Pritchard & Scott, (1981)). For, in this case,

$$f(0) = f'(0) = f''(0) = f'''(0) = 0,$$

$$g(0) = h(0) = 0$$

and

$$\varphi'(0) = 2, \qquad \varphi''(0) = 0.$$

Moreover, referring back to Section 2, in these circumstances

$$h'(0) = -g'(0) = -g''(0) = 0,$$

whether g is determined from the differential equation (2.13) or from an asymptotic expression for g. Thus all the consistency conditions are trivially met.

Proof. Setting $\tau = 0$ in (4.8) and $\xi = 0$ in (4.9) leads to the conclusion that $v(\xi, 0) = f(\xi)$, for $\xi > 0$, and $v(0, \tau) = g(\varphi(\tau))$, for any $\tau > 0$. To verify the second boundary condition, differentiate equation (4.9) with respect to ξ and then set $\xi = 0$. For $0 < \xi < \tau$, this gives

$$\begin{aligned} v_\xi(\xi,\tau) = &-\frac{1}{\sqrt{2}} \exp((\tau-\xi))/\sqrt{2}) \int_{\tau-\xi}^{\tau} \exp(-s/\sqrt{2}) F(s) ds \\ &+ h(\varphi(\tau-\xi)) + \frac{1}{4}\int_{\tau-\xi}^{\tau} \frac{1}{a(s)} \left(v+v^2\right)(\xi-(\tau-s),s)ds \\ &- \frac{1}{\sqrt{2}}\frac{1}{4}\int_{\tau-\xi}^{\tau} \frac{ds}{a(s)} \int_0^\infty \Big[\exp\left(-\left|\xi-y-(\tau-s)\right|/\sqrt{2}\right) \\ &+ \exp\left(-\left(\xi+y-(\tau-s)\right)/\sqrt{2}\right)\Big] \left(v+v^2\right)(y,s)dy. \end{aligned} \tag{4.20}$$

In consequence one sees readily that, $v_\xi(0, \tau) = h(\varphi(\tau))$, for $0 \le \tau \le S$. Also, it is easy to check using (4.8) and (4.9) that, thanks to the first consistency condition (4.17), the function $v(\xi, \tau)$ is continuous on the line segment $L = \{(\xi,\tau) \mid \xi = \tau,\ 0 \le \tau \le S\}$, which is to say, $v(\tau+0,\tau) = v(\tau-0,\tau)$. A similar calculation using (4.8) yields

$$v_\xi(\xi,\tau) = f'(\xi-\tau) - \frac{1}{\sqrt{2}}\exp(-(\xi-\tau)/\sqrt{2})\int_0^\tau \exp(-s/\sqrt{2})F(s)ds$$
$$-\frac{1}{\sqrt{2}}\frac{1}{4}\int_0^\tau \frac{ds}{a(s)}\int_0^\infty \Big[+\exp\Big(-|\xi-y-(\tau-s)|/\sqrt{2}\Big)$$
$$+\exp\Big(-(\xi+y-(\tau-s))/\sqrt{2}\Big)\Big](v+v^2)(y,s)dy$$
$$+\frac{1}{4}\int_0^\tau \frac{1}{a(s)}\left(v+v^2\right)(\xi-(\tau-s),s)ds. \tag{4.21}$$

provided $\xi > \tau$. Therefore,

$$v_\xi(\tau+0,\tau) = f'(0) - \frac{1}{\sqrt{2}}\int_0^\tau \exp(-s/\sqrt{2})F(s)ds$$
$$-\frac{1}{\sqrt{2}}\frac{1}{4}\int_0^\tau \frac{ds}{a(s)}\int_0^\infty \Big[\exp\Big(-|y-s|/\sqrt{2}\Big)+\exp\Big(-(y+s)/\sqrt{2}\Big)\Big]$$
$$\times\left(v+v^2\right)(y,s)dy + \frac{1}{4}\int_0^\tau \frac{1}{a(s)}\left(v+v^2\right)(s,s)ds.$$

On the other hand, (4.20) yields

$$v_\xi(\tau-0,\tau) = h(\varphi(0)) - \frac{1}{\sqrt{2}}\int_0^\tau \exp(-s/\sqrt{2})F(s)ds$$
$$-\frac{1}{\sqrt{2}}\frac{1}{4}\int_0^\tau \frac{ds}{a(s)}\int_0^\infty \Big[\exp\Big(-|y-s|/\sqrt{2}\Big)+\exp\Big(-(y+s)/\sqrt{2}\Big)\Big]$$
$$\times\left(v+v^2\right)(y,s)dy + \frac{1}{4}\int_0^\tau \frac{1}{a(s)}\left(v+v^2\right)(s,s)ds.$$

