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WELL-POSEDNESS FOR REGULARIZED NONLINEAR DISPERSIVE WAVE EQUATIONS

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Abstract. In this essay, we study the initial-value problem

$$\left.\begin{aligned} &u_t + u_x + g(u)_x + Lu_t = 0, \qquad x \in \mathbb{R}, \quad t > 0,\\ &u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \end{aligned}\right\} \tag{0.1}$$

where $u = u(x,t)$ is a real-valued function, L is a Fourier multiplier operator with real symbol $\alpha(\xi)$, say, and g is a smooth, real-valued function of a real variable. Equations of this form arise as models of wave propagation in a variety of physical contexts. Here, fundamental issues of local and global well-posedness are established for L_p, H^s and bore-like or kink-like initial data. In the special case where $\alpha(\xi) = |\xi|^r$ wherein $r > 1$ and $g(u) = \frac{1}{2}u^2$, (0.1) is globally well-posed in time if s and r satisfy a simple algebraic relation.

1. **Introduction and notation.** The regularized long-wave equation or BBM-equation

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{1.1}$$

was put forward by Peregrine (1966, 1967) and Benjamin *et al.* (1972) as an alternative model to the Korteweg-de Vries equation for small-amplitude, long wavelength surface water waves. In the analysis following the derivation of (1.1), Benjamin *et al.* (1972) proved (1.1) it to be globally well posed in the Sobolev class $H^1(\mathbb{R})$ and in spaces such as $C_b^k(\mathbb{R}) \cap H^s(\mathbb{R})$ provided $s \geq 1$. Bona and Tzvetkov (2009) recently showed that (1.1) is globally well-posed in $L_2(\mathbb{R})$ and that this result is sharp in a certain sense.

The Benjamin-Ono equation

$$u_t + u_x + uu_x - Hu_{xx} = 0, \tag{1.2}$$

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where H is the Hilbert transform, is a model equation derived by Benjamin (1967) for a class of internal water waves. Just as the BBM-equation is an alternative to KdV-equation, Kalisch and Bona (2000) remarked that the regularized Benjamin-Ono equation

$$u_t + u_x + uu_x + Hu_{xt} = 0 \tag{1.3}$$

is formally equivalent to the Benjamin-Ono equation (1.2) (and see the rigorous theory in Albert and Bona 1991). They also showed that (1.3) is well-posed in $H^s(\mathbb{R})$ locally in time for $s > \frac{1}{2}$ and globally in time for $s \geq \frac{3}{2}$.

Notice that the symbols of the operators $-\partial_x^2$ in (1.1) and $H\partial_x$ in more recent work (1.3) are $|\xi|^2$ and $|\xi|$, respectively. It seems natural to inquire about the initial-value problem

$$\left.\begin{aligned} &u_t + u_x + uu_x + D^r u_t = 0, \qquad x \in \mathbb{R}, \quad t > 0, \\ &u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \end{aligned}\right\} \tag{1.4}$$

where $D = (-\partial_x^2)^{\frac{1}{2}}$ and $r \geq 1$. Well-posedness issues remain interesting for nonhomogenous symbols as well. Indeed, a considerable range of symbols arise in practice, so well-posedness issues for the generalized class (0.1) are not just of mathematical interest.

To state the main results of the present study, it is helpful to introduce our notation, which is mostly standard.

For $1 \leq r < \infty$, $L_r = L_r(\mathbb{R})$ connotes the r^{th}-power Lebesgue-integrable functions with the usual modification for the case $r = \infty$. The norm of a function $f \in L_r$ is written $|f|_r$. The Sobolev class $H^s = H^s(\mathbb{R})$ is the class of tempered distributions whose Fourier transform $\widehat{f}$ is a measurable function such that

$$\|f\|_s^2 = \int_{-\infty}^{\infty} (1 + |\xi|)^{2s} |\widehat{f}(\xi)|^2 \, d\xi < +\infty$$

where $\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} \, d\xi$. Note that $\|f\|_0 = |f|_2$ and the latter notation will be preferred. If X is any Banach space and $T > 0$, $C(0,T;X)$ is the class of continuous functions from $[0,T]$ into X with its usual norm

$$\|u\|_{C(0,T;X)} = \max_{0 \leq t \leq T} \|u(t)\|_X.$$

If $S \subset X$ is a subset, then $C(0,T;S)$ is the collection of elements u in $C(0,T;X)$ such that $u(t) \in S$ for $0 \leq t \leq T$. When $T = \infty$, $C(0,\infty;X)$ is a Fréchet space with defining set of semi-norms

$$p_n(u) = \max_{0 \leq t \leq n} \|u(t)\|_X, \quad n = 1, 2, \cdots.$$

The subspace $C_b(0,\infty;X)$ of elements of $C(0,\infty;X)$ which are uniformly bounded is a Banach space with norm

$$\|u\|_{C_b(0,\infty;X)} = \sup_{t \geq 0} \|u(t)\|_X.$$

The Banach space $C^1(0,T;X)$ is the subspace of $C(0,T;X)$ for which the limit

$$u'(t) = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h}$$

exists in $C(0,T;X)$. It is equipped with the obvious norm. Inductively, one defines $C^k(0,T;X)$ and, by analogy, $C^k(0,\infty;X)$ and $C_b^k(0,\infty;X)$.

The notion of well-posedness in view is a standard one.

Definition 1. An evolution equation

$$u_t = Au, \quad u(0) = u_0$$

is said to be locally (in time) well-posed in a Banach space X if for any $u_0 \in X$, there is a positive number T such that the equation possesses a unique solution u which lies in $C(0,T;X)$. Moreover, the solution u must depend continuously on u_0. That is to say, the mapping $u_0 \mapsto u$ from X to $C(0,T;X)$ must be continuous. This correspondence is sometimes called the flow map. The evolution equation is well-posed globally in time if T can be chosen arbitrarily large.

Here is a sample of the outcome of our investigation, stated in rough form.

Theorem 1.1. *The initial-value problem* (1.4) *is locally well-posed in* H^s *if* r *and* s *satisfy one of the following conditions:*

(a) $r \geq 1$ *and* $s > \frac{1}{2}$;

(b) $r > \frac{5}{4}$ *and* $s > \frac{1}{4}$;

(c) $r > \frac{3}{2}$ *and* $s \geq 0$.

In addition, if $r > 1$ *and* $s \geq 1 - \frac{r}{2}$, *then the well-posedness is global.*

The plan for the remainder of the paper is the following. In Section 2, the pure initial-value problem is converted into an integral equation. Local existence is then established for this integral equation by an application of the contraction-mapping principle in appropriate L_p-spaces. For a restricted class of the equations possessing a local well-posedness theory, an *a priori* bound is derived that leads to global well posedness. In Section 3, the same technique of contraction mapping is used to give local well-posedness results in L_2-based Sobolev spaces. The proof of global well-posedness is inspired by the work of Bona and Tzvetkov (2009) concerned with (1.1). In Section 4, bore-like initial data is countenanced, and similar theory is derived in this case. The paper concludes with a brief summary and an interesting further avenue that might be worth investigating.

2. **Local well-posedness in L_p spaces.**

2.1. **Associated Integral Equation.**

The theory begins by converting the original initial-value problem into an associated integral equation. For this, we operate formally and consider afterward the issue of whether or not solutions of the integral equation are indeed solutions of the initial-value problem.

