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COMPARISON OF QUARTER-PLANE AND TWO-POINT BOUNDARY VALUE PROBLEMS: THE BBM-EQUATION

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Abstract. The focus of the present study is the BBM equation which models unidirectional propagation of small amplitude long waves in shallow water and other dispersive media. Interest will be turned to the two-point boundary value problem wherein the wave motion is specified at both ends of a finite stretch of the medium of propagation. The principal new result is an exact theory of convergence of the two-point boundary value problem to the quarter-plane boundary value problem in which a semi-infinite stretch of the medium is disturbed at its finite end. The latter problem has been featured in modeling waves generated by a wavemaker in a flume and in describing the evolution of long crested, deep water waves propagating into the near shore zone of large bodies of water. In addition to their intrinsic interest, our results provide justification for the use of the two-point boundary value problem in numerical studies of the quarter plane problem.

1. **Introduction.** Considered here are small amplitude long waves on the surface of an ideal fluid of finite depth over a featureless, horizontal bottom under the force of gravity. When such wave motion is long crested, it may propagate essentially in, say, the x-direction and without significant variation in the y-direction of a standard xyz-Cartesian frame in which gravity acts in the negative z-direction. For such waves, the full three-dimensional Euler equations can be reduced to approximate models featuring only one independent spatial variable. Such models go back at least to

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the middle of the 19^{th} century and are included in works by Airy (1845) and Stokes (1847) in the first half of the century. The model featured in the present study has its roots in the work of Boussinesq (1871), (1872), (1877) and later, Korteweg and de Vries (1895). More detailed historical accounts and derivations can be found in modern works (eg. Bona, Chen and Saut 2002, Miura 1976, Whitham 1974).

It suffices for describing the issue at hand to remind the reader that if x denotes the coordinate in the direction of propagation and h_0 the undisturbed depth, then the crucial dependent variable is $\eta(x,t) = h(x,t) - h_0$ where t is proportional to elapsed time and $h(x,t)$ is the depth of the water column over the spatial point x at time t. It is assumed that the waves propagate in the direction of increasing values of x, that the amplitude a of the waves is small compared to the undisturbed depth h_0, that typical wavelengths λ of the motion are long compared to h_0, so

$$\alpha = \frac{a}{h_0} << 1 \quad \text{and} \quad \beta = \frac{h_0}{\lambda} << 1,$$

and that the Stokes number

$$S = \frac{\alpha}{\beta^2} = \frac{a\lambda^2}{h_0^3} \tag{1.1}$$

is of order one. The latter presumption implies a balance is struck between nonlinear and dispersive effects. Under these assumptions, the non-dimensionalized evolution equations

$$\eta_t + \eta_x + \eta\eta_x + \eta_{xxx} = 0 \tag{1.2}$$

and

$$\eta_t + \eta_x + \eta\eta_x - \eta_{xxt} = 0 \tag{1.3}$$

are formal reductions of the two-dimensional Euler equations. The former is the classical Korteweg-de Vries (1895) equation first derived by Boussinesq (1877), while the latter is the regularized long wave or BBM equation written by Peregrine (1967) in his study of bore propagation and first analyzed by Benjamin *et al.* (1972). Both these equations are written in nondimensional, laboratory coordinates, so the small amplitude, long wavelength assumptions reside implicitly in η, and hence should be explicit in the auxiliary data attached to the evolution equation if physically relevant solutions are to be considered. In (1.2) and (1.3) it has been assumed that $S = 1$ and the horizontal variable is scaled by λ, the vertical coordinate by h_0, the deviation η of the free surface by a and time by $(h_0/g)^{\frac{1}{2}}$.

Attention is turned to the just mentioned auxiliary data. It is standard in mathematical studies of these equations to focus upon the pure initial value problem in which η is specified for all the relevant values of x at a given value of t, normally taken to be $t = 0$. That is,

$$\eta(x,0) = f(x) \tag{1.4}$$

is specified for all values of x and values of $t > 0$ represent time elapsed since the inception of the motion as described by (1.4). Of course, if one wishes to be more explicit about the small amplitude, long wavelength assumption, then f can be taken in the form $f(x) = \alpha F(\beta x)$ where F is independent of α and β. The formulation (1.4) does not inquire as to how the motion was truly initiated, but imagines a snapshot taken of a disturbance already generated and then uses (1.2) or (1.3) to predict the further evolution of the waves. The initial-value problems (1.2)-(1.4) and (1.3)-(1.4) have a distinguished history both analytically and in experimental studies and applications, which we do not enter here.

Another natural formulation for both (1.2) and (1.3) is the quarter-plane or half-line problem. This problem, put forward by Bona and Bryant (1973), is concerned with waves propagating into an undisturbed stretch of the medium of propagation. One imagines measuring the waves as they come into the relevant portion of the medium at some fixed spatial point, say $x = 0$. This leads to the boundary condition

$$\eta(0,t) = g(t) \qquad \text{for} \quad t \geq 0. \tag{1.5}$$

As in (1.4), if one wishes to make the small amplitude, long wavelength presumption apparent, one might take $g(t) = \alpha G(\beta t)$ where G is independent of α and β. Since both (1.2) and (1.3) are written to describe waves propagating in the positive direction along the x-axis, it is not particularly desirable to impose a boundary condition at a finite point to the right of $x = 0$. To do so can lead to reflected waves which neither (1.2) nor (1.3) is capable of approximating accurately. (For such motions, systems of equations are useful; see, for example, Bona, Chen and Saut 2002, 2004.) This point leads one to pose the problem for all $x \geq 0$, thus placing the issue of a boundary condition at the right-hand end-point at ∞. The equations (1.2) or (1.3) along with the boundary condition (1.5) must be supplemented by an initial condition as in (1.4), *viz.*

$$\eta(x,0) = f(x) \qquad \text{for} \quad x \geq 0. \tag{1.6}$$

In practice, it is often the case that $f \equiv 0$, corresponding to an initially undisturbed medium, but the mathematical theory does not require this. Function class restrictions on u which imply at least a weak form of boundedness as $x \to +\infty$, suffice to guarantee that (1.2)-(1.5)-(1.6) and (1.3)-(1.5)-(1.6) constitute well-posed problems. These restrictions are implied by the corresponding restriction on the initial data f.

The initial boundary value problems (1.2)-(1.5)-(1.6) and (1.3)-(1.5)-(1.6), sometimes in a modified form that includes some kind of dissipation, have been used to test the predictive power of (1.2) and (1.3) in laboratory settings (see, for example, Hammack 1973, Hammack and Segur 1974 and Bona, Pritchard and Scott 1981). However, when comparison between experimentally produced waves are made with model predictions, one often has to resort to numerical approximation of its solution. For this, a bounded domain is normally used, though there is theory for numerical schemes directly approximating the initial boundary value problem for (1.3)-(1.5)-(1.6) (see e.g. Guo and Shen 2000). There is also available analytical theory for the two-point boundary value problem wherein (1.2) or (1.3) is posed on a finite spatial interval with an initial condition and suitable boundary conditions. In the case of (1.3), this was first developed by Bona and Dougalis (1980) who showed that (1.3) is globally well-posed with the auxiliary specifications

$$\begin{aligned} &\eta(x,0) = f(x), && \text{for} \quad 0 \leq x \leq L, \\ &\eta(0,t) = g(t), \quad \eta(L,t) = h(t), && \text{for} \quad t \geq 0, \end{aligned} \tag{1.7}$$

when f, g and h are suitably restricted. In the comparisons with experiments mentioned above, f and h are taken to be zero and both the experiments and the numerical simulations are only carried out on a time interval during which there is no appreciable motion at the right-hand end of the domain of propagation. (In the experiments, the waves were generated by a flap-type wavemaker and the boundary data g in (1.5) or (1.7) was determined by measurement.) Numerical schemes for this problem were put forward and tested in Bona, Pritchard and Scott (1985). More recent work appears in Bona and Chen (1998). Theory based directly on the motion of the wavemaker rather than on an auxiliary measurement has recently been developed by Bona and Varlamov (2005).