Hence, due to the second consistency condition (4.17), the jump across the line segment L vanishes, *viz.*

$$[v_\xi]_L \equiv v_\xi(\tau+0,\tau) - v_\xi(\tau-0,\tau) = f'(0) - h(\varphi(0)) = 0.$$

In the region $\xi > \tau$, $0 \le \tau \le S$, (4.8) implies

$$v_\tau(\xi,\tau) = -f'(\xi-\tau) + \frac{1}{\sqrt{2}}\exp(-(\xi-\tau)/\sqrt{2})\int_0^\tau \exp(-s/\sqrt{2})F(s)ds$$
$$+\exp(-\xi/\sqrt{2})F(\tau) + \frac{1}{4}\frac{1}{a(\tau)}\int_0^\infty \Big[\exp\Big(-|\xi-y|/\sqrt{2}\Big)\operatorname{sgn}(\xi-y)$$
$$+\exp\Big(-(\xi+y)/\sqrt{2}\Big)\Big]\left(v+v^2\right)(y,\tau)dy$$

$$+\frac{1}{\sqrt{2}}\frac{1}{4}\int_0^\tau \frac{ds}{a(s)}\int_0^\infty \left[\exp\left(-\left|\xi-y-(\tau-s)\right|/\sqrt{2}\right)\right.$$

$$\left.+\exp\left(-\left(\xi+y-(\tau-s)\right)/\sqrt{2}\right)\right]\left(v+v^2\right)(y,s)dy$$

$$-\frac{1}{4}\int_0^\tau \frac{1}{a(s)}\left(v+v^2\right)(\xi-(\tau-s),s)ds. \tag{4.22}$$

In the domain $\xi<\tau$, $0\le\tau\le S$, there obtains

$$v_\tau(\xi,\tau)=g'\left(\varphi(\tau-\xi)\right)\cdot\varphi'(\tau-\xi)+\frac{1}{\sqrt{2}}\exp((\tau-\xi)/\sqrt{2})\int_{\tau-\xi}^\tau \exp(-s/\sqrt{2})F(s)ds$$

$$+\exp(-\xi/\sqrt{2})F(\tau)-F(\tau-\xi)$$

$$+\frac{1}{4}\frac{1}{a(\tau)}\int_0^\infty\left[\exp\left(-\left|\xi-y\right|/\sqrt{2}\right)\operatorname{sgn}(\xi-y)\right.$$

$$\left.+\exp\left(-(\xi+y)/\sqrt{2}\right)\right]\left(v+v^2\right)(y,\tau)dy$$

$$+\frac{1}{\sqrt{2}}\frac{1}{4}\int_{\tau-\xi}^\tau \frac{ds}{a(s)}\int_0^\infty \left[\exp\left(-\left|\xi-y-(\tau-s)\right|/\sqrt{2}\right)\right.$$

$$\left.+\exp\left(-\left(\xi+y-(\tau-s)\right)/\sqrt{2}\right)\right]\left(v+v^2\right)(y,s)dy$$

$$-\frac{1}{4}\int_0^\tau \frac{\left(v+v^2\right)(\xi-(\tau-s),s)}{a(s)}ds. \tag{4.23}$$

Calculating the jump of $v_\tau(\xi,\varphi(\tau))$ on the line segment L, it is seen that

$$[v_\tau]_L=-f'(0)+h(0)=0.$$

The formulas (4.21)-(4.22) yield, for the entire strip $\xi\ge0$, $0\le\tau\le S$

$$v_\tau+v_\xi=\exp(-\xi/\sqrt{2})F(\tau)+\frac{1}{4}\frac{1}{a(\tau)}\int_0^\infty\left[e^{-|\xi-y|/\sqrt{2}}\operatorname{sgn}(\xi-y)+e^{-(\xi+y)/\sqrt{2}}\right]$$

$$\times\left(v+v^2\right)(y,\tau)dy. \tag{4.24}$$

Differentiation of (4.20) for $0<\xi<\tau$, $0\le\tau\le S$ leads to

$$v_{\xi\xi}=-\frac{1}{\sqrt{2}}\left[g'\left(\varphi(\tau-\xi)\right)\cdot\varphi'(\tau-\xi)+h\left(\varphi(\tau-\xi)\right)\right]-h'\left(\varphi(\tau-\xi)\right)\varphi'(\tau-\xi)$$