Write the evolution equation (0.1) posed on all of $\mathbb{R}$ in the form

$$(I + L)u_t(x,t) = -\big(u(x,t) + g(u(x,t))\big)_x \tag{2.1}$$

and take the Fourier transform with respect to the spatial variable x. Denoting the Fourier transform of u with respect to x by $\widehat{u}$, there appears the formal relation

$$\big(1 + \alpha(\xi)\big)\widehat{u}_t(\xi,t) = -i\xi\big(\widehat{u}(\xi,t) + \widehat{g(u)}(\xi,t)\big).$$

Dividing by $1 + \alpha$ and taking the inverse Fourier transform leads to the integral equation

$$u_t(x,t) = K * \big(u + g(u)\big)(x,t) \tag{2.2}$$

where the kernel K is given as

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-i\xi}{1+\alpha(\xi)}\, e^{ix\xi}\, d\xi$$

or

$$\widehat{K}(\xi) = \frac{-i\xi}{\sqrt{2\pi}\big(1+\alpha(\xi)\big)}.$$

Of course the convolution may have to be interpreted in the sense of tempered distributions. A formal integration in the temporal variable then leads to the BBM-type integral equation

$$u(x,t) = u_0(x) + \int_0^t \int_{-\infty}^{\infty} K(x-y)\Big(u(y,s) + g(u(y,s))\Big)\,dy\,ds \tag{2.3}$$

where $u_0(x) = u(x,0)$ is the initial data. For classes of functions $v : \mathbb{R}\times[0,T] \to \mathbb{R}$ to be delineated presently, let $w = \mathcal{A}(v)$ be the function obtained from v by replacing u with v on the right-hand side of (2.3). The equation (2.3) then takes the form

$$u = \mathcal{A}(u). \tag{2.4}$$

In terms of the integral equation, a solution is thus seen to comprise a fixed point of the nonlinear operator $\mathcal{A}$.

2.2. **Local well-posedness in L_p spaces.** Assumptions on g and the symbol α of L are now provided. As mentioned earlier, our goal is to work in relatively large function spaces.

(H1) The function $g : \mathbb{R} \to \mathbb{R}$ is C^1, $g(0) = g'(0) = 0$ and there is a $p > 1$ and a constant C_0 such that

$$\big|1 + g'(z)\big| \le C_0(1 + |z|^{p-1})$$

for all $z \in \mathbb{R}$. (The assumptions $g(0) = g'(0) = 0$ are innocuous; since g appears only differentiated, the value $g(0)$ is irrelevant. The value of $g'(0)$ can be absorbed into the linear convection term u_x.)

(H2) The symbol α is a real-valued, even, continuous function, having the property that the tempered distribution K whose Fourier transform is $-i\xi/\sqrt{2\pi}\big(1+\alpha(\xi)\big)$ is given by a measurable function lying in $L_1(\mathbb{R}) \cap L_r(\mathbb{R})$ for some $r > 1$.

Examples: If $L = -\partial_x^2$, then

$$K(x) = \frac{1}{2}\,\mathrm{sgn}(x)e^{-|x|}$$

as one ascertains by a direct calculation using the Residue Theorem (see Benjamin *et al.* 1972). Clearly, this version of K satisfies (H2) for any positive value of r, including $r = +\infty$.

If $L = D^r$ with $r > 1$ where $\widehat{D^r h}(\xi) = |\xi|^r \widehat{h}(\xi)$, then

$$K(x) = \frac{1}{\sqrt{2\pi}}\,\mathcal{F}^{-1}\Big\{\frac{i\xi}{1+|\xi|^r}\Big\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\xi e^{ix\xi}}{1+|\xi|^r}\,d\xi = -\frac{1}{\pi}\int_0^{\infty} \frac{\xi\sin(x\xi)}{1+\xi^r}\,d\xi$$

where $\mathcal{F}$ connotes the Fourier transform in the spatial variable x and $\mathcal{F}^{-1}$ is its inverse. It thus follows that K is odd and in particular $K(0) = 0$. For any $x > 0$, integration by parts twice yields

$$\begin{aligned} K(x) &= -\frac{1}{\pi x}\int_0^{\infty} \frac{1-(r-1)\xi^r}{(1+\xi^r)^2}\cos(x\xi)\,d\xi \\ &= -\frac{1}{\pi x^2}\int_0^{\infty} \frac{r(r+1)\xi^{r-1} - r(r-1)\xi^{2r-1}}{(1+\xi^r)^3}\sin(x\xi)\,d\xi. \end{aligned}$$

It is thereby concluded that $K(x) = O(x^{-2})$ as $x \to \infty$. On the other hand, for $x > 0$, $K(x)$ may also be represented in the form

$$\begin{aligned} K(x) &= -\frac{1}{\pi}\int_0^{\frac{1}{x}} \frac{\xi \sin(\xi x)}{1+\xi^r}\,d\xi - \frac{1}{\pi}\int_{\frac{1}{x}}^{\infty} \frac{\xi \sin(\xi x)}{1+\xi^r}\,d\xi \\ &= -\frac{1}{\pi}\int_0^{\frac{1}{x}} \frac{\xi \sin(\xi x)}{1+\xi^r}\,d\xi - \frac{1}{\pi}\frac{x^r \cos 1}{x^2(x^r+1)} - \frac{1}{\pi x}\int_{\frac{1}{x}}^{\infty} \frac{1-(r-1)\xi^r}{(1+\xi^r)^2}\cos(x\xi)\,d\xi \\ &= -\frac{1}{\pi}\int_0^{\frac{1}{x}} \frac{\xi \sin(\xi x)}{1+\xi^r}\,d\xi - \frac{x^r \cos 1}{\pi x^2(x^r+1)} - \frac{x^r}{\pi x^2}\int_1^{\infty} \frac{x^r-(r-1)y^r}{(x^r+y^r)^2}\cos y\,dy. \end{aligned}$$

It follows immediately that

$$|K(x)| \le \frac{1}{\pi}\int_0^{\frac{1}{x}} \frac{\xi^2 x}{1+\xi^r}\,d\xi + x^{r-2}(x^r+1) = \frac{1}{\pi}\int_0^1 \frac{\xi^2}{x^{2-r}(x^r+\xi^r)}\,d\xi + x^{r-2}(x^r+1).$$

It is straightforward to see that

$$|K(x)| = O\Big(x^{r-2}\Big)$$

as $x \to 0$. These considerations imply $K \in L_1 \cap L_q$ for any $q < 1/(2-r)$ if $r < 2$, whilst $K \in L_1 \bigcap L_\infty$ if $r \ge 2$.

Here is a local existence result for (2.3).

Theorem 2.1. *Consider the integral equation (2.3) and suppose the nonlinear function g and the integral kernel K satisfy hypotheses (H1) and (H2). Then (2.3) is locally well-posed in L_q for any q with*

$$q \ge \max\Big\{p,\ \frac{r(p-1)}{r-1}\Big\}, \tag{2.5}$$

where p and r are the values specified in (H1) and (H2). Moreover, the flow map $\mathcal{G} : u_0 \mapsto u$, that associates to the initial data u_0 the unique solution u, is C^1.

Proof. It is shown that the operator $\mathcal{A}$ in (2.4) is a contraction mapping of $C(0, T; B)$ for some $T > 0$, where B is a closed ball in L_q.

The condition on q in (2.5) implies that $1 < q/(q-p+1) \le r$, hence, $K \in L_{q/(q-p+1)}$. For any $u \in L_q$, note that

$$|g(u) + u| = \Big|\int_0^1 \Big(1 + g'(su)\Big)\,ds\,u\Big| \le C_0\Big(|u| + |u|^p\Big), \tag{2.6}$$

where C_0 is the constant appearing in (H1). By Young's inequality,

$$|K * (g(u)+u)|_q \le C_0\big(|K|_1|u|_q + |K|_{q/(q-p+1)}|u|_q^p\big), \tag{2.7}$$

which is to say,

$$K * (g(u) + u) \in L_q\,.$$

It follows that $\mathcal{A}$ maps $C(0, T; L_q)$ to itself, for any $T > 0$.