Study of the KdV equation posed on a finite interval began with the work of B. A. Bubnov (1979). A review may be found in the recent paper of Bona, Sun and Zhang (2003). For the Korteweg-de Vries equation (1.2), well-posedness holds for the auxiliary specifications

$$\begin{aligned} &\eta(x,0) = f(x), && \text{for } 0 \le x \le L, \\ \eta(0,t) = g(t), \quad &\eta(L,t) = h(t), \quad \eta_x(L,t) = r(t), && \text{for } t \ge 0, \end{aligned} \tag{1.8}$$

where f, g, h and r are drawn from reasonable function classes. It is also the case that the problem (1.2) posed with

$$\begin{aligned} &\eta(x,0) = f(x), && \text{for } 0 \le x \le L, \\ \eta(0,t) = g(t), \quad &\eta_x(L,t) = h(t), \quad \eta_{xx}(L,t) = r(t), && \text{for } t \ge 0 \end{aligned} \tag{1.9}$$

is well-posed in reasonable function classes as Colin and Ghidaglia (2001) showed.

A natural question arises within the circle of ideas just reviewed. What is the relationship between the two-point boundary value problems for (1.2) or (1.3) and the quarter-plane problem for the same equations? It has been assumed, in using a finite interval for numerical simulations, that these problems yield essentially the same answer in the appropriate part of space-time, if $h \equiv 0$ (and $r \equiv 0$ in the case of (1.2)). However, the only theory that has come to our attention is the work of Colin and Gisclon (2001) connected with (1.9).

It is our purpose to bring forward exact theory comparing the two types of problems in view here. The present paper deals with the BBM-equation (1.3), as the title suggests. A companion paper will consider the same issue for the Korteweg-de Vries equation posed as in (1.8).

The plan of the paper is as follows. Section 2 is devoted to the quarter plane problem. We review existing theory briefly and then extend this theory in a way that is useful for the present goals. Similar theory is worked out for the two-point boundary value problem in Section 3, while the main comparison results are derived in Section 4.

To give the study focus, the main result is here stated informally. Detailed assumptions can be found spelled out in Section 4.

Theorem 1.1. *Let u be the solution of the BBM-equation* (1.3) *posed for $x, t \ge 0$ with zero initial condition and the boundary condition described in* (1.5) *and let v be the solution of the two-point boundary-value problem for the BBM-equation* (1.3) *posed for $0 \le x \le L$ and $t \ge 0$ with zero initial condition and the boundary conditions described in* (1.7) *with $h \equiv 0$. For any $\lambda \in (0,1)$ there is a positive increasing function $\gamma(t)$ only dependent on the values of λ, $\int_0^t |g(s)|\,ds$ and $\int_0^t g^2(s)\,ds$ such that*

$$\|u(\cdot,t) - v(\cdot,t)\|_{H^1[0,L]} \le \gamma(t)e^{-\lambda L}.$$

Remark: A more precise appreciation of the function γ appears in Section 4, but note the exponential approach of the two solutions as L becomes large.

2. **The Quarter-Plane Problem.** For the readers' convenience, we commence by collecting together the main notation to be used throughout. The positive real axis $[0,\infty)$ is denoted by $\mathbb{R}^+$. Throughout this paper, I is used to denote the interval $[0,T]$ if T is finite and $[0,\infty)$ if $T = \infty$. The class $C(I)$ is the continuous functions

defined on I, $C_b(I)$ is the subset of $C(I)$ consisting of all bounded continuous functions on I, while $C_0(\mathbb{R}^+)$ is the subset of bounded and continuous functions that vanish at $+\infty$. For $p \geq 1$, $L_p = L_p(\mathbb{R}^+)$ is the Lebesque space with its usual norm; the notation $|\cdot|_p = \|\cdot\|_{L_p}$ will be used. The norm on $C_b(\mathbb{R}^+)$ and $C_0(\mathbb{R}^+)$ is $|\cdot|_\infty$. For any real number s, $H^s = H^s(\mathbb{R}^+)$ is the usual L_2–based Sobolev space with its norm abbreviated by $\|f\|_s = \|f\|_{H^s}$ and $H_l^s = H^s(0,l)$ with its usual quotient norm denoted by $\|f\|_{H_l^s}$. If J is an interval in $\mathbb{R}$ and X a Banach space, then $C(J;X)$ consists of all continuous functions defined on J with images in X and $C_b(J;X)$ is the subspace of functions $f \in C(J;X)$ such that $\sup_{t\in J}\|f(t)\|_X < \infty$. If $j \geq 1$ is an integer, then $C^j(J;X)$ is the subset of $C(J;X)$ whose elements are j-times differentiable with respect to the variable t. These spaces carry their usual norms.

Considered now is the quarter-plane problem for the BBM-equation

$$\left.\begin{array}{ll} u_t + u_x + uu_x - u_{xxt} = 0, & x > 0, \quad t > 0, \\ u(0,t) = g(t), & t \geq 0, \\ u(x,0) = 0, & x \geq 0, \end{array}\right\} \tag{2.1}$$

repeated here for easy reference, with an undisturbed initial medium and with the natural compatibility condition

$$u(0,0) = g(0) = 0.$$

Write the BBM-equation as

$$u_t - u_{xxt} = -u_x - uu_x,$$

and formally solve for u_t (see Benjamin *et al.* 1972 or Bona and Luo 1995) to obtain

$$u_t(x,t) = g'(t)e^{-x} - \int_0^\infty P(x,y)\big(u_y(y,t) + u(y,t)u_y(y,t)\big)\,dy, \tag{2.2}$$

where

$$P(x,y) = \frac{1}{2}\Big(-e^{-(x+y)} + e^{-|x-y|}\Big). \tag{2.3}$$

Since $\lim_{y\to\infty} P(x,y) = 0$ and $P(x,0) = 0$, integrating by parts on the right-hand side of (2.2) yields

$$u_t(x,t) = g'(t)e^{-x} + \int_0^\infty K(x,y)\Big(u(y,t) + \frac{1}{2}u^2(y,t)\Big)\,dy \tag{2.4}$$

where

$$K(x,y) = \frac{1}{2}\Big(e^{-(x+y)} + \operatorname{sgn}(x-y)e^{-|x-y|}\Big). \tag{2.5}$$

Formally integrating with respect to the temporal variable over $[0,t]$, and recalling that $g(0) = 0$, one obtains the integral equation

$$u(x,t) = g(t)e^{-x} + \int_0^t \int_0^\infty K(x,y)\Big(u(y,s) + \frac{1}{2}u^2(y,s)\Big)\,dy\,ds. \tag{2.6}$$

Note that if $u(x,0) = f(x)$ is not the zero function, then f will also appear on the right-hand side of (??), *viz.*

$$u(x,t) = f(x) + g(t)e^{-x} + \int_0^t \int_0^\infty K(x,y)\Big(u(y,s) + \frac{1}{2}u^2(y,s)\Big)\,dy\,ds. \tag{2.7}$$

The main result of this section is now stated.

Theorem 2.1. *Let $I = [0,T]$ where T is positive or $I = [0,\infty)$ if $T = \infty$ and suppose the boundary value $g \in C(I)$. Then (2.1) is globally well-posed in the sense that there is a unique distributional solution $u \in C(I; H^1(\mathbb{R}^+))$ which depends continuously in $C(I; H^1(\mathbb{R}^+))$ on $g \in C(I)$. Moreover, u is C^∞ in the spatial variable x. If $g \in C^k(I)$ for some $k \geq 1$, then the solution u is a classical solution. Furthermore, for any given $\lambda \in (0,1)$, there are functions $A(t) = A_{g,\lambda}(t)$ and $B(t) = B_{g,\lambda}(t)$ which only depend on the values of λ, $\int_0^t |g(s)|\,ds$ and $\int_0^t g^2(s)\,ds$, such that*

$$|u(x,t)| \leq |g(t)|e^{-x} + A(t)e^{B(t)\,t-\lambda x}$$

for any $(x,t) \in \mathbb{R}^+ \times I$. Furthermore, if g lies in $C_b(I) \cap L_1(I)$, then in the above inequality, $|g(t)|$, $A(t)$ and $B(t)$ can be replaced by three constants only dependent on the values of $|g|_\infty$, $|g|_1$ and $|g|_2$.