$$+\frac{g\left(\varphi(\tau-\xi)\right)+g^2\left(\varphi(\tau-\xi)\right)}{4a(\tau-\xi)}+\frac{1}{2}\exp((\tau-\xi))/\sqrt{2})\int_{\tau-\xi}^\tau \exp(-s/\sqrt{2})F(s)ds$$

$$+\frac{1}{4}\int_{\tau-\xi}^\tau \frac{1}{a(s)}\partial_\xi\left(v+v^2\right)(\xi-(\tau-s),s)ds$$

$$-\frac{1}{4\sqrt{2}}\int_{\tau-\xi}^{\tau}\frac{ds}{a(s)}\int_0^\infty dy\left[\exp\left(-\left|\xi-y-(\tau-s)\right|/\sqrt{2}\right)\operatorname{sgn}\left(\xi-y-(\tau-s)\right)\right.$$

$$\left.+\exp\left(-\left(\xi+y-(\tau-s)\right)/\sqrt{2}\right)\right]\times\left(v+v^2\right)(y,s). \tag{4.25}$$

On the other hand, differentiation of (4.21) yields for $\xi > \tau$, $0 \le \tau \le S$

$$v_{\xi\xi} = f''(\xi-\tau)+\frac{1}{2}\exp(-(\xi-\tau)/\sqrt{2})\int_0^\tau \exp(-s/\sqrt{2})F(s)ds$$

$$+\frac{1}{4}\int_0^\tau\frac{1}{a(s)}\partial_\xi\left(v+v^2\right)(\xi-(\tau-s),s)ds$$

$$-\frac{1}{4\sqrt{2}}\int_0^\tau\frac{ds}{a(s)}\int_0^\infty dy\left[\exp\left(-\left|\xi-y-(\tau-s)\right|/\sqrt{2}\right)\operatorname{sgn}\left(\xi-y-(\tau-s)\right)\right.$$

$$\left.+\exp\left(-\left(\xi+y-(\tau-s)\right)/\sqrt{2}\right)\right]\times\left(v+v^2\right)(y,s). \tag{4.26}$$

Therefore, due to the consistency condition (4.18) the jump of $v_{\xi\xi}$ on the line segment L is zero. Indeed, (4.25) and (4.26) imply that

$$[v_{\xi\xi}]_L \equiv v_{\xi\xi}\left(\tau+0,\varphi(\tau)\right)-v_{\xi\xi}\left(\tau-0,\varphi(\tau)\right)$$

$$= f''(0)+\frac{1}{\sqrt{2}}\left[g'(0)\varphi'(0)+h(0)\right]+h'(0)\varphi'(0)-\frac{g(0)\left[1+g(0)\right]}{4\varphi'(0)} = 0.$$

Now we are prepared to compute the third derivatives $v_{\xi\xi\xi}$ and $v_{\xi\xi\tau}$. For $0 < \xi < \tau$, $0 \le \tau \le S$ differentiation of (4.25) yields

$$v_{\xi\xi\xi} = -\frac{1}{2\sqrt{2}}\exp((\tau-\xi))/\sqrt{2})\int_{\tau-\xi}^\tau \exp(-s/\sqrt{2})F(s)ds+\frac{1}{2}F(\tau-\xi)$$

$$-\frac{1}{\sqrt{2}}\partial_\xi F(\tau-\xi)+h''\left(\varphi(\tau-\xi)\right)\left(\varphi'(\tau-\xi)\right)^2+h'\left(\varphi(\tau-\xi)\right)\varphi''(\tau-\xi)$$

$$+\frac{a'(\tau-\xi)g(\varphi(\tau-\xi))\left[1+\left(g(\varphi(\tau-\xi))\right)^2\right]}{4\left(a(\tau-\xi)\right)^2}$$

$$-\frac{g'(\varphi(\tau-\xi))\varphi'(\tau-\xi)\left[1+2g(\varphi(\tau-\xi))\right]}{4a(\tau-\xi)}$$

$$+\frac{\left(v_\xi+2vv_\xi\right)(0,(\varphi(\tau-\xi))}{4a(\tau-\xi)}+\frac{1}{4}\int_0^\tau\frac{1}{a(s)}\partial_\xi^2\left(v+v^2\right)(\xi-(\tau-s),s)ds$$

$$-\frac{1}{8}\int_{\tau-\xi}^\tau\frac{ds}{a(s)}\int_0^\infty dy\left[\exp\left(-\left|\xi-y-(\tau-s)\right|/\sqrt{2}\right)\right.$$

$$\left.+\exp\left(-\left(\xi+y-(\tau-s)\right)/\sqrt{2}\right)\right]\times\left(v+v^2\right)(y,s)$$