It is now shown that $\mathcal{A}$ is contractive on a suitable subset of $C(0; T; L_q)$ provided T is well chosen. Specify the constants β and T by

$$\beta = 2|u_0|_q \quad \text{and} \quad T = \frac{1}{2C_0(|K|_1 + |K|_{q/(q-p+1)}\beta^{p-1})}.$$

Define the space X to be

$$X = X_{T,\beta} = C(0, T; B_\beta)$$

where $B_\beta = \{u \in L_q : |u|_q \leq \beta\}$. The set X is a complete metric space with the distance d induced by the norm on $C(0,T;L_q)$. For any $u \in X$,

$$d(\mathcal{A}u, 0) = \|\mathcal{A}u\|_X \leq |u_0|_q + TC_0\Big(|K|_1\beta + |K|_{q/(q-p+1)}\beta^p\Big) \leq \beta.$$

That is to say, $\mathcal{A}$ maps X to itself. Moreover, if $u, v \in X$, then

$$\begin{aligned}
&\Big|K * \big(u - v + g(u) - g(v)\big)\Big|_q \\
&= \Big|K * \int_0^1 \big(1 + g'(v + s(u-v))\big)\, ds (u-v)\Big|_q \\
&\leq \Big||K| * \int_0^1 C_0(1 + |v + s(u-v)|^{p-1})\, ds |u-v|\Big|_q \qquad (2.8)\\
&\leq C_0 \int_0^1 \Big\{|K|_1|u-v|_q + |K|_{q/(q-p+1)}|v + s(u-v)|_q^{p-1}|u-v|_q\Big\}\, ds \\
&\leq C_0\Big\{|K|_1 + |K|_{q/(q-p+1)}\beta^{p-1}\Big\}|u-v|_q.
\end{aligned}$$

In consequence, it appears that

$$\begin{aligned}
&\big|\mathcal{A}u(\cdot,t) - \mathcal{A}v(\cdot,t)\big|_q \\
&= \Big|\int_0^t K * \Big(u(\cdot,\tau) - v(\cdot,\tau) + g(u(\cdot,\tau)) - g(v(\cdot,\tau))\Big)\, d\tau\Big|_q \\
&\leq \int_0^t \Big|K * \Big(u(\cdot,\tau) - v(\cdot,\tau) + g(u(\cdot,\tau)) - g(v(\cdot,\tau))\Big)(\cdot,\tau)\Big|_q\, d\tau \\
&\leq C_0 \int_0^t \Big\{|K|_1 + |K|_{q/(q-p+1)}\beta^{p-1}\Big\}|u(\cdot,\tau) - v(\cdot,\tau)|_q\, d\tau.
\end{aligned}$$

Taking the maximum in this inequality for $t \in [0,T]$ yields

$$\begin{aligned}
&d(\mathcal{A}u, \mathcal{A}v) = ||\mathcal{A}u - \mathcal{A}v||_X \\
\leq &C_0T(|K|_1 + |K|_{q/(q-p+1)}\beta^{p-1})||u-v||_X \leq \frac{1}{2}||u-v||_X = \frac{1}{2}d(u,v). \qquad (2.9)
\end{aligned}$$

This, together with the fact that $\mathcal{A}$ maps X to itself, demonstrates that $\mathcal{A}$ is contractive. The contraction mapping principle then comes to our rescue and existence and uniqueness in $C(0,T;L_q)$ follow readily.

Define an operator $\mathcal{B}$ as

$$\mathcal{B}v = \int_0^t K * \big(v + g(v)\big)\, ds$$

(so that $\mathcal{A}$ and $\mathcal{B}$ are related by $\mathcal{A}v = u_0 + \mathcal{B}v$). A calculation shows $\mathcal{B}$ is Fréchet differentiable and that for $v, h \in C(0,T;L_q)$,

$$\mathcal{B}'(v)h = \int_0^t \int_{-\infty}^{\infty} K(x-y)\big(h + g'(v)h\big)\, dy\, d\tau = \int_0^t \int_{-\infty}^{\infty} K(x-y)\big(1 + g'(v)\big)h\, dy\, d\tau.$$

For $\phi, \psi \in C(0,T;L_q)$, define

$$\mathcal{H}(\phi, \psi) = \psi - \phi - \mathcal{B}\psi,$$

so that when $\phi = u_0$ and $\psi = u$ where u is the fixed point of the operator $\mathcal{A}$ corresponding to initial data u_0, then

$$\mathcal{H}(u_0, u) = u - u_0 - \mathcal{B}(u) = u - \mathcal{A}u = 0$$

and

$$D_\psi \mathcal{H}(u_0, u)h = h - \int_0^t \int_{-\infty}^{\infty} K(x-y)\big(1+g'(u)\big)h\,dy\,d\tau = h - \mathcal{B}'(u)h.$$

Moreover, because of the choice of T and β, it is seen that

$$\begin{aligned}
|\mathcal{B}'(u)h|_q &= \Big|\int_0^t \int_{-\infty}^{\infty} K(\cdot - y)\big(1+g'(u)\big)h\,dy\,d\tau\Big|_q \\
&\le T \sup_{0\le t\le T} |K * \big((1+g'(u)h\big)|_q \\
&\le C_0 T \sup_{0\le t\le T} \Big(|K|_1 |h(\cdot,t)|_q + |K|_{\frac{q}{q-p+1}} |u(\cdot,t)|_q^{p-1} |h(\cdot,t)|_q\Big) \\
&\le C_0 T \Big(|K|_1 + |K|_{\frac{q}{q-p+1}}\, \beta^{p-1}\Big)\|h\|_X \\
&= \frac{1}{2}\|h\|_X.
\end{aligned}$$

Hence, $D_\psi \mathcal{H}(u_0, u) = I - \mathcal{B}'(u)$ is invertible and therefore, by the implicit function theorem, the flow map $\mathcal{G}(u_0) = u$ is a C^1 map, and

$$D_{u_0} u = -(I - \mathcal{B}'(u))^{-1} D_\phi \mathcal{H}(u_0, u).$$

□

Corollary 1. *If $g \in C^k$ and $g^{(j)}$ is bounded by a polynomial of degree $p-j$ for $j = 1, \cdots, k$, then the flow map $u_0 \mapsto u = \mathcal{G}(u_0)$ is a C^k-map from L_q to $C(0,T;L_q)$. If g is a polynomial, then the flow map $\mathcal{G}$ is real analytic.*

Proposition 1. *(Temporal Regularity) Suppose g in (2.3) is a polynomial of degree p. Then the solution $u \in C^\infty(0,T;L_q)$, or what is the same,*

$$\frac{\partial^k u}{\partial t^k} \in C(0,T;L_q)$$

for all $k \ge 0$.

Proof. Because of Theorem 2.1, there is $T > 0$ such that $u \in C(0,T;L_q)$ is the fixed point of the operator $\mathcal{A}$ as in (2.3).

Our earlier considerations have revealed that if v is defined to be

$$v(x,t) = K * \big(u + g(u)\big)(x,t) \in C(0,T;L_q),$$

then from (2.2),

$$u(x,t) = u_0 + \int_0^t v(x,s)\,ds \in C^1(0,T;L_q).$$

Once u is known to lie in $C^1(0,T;L_q)$, estimates which are by now familiar show that $v \in C^1(0,T;L_q)$ and that

$$u_{tt} = v_t = K * \big(u_t + g'(u)\,u_t\big). \tag{2.10}$$

A straightforward induction now finishes the proof. Note that the formula for $\partial_t^k u$ is easily determined to be

$$\partial_t^k u = K * \Big(\partial_t^{k-1} u + g'(u)\partial_t^{k-1} u + \cdots\Big),$$

$k = 1, 2 \cdots$, where all the terms under the convolution are composed of monomials in u, u_t, $\cdots$, $\partial_t^{k-1} u$ of degree at most p. □