The proof will be provided at the end of this section. The result will follow from preliminary ruminations which we begin now.

Theorem 2.2. *(Local Existence) If $g \in C(I)$, then there is a positive, finite value T_0 and an associated interval $I_0 = [0,T_0] \subset I$ such that integral equation (2.7) has a unique solution u, say, lying in the space $C(I_0; C_0)$. Moreover,*

$$\lim_{t\to 0^+} u(x,t) = 0 \tag{2.8}$$

in $C_b(\mathbb{R}^+)$ and

$$\lim_{x\to 0^+} u(x,t) = g(t) \tag{2.9}$$

in $C(I_0)$. The solution u depends continuously in $C(I_0; C_0)$ on $g \in C(I_0)$.

Proof. The first part of the proof is made via the contraction mapping principle in the space $C(I_0; C_0(\mathbb{R}^+))$ applied to the operator which is defined by the right-hand side of (2.7) (see Bona and Luo 1995). If u solves (2.4), then u assumes the initial and boundary conditions expressed in (2.8) and (2.9) because of the form of K. The continuous dependence of u on g follows because u is obtained by a contraction mapping argument in which the operator clearly depends continuously on g. More precisely, suppose u and v are solutions of (2.7) corresponding to g and h, respectively, and suppose u to be obtained via the contraction mapping principle. In an obvious notation, write

$$u = A_g(u) \quad \text{and} \quad v = A_h(v)$$

where A_g connotes the operator on the right-hand side of (2.7), and similarly for A_h. Notice that

$$u - v = A_g(u) - A_h(v) = A_g(u) - A_g(v) + A_g(v) - A_h(v),$$

and that, consequently,

$$\begin{aligned} \|u-v\|_{C(I_1;C_0(\mathbb{R}^+))} &\leq \|A_g(u) - A_g(v)\|_{C(I_1;C_0(\mathbb{R}^+))} + |g-h|_\infty \\ &\leq \theta\|u-v\|_{C(I_1;C_0(\mathbb{R}^+))} + |g-h|_\infty, \end{aligned}$$

where $\theta < 1$ is the contraction constant associated to A_g and $I_1 = [0,T_1]$ with $0 < T_1 \leq T_0$ small enough that v lies within the ball where A_g is contractive. It follows instantly that

$$\|u-v\|_{C(I_1;C_0(\mathbb{R}^+))} \leq \frac{1}{1-\theta}|g-h|_\infty$$

showing that the solution mapping $g \mapsto u$ is in fact locally Lipschitz continuous. (Indeed, this mapping is analytic, but we do not stop off to pursue this point here.

See Zhang 1995 for theory for (1.2) in this regard.) A straightforward iteration of this argument yields the result on $[0, T_0]$. □

The same type of argument establishes the next result.

Theorem 2.3. *If $g \in C(I)$, then there is $I_0 = [0, T_0] \subseteq I$ such that the integral equation (2.7) has a unique solution u, say, lying in the space $C(I_0; H^1(\mathbb{R}^+))$ and satisfying the initial condition (2.8) in $H^1(\mathbb{R}^+)$ and the boundary condition (2.9) in $C(I_0)$. The solution u depends continuously in $C(I_0; H^1(\mathbb{R}^+))$ on $g \in C(I_0)$.*

Corollary 1. *(Regularity) If $u \in C(I_0; C_b(\mathbb{R}^+))$ solves (2.7) where $g \in C(I_0)$, then u is C^∞ in the spatial variable x and $\partial_x^j u \in C(I_0; H^1(\mathbb{R}^+))$ for $j = 0, 1, 2, \cdots$. Moreover, u comprises a distributional solution of the BBM-equation on $\mathbb{R}^+ \times I_0$. If $g \in C^k(I_0)$, for some $k \geq 1$, then u is a classical solution of (1.3) on $\mathbb{R}^+ \times I_0$ and $\partial_t^i \partial_x^j u \in C(I_0; H^1(\mathbb{R}^+))$ for $0 \leq i \leq k$ and $0 \leq j$.*

Proof. It is straightforward to deduce that if $u \in C(I_0; H^1(\mathbb{R}^+))$ solves (2.7), then u_x exists and

$$\begin{aligned} u_x(x,t) =& -g(t)e^{-x} + \int_0^t \Big(u(x,s) + \frac{1}{2}u^2(x,s)\Big)\, ds \\ &- \frac{1}{2}\int_0^t \int_0^\infty \big(e^{-(x+y)} + e^{-|x-y|}\big)\Big(u(y,s) + \frac{1}{2}u^2(y,s)\Big)\, dy\, ds. \end{aligned} \tag{2.10}$$

Examination of the right-hand side reveals u_x lies in $C(I_0; H^1(\mathbb{R}^+))$ and thus u_{xx} exists because of the extra spatial smoothness; moreover,

$$\begin{aligned} u_{xx} =& g(t)e^{-x} + \int_0^t \Big(u_x(x,s) + u(x,s)u_x(x,s)\Big)\, ds \\ &+ \int_0^t \int_0^\infty K(x,y)\Big(u(y,s) + \frac{1}{2}u^2(y,s)\Big)\, dy\, ds. \end{aligned} \tag{2.11}$$

A perusal of (2.11) indicates that $u_{xx} \in C(I_0; H^1(\mathbb{R}^+))$. Inductively, it is shown that $\partial_x^j u \subset C(I_0; H^1(\mathbb{R}^+))$ for every $j \geq 0$, hence, u is C^∞ in x. Formulas (2.11) and (2.7) combined imply that u is a distributional solution of the BBM-equation. Indeed, the combination

$$u(x,t) - u_{xx}(x,t) = -\int_0^t \Big(u_x(x,s) + u(x,s)u_x(x,s)\Big)\, ds$$

lies in $C^1([0, T_0]; C_0^k(\mathbb{R}^+))$ for any $k \geq 0$ and the combination

$$\partial_t(u - u_{xx}) + u_x + uu_x$$

is comprised of continuous functions whose sum is identically zero.

The further temporal and spatial regularity follows by a continued bootstrap argument in case g has some differentiability. □

Notice that the difference $u(x,t) - g(t)e^{-x}$, which according to (2.7) is equal to a double integral, is one order smoother than either u or g in the temporal variable t and is C^∞ in the spatial variable x. Moreover, it has zero initial value and vanishes at $x = 0$ and in the limit as $x \to \infty$, for any $t > 0$. This observation leads one to introduce a new dependent variable

$$U(x,t) = u(x,t) - g(t)e^{-x}. \tag{2.12}$$

Writing (2.1) and (2.7), respectively, in terms of U leads to the equations

$$\left.\begin{aligned} U_t + U_x + UU_x - U_{xxt} = -\Big(g(t)e^{-x} + g(t)e^{-x}U + \frac{1}{2}g^2(t)e^{-2x}\Big)_x, \quad & x > 0, t > 0, \\ U(0,t) = 0, \quad U(x,0) = 0, \quad & x \geq 0, t \geq 0 \end{aligned}\right\} \tag{2.13}$$

and

$$\begin{aligned} U(x,t) = & \int_0^t \int_0^\infty K(x,y)\Big(U(y,s) + g(s)e^{-y}U(y,s) + \frac{1}{2}U^2(y,s)\Big)\,dy\,ds \\ & + \frac{1}{2}xe^{-x}\int_0^t g(s)\,ds + \Big(\frac{1}{3}e^{-x} - \frac{1}{3}e^{-2x}\Big)\int_0^t g^2(s)\,ds. \end{aligned} \tag{2.14}$$

It is helpful to understand the relationship between solutions of (2.13) and (2.14). Note, first, that if $g \in C(I)$, then (2.14) has a unique solution $U \in C(I_0; H^1(\mathbb{R}^+))$ in light of Theorem 2.3, where $I_0 = [0, T_0] \subseteq I$. Since the integrals $\int_0^t g(s)\,ds$ and $\int_0^t g^2(s)\,ds$ are continuously differentiable with respect to t, it follows immediately that such a solution of (2.14) is a classical solution of (2.13). That is, all the terms in (2.13) exist classically, are continuous functions of (x,t), and the equation is satisfied pointwise.