$$-\frac{1}{4\sqrt{2}}\int_{\tau-\xi}^{\tau}\frac{1}{a(s)}\left(v+v^2\right)(\xi-(\tau-s),s)ds, \tag{4.27}$$

and for $\xi > \tau$, $0 \le \tau \le S$ differentiation of (4.26) leads to

$$\begin{aligned} v_{\xi\xi\xi} = f'''(\xi-\tau) - \frac{1}{2\sqrt{2}}\exp(-(\xi-\tau)/\sqrt{2})\int_0^{\tau}\exp(-s/\sqrt{2})F(s)ds \\ +\frac{1}{4}\int_0^{\tau}\frac{1}{a(s)}\partial_{\xi}^2\left(v+v^2\right)(\xi-(\tau-s),s)ds \\ -\frac{1}{8}\int_0^{\tau}\frac{ds}{a(s)}\int_0^{\infty}dy\left[\exp\left(-\left|\xi-y-(\tau-s)\right|/\sqrt{2}\right)\right. \\ \left.+\exp\left(-\left(\xi+y-(\tau-s)\right)/\sqrt{2}\right)\right]\times\left(v+v^2\right)(y,s) \\ -\frac{1}{4\sqrt{2}}\int_0^{\tau}\frac{1}{a(s)}\left(v+v^2\right)(\xi-(\tau-s),s)ds. \end{aligned} \tag{4.28}$$

Therefore, according to the compatibility condition (4.19), $v_{\xi\xi\xi}(\xi,\varphi(\tau))$ is continuous on the line segment L since

$$\begin{aligned} \left[v_{\xi\xi\xi}\right]_L \equiv v_{\xi\xi\xi}(\tau+0,\tau) - v_{\xi\xi\xi}(\tau-0,\tau) \\ = f'''(0) - \frac{1}{\sqrt{2}}\left[g''(0)\left(\varphi'(0)\right)^2 + g'(0)\varphi''(0) + h'(0)\varphi'(0)\right] \\ -\frac{1}{2}\left[g'(0)\varphi'(0)+h(0)\right] - h''(0)\left(\varphi'(0)\right)^2 - h'(0)\varphi''(0) \\ +\frac{\varphi''(0)g(0)\left[1+g(0)\right]}{4\left(\varphi'(0)\right)^2} - \left(\frac{h(0)}{4\varphi'(0)}+\frac{g'(0)}{4}\right)\left[1+2g(0)\right] = 0. \end{aligned}$$

Next, we calculate the mixed derivative $v_{\xi\xi\tau}$ and show that it is continuous on L thanks to the same consistency condition (4.19). Indeed, for $0 < \xi < \tau$, $0 \le \tau \le S$, it is straigthforwardly ascertained that

$$\begin{aligned} v_{\xi\xi\tau} = -\frac{1}{\sqrt{2}}\left[g''\left(\varphi(\tau-\xi)\right)\cdot\left(\varphi'(\tau-\xi)\right)^2 + g'(\varphi(\tau-\xi))\cdot\varphi''(\tau-\xi)\right. \\ \left.+h'(\varphi(\tau-\xi))\cdot\varphi'(\tau-\xi)\right] \\ -h''(\varphi(\tau-\xi))\cdot\left(\varphi'(\tau-\xi)\right)^2 - h'(\varphi(\tau-\xi))\cdot\varphi''(\tau-\xi) \\ +\frac{a'(\tau-\xi)}{4\left(a(\tau-\xi)\right)^2}\frac{g(\varphi(\tau-\xi)) + \left(g(\varphi(\tau-\xi))\right)^2}{g(\varphi(\tau-\xi))} \\ -\frac{1}{4a(\tau-\xi)}\frac{g'(\varphi(\tau-\xi))\varphi'(\tau-\xi)\left[1+2g(\varphi(\tau-\xi))\right]}{g(\varphi(\tau-\xi))} \\ +\frac{1}{2\sqrt{2}}\exp(-(\xi-\tau))/\sqrt{2})\int_{\tau-\xi}^{\tau}\exp(-s/\sqrt{2})F(s)ds \end{aligned}$$

$$+\frac{1}{2}e^{-\xi/\sqrt{2}}F(\tau)-\frac{1}{2}F(\tau-\xi)$$

$$+\frac{1}{4}\left[\frac{\partial_\xi\left(v+v^2\right)(\xi,\tau)}{a(\tau)}-\frac{\partial_\xi\left(v+v^2\right)(0,\tau-\xi)}{a(\tau-\xi)}\right]$$