Another point worth mentioning is the smoothing associated with taking a temporal derivative. Indeed, since $\widehat{u_t} = \frac{-i\xi}{1+\alpha(\xi)}\big(\widehat{u} + \widehat{g(u)}\big)$, u_t is smoother than $u + g(u)$ if α grows super-linearly at infinity. For simplicity, let the nonlinear function g be homogeneous, say $g(z) = z^p$. For the dispersion α, suppose there is a positive number $s > 1 + \frac{p-1}{q}$ such that

$$\liminf_{|\xi|\to\infty} \frac{\alpha(\xi)}{|\xi|^s} > 0.$$

Then, for any ϵ in the range $[0, s-1-\frac{p-1}{q})$,

$$\begin{aligned}
\Big|(1+|\xi|)^\epsilon \widehat{u_t}(\xi,t)\Big|_{q/(q-1)} &= \Big|\frac{i\xi(1+|\xi|)^\epsilon}{1+\alpha(\xi)}\Big(\widehat{u}(\xi,t) + \widehat{u^p}(\xi,t)\Big)\Big|_{q/(q-1)} \\
&\le \gamma_1 |\widehat{u}(\cdot,t)|_{q/(q-1)} + \gamma_2 |\widehat{u^p}(\cdot,t)|_{q/(q-p)} \\
&\le \gamma_1 |\widehat{u}(\cdot,t)|_{q/(q-1)} + \gamma_2 |\widehat{u}(\cdot,t)|^p_{q/(q-1)}
\end{aligned}$$

where the numbers γ_1 and γ_2 are determined to be

$$\gamma_1 = \sup_{\xi\in\mathbb{R}} \frac{|\xi|(1+|\xi|)^\epsilon}{1+\alpha(\xi)} \quad \text{and} \quad \gamma_2^{\frac{q}{p-1}} = \int_{\mathbb{R}} \Big(\frac{|\xi|(1+|\xi|)^\epsilon}{1+\alpha(\xi)}\Big)^{\frac{q}{p-1}} d\xi.$$

Thus, $u_t \in C\big(0,T; W^\epsilon_{q/(q-1)}\big)$ where for $r \ge 1$, $W_r^\epsilon = \{u \in L_r : (1+\xi^2)^{\frac{\epsilon}{2}}\widehat{u} \in L_r\}$. In particular, for the original BBM-equation where $s = 2$ and $p = 2$, if the initial data $u_0 \in L_2$, then the solution $u \in C(0,\infty; L_2)$ as proved by Bona and Tzvetkov (2009). In this case, it is concluded from the above ruminations that the time derivative u_t lies in $C(0,\infty; H^1)$ and so is spatially smoother than u.

Proposition 2. *(Spatial Regularity) Let $u \in C(0,T;L_q)$ be the solution whose existence is guaranteed in Theorem 2.1. Furthermore, suppose that $u_0^{(j)} \in C_b \bigcap L_q$ for $j = 1, 2, \cdots, k$. Presume also that g is polynomial of degree p. Then $u \in C(0,T; C_b^k \cap W_q^k)$. Moreover, the flow map $\mathcal{G} : u_0 \mapsto u$ is continuous from $u_0 \in C_b^k \bigcap W_q^k$ to $u \in C(0,T; C_b^k \cap W_q^k)$.*

Proof. The contraction mapping argument used to prove existence of solutions in L_q is readily adapted to the space $W_q^k \cap C_b^k$. It follows therefore that at least on some, possibly shorter time interval $[0, T']$, the solution u lies in $C(0,T'; W_q^k \cap C_b^k)$. It remains to see that we can take $T' = T$, which is to say, so long as the solution remains in L_q, it must also lie in the smaller space $W_q^k \cap C_b^k$.

It suffices to provide *a priori* bounds on the relevant norms. These are derived using the inequalities

$$|K * (u_1 u_2 \cdots u_p)|_q \le |K|_{\frac{q}{q-p+1}} |u_1|_q \cdots |u_p|_q$$

and

$$|K * (u_1 u_2 \cdots u_p)|_\infty \le |K|_{\frac{q}{q-p+1}} |u_1|_q \cdots |u_{p-1}|_q |u_p|_\infty.$$

The derivation of the relevant *a priori* bounds proceeds by induction on k. First, note that by hypothesis (H1), (or indeed by the fact that g is a polynomial of degree p that vanishes at the origin), there is a constant C such that $|z+g(z)| \le C(|z|+|z|^p)$ for all $z \in \mathbb{R}$. For $k = 0$, proceed as follows. If $u_0 \in L_q \cap C_b$, then since

$$u = u_0 + \int_0^t K * \Big(u + g(u)\Big)\, ds,$$

it follows immediately that

$$|u(\cdot,t)|_\infty \le |u_0|_\infty + C\int_0^t \Big(|K|_1|u(\cdot,s)|_\infty + |K|_{\frac{q}{q-p+1}}|u(\cdot,s)|_q^{p-1}|u(\cdot,s)|_\infty\Big)\,ds.$$

Thus, as long as $|u(\cdot,s)|_q$ remains bounded, Gronwall's Lemma provides the desired L_∞-bound.

If $k=1$, argue as follows. Let $v=\partial_x u$ so that v satisfies the integral equation

$$v(x,t) = u_0'(x) + \int_0^t K * \big(v + g'(u)v\big)\,ds.$$

Since $|1+g'(z)| \le C_0(1+|z|^{p-1})$ for all $z\in\mathbb{R}$ on account of Hypothesis (H1), it follows that at least for $0\le t\le T'$,

$$|v(\cdot,t)|_q \le |u_0'|_q + C_0\int_0^t \Big(|K|_1|v(\cdot,s)|_q + |K|_{\frac{q}{q-p+1}}|u(\cdot,s)|_q^{p-1}|v(\cdot,s)|_q\Big)\,ds \tag{2.11}$$

and

$$|v(\cdot,t)|_\infty \le |u_0'|_\infty + C_0\int_0^t \Big(|K|_1|v(\cdot,s)|_\infty + |K|_{\frac{q}{q-p+1}}|u(\cdot,s)|_q^{p-1}|v(\cdot,s)|_\infty\Big)\,ds. \tag{2.12}$$

Again, Gronwall's Lemma comes to our rescue and the function $v=\partial_x u$ is seen to possess an *a priori* bound as long as $|u(\cdot,s)|_q$ remains bounded.

If $k=2$, then if $w=\partial_x^2 u$, it follows that w satisfies the integral equation

$$w = u_0'' + \int_0^t K * \big(w + g'(u)w + g''(u)v^2\big)\,ds$$

where $v=\partial_x u$ as before. Since g'' is a polynomial of degree $p-2$, there is a constant $C_1>0$ such that $g''(z)$ is bounded by $C_1(1+|z|^{p-2})$ for any $z\in\mathbb{R}$. As in (2.11) and (2.12), one sees that

$$\begin{aligned}|w(\cdot,t)|_q \le |u_0''|_q &+ C_0\int_0^t \Big(|K|_1|w(\cdot,s)|_q + |K|_{\frac{q}{q-p+1}}|u(\cdot,s)|_q^{p-1}|w(\cdot,s)|_q\Big)\,ds\\ &+ C_1\int_0^t \Big(|K|_1|v(\cdot,s)|_\infty|v(\cdot,s)|_q + |K|_{\frac{q}{q-p+1}}|u(\cdot,s)|_q^{p-2}|v(\cdot,s)|_q^2\Big)\,ds\end{aligned}$$

and

$$\begin{aligned}|w(\cdot,t)|_\infty \le &|u_0''|_\infty + C_0\int_0^t \Big(|K|_1|v(\cdot,s)|_\infty + |K|_{\frac{q}{q-p+1}}|u(\cdot,s)|_q^{p-1}|w(\cdot,s)|_\infty\Big)\,ds\\ &+ C_1\int_0^t \Big(|K|_1|v(\cdot,s)|_\infty^2 + |K|_{\frac{q}{q-p+1}}|u(\cdot,s)|_q^{p-2}|v(\cdot,s)|_q|v(\cdot,s)|_\infty\Big)\,ds.\end{aligned}$$

Since v is already known to lie in $L_q\cap C_b$, the last two inequalities allow another application of Gronwall's Lemma and *a priori* bounds in $W_q^2\cap C_b^2$ result.