Suppose instead that $U \in C(I_0; H^1(\mathbb{R}^+))$ is a distributional solution of (2.13). Then the combination $U_t - U_{xxt}$ lies in $C(I_0; L_2(\mathbb{R}^+))$. Moreover, as $H^1(\mathbb{R}^+) \subset C_0(\mathbb{R}^+)$, U takes on the boundary condition at $x = 0$ specified in (2.13) in the strong sense of bounded continuous functions. Thus the distribution $V = U_t$ satisfies $(I - \partial_x^2)V = F$ where F lies in $C(I_0; L_2(\mathbb{R}^+))$. Elementary considerations show that V must satisfy the equation

$$V(x,t) = \int_0^\infty P(x,y)F(y,t)\,dy. \tag{2.15}$$

The right-hand side of (2.15) lies in $C(I_0; H^2(\mathbb{R}^+))$. Straightforward machinations then imply that U satisfies (2.14). Thus, it is ascertained that (2.13) and (2.14) are equivalent at least in the context of solutions in $C(I_0; H^1(\mathbb{R}^+))$. As solutions of (2.14) in this function class are unique, so too are such distributional solutions of (2.13).

Attention is now turned to the provision of *a priori* deduced bounds that imply the local well-posedness is in fact global. Standard energy estimates come to the fore in this endeavour. (Note that the solution U has sufficient regularity to justify the various formal calculations embarked upon now.) Multiply both sides of (2.13) by $2U$ and integrate over $\mathbb{R}^+$ with respect to x. After suitable integrations by parts, there appears

$$\begin{aligned} & \frac{d}{dt}\int_0^\infty \Big(U^2(x,t) + U_x^2(x,t)\Big)\,dx \\ & \quad = -\int_0^\infty 2U\Big(g(t)e^{-x} + g(t)e^{-x}U + \frac{1}{2}g^2(t)e^{-2x}\Big)_x\,dx. \end{aligned}$$

Further integrations by parts and straightforward estimates yield

$$\begin{aligned}\frac{d}{dt}\|U(\cdot,t)\|_1^2 &= g(t)\int_0^\infty e^{-x}U^2(x,t)\,dx + 2g(t)\int_0^\infty e^{-x}U(x,t)\,dx \\ &\quad + 2g^2(t)\int_0^\infty e^{-2x}U(x,t)\,dx \\ &\le |g(t)|\|U(\cdot,t)\|_1^2 + |g(t)|\|U(\cdot,t)\|_1 + \frac{1}{2}g^2(t)\|U(\cdot,t)\|_1.\end{aligned}$$

It follows that

$$\frac{d}{dt}\|U(\cdot,t)\|_1 \le \frac{1}{2}|g(t)|\|U(\cdot,t)\|_1 + \frac{1}{2}|g(t)| + \frac{1}{4}g^2(t).$$

Applying the Gronwall lemma, it is concluded that

$$\begin{aligned}\|U(\cdot,t)\|_1 &\le \int_0^t \Big(\frac{1}{2}|g(s)| + \frac{1}{4}g^2(s)\Big)e^{\frac{1}{2}\int_s^t |g(\tau)|\,d\tau}\,ds \\ &\le e^{\frac{1}{2}\int_0^t |g(\tau)|\,d\tau}\int_0^t \Big(\frac{1}{2}|g(s)| + \frac{1}{4}g^2(s)\Big)\,ds.\end{aligned} \tag{2.16}$$

Using the bound in (2.16), the H^1-norm of U_t can also be estimated. Multiply (2.13) by U_t and integrate over $\mathbb{R}^+$ with respect to x to reach the relation

$$\begin{aligned}\|U_t(\cdot,t)\|_1^2 &= -\int_0^\infty U_t(x,t)\Big(U + \frac{1}{2}U^2 + g(t)e^{-x} + g(t)e^{-x}U + \frac{1}{2}g^2(t)e^{-2x}\Big)_x\,dx \\ &\le |U_{tx}(\cdot,t)|_2|U + \frac{1}{2}U^2 + g(t)e^{-x} + g(t)e^{-x}U + \frac{1}{2}g^2(t)e^{-2x}|_2 \\ &\le \|U_t(\cdot,t)\|_1\Big((1+|g(t)|)\|U(\cdot,t)\|_1 + \frac{1}{2}\|U(\cdot,t)\|_1^2 + \frac{1}{2}|g(t)| + \frac{1}{4}g^2(t)\Big).\end{aligned}$$

As a consequence, it is deduced that

$$\|U_t(\cdot,t)\|_1 \le (1+|g(t)|)\|U(\cdot,t)\|_1 + \frac{1}{2}\|U(\cdot,t)\|_1^2 + \frac{1}{2}|g(t)| + \frac{1}{4}g^2(t). \tag{2.17}$$

The bounds in (2.16) and (2.17) together with a standard iteration of the local well-posedness theory allow the following conclusion.

Theorem 2.4. *The initial-boundary-value problem* (2.13) *is globally well-posed for data $g \in C(I)$. The solution U respects the bounds in* (2.16) *and* (2.17), *$\partial_x^j U$ and $\partial_x^j U_t$ lie in $C(I; H^1(\mathbb{R}^+))$ for all $j \ge 0$. Moreover, if g lies in $C(I) \cap L_1(I)$, then $U \in C_b(I; H^1(\mathbb{R}^+))$ and*

$$\|U(\cdot,t)\|_1 \le e^{\frac{1}{2}|g|_1}\Big(\frac{1}{2}|g|_1 + \frac{1}{4}|g|_2\Big), \tag{2.18}$$

uniformly for $t \in I$.

Theorem 2.5. *If $g \in C^k(I)$ for some $k \ge 0$, then the solution U of* (2.13) *has the properties*

$$\partial_t^i \partial_x^j U \in C(I; H^1)$$

for every $j \ge 0$ and $0 \le i \le k+1$.

The proof is similar to that of Corollary 1.

Corollary 2. *If $g \in C(I)$, then the solution u of* (2.7) *lies in the space $C(I; H^\infty(\mathbb{R}^+))$ and*

$$\|u(\cdot,t)\|_1 \le |g(t)| + \int_0^t \Big(\frac{1}{2}|g(s)| + \frac{1}{4}g^2(s)\Big)e^{\frac{1}{2}\int_s^t |g(\tau)|\,d\tau}\,ds. \tag{2.19}$$

If, moreover, $g \in C_b(I) \cap L_1(I)$, then the solution u of (2.7) *lies in $C_b(I; H^1(\mathbb{R}^+))$ and*

$$\|u(\cdot,t)\|_1 \le |g|_\infty + e^{\frac{1}{2}|g|_1}\Big(\frac{1}{2}|g|_1 + \frac{1}{4}|g|_2^2\Big).$$

Theorem 2.6. *Suppose $g \in C(I)$ and $\lambda \in (0,1)$. Define A and B by*

$$A(t) = A_g(t) = e^{\frac{1}{1-\lambda}\int_0^t |g(s)|\,ds} \int_0^t \Big(\frac{1}{2e(1-\lambda)}|g(s)| + \frac{1}{12}g^2(s)\Big)\,ds \tag{2.20}$$

and

$$B(t) = B_g(t) = 1 + \int_0^t \Big(\frac{1}{4}|g(s)| + \frac{1}{8}g^2(s)\Big)\,ds. \tag{2.21}$$