$$+\frac{1}{4}\int_{\tau-\xi}^{\tau}\frac{1}{a(s)}\partial^2_{\xi\tau}\left(v+v^2\right)(\xi-(\tau-s),s)ds$$

$$-\frac{1}{4\sqrt{2}a(\tau)}\int_0^\infty\left[e^{-|\xi-y|/\sqrt{2}}\text{sgn}(\xi-y)+e^{-(\xi+y)/\sqrt{2}}\right]\left(v+v^2\right)(y,\tau)dy$$

$$+\frac{1}{8}\int_{\tau-\xi}^{\tau}\frac{ds}{a(s)}\int_0^\infty dy\left[\exp\left(-|\xi-y-(\tau-s)|/\sqrt{2}\right)\right.$$

$$\left.+\exp\left(-(\xi+y-(\tau-s))/\sqrt{2}\right)\right]\times\left(v+v^2\right)(y,s)$$

$$+\frac{1}{4\sqrt{2}}\int_{\tau-\xi}^{\tau}\frac{\left(v+v^2\right)(\xi-(\tau-s),s)}{a(s)}ds. \tag{4.29}$$

For $\xi>\tau$, $0\le\tau\le S$ the result is

$$v_{\xi\xi\tau}=-f'''(0)++\frac{1}{2\sqrt{2}}\exp(-(\xi-\tau))/\sqrt{2})\int_{\tau-\xi}^{\tau}\exp(-s/\sqrt{2})F(s)ds$$

$$+\frac{1}{2}e^{-\xi/\sqrt{2}}F(\tau)+\frac{1}{4}\frac{\partial_\xi\left(v+v^2\right)(\xi,\tau)}{a(\tau)}$$

$$+\frac{1}{4}\int_0^{\tau}\frac{1}{a(s)}\partial^2_{\xi\tau}\left(v+v^2\right)(\xi-(\tau-s),s)ds$$

$$-\frac{1}{4\sqrt{2}a(\tau)}\int_0^\infty\left[e^{-|\xi-y|/\sqrt{2}}\text{sgn}(\xi-y)+e^{-(\xi+y)/\sqrt{2}}\right]\left(v+v^2\right)(y,\tau)dy$$

$$+\frac{1}{8}\int_0^{\tau}\frac{ds}{a(s)}\int_0^\infty dy\left[e^{-|\xi-y-(\tau-s)|/\sqrt{2}}+e^{-(\xi+y-(\tau-s))/\sqrt{2}}\times\left(v+v^2\right)(y,s)\right.$$

$$+\frac{1}{4\sqrt{2}}\int_0^{\tau}\frac{\left(v+v^2\right)(\xi-(\tau-s),s)}{a(s)}ds. \tag{4.30}$$

The formulas (4.29) and (4.30) allow to verify that the jump of $v_{\xi\xi\tau}$ on the line segment L is zero due to (4.19). Thus, the last consistency condition guarantees the continuity of both $v_{\xi\xi\xi}$ and $v_{\xi\xi\tau}$ in the entire strip $\xi>0$, $0<\tau<S$.

Finally, calculating $v_{\xi\xi\xi} + v_{\xi\xi\tau}$ for $\xi > 0,\ 0 < \tau < S$ on the basis of (4.27), (4.28), (4.29) and (4.30) and taking into account (4.24) shows that the equation

$$v_\tau + v_\xi - 2\left(v_{\xi\xi\xi} + v_{\xi\xi\tau}\right) = -\frac{1}{2a(\tau)}\partial_\xi\left(v + v^2\right)$$

is satisfied in $\xi > 0,\ 0 < \tau < S$ in the sense of continuous functions. □

Remarks. 1. Further regularity of γ, f, g and h results in further regularity of the solution v. This follows by a straightforward continuation of the arguments put forth in Lemma 2. A precise statement is provided in Theorem 4.

2. If $f(\xi) \to 0$ as $\xi \to +\infty$, then the solution v has the same property for all $\tau \in [0, S]$. This follows from the following two observations. First, if $\overset{\circ}{C}_S \subset C_S$ is the subset of functions $v \in C_S$ such that $v(\xi,\tau) \to 0$ as $\xi \to +\infty$ for each $\tau \in [0, S]$, then $\overset{\circ}{C}_S$ is a closed subspace of C_S. Second, if $v \in \overset{\circ}{C}_S$, then $Av \in \overset{\circ}{C}_S$. This is a consequence of the representation (4.13) for $\xi \geq \tau > 0$, where the kernel $M(\xi,\tau,y,s)$ decays exponentially as $\xi \to \infty$ and so lies in $L_1(\mathbf{R}^+)$. Once these two points are appreciated, the stated result follows from the Corollary to Lemma 1.