A tedious, but straightforward induction concludes the proof of the proposition. □

Remark: With a little more effort, one can show that the iteration starting at the initial data u_0

$$u_{n+1} = \mathcal{A}u_n, \quad n=0,1,2,\cdots,$$

which is known to converge in $C(0,T;L_q)$, converges also in $C(0,T;W_q^k\cap C_b^k)$, even though the mapping $\mathcal{A}$ need not to be contractive in this smaller space. For example, suppose $u_0\in W_q^1$ and let $\{u_n\}$ be the sequence of iterates defined above.

On the interval $[0, T]$, it is known that $u_n \to u$ in $C(0, T; L_q)$. Moreover, it follows from the contraction mapping principle that there is a θ with $0 < \theta < 1$ such that

$$\sup_{0\le t\le T} |u_{n+1}(\cdot, t) - u_n(\cdot, t)|_q \le D\theta^n, \quad n = 1, 2, \cdots, \tag{2.13}$$

where $D = |u_0|_q$. Of course, $u_n \in C(0, T; W_q^1)$, but it is not immediately clear that $u_n \to u$ in $C(0, T; W_q^1)$. However, if it is the case that

$$\sum_{n=0}^{\infty} \sup_{0\le t\le T} |v_{n+1}(\cdot, t) - v_n(\cdot, t)|_q < +\infty \tag{2.14}$$

where $v = \partial_x u$ as before, then $\{v_n\}$ is Cauchy in $C(0, T; L_q)$ and thus $u_n \to u$ in $C(0, T; W_q^1)$.

The boundedness of the sum in (2.14) will follow from the following inequality; for $n = 0, 1, \cdots$,

$$\begin{aligned} |v_{n+1}(\cdot, t) - v_n(\cdot, t)|_q \le\ & C \int_0^t |K|_1 |v_n(\cdot, s) - v_{n-1}(\cdot,)|_q \, ds \\ &+ C \int_0^t |K|_{\frac{q}{q-p+1}} \Big(|u_n(\cdot, s) - v_n(\cdot, s)|_q^{p-1} |v_n(\cdot, s)|_q \\ &\qquad + |u_{n-1}(\cdot, s)|_q^{p-1} |v_{n+1}(\cdot, s) - v_n(\cdot, s)|_q \Big) \, ds \end{aligned}$$

where C is a constant independent of n of course. As it is already known from Proposition 2.4 that $\|u_n\|_{C(0,T;W_q^1)} \le M$ for all n, this inequality can be extended to

$$|v_{n+1}(\cdot, t) - v_n(\cdot, t)|_q \le A \int_0^t |v_n(\cdot, s) - v_{n-1}(\cdot, s)|_q \, ds + B \int_0^t |u_n(\cdot, s) - u_{n-1}(\cdot, s)|_q \, ds \tag{2.15}$$

where A and B depend only on M, C, and norms of the kernel K. For $n = 0, 1, 2, \cdots$, define σ_n and τ_n by

$$\sigma_n = \sup_{0\le t\le T} |v_{n+1}(\cdot, t) - v_n(\cdot, t)|_q$$

and

$$\tau_n = \sup_{0\le t\le T} |u_{n+1}(\cdot, t) - u_n(\cdot, t)|_q.$$

For $0 \le t \le T$,

$$|v_1(\cdot, t) - v_0(\cdot, t)|_q \le A|u_0'|_q t + B\tau_1 t,$$

$$|v_2(\cdot, t) - v_1(\cdot, t)|_q \le A(A|u_0'|_q + B\tau_1)\frac{t^2}{2} + B\tau_2 t,$$

and, inductively, it is seen that

$$\begin{aligned} &|v_{n+1}(\cdot, t) - v_n(\cdot, t)|_q \\ \le\ &A^n(A|u_0'|_q + B\tau_1)\frac{t^n}{n!} + A^{n-1}B\tau_2\frac{t^{n-1}}{(n-1)!} + A^{n-2}B\tau_3\frac{t^{n-2}}{(n-2)!} + \cdots + B\tau_{n+1}t, \end{aligned}$$

for $n = 2, 3, \cdots$. Taking the supremum of the last inequality for $t \in [0, T]$ yields

$$\sigma_n \le (A|u_0'|_q + B\tau_1)\frac{(AT)^n}{n!} + B\tau_2\frac{(AT)^{n-1}}{(n-1)!} + B\tau_3\frac{(AT)^{n-2}}{(n-2)!} + \cdots + B\tau_{n+1}T.$$

Because of (2.13), it is readily deduced from this inequality that

$$\sum_{n=0}^{\infty} \sigma_n < +\infty,$$

whence the desired conclusion. This argument is straightforwardly generalized to the full setting $W_q^k \cap C_b^k$ by induction on k.

Theorem 2.2. *In Theorem 2.1, suppose that the relationship between r and p is further restricted by requiring*

$$r \geq \frac{p+1}{2}.$$

Then the integral equation (2.3) is locally well posed in $L_2 \cap L_{p+1}$, so, if the initial data $u_0 \in L_2 \cap L_{p+1}$, then there is a $T > 0$ such that (2.3) has an unique solution u lying in $C(0,T; L_2 \cap L_{p+1})$, and the mapping $u_0 \mapsto u$ is Lipschitz from the space $L_2 \cap L_{p+1}$ to $C(0,T; L_2 \cap L_{p+1})$. Moreover, its L_2-norm is bounded by

$$|u(\cdot,t)|_2^2 \leq e^{Ct}|u_0|_2^2 + 2\int_0^t e^{C(t-\tau)}|u(\cdot,\tau)|_{p+1}^{p+1}\, d\tau,$$

where $C = 2C_0|K|_1$.

Proof. It is already understood that there is a unique solution $u \in C(0,T; L_q)$ whenever $q \geq p+1$. The result at hand follows by arguments that are, by now, familiar, from the inequality

$$|u(\cdot,t)|_2 \leq |u_0|_2 + \int_0^t |K|_1 |u(\cdot,s)|_2\, ds + C\int_0^t |K|_{\frac{2p+2}{p+3}} |u(\cdot,s)|_{p+1}^p\, ds.$$

Moreover, the sequence of iterates $u_{n+1} = \mathcal{A}u_n$, $n = 0, 1, \cdots$, starting anywhere in the appropriate ball around the origin in L_{p+1} also converges in $C(0,T; L_2)$. □

Lemma 1. *Let $u \in C(0,T; L_2 \cap L_{p+1})$ be a solution of (2.3). The functional*

$$\int_{-\infty}^{\infty} \big(F(u) + \frac{1}{2}u^2\big)\, dx$$

is bounded and independent of t, where F is the primitive of g given by $F(z) = \int_0^z g(z)\, dz$.

Proof. For smooth solutions, we have that

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{\infty} \Big(F(u(x,t)) + \frac{1}{2}u^2(x,t)\Big)\, dx \\
&= \int_{-\infty}^{\infty} (g(u) + u)u_t\, dx \\
&= -\int_{-\infty}^{\infty} (g(u)+u)(I+L)^{-1}\partial_x\big(g(u)+u\big)\, dx.
\end{aligned}$$

As $(I+L)^{-1}\partial_x$ is skew-adjoint, the right-hand side is obviously zero. For solutions in the advertised class, the result follows from the regularity theory, the continuous dependence of solutions on the initial data and density of, say, $\mathcal{D}(\mathbb{R})$ in $L_2 \bigcap L_{p+1}$. □

Corollary 2. *Let $u \in C(0,T;L_2 \bigcap L_{p+1})$ be the solution in Lemma 1. If there is a positive number γ such that the function F satisfies $2F(x)+x^2 > \gamma(x^2+|x|^{p+1})$ for all $x \in \mathbb{R}$, then,*

$$\int_{-\infty}^{\infty} (u^2 + |u|^{p+1})\, dx \le \frac{1}{\gamma} \int_{-\infty}^{\infty} (2F(u) + u^2)\, dx.$$

In consequence, the local existence result can be iterated to produce a solution u of (2.3) which lies in $C(0,\infty;L_2 \bigcap L_{p+1})$.