If U is the solution of (2.13) *with auxiliary data g, then*

$$|U(x,t)| \le A(t)e^{\frac{B(t)}{1-\lambda}t - \lambda x}. \tag{2.22}$$

If, in addition, $g \in C(I) \cap L_1(I)$, then $A(t)$ and $B(t)$ are uniformly bounded above by

$$\bar{A} = e^{\frac{|g|_1}{1-\lambda}}\Big(\frac{1}{2e(1-\lambda)}|g|_1 + \frac{1}{12}|g|_2^2\Big)$$

and

$$\bar{B} = 1 + \frac{1}{4}|g|_1 + \frac{1}{8}|g|_2^2,$$

respectively, for $t \in I$. In this case, U respects the bound

$$|U(x,t)| \le \bar{A}e^{\frac{\bar{B}}{1-\lambda}t - \lambda x}. \tag{2.23}$$

Proof. Write $U = e^{-\lambda x}\tilde{U}$, so that $U^2 = e^{-\lambda x}U\tilde{U}$. The integral equation (2.14) and elementary considerations yield the inequality

$$\begin{aligned}
\Big|\tilde{U}(x,t)\Big| \le& \int_0^t \Big|\int_0^\infty K(x,y)e^{\lambda(x-y)}\Big(\tilde{U} + g(s)e^{-y}\tilde{U} + \frac{1}{2}U\tilde{U}\Big)\,dy\Big|\,ds \\
&+ \frac{1}{2}xe^{-(1-\lambda)x}\Big|\int_0^t g(s)\,ds\Big| + \frac{1}{12}(e^{-(1-\lambda)x} - e^{-(2-\lambda)x})\int_0^t g^2(s)\,ds \\
\le& \int_0^\infty (1 + \frac{1}{2}|U(\cdot,s)|_\infty)\Big|K(x,y)\Big|e^{\lambda(x-y)}\,dy \int_0^t \Big|\tilde{U}(\cdot,s)\Big|_\infty\,ds \\
&+ \int_0^\infty \Big|K(x,y)\Big|e^{\lambda(x-y)-y}\,dy \int_0^t \Big|g(s)\tilde{U}(\cdot,s)\Big|_\infty\,ds \\
&+ \frac{1}{2e(1-\lambda)}\Big|\int_0^t g(s)\,ds\Big| + \frac{1}{12}\int_0^t g^2(s)\,ds.
\end{aligned}$$

Direct calculation reveals that

$$\begin{aligned}
\int_0^\infty \Big|K(x,y)e^{\lambda(x-y)}\Big|\,dy &= \frac{1}{1-\lambda^2} - \frac{\lambda}{1-\lambda^2}e^{-(1-\lambda)x} - \frac{1}{1+\lambda}e^{-2x} \\
&\le \frac{1}{1-\lambda^2} \le \frac{1}{1-\lambda}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty \Big|K(x,y)e^{\lambda(x-y)}e^{-y}\Big|\,dy &= \frac{1+\lambda}{\lambda(2+\lambda)}\Big(e^{-(1-\lambda)x} - e^{-x}\Big) \\
&< \frac{1}{1-\lambda},
\end{aligned}$$

whence,

$$\begin{aligned}\Big|\tilde U(\cdot,t)\Big|_\infty &\le \frac{1}{1-\lambda}\int_0^t \Big(1+\frac12|U(\cdot,s)|_\infty+|g(s)|\Big)\Big|\tilde U(\cdot,s)\Big|_\infty\,ds\\ &\quad+\frac{1}{2e(1-\lambda)}\int_0^t |g(s)|\,ds+\frac{1}{12}\int_0^t g^2(s)\,ds\\ &=\int_0^t b(s)\Big|\tilde U(\cdot,s)\Big|_\infty\,ds+\int_0^t a(s)\,ds\end{aligned}$$

where

$$b(s)=\frac{1}{1-\lambda}\Big(1+\frac12|U(\cdot,s)|_\infty+|g(s)|\Big)$$

and

$$a(s)=\frac{1}{2e(1-\lambda)}|g(s)|+\frac{1}{12}g^2(s).$$

Applying the Gronwall lemma to the last inequality, it transpires that

$$\Big|\tilde U(\cdot,t)\Big|_\infty\le\int_0^t a(t-s)e^{\int_s^t b(\tau)\,d\tau}\,ds\le e^{\int_0^t b(\tau)\,d\tau}\int_0^t a(s)\,ds.$$

Since $|U(\cdot,s)|_\infty\le\|U(\cdot,s)\|_1\le\int_0^s\left\{\frac12|g(\tau)|+\frac14g^2(\tau)\right\}d\tau$, it is the case that

$$\begin{aligned}\int_0^t b(\tau)\,d\tau &\le\frac{1}{1-\lambda}\Big(t+\int_0^t|g(s)|\,ds+\frac12\int_0^t\int_0^s\{\frac12|g(\tau)|+\frac14g^2(\tau)\}\,d\tau\,ds\Big)\\ &=\frac{1}{1-\lambda}\Big(t+\int_0^t|g(s)|\,ds+\frac12\int_0^t(t-\tau)\{\frac12|g(\tau)|+\frac14g^2(\tau)\}\,d\tau\Big)\\ &\le\frac{1}{1-\lambda}\Big(t+\int_0^t|g(s)|\,ds+t\int_0^t\{\frac14|g(\tau)|+\frac18g^2(\tau)\}\,d\tau\Big)\\ &=\frac{1}{1-\lambda}\int_0^t|g(s)|\,ds+\frac{B(t)}{1-\lambda}\,t.\end{aligned}$$

In consequence, it follows that

$$\Big|\tilde U(\cdot,t)\Big|_\infty\le A(t)e^{\frac{B(t)}{1-\lambda}t},$$

and therefore,

$$\Big|U(x,t)\Big|\le A(t)e^{\frac{B(t)}{1-\lambda}t-\lambda x}.$$

The first part of the theorem is proved. The second part follows immediately from the first part and the extra assumptions on g. □

Corollary 3. *If $g\in C(I)$, then for any $\lambda\in(0,1)$ and $j\ge1$,*

$$\lim_{x\to\infty}e^{\lambda x}\partial_x^jU(x,t)=0$$

and

$$\lim_{x\to\infty}e^{\lambda x}\partial_x^jU_t(x,t)=0,$$

uniformly for t in any compact subset of I.

Proof. Taking the derivative with respect to x on both sides of (2.14), there appears the formula

$$\begin{aligned} U_x(x,t) = &\int_0^t \Big(U(x,s) + g(s)e^{-x}U(x,s) + \frac{1}{2}U^2(x,s)\Big)\, ds \\ &- \int_0^t \int_{-\infty}^{\infty} M(x,y)\Big(U(y,s) + g(s)e^{-y}U(y,s) + \frac{1}{2}U^2(y,s)\Big)\, ds\, dy \\ &+ \int_0^t \Big(\frac{1}{2}g(s)(1-x)e^{-x} - \frac{1}{3}g^2(s)(e^{-x} - 2e^{-2x})\Big)\, ds \end{aligned} \tag{2.24}$$

where

$$M(x,y) = \frac{1}{2}(e^{-(x+y)} + e^{-|x-y|}). \tag{2.25}$$

Taking the t-derivative of (2.24) gives

$$\begin{aligned} U_{tx}(x,t) = &U(x,t) + g(t)e^{-x}U(x,t) + \frac{1}{2}U^2(x,t) \\ &- \int_{-\infty}^{\infty} M(x,y)\Big(U(y,t) + g(t)e^{-y}U(y,t) + \frac{1}{2}U^2(y,t)\Big)\, dy \\ &+ \frac{1}{2}g(t)(1-x)e^{-x} - \frac{1}{3}g^2(t)(e^{-x} - 2e^{-2x}). \end{aligned} \tag{2.26}$$