3. If instead of assuming $f \in C_b(\mathbf{R}^+)$, it is supposed that $f \in H^1(\mathbf{R}^+) \subset C_b(\mathbf{R}^+)$, it follows that the solution v of (4.8)-(4.9)-(4.10) lies in $C(0, S; H^1)$. This follows readily from the analog of (4.12) in an H^1-setting. Similarly, if $f \in H^k(\mathbf{R}^+)$ for $k > 1$ and γ, g and h are suitably more regular, then the solution v will lie in $C\left(0, S; H^k\right)$.

4. In all the above scenarios, the solution depends continuously on the data. This follows immediately because the solution is obtained by way of the contraction mapping principle.

5. If we happen to know that the solution v corresponding to auxiliary data γ, f, g, and h remains bounded on any bounded time interval for which it exists, and that g, h, and γ are defined on all $\mathbf{R}^+$, it follows by standard arguments and the local existence theorem that v may be uniquely extended to a globally defined solution, that is, a function $v : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}$ that solves (4.4) everywhere in $(0,\infty) \times (0,\infty)$.

Once v is determined as in Lemma 2, the solution u of the moving boundary problem (4.1) is obtained from the transformation (4.2). The following theorem about (4.1) thereby emerges.

Theorem 3. *Let γ, g, h and f be as in Lemma 2. Then there is a $T > 0$ and a unique function u defined on Ω_T which is a classical solution of the boundary-value problem (4.1).*

The remarks following the proof of Lemma 2 all have obvious counterparts for (4.1) that follow directly from the change of variables (4.2).

5. A PRIORI BOUNDS

In this section, an "energy-type" relation is derived which allows one to extend the local solvability result for (4.1) to arbitrary time intervals $[0,T]$. To begin, define U by the relation

$$u(x,t) = \overline{u}(x,t) + U(x,t) \tag{5.1}$$

where

$$\overline{u}(x,t) = [g(t) + (x-\gamma(t))\,h(t)]\,\overline{\omega}(x,t)$$

and

$$\overline{\omega}(x,t) = \exp\left(-\left(x-\gamma(t)\right)^2\right).$$

Note that $\overline{\omega}(\gamma(t),t) = 1$ and $\overline{\omega}_x(\gamma(t),t) = 0$, so that the function $\overline{u}(x,t)$ satisfies the boundary conditions in (4.1), and the function $U(x,t)$ therefore satisfies the homogeneous boundary conditions $U(\gamma(t),t) = 0$, $U_x(\gamma(t),t) = 0$. Also, $U(x,t) \to 0$ as $x \to \infty$ if the same is true of u since the functions $g(t)$ and $h(t)$ are bounded on bounded time intervals.

Substituting (5.1) into equation (4.1) yields an equation for U, namely

$$\overline{u}_t + U_t + \overline{u}_x + U_x - (\overline{u}+U)_{xxx} - 2(\overline{u}+U)_{xxt} + \frac{1}{2}\left[(\overline{u}+U)^2\right]_x = 0,$$

which may be rewritten in the form

$$U_t + (1+\overline{u})U_x + UU_x + \overline{u}_x U - U_{xxx} - 2U_{xxt} = \bar{f}(x,t) \tag{5.2}$$

where

$$\bar{f}(x,t) = -\overline{u}_t - \overline{u}_x - \overline{u}\overline{u}_x + \overline{u}_{xxx} + 2\overline{u}_{xxt}.$$

Note that the functions $\overline{u}_t$ and $\overline{u}_{xxt}$ include the first derivatives of the boundary data g and h.

Presume for the moment that the auxiliary data γ, g and h are in C_b^3, say, and that $f \in H^k(\mathbf{R}^+)$ for some $k \geq 4$. This implies that the functions $v(x,t) = u\left(x-\gamma(t), t/2-\gamma(t)\right)$ and $V(x,t) = U\left(x-\gamma(t), t/2-\gamma(t)\right)$ lie in $C_b^3(\mathbf{R}^+ \times [0,T])$ and, for each $t \in [0,T]$, that they lie in $H^k(\mathbf{R}^+)$. This level of regularity more than suffices to justify the formal calculations that appear below. Once the final inequalities are obtained, the continuous dependence results come to our aid and it is adduced that the bounds continue to hold for weaker levels of regularity of the data.