The next result is a special case of Theorem 2.1 and Corollary 2.7.

Corollary 3. *Let $p \ge 1$ be any integer. The generalized BBM-equation*

$$u_t + u_x + u^{p-1}u_x - u_{xxt} = 0, \quad x \in \mathbb{R},\, t > 0,$$

is locally well-posed in L_q for any $q \ge p$. That is, if the initial data $u(\cdot,0) = u_0 \in L_q$, then there exists a positive number $T = T(|u_0|_q)$ such that the above equation has an unique solution $u \in C(0,T;L_q)$ which is continuously dependent on u_0. If $p \ge 3$ is an odd integer and the initial data $u_0 \in L_2 \cap L_{p+1}$, then the solution u is globally defined and lies in $C_b(0,\infty;L_2 \cap L_{p+1})$.

Remark: The issue of global solutions in $L_2 \cap L_{p+1}$ remains open for even, positive integers p. In the case $p = 2$, that is the BBM-equation (1.1), it is known that the problem is globally well posed in L_2 (see Bona and Tzvetkov 2009).

3. **Initial data in L_2-based Sobolev classes.** In this section, attention is turned to the initial-value problem (0.1) in the L_2-based Sobolev spaces H^s. We discuss a general relation between s, the properties of the dispersion relation α and the nonlinearity g which guarantees well-posedness locally and globally in time. In a special case when g is simply quadratic, $g(u) = \frac{1}{2}u^2$ say, we are interested in how small s can be and have well-posedness globally in time. The technique for the latter analysis is based on theory developed in Bona and Tzvetkov (2009).

The following assumptions will be in force throughout this section.

(A1) $g : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree p with $g(0) = g'(0) = 0$.

(A2) $\alpha : \mathbb{R} \to \mathbb{R}$ is continuous, even, vanishing at zero, and there is an $r \ge 1$ such that

$$0 < \gamma_0 = \inf_{\xi\in\mathbb{R}} \frac{1+\alpha(\xi)}{(1+|\xi|)^r} \le \sup_{\xi\in\mathbb{R}} \frac{1+\alpha(\xi)}{(1+|\xi|)^r} = \tilde{\gamma} < \infty.$$

Theorem 3.1. *The initial value problem (0.1) is locally well-posed in H^s provided that r, s and p satisfy any of the following three criteria:*

(1) $r \ge 1$ and $s > \frac{1}{2}$;

(2) $r > \frac{3}{2} - \frac{1}{2p}$ and $s > \frac{1}{2} - \frac{1}{2p}$;

(3) $r > 2 - \frac{1}{p}$ and $s > \frac{1}{2} - \frac{1}{p}$ for $p > 2$ or $s \ge 0$ for $p = 2$.

Proof. Take the Fourier transform in (0.1) with respect to the spatial variable x and divide both sides of the outcome by $1+\alpha(\xi)$ to obtain

$$\widehat{u}_t(\xi,t) = \frac{-i\xi}{1+\alpha(\xi)}\Big(\widehat{u}(\xi,t) + \widehat{g(u)}(\xi,t)\Big).$$

After an integration with respect to t, there appears

$$\widehat{u}(\xi,t) = \widehat{u}_0(\xi) - \int_0^t \frac{i\xi}{1+\alpha(\xi)}\Big(\widehat{u}(\xi,\tau) + \widehat{g(u)}(\xi,\tau)\Big)\, d\tau. \tag{3.1}$$

For any $T > 0$, define an operator $A : C(0,T;H^s) \to C(0,T;H^s)$ by

$$\widehat{Au}(\xi,t) = \widehat{u}_0(\xi) - \int_0^t \frac{i\xi}{1+\alpha(\xi)}\Big(\widehat{u}(\xi,\tau) + \widehat{g(u)}(\xi,\tau)\Big)\,d\tau.$$

In case (1) where $r \geq 1$ and $s > \frac{1}{2}$, $g(u) \in H^s$ if $u \in H^s$ since g is a polynomial and H^s is a Banach algebra. In case (2), $r > \frac{3}{2} - \frac{1}{2p}$ and $s > \frac{1}{2} - \frac{1}{2p}$, $u \in H^s$ implies $u^q \in H^{s-\frac{1}{2}+\frac{1}{2p}}$ for any $q \leq p$ and hence $g(u) \in H^{s-\frac{1}{2}+\frac{1}{2p}}$. Since $\alpha(\xi)$ has the growth rate $|\xi|^r$ as $\xi \to \infty$, in both cases (1) and (2), A is therefore seen to map $C(0,\infty;H^s)$ to itself.

In case (3) where $p = 2$, $r > \frac{3}{2}$ and $s \geq 0$, if $u \in H^s$, then for any $\xi \in \mathbb{R}$,

$$(1+|\xi|)^s|\widehat{u^2}(\xi)| \leq \Big((1+|\cdot|)^s|\widehat{u}|\Big)*\Big((1+|\cdot|)^s|\widehat{u}|\Big)(\xi) \leq \int_{-\infty}^{\infty}(1+|\xi|)^{2s}|\widehat{u}(\xi)|^2\,d\xi = \|u\|_s^2.$$

In case (3) where $p \geq 3$, $r > 2 - \frac{1}{p}$ and $s > \frac{1}{2} - \frac{1}{p}$, let $\epsilon \in (0, s - \frac{1}{2} + \frac{1}{p}) \bigcap (0, r - 2 + \frac{1}{p})$ be sufficiently small. Then, for any $u \in H^s$ and $\xi \in \mathbb{R}$,

$$\begin{aligned}(1+|\xi|)^{s-\frac{1}{2}+\frac{1}{p}-\epsilon}|\widehat{u^p}(\xi)| &\leq \Big((1+|\cdot|)^{s-\frac{1}{2}+\frac{1}{p}-\epsilon}|\widehat{u}|\Big)*\cdots*\Big((1+|\cdot|)^{s-\frac{1}{2}+\frac{1}{p}-\epsilon}|\widehat{u}|\Big)(\xi)\\ &\leq \Big\{\int_{-\infty}^{\infty}\Big((1+|\xi|)^{s-\frac{1}{2}+\frac{1}{p}-\epsilon}|\widehat{u}(\xi)|\Big)^{\frac{p}{p-1}}\,d\xi\Big\}^{p-1}\\ &\leq \Big\{\int_{-\infty}^{\infty}(1+|\xi|)^{-1-\frac{2p\epsilon}{p-2}}\,d\xi\Big\}^{\frac{p-2}{2}}\|u\|_s^p.\end{aligned}$$

The second inequality holds since $|f_1 * \cdots * f_p|_\infty \leq |f_1|_q|f_2|_q\cdots|f_p|_q$ where $q = p/(p-1)$. These relations in turn lead to the inequality

$$\int_{-\infty}^{\infty}(1+|\xi|)^{2s}\frac{|i\xi|^2}{(1+\alpha(\xi))^2}|\widehat{u^p}(\xi)|^2\,d\xi \leq \frac{1}{\gamma_0^2}\int_{-\infty}^{\infty}(1+|\xi|)^{2s-2r+2}|\widehat{u^p}(\xi)|^2\,d\xi \leq C\|u\|_s^{2p},$$

where γ_0 is as in (A2) and C is a constant which need not be any larger than

$$\frac{1}{\gamma_0^2}\Big(\int(1+|\xi|)^{-2r+3-\frac{2}{p}+2\epsilon}\,d\xi\Big)\Big(\int_{-\infty}^{\infty}(1+|\xi|)^{-1-\frac{2p\epsilon}{p-2}}\,d\xi\Big)^{p-2}.$$

It is concluded that $Au \in C(0,\infty;H^s)$ if $u \in C(0,\infty;H^s)$ in both the situations comprising case (3).