The inequality (2.22) implies that

$$\lim_{x\to\infty} e^{\lambda x}U_x(x,t) = 0$$

and

$$\lim_{x\to\infty} e^{\lambda x}U_{tx}(x,t) = 0,$$

and these limits are uniform for t in any compact subset of I. Note that

$$\begin{aligned} U_{xx}(x,t) = &U(x,t) + \int_0^t (U_x(x,s) + U(x,s)U_x(x,s))\, ds \\ &+ \int_0^t \Big(g(s)e^{-x} + g(s)e^{-x}U(x,s) + \frac{1}{2}g^2(s)e^{-2x}\Big)_x\, ds \end{aligned}$$

and

$$\begin{aligned} U_{xxt}(x,t) = &U_t(x,t) + U_x(x,t) + U(x,s)U_x(x,t) \\ &+ \Big(g(t)e^{-x} + g(t)e^{-x}U(x,t) + \frac{1}{2}g^2(t)e^{-2x}\Big)_x. \end{aligned}$$

The inequality (2.22) thus implies

$$\lim_{x\to\infty} e^{\lambda x}U_{xx}(x,t) = 0$$

and

$$\lim_{x\to\infty} e^{\lambda x}U_{xxt}(x,t) = 0,$$

uniformly on compact subsets of I. □

3. The Two-point Boundary-Value Problem. Considered in this section is the two-point boundary value problem

$$\left.\begin{aligned}&v_t + v_x + vv_x - v_{xxt} = 0, && 0 \le x \le L, \quad t > 0,\\ &v(0,t) = g(t),\ v(L,t) = h(t), && t \ge 0,\\ &v(x,0) = 0, && 0 \le x \le L\end{aligned}\right\} \tag{3.1}$$

together with the compatibility conditions

$$v(0,0) = g(0) = v(L,0) = h(0) = 0.$$

The main result is as follows.

Theorem 3.1. *If $g, h \in C(I)$, then there is a unique distributional solution v of* (3.1) *which lies in the space $C(I; C^\infty([0,L]))$. The solution v depends continuously on g and h. If $g, h \in C^k(I)$ for some $k \ge 1$, then v satisfies* (3.1) *in the classical sense on $[0,L] \times I$.*

This theorem is a consequence of the last corollary in this section. Its proof is the object of the rest of the section.

Solving for v_t in (3.1) as in Bona and Dougalis (1980) leads to

$$v_t(x,t) = g'(t)\phi_1(x) + h'(t)\phi_2(x) - \int_0^L P_L(x,y)\big(v_y(y,t) + v(y,t)v_y(y,t)\big)\,dy, \tag{3.2}$$

where

$$\phi_1(x) = \frac{e^{L-x} - e^{-L+x}}{e^L - e^{-L}}, \qquad \phi_2(x) = \frac{e^x - e^{-x}}{e^L - e^{-L}} \tag{3.3}$$

and

$$P_L(x,y) = \frac{1}{2(e^{2L}-1)}\Big(-e^{x+y} + e^{|x-y|} - e^{2L-(x+y)} + e^{2L-|x-y|}\Big). \tag{3.4}$$

Since $P_L(x,L) = P_L(x,0) = 0$, integrating by parts on the right-hand side of (3.2) yields

$$v_t(x,t) = g'(t)\phi_1(x) + h'(t)\phi_2(x) + \int_0^L K_L(x,y)\big(v(y,t) + \tfrac{1}{2}v^2(y,t)\big)\,dy \tag{3.5}$$

where

$$K_L(x,y) = \frac{1}{2(e^{2L}-1)}\Big(-e^{x+y} - \operatorname{sgn}(x-y)e^{|x-y|} + e^{2L-(x+y)} + \operatorname{sgn}(x-y)e^{2L-|x-y|}\Big). \tag{3.6}$$

Integrate both sides of (3.5) with respect to the temporal variable t and use the facts that $g(0) = h(0) = 0$ to determine that

$$\begin{aligned} v(x,t) =& g(t)\phi_1(x) + h(t)\phi_2(x)\\ &+ \int_0^t \int_0^L K_L(x,y)\big(v(y,\tau) + \tfrac{1}{2}v^2(y,\tau)\big)\,dy\,d\tau. \end{aligned} \tag{3.7}$$

Theorem 3.2. *(Local Existence) If $g, h \in C(I)$, then there exists $I_0 = [0, T_0] \subset I$ such that the integral equation* (3.7) *has a unique solution v, say, lying in $C(I_0; C([0,L]))$. Moreover,*

$$\lim_{t \to 0^+} v(x,t) = 0$$

in $C([0,L])$ and

$$\lim_{x \to 0^+} v(x,t) = g(t) \qquad \text{and} \qquad \lim_{x \to L^-} v(x,t) = h(t)$$

in $C(I_0)$. The solution depends continuously in $C(I_0; C([0,L]))$ on $g, h \in C(I_0)$.

Proof. See Bona and Dougalis (1980) or the proofs of Theorems 2.2 and 2.3. □

Theorem 3.3. *(Regularity) If $v \in C(I_0; C([0,L]))$ is the solution of the integral equation (3.7) where $g, h \in C(I_0)$, then v is C^∞ in x and for every $j \geq 0$, $\partial_x^j v \in C(I_0; C([0,L]))$. Moreover, the function v is a distributional solution of the BBM-equation on $[0,L] \times I_0$. If $g, h \in C^k(I_0)$ for some $k \geq 1$, then u comprises a classical solution of the BBM-equation (3.1) on $[0,L] \times I_0$ and $\partial_t^i \partial_x^j v \in C(I_0; C[0,L])$ for $0 \leq i \leq k$ and $0 \leq j$.*

Proof. The proof follows the lines laid out in the proof of Corollary 1. □

As in Section 2, it is propitious to introduce an intermediate variable V given by the formula

$$V(x,t) = v(x,t) - [g(t)\phi_1(x) + h(t)\phi_2(x)] = v(x,t) - \mu(x,t) \tag{3.8}$$

where

$$\mu(x,t) = g(t)\phi_1(x) + h(t)\phi_2(x) \tag{3.9}$$

and ϕ_1 and ϕ_2 are defined in (3.3). Then $V(0,t) = V(L,t) = 0$ for all $t \in I$ and

$$V(x,t) = \int_0^t \int_0^L K_L(x,y)\big(v(y,\tau) + \frac{1}{2}v^2(y,\tau)\big)\, dy\, d\tau. \tag{3.10}$$

This representation makes it clear that V is one order smoother in both space and time than g, h and v. Additionally, V satisfies the equation

$$\left.\begin{aligned} V_t + V_x + VV_x - V_{xxt} &= -\Big(\mu + \mu V + \frac{1}{2}\mu^2\Big)_x, \quad 0 < x < L, \quad t \in I_0 \\ V(0,t) = V(L,t) &= 0, \qquad V(x,0) = 0, \quad 0 \leq x \leq L, \quad t \in I_0. \end{aligned}\right\} \tag{3.11}$$

To extend the existence time interval from $I_0 = [0, T_0]$ to I, a standard energy method is used. Multiply both sides of (3.11) by $2V$ and integrate over $[0,L]$ with respect to x to obtain

$$\begin{aligned} \frac{d}{dt}\int_0^L \Big(V^2(x,t) + V_x^2(x,t)\Big)\, dx &= -\int_0^L 2V(x,t)\Big(\mu + \mu V + \frac{1}{2}\mu^2\Big)_x\, dx \\ &\leq |\mu(\cdot,t)|_\infty \|V(\cdot,t)\|_{H_L^1}^2 + 2|\mu(\cdot,t)|_2\, |V(\cdot,t)|_2 + |\mu(\cdot,t)|_\infty |\mu(\cdot,t)|_2\, |V_x(\cdot,t)|_2 \\ &\leq c(t)\|V(\cdot,t)\|_{H_L^1}^2 + 2\Big(c(t) + c^2(t)\Big)\|V(\cdot,t)\|_{H_L^1} \end{aligned} \tag{3.12}$$

where

$$c(t) = |g(t)| + |h(t)|.$$

The last inequality reduces to

$$\frac{d}{dt}\|V(\cdot,t)\|_{H_L^1} \leq \frac{1}{2}c(t)\|V(\cdot,t)\|_{H_L^1} + c(t) + c^2(t). \tag{3.13}$$