Multiplying the equation (5.2) by U, integrating with respect to x over $[\gamma(t), \infty)$ for each $t \in [0, T]$, and integrating by parts in x leads to

$$\frac{1}{2}\frac{d}{dt}\int_{\gamma(t)}^{\infty} \left(|U|^2 + 2\,|U_x|^2\right) dx + \frac{1}{2}\int_{\gamma(t)}^{\infty} \overline{u}_x\, |U|^2\, dx = \int_{\gamma(t)}^{\infty} \bar{f} U dx. \tag{5.3}$$

Integrating the last relation in the temporal variable from 0 to t and applying the elementary inequality

$$|ab| \le \frac{1}{2}\left(a^2 + b^2\right)$$

to the last term on the right-hand side of the result yields

$$\frac{1}{2}\int_{\gamma(t)}^{\infty} \left(|U|^2 + 2\,|U_x|^2\right) dx + \frac{\kappa_1}{2}\int_0^t d\tau \int_{\gamma(\tau)}^{\infty} |U|^2\, dx$$

$$\le C_1 + \frac{1}{2}\int_0^t d\tau \int_{\gamma(\tau)}^{\infty} |U|^2\, dx + \frac{1}{2}\int_0^t d\tau \int_{\gamma(\tau)}^{\infty} \left|\bar{f}(x,\tau)\right|^2 dx \tag{5.4}$$

where

$$\kappa_1 = \inf_{(x,t)\in\Omega_T} \overline{u}_x(x,t), \quad \Omega_T = \mathbf{R}^+ \times [0,T],$$

$$C_1 = \frac{1}{2}\int_0^{\infty} \left(|U|^2 + 2\,|U_x|^2\right)(x,0) dx,$$

$$U(x,0) = f(x) - [g(0) + xh(0)] \exp\left(-x^2\right),$$

$$U_x(x,0) = f'(x) - \left[(1-2x^2)h(0) - 2xg(0)\right] \exp\left(-x^2\right),$$

and

$$\left|\bar{f}(x,t)\right|^2 = \overline{u}_t^2 + \overline{u}_x^2 - \left(\overline{u}\overline{u}_x\right)^2 + \overline{u}_{xxx}^2 + 4\overline{u}_{xxt}^2$$
$$+2\left(\overline{u}_t\overline{u}_x + \overline{u}_t\overline{u}_x\overline{u} + \overline{u}_x^2\overline{u} + \overline{u}_t\overline{u}_{xxx} + 2\overline{u}_t\overline{u}_{xxt}\right).$$

Since

$$\overline{u}_x = h(t)\left(1-2\xi^2\right)e^{-\xi^2} - 2g(t)\xi e^{-\xi^2} = h(t)f_1(\xi) - g(t)f_2(\xi),$$

where $f_1(\xi)$ and $f_2(\xi)$ are bounded independently of $\xi \in \mathbf{R}^+$ and $g(t)$ and $h(t)$ are bounded for $t \in [0,T]$, it is clear that κ_1 is finite, though of course it need not be positive. Dropping the nonnegative term $\int_{\gamma(t)}^{\infty} |U_x|^2\, dx$ on the left-hand side of (5.4) and letting $\kappa = \kappa_1 - 1$, there obtains the inequality

$$\int_{\gamma(t)}^{\infty} |U|^2\, dx + \kappa \int_0^t d\tau \int_{\gamma(\tau)}^{\infty} |U|^2\, dx \le 2C_1 + \bar{F}(t),$$

where

$$\bar{F}(t) = \int_0^t d\tau \int_{\gamma(\tau)}^{\infty} \left|\bar{f}(x,\tau)\right|^2 dx.$$

Introducing the notation

$$\bar{U}(t) = \int_0^t d\tau \int_{\gamma(\tau)}^{\infty} |U(x,\tau)|^2 \, dx,$$

the last inequality may be written

$$\bar{U}'(t) + \kappa \bar{U}(t) \leq 2C_1 + \bar{F}(t), \qquad \text{for} \quad t \in [0,T],$$

with

$$\bar{U}(0) = 0.$$

Gronwall's lemma thus implies

$$\bar{U}(t) \leq 2C_1 \frac{1-e^{-\kappa t}}{\kappa} + \int_0^t \exp\left(-\kappa(t-\tau)\right) \bar{F}(\tau) d\tau. \tag{5.5}$$