Following the steps laid out in the proof of Theorem 2.1, it can be shown that in all three cases, when $T > 0$ is chosen sufficiently small, the operator A is contractive in $C(0,T;B_{2\|u_0\|_s})$, where $B_{2\|u_0\|_s} = \{u : u \in H^s, \|u\|_s \leq 2\|u_0\|_s\}$. The contraction mapping principle completes the proof. ☐

Remark: It is worth noting that if u is a solution of (0.1) and $u \in C(0,T;H^{\frac{r}{2}})$, then the functional

$$\int_{-\infty}^{\infty}\Big(u^2(x) + uLu(x)\Big)\,dx = \int_{-\infty}^{\infty}(1+\alpha(\xi))|\widehat{u}(\xi)|^2\,d\xi$$

is independent of time. As a consequence, hypothesis (A2) implies that

$$\|u(\cdot,t)\|_{\frac{r}{2}}^2 \leq \frac{\tilde{\gamma}}{\gamma_0}\|u_0\|_{\frac{r}{2}}^2. \tag{3.2}$$

Lemma 2. *If $u \in H^s$ where $s > \frac{1}{2}$, then for any integer $p \geq 2$, $u^p \in H^s$ and*

$$\|u^p\|_s \leq C|\widehat{u}|_1^{p-1}\|u\|_s$$

where $C > 0$ is a constant dependent only on s and p.

Proof. Elementary considerations reveal that there is a constant c dependent only on $s \geq 0$ and $p \geq 1$ such that

$$(1 + x_1 + \cdots + x_p)^s \leq c(1 + x_1^s + \cdots + x_p^s)$$

for all $x_1, x_2, \cdots, x_p \geq 0$. It follows that

$$\begin{aligned}
\|u^p\|_s^2 &= \int_{-\infty}^{\infty} (1 + |\xi|)^{2s} |\widehat{u^p}(\xi)|^2 \, d\xi \\
&= \int_{-\infty}^{\infty} (1 + |\xi|)^{2s} \big|\widehat{u} * \cdots * \widehat{u}(\xi)\big|^2 \, d\xi \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Big(1 + |\xi - \xi_1 - \cdots - \xi_{p-1} + \xi_1 + \cdots + \xi_{p-1}|\Big)^{2s} \\
&\qquad \big|\widehat{u}(\xi - \xi_1 - \cdots - \xi_{p-1})\widehat{u}(\xi_1) \cdots \widehat{u}(\xi_{p-1})\big|^2 \, d\xi_1 \cdots d\xi_{p-1} \, d\xi \\
&\leq c \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Big(1 + \big|\xi - \xi_1 - \cdots - \xi_{p-1}\big|^{2s} + \big|\xi_1\big|^{2s} \cdots + \big|\xi_{p-1}\big|^{2s}\Big) \\
&\qquad \Big|\widehat{u}(\xi - \xi_1 - \cdots - \xi_{p-1})\widehat{u}(\xi_1) \cdots \widehat{u}(\xi_{p-1)}\Big|^2 \, d\xi_1 \cdots d\xi_{p-1} \, d\xi \\
&\leq c \,\big|\widehat{u} * \cdots \widehat{u}\big|_2^2 + p\, c \,\big|\widehat{D^s u} * \widehat{u} \cdots \widehat{u}\big|_2^2.
\end{aligned}$$

Applying Young's inequality to the right-hand side of the last inequality yields

$$\|u^p\|_s^2 \leq c \,|\widehat{u}|_1^{2(p-1)} \,|\widehat{u}|_2^2 + p\, c \,|\widehat{u}|_1^{2(p-1)} \,|\widehat{D^s u}|_2^2 \leq c\,(p+1)|\widehat{u}|_1^{2(p-1)}\|u\|_s^2,$$

thereby establishing the lemma. ∎

Theorem 3.2. *Suppose g satisfies (A1) and in (A2), $r > 1$. Then the initial-value problem (0.1) is globally well-posed in H^s if $s \geq \frac{r}{2}$.*

Proof. Since $s \geq \frac{r}{2}$ satisfies all three conditions in Theorem 3.1, (0.1) is well-posed locally in time and there is a $T > 0$ such that $u \in C(0, T; H^s)$. It remains to show that T can be taken arbitrarily large.

In the case $s = \frac{r}{2}$, The bound in formula (3.2) implies that the solution can be extended from $C(0, T; H^s)$ to $C(0, \infty; H^s)$.

When $s > \frac{r}{2}$, multiply both sides of (0.1) by $2(I + D)^{2s-r}u(x, t)$ and integrate over $\mathbb{R}$ with respect to x to obtain

$$\begin{aligned}
&2 \int_{-\infty}^{\infty} \Big((I + D)^{2s-r}u(x, t)\Big)\Big((I + L)u_t(x, t)\Big) \, dx \\
&= -\,2 \int_{-\infty}^{\infty} \Big((I + D)^{2s-r}u(x, t)\Big)\Big(u(x, t) + (g(u))(x, t)\Big)_x \, dx \\
&= -\,2 \int_{-\infty}^{\infty} i\xi(1 + |\xi|)^{2s-r}\Big(|\widehat{u}(\xi, t)|^2 + \bar{\widehat{u}}(\xi, t)\widehat{g(u)}(\xi, t)\Big) \, d\xi \\
&= -\,2 \int_{-\infty}^{\infty} i(\xi(1 + |\xi|)^{2s-r}\bar{\widehat{u}}(\xi, t)\widehat{g(u)}(\xi, t) \, d\xi.
\end{aligned}$$

The last expression may be written as

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{\infty}(1+\alpha(\xi))(1+|\xi|)^{2s-r}|\widehat{u}(\xi,t)|^2\,d\xi\\
\leq& 2\int_{-\infty}^{\infty}(1+|\xi|)^{2s-r+1}|\widehat{u}(\xi,t)||\widehat{g(u)}(\xi,t)|\,d\xi\\
\leq& 2\,\|u(\cdot,t)\|_s\|g(u)\|_{s-r+1}\\
\leq& 2\,\|u(\cdot,t)\|_s\|g(u)\|_s,
\end{aligned}$$

at least for smooth solutions. Since g is a polynomial of degree p satisfying (A1), Lemma 2 implies that

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty}(1+\alpha(\xi))(1+|\xi|)^{2s-r}|\widehat{u}(\xi,t)|^2\,d\xi &\leq \tilde{g}(|\widehat{u}(\cdot,t)|_1)\|u(\cdot,t)\|_s^2\\
&\leq c\,\tilde{g}(\|u(\cdot,t)\|_{\frac{r}{2}})\|u(\cdot,t)\|_s^2\\
&\leq c\,\tilde{g}\Big(\frac{\tilde{\gamma}}{\gamma_0}\|u_0\|_{\frac{r}{2}}\Big)\|u(\cdot,t)\|_s^2,
\end{aligned}$$

where $\tilde{g}$ is a polynomial of degree $p-1$ with non-negative coefficients and c depends only on r. Integrating the last inequality with respect to t, it follows that

$$\begin{aligned}
\int_{-\infty}^{\infty}(1+\alpha(\xi))(1+|\xi|)^{2s-r}|\widehat{u}(\xi,t)|^2\,d\xi \leq& \int_{-\infty}^{\infty}(1+\alpha(\xi))(1+|\xi|)^{2s-r}|\widehat{u}_0(\xi)|^2\,d\xi\\
&+C\left(\|u_0\|_{\frac{r}{2}},g\right)\int_0^t\|u(\cdot,\tau)\|_s^2\,d\tau.
\end{aligned}$$

Applying (A2) again yields

$$\gamma_0\|u(\cdot,t)\|_s^2 \leq \gamma_1\|u_0\|_s^2 + C\left(\|u_0\|_{\frac{r}{2}},g\right)\int_0^t\|u(\cdot,\tau)\|_s^2\,d\tau.$$

By the Gronwall lemma, there are two constants c_1 and c_2 in which c_1 is dependent only on $\|u_0\|_s$ and c_2 only on $\|u_0\|_{\frac{r}{2}}$ such that

$$\|u(\cdot,t)\|_s \leq c_1 e^{c_2 t}.$$

This *a priori* bound allows us to iterate the local theory and achieve a globally defined solution. □

It is natural to wonder whether there is global well-posedness for values of $s<\frac{r}{2}$. If so, how small can s be and still maintain global well-posedness? We have results in case g is quadratic. This result follows closely the argument in Bona and Tzvetkov (2009).