Solving this inequality yields the upper bound

$$\begin{aligned} \|V(\cdot,t)\|_{H_L^1} &\leq \int_0^t \Big(c(s) + c^2(s)\Big) e^{\frac{1}{2}\int_s^t c(\tau)\, d\tau}\, ds \\ &\leq \int_0^t \Big(c(s) + c^2(s)\Big)\, ds\, e^{\frac{1}{2}\int_0^t c(\tau)\, d\tau}. \end{aligned} \tag{3.14}$$

Multiply (3.11) by V_t and integrate over $[0, L]$ with respect to x to reach the inequality

$$\begin{aligned}\|V_t(\cdot,t)\|^2_{H^1_L} - V_t(L,t)V_{xt}(L,t) + V_t(0,t)V_{xt}(0,t) \\ = -\int_0^L V_t\Big(V + \frac{1}{2}V^2 + \mu + \mu V + \frac{1}{2}\mu^2\Big)_x\, dx \\ \le \|V_{xt}\|_{L_2(0,L)} \|V + \frac{1}{2}V^2 + \mu + \mu V + \frac{1}{2}\mu^2\|_{L_2(0,L)} \\ \le \|V_t(\cdot,t)\|_{H^1_L} \|V + \frac{1}{2}V^2 + \mu + \mu V + \frac{1}{2}\mu^2\|_{L_2(0,L)}.\end{aligned}$$

Since $K_L(0,y) = K_L(L,y) = 0$ for every $y \in (0, L]$, $V_t(L,t) = V_t(0,t) = 0$ for all $t \in I$, and so it follows that

$$\|V_t(\cdot,t)\|_{H^1_L} \le \Big(1 + c(t)\Big)\|V(\cdot,t)\|_{H^1_L} + \frac{1}{2}\|V(\cdot,t)\|^2_{H^1_L} + c(t) + \frac{1}{2}c^2(t). \qquad (3.15)$$

Applying (3.14) in (3.15) yields an *a priori* bound on $\|V_t(\cdot,t)\|_{H^1_L}$. The associated *a priori* bounds in $C(I; H^1_L)$ allow iteration of the local result to obtain a solution defined on all of I. The regularity Theorem 3.3 then immediately allows inference of the following result.

Theorem 3.4. *The initial-boundary-value problem* (3.11) *is globally well-posed in* $C^\infty([0, L])$. *That is, corresponding to given* $g, h \in C(I)$, *there is a unique solution* $V \in C(I; H^1_L)$ *and* $V, V_t \in C(I; C^\infty([0, L]))$. *The solution* V *respects the bounds in* (3.14) *and* (3.15) *and depends continuously on variations of* g *and* h *within their function classes. In addition, if* $g, h \in C(I) \cap L_1(I)$, *then the* H^1_L*-norm of* V *is uniformly bounded, viz.*

$$\|V(\cdot,t)\|_{H^1_L} \le (|g|_1 + |h|_1 + 2|g|_2^2 + 2|h|_2^2)\, e^{\frac{1}{2}(|g|_1+|h|_1)}.$$

Corollary 4. *(Global Well-posedness) The initial-boundary-value problem* (3.7) *is globally well-posed. That is, corresponding to* $g, h \in C(I)$, *there is a unique solution* v *of* (3.7) *in* $C(I; H^1_L)$. *The solution* v *lies in* $C(I; C^\infty([0, L]))$, *depends continuously in this class upon variations of* g *and* h *in* $C(I)$ *and respects the inequality*

$$\|v(\cdot,t)\|_{H^1_L} \le |g(t)| + \int_0^t \Big(c(s) + c^2(s)\Big)\, ds\, e^{\frac{1}{2}\int_0^t c(\tau)\, d\tau}$$

where $c(s) = |g(s)| + |h(s)|$. *Moreover, if* $g, h \in C_b(I) \cap L_1(I)$, *then* v *satisfies the time-independent bound*

$$\|v(\cdot,t)\|_{H^1_L} \le |g|_\infty + \Big(|g|_1 + |h|_1 + 2|g|_2^2 + 2|h|_2^2\Big)\, e^{\frac{1}{2}(|g|_1+|h|_1)}.$$

Theorem 3.1 now follows.

4. **Comparison Results.** Let u be the solution of the quarter plane problem (2.1) for the BBM equation (2.1) where g is supposed to lie in $\in C(I)$ with the compatibility condition $g(0) = 0$. Then, the value $u(L,t)$ is well defined, where $L > 0$ is as in the last section. Let v be the solution of the two-point boundary-value problem (3.1) where g is the same as the boundary condition for u. The goal of this section is to develop estimates of the difference between u and v on the spatial interval $[0, L]$. Interest will be especially focused on the situation obtaining when $h \equiv 0$, as this case has considerable practical interest.

To begin, introduce a new dependent variable

$$Q(x,t) = U(x,t) - U(L,t)\phi_2(x) = u(x,t) - k_L(x,t) \tag{4.1}$$

where U is the solution of (2.13) or (2.14),

$$k_L(x,t) = g(t)e^{-x} + U(L,t)\phi_2(x) = g(t)e^{-x} + \Big(u(L,t) - g(t)e^{-L}\Big)\phi_2(x) \tag{4.2}$$

and ϕ_2 is defined in (3.3). A simple calculation shows that Q satisfies the initial-boundary-value problem

$$\left.\begin{aligned} Q_t + Q_x + QQ_x - Q_{xxt} &= -\Big(k_L + k_L Q + \frac{1}{2}k_L^2\Big)_x, \quad 0 < x < L,\, t > 0,\\ Q(0,t) = Q(L,t) &= 0, \quad Q(x,0) = 0, \quad 0 < x < L,\, t > 0. \end{aligned}\right\} \tag{4.3}$$

Because of the theory developed for U, Q is a classical solution which is C^∞ in the spatial variable x. The difference between Q and V, the solution of (3.8), is a useful quantity to understand. Once this is appropriately bounded, the identity $v - u = V - Q - (k_L - \mu)$ allows one to make a further estimate of the desired sort. Denote by W the difference

$$W = V - Q = (v - u) + (k_L - \mu). \tag{4.4}$$

Then, W is differentiable in $t \in I$ and C^∞ in $x \in [0, L]$ provided $g, h \in C(I)$, and W satisfies the initial-boundary-value problem

$$\left.\begin{aligned} W_t + W_x + \frac{1}{2}\Big[(2Q + W)W\Big]_x - W_{xxt} &= -\Big(\mu + \mu(W + Q) + \frac{1}{2}\mu^2\Big)_x \\ &\quad + \Big(k_L + k_L Q + \frac{1}{2}k_L^2\Big)_x, \qquad 0 < x < L, \quad t > 0,\\ W(0,t) = W(L,t) = 0, \quad W(x,0) &= 0, \qquad 0 \le x \le L, \quad t \ge 0. \end{aligned}\right\} \tag{4.5}$$

Multiply (4.5) by $2W$ and integrate over $[0, L]$; after integrations by parts there appears

$$\begin{aligned} \frac{d}{dt}\int_0^L &\Big(W^2(x,t) + W_x^2(x,t)\Big)\,dx \\ &= 2\int_0^L QWW_x\,dx + \int_0^L 2W\Big((k_L - \mu) + (k_L - \mu)Q + \frac{1}{2}(k_L^2 - \mu^2) - \mu W\Big)_x dx \\ &= 2\int_0^L (Q + \mu)WW_x\,dx + 2\int_0^L W\Big((k_L - \mu) + (k_L - \mu)Q + \frac{1}{2}(k_L^2 - \mu^2)\Big)_x dx \\ &\le |Q(\cdot,t) + \mu(\cdot,t)|_\infty \|W(\cdot,t)\|_{H_L^1}^2 \\ &\quad + 2\Big|1 + Q(\cdot,t) + \frac{1}{2}\Big(k_L(\cdot,t) + \mu(\cdot,t)\Big)\Big|_\infty \|k_L(\cdot,t) - \mu(\cdot,t)\|\|W_x(\cdot,t)\| \end{aligned} \tag{4.6}$$

where

$$Q(x,t) + \mu(x,t) = U(x,t) - U(L,t)\phi_2(x) + g(t)\phi_1(x) + h(t)\phi_2(x).$$