Note that the constant κ in (5.5) is not positive, so this estimate does not yield other than an exponential growth of $\bar{U}(t)$ with time. However, combining (5.4) with (5.5) allows one to conclude that

$$\|U(\cdot,t)\|_{H^1(\mathbf{R}^+)}$$

is uniformly bounded for all $t \in [0,T]$. It follows at once that $u(\cdot,t)$ is bounded on $[0,T]$ in $H^1(\mathbf{R}^+)$ since

$$\|u(\cdot,t)\|_{H^1(\mathbf{R}^+)} \leq \|\overline{u}(\cdot,t)\|_{H^1(\mathbf{R}^+)} + \|U(\cdot,t)\|_{H^1(\mathbf{R}^+)}. \tag{5.6}$$

Theorem 4. *Let $T > 0$ be arbitrary and suppose $f \in H^1(\mathbf{R}^+) \cap C_b^{k-1}(\mathbf{R}^+)$ and γ, g, $h \in C^k(0,T)$ where $k \geq 2$. Then there exists a unique solution u of the integral equation (4.8)-(4.9)-(4.10) such that $v(x,t) = u(x-\gamma(t), t/2-\gamma(t)))$ lies in $C(0,T; H^1(\mathbf{R}^+))$ and in $C\left(0,T; C_b^{k-1}(\mathbf{R}^+)\right)$.*

Moreover, the mapping that associates to (f, g, h, γ) the solution u is a locally Lipschitz mapping from $H^1(\mathbf{R}^+) \times C^1(0,T) \times C^1(0,T) \times C^1(0,T)$ to the space X_γ^1 where

$$X_\gamma^j = \Big\{ v : \Omega_T \hookrightarrow \mathbf{R} : \quad v(\cdot,t) \in H^1((\gamma(t),\infty)) \cap C_b^{k-1}((\gamma(t),\infty))$$
$$0 \leq t \leq T \text{ and the correspondence } t \hookrightarrow v(x-\gamma(t),t)$$
$$\text{is continuous from } [0,T] \text{ to } H^j(\mathbf{R}^+) \Big\},$$

$j = 0, 1, 2, \cdots$.

Proof. Existence. By a straightforward iteration of the existence proof of Lemma 1, we infer that there is an increasing sequence of times $\{T_k\}_{k=0}^{\infty}$

over which the solution in the small can be extended. The terms of this sequence are determined by (4.16) to be

$$T_k = T_{k-1} + \frac{1}{C_a^1 \left[2 + 8r\left(T_k\right)\right]},$$

where

$$r\left(T_k\right) = \sup_{\xi \geq 0} |u(\xi, T_{k-1})| + 3\, \|g\|_{C(0,T_k)} + \|h\|_{C(0,T_k)}\,.$$

Of course,

$$3\, \|g\|_{C(0,T_k)} + \|h\|_{C(0,T_k)} \leq 3\, \|g\|_{C(0,T)} + \|h\|_{C(0,T)}$$

and the latter quantity is bounded. Moreover it is elementary that

$$\sup_{\xi \geq \gamma(T_{k-1})} |u(\xi, T_{k-1})| \leq \|u(\cdot, T_k)\|_{H^1(\gamma(T_{k-1}),\infty)}$$

and the latter quantity is bounded a priori on $[0, T]$ according to the last proposition. In consequence, $r\left(T_k\right)$ is bounded independently of $T_k \in [0, T]$, from which one deduces a positive lower bound on the difference $T_k - T_{k-1}$. This in turn means the solution can be extended to $[0, T]$ by a finite number of applications of the contraction mapping argument.

Uniqueness. Assume that on the time interval $[0, T]$ there exist two solutions u_1 and u_2 of the problem (4.1) in the space X_γ^1. This corresponds to existence of two solutions v_1 and v_2 in the space $C(0, T; H^1)$. Let $w = u_1 - u_2$ and correspondingly $W = U_1 - U_2$. Then (5.5) immediately implies that $W = 0$ a.e. on $[0, T]$ whence $u_1 = u_2$.

Continuous dependence follows from the usual procedure of approximating the initial and boundary data by smooth functions, making the calculations for the associated smooth solutions and then using the continuous dependence theory to obtain the final result for less smooth data.

6. Conclusion

Developed herein is a theory for water waves generated by a wavemaker which is at the same KdV- or Boussinesq-level of approximation as in earlier works (e.g., Bona & Bryant, 1973). However, the present work relies, for the initiation of the waves, upon direct measurement of the wavemaker motion rather than an auxiliary measurement taken downstream. A natural further development would be to create a numerical scheme for the approximation of solutions of the initial-boundary-value problem (4.1) together with a scheme to solve the Emden-Fowler type equation (2.17), (2.18). This is the first step toward determining the quantitative predictive power of the conception put forward here. Such a program will be the subject of a future work.

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