Notice that

$$\begin{aligned} |Q(\cdot,t) + \mu(\cdot,t)|_\infty &\le 2|U(\cdot,t)|_\infty + |g(t)| + |h(t)| \\ &\le \int_0^t \Big(|g(s)| + \frac{1}{2}g^2(s)\Big)\,ds + |g(t)| + |h(t)|, \end{aligned}$$

Also, we have

$$\begin{aligned}1 + Q(x,t) + &\frac{1}{2}\Big(k_L(x,t) + \mu(x,t)\Big)\\ =&1 + U(x,t) - U(L,t)\phi_2(x)\\ &+ \frac{1}{2}\Big(g(t)e^{-x} + U(L,t)\phi_2(x) + g(t)\phi_1(x) + h(t)\phi_2(x)\Big)\\ =&1 + U(x,t) - \frac{1}{2}U(L,t)\phi_2(x) + \frac{1}{2}\Big(g(t)e^{-x} + g(t)\phi_1(x) + h(t)\phi_2(x)\Big),\end{aligned}$$

with

$$\begin{aligned}\Big|1+&Q(\cdot,t) + \frac{1}{2}\Big(k_L(\cdot,t) + \mu(\cdot,t)\Big)\Big|_\infty\\ &\le 1 + |U(\cdot,t)|_\infty + |g(t)| + \frac{1}{2}|h(t)|\\ &\le 1 + |g(t)| + \frac{1}{2}|h(t)| + \int_0^t \Big(\frac{3}{4}|g(s)| + \frac{3}{8}g^2(s)\Big)\,ds\end{aligned}$$

and

$$\|k_L(\cdot,t) - \mu(\cdot,t)\| \le |g(t)|e^{-L} + \big|U(L,t) - h(t)\big|.$$

Define the quantities

$$C(t) = \int_0^t \Big(|g(s)| + \frac{1}{2}g^2(s)\Big)\,ds + |g(t)| + |h(t)|, \tag{4.7}$$

$$D(t) = 1 + |g(t)| + \frac{1}{2}|h(t)| + \int_0^t \Big(\frac{3}{4}|g(s)| + \frac{3}{8}g^2(s)\Big)\,ds \tag{4.8}$$

and

$$E(t) = |g(t)|e^{-L} + |U(L,t) - h(t)|. \tag{4.9}$$

With these definitions, it transpires that

$$\frac{d}{dt}\|W(\cdot,t)\|_{H_L^1} \le C(t)\|W(\cdot,t)\|_{H_L^1} + D(t)E(t),$$

whence,

$$\|W(\cdot,t)\|_{H_L^1} \le \int_0^t D(s)E(s)e^{\int_s^t C(\tau)\,d\tau}\,ds \le e^{\int_0^t C(\tau)\,d\tau}\int_0^t D(s)E(s)\,ds. \tag{4.10}$$

Theorem 4.1. *Let $g \in C(I)$ and $h \equiv 0$, and let $\lambda \in (0,1)$ be fixed. Then there is a positive function $\gamma(t)$ which only depends on λ and the values of g on $[0,t]$ such that*

$$\|u(\cdot,t) - v(\cdot,t)\|_{H_L^1} \le \gamma(t)e^{-\lambda L}.$$

Proof. When $h \equiv 0$, Theorem 2.6 implies

$$|U(L,t)| = |u(L,t) - g(t)e^{-L}| \le A(t)e^{\frac{B(t)}{1-\lambda} - \lambda L}$$

where A and B are defined in (2.20) and (2.21), respectively. It follows that

$$E(t) \le \Big(|g(t)| + A(t)e^{\frac{B(t)}{1-\lambda}t}\Big)e^{-\lambda L},$$

and thus (4.10) gives

$$\|W(\cdot,t)\|_{H_L^1} \le e^{\int_0^t C(\tau)\,d\tau}\int_0^t D(s)\Big(|g(s)| + A(s)e^{\frac{B(s)}{1-\lambda}s}\Big)\,ds\,e^{-\lambda L}$$

where C, D and E are as in (4.7), (4.8) and (4.9), respectively. The definition of W therefore entails that,

$$\begin{aligned}\|u(\cdot,t)-v(\cdot,t)\|_{H^1_L} \leq& \|W(\cdot,t)\|_{H^1_L} + \|k_L(\cdot,t)-\mu(\cdot,t)\|_{H^1_L}\\ \leq& e^{\int_0^t C(\tau)\,d\tau}\int_0^t D(s)\Big(|g(s)|+A(s)e^{\frac{B(s)}{1-\lambda}s}\Big)\,ds\,e^{-\lambda L}\\ &+2|g(t)|e^{-L}+2|U(L,t)|\\ \leq& e^{\int_0^t C(\tau)\,d\tau}\int_0^t D(s)\Big(|g(s)|+A(s)e^{\frac{B(s)}{1-\lambda}s}\Big)\,ds\,e^{-\lambda L}\\ &+2\Big(|g(t)|+A(t)e^{\frac{B(t)}{1-\lambda}t}\Big)e^{-\lambda L}.\end{aligned}$$

Define γ to be

$$\gamma(t)=e^{\int_0^t C(\tau)\,d\tau}\int_0^t D(s)\Big(|g(s)|+A(s)e^{\frac{B(s)}{1-\lambda}s}\Big)\,ds+2\Big(|g(t)|+A(t)e^{\frac{B(t)}{1-\lambda}t}\Big). \tag{4.11}$$

With this definition, it follows at once that

$$\|u(\cdot,t)-v(\cdot,t)\|_{H^1_L}\leq\gamma(t)e^{-\lambda L}.$$

The theorem is proved. □

Corollary 5. *Let $g\in C(I)$ and $h\equiv 0$. View $v(x,t)=v_L(x,t)$ as function of L as well. Then, for any fixed point $(x,t)\in\mathbb{R}^+\times I$,*

$$\lim_{L\to\infty} v_L(x,t)=u(x,t)$$

where u is the solution of the quarter-plane problem (2.1).

The latter convergence is uniform on compact sets. More precisely, we have the following.

Corollary 6. *Let $g\in C(I)$ and $\lambda\in(0,1)$ be fixed and let $h\equiv 0$. For any $\epsilon>0$ and any finite time interval $[0,T_0]\subset I$, if L is chosen greater than $\frac{1}{\lambda}\ln\frac{\gamma(T_0)}{\epsilon}$, then*

$$|u(x,t)-v(x,t)|\leq\epsilon$$

uniformly on $[0,L]\times[0,T_0]$.

Remark: If the boundary datum g has the form $g(t)=\alpha G(\beta t)$ where G and α/β^2 are both order one, then elementary estimates show that $\gamma(t)$ is of order e^{1/β^4}. In consequence, our estimate indicates that L needs to be taken to be of order $\gtrsim 1/\beta^4$ if the time scale of interest is $1/\beta^3$. As $1/\beta^3$ is, in these variables, the time scale over which nonlinear and dispersive effects can make an order one relative contribution to the wave profile, and as the wavelength is of order $1/\beta$, this result is not too bad despite the crudeness of some of the estimates. In any event, the principle that has been established is that if boundary data g is provided on $[0,T]$, then there are values of L large enough that the solution of the two-point boundary-value problem with $h\equiv 0$ approximates well the associated solution of the quarter-plane problem.

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