Lemma 3. *Consider the initial-value problem*

$$\left.\begin{aligned}
&v_t+v_x+vv_x+Lv_t=0,\quad x\in\mathbb{R},\quad t>0,\\
&v(x,0)=v_0(x),\quad x\in\mathbb{R}.
\end{aligned}\right\}\tag{3.3}$$

If

(i) $r\geq 1$ in hypothesis (A2) and $s>\frac{1}{2}$, or
(ii) $r>\frac{5}{4}$, and $s>\frac{1}{4}$, or
(iii) $r>\frac{3}{2}$ and $s\geq 0$,

then, for any $T>0$, there is an $\epsilon=\epsilon(T)>0$ such that if $v_0\in H^s$ and $\|v_0\|_s\leq\epsilon$, then v exists and lies in $C(0,T';H^s)$ with $T'\geq T$.

Proof. The well-posedness of (3.3) in H^s locally in time is proved in Theorem 3.1 for all three cases. It remains to show how the time interval of existence depends on the magnitude of the initial data. Apply the operator $2(I+D)^{2s-r}$ to both sides of (3.3) and integrate the result over $\mathbb{R}$. This leads to the formula

$$\frac{d}{dt}\int_{-\infty}^{\infty}(I+D)^{2s-r}v(x,t)\,(I+L)v(x,t)\,dx = -\int_{-\infty}^{\infty}(I+D)^{2s-r}v(x,t)(v^2(x,t))_x\,dx$$
$$= -\int_{-\infty}^{\infty} i\xi(1+|\xi|)^{2s-r}\bar{\widehat{v}}(\xi,t)\widehat{v^2}(\xi,t)\,d\xi.$$

Notice that the term

$$\int_{-\infty}^{\infty} v_x(x,t)(I+D)^{2s-r}v(x,t)\,dx$$

that should apparently appear vanishes since $(I+D)^{2s-r}$ is self-adjoint and ∂_x is skew-adjoint. It follows that

$$\frac{d}{dt}\int_{-\infty}^{\infty}(I+D)^{2s-r}v(x,t)(I+L)v(x,t)\,dx \le \int_{-\infty}^{\infty}(1+|\xi|)^{2s-r+1}|\widehat{v}(\xi,t)||\widehat{v^2}(\xi,t)|\,d\xi. \tag{3.4}$$

The argument given in the proof of Theorem 3.1, in all of the three cases (i), (ii) and (iii), shows the right-hand side of the last inequality to be bounded by $c\,\|v(\cdot,t)\|_s^3$ where $c = c(s,r)$ is a positive constant depending only on s and r. That is,

$$\frac{d}{dt}\int_{-\infty}^{\infty}(I+D)^{2s-r}v(x,t)(I+L)v^2(x,t)\,d\xi \le c\,\|v(\cdot,t)\|_s^3.$$

Define $z(t)$ to be the non-negative quantity

$$z(t) = \Big(\int_{-\infty}^{\infty}(I+D)^{2s-r}v(x,t)\,(I+L)v(x,t)\,dx\Big)^{\frac{1}{2}}.$$

Because of condition (A2) on the symbol α of the dispersion operator L, $z(t)$ is equivalent to $\|v(\cdot,t)\|_s$; indeed, because of (A2),

$$\sqrt{\gamma_0}\|v(\cdot,t)\|_s \le z(t) \le \sqrt{\tilde{\gamma}}\|v(\cdot,t)\|_s.$$

In consequence, (3.4) can be extended to

$$\frac{d}{dt}z^2(t) \le c\,\|v(\cdot,t)\|_s^3 \le \frac{c}{\sqrt{\gamma_0^3}}z^3(t).$$

Solving this differential inequality for an upper bound, it is determined that

$$z(t) \le \frac{z(0)}{1-\frac{c}{2\sqrt{\gamma_0^3}}z(0)t}$$

at least as long as $t < 2\sqrt{\gamma_0^3}/cz(0)$. It thus follows that if we choose

$$\epsilon(T) = \frac{\sqrt{\gamma_0^3}}{\sqrt{\tilde{\gamma}}cT},$$

then $z(t) \le 2z(0)$ for $0 \le t \le T$. Using hypothesis (A2), the latter inequality may be extended to

$$\|v(\cdot,t)\|_s \le 2\frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma_0}}\|v_0\|_s$$

provided $0 \le t \le T$. The result follows. ∎

Lemma 4. *Let $v \in C(0,T;H^s)$ be the solution of (3.3) obtained in Lemma 3. Consider the initial-value problem*

$$\left.\begin{aligned} &w_t + w_x + \frac{1}{2}(2vw + w^2)_x + Lw_t = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ &w(x,0) = w_0(x), \quad x \in \mathbb{R}. \end{aligned}\right\} \tag{3.5}$$

If $s \geq 1 - \frac{r}{2}$, then (3.5) is locally well-posed in $H^{\frac{r}{2}}$. Moreover, the solution lies in $C(0,T;H^{\frac{r}{2}})$ on any time interval $[0,T]$ for which v exists.

Proof. As before, (3.5) can be converted to an integral equation, *viz.*

$$w(x,t) = w_0(x) + \frac{1}{2}\int_0^t K * (2w + 2vw + w^2)(x,\tau)\,d\tau$$

where the convolution kernel K is defined as before via its Fourier transform, $\widehat{K}(\xi) = \frac{-i\xi}{\sqrt{2\pi}(1+\alpha(\xi))}$. Define an operator A by

$$Aw(x,t) = w_0(x) + \frac{1}{2}\int_0^t K * (2w + 2vw + w^2)(x,\tau)\,d\tau$$

for $w \in C(0,T;H^{\frac{r}{2}})$. Since $\frac{r}{2} > \frac{1}{2}$, it is straightforward to see that $K * w$ and $K * (w^2)$ lie $H^{\frac{r}{2}}$ if $w \in H^{\frac{r}{2}}$. Likewise, it is not hard to verify that $vw \in H^s$ if $v \in H^s$ and $w \in H^{\frac{r}{2}}$ since $s < \frac{r}{2}$. It follows that $K * (vw) \in H^{r+s-1} \subset H^{\frac{r}{2}}$ if $s \geq 1 - \frac{r}{2}$. Therefore, the operator A maps the space $C(0,T;H^{\frac{r}{2}})$ to itself. Just as in the proof of Theorem 2.1, when $T' < T$ is chosen sufficiently small, A is a contraction mapping of a suitable ball about zero in $C(0,T';H^{\frac{r}{2}})$. That is to say, (3.5) is locally well-posed in $H^{\frac{r}{2}}$.

It remains to extend the time interval from $[0,T']$ to $[0,T]$. An *a priori* bound together with the local well-posedness suffices to establish this fact. Multiply both sides of (3.5) by $2w$ and integrate with respect to x over $\mathbb{R}$; after integration by parts, there appears

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{\infty} w(x,t)(I+L)w(x,t)\,dx &= -\int_{-\infty}^{\infty} v_x(x,t)w^2(x,t)\,dx \\ &\leq \int_{-\infty}^{\infty} |\xi||\widehat{v}(\xi,t)|\,|\widehat{w^2}(\xi,t)|\,dx \\ &\leq \big||\xi|^s\widehat{v}(\cdot,t)\big|_2\,\big||\xi|^{1-s}\widehat{w^2}(\cdot,t)\big|_2 \\ &\leq c\,\|v(\cdot,t)\|_s\|w(\cdot,t)\|_{\frac{r}{2}}^2 \end{aligned}$$

where c is a constant dependent only on r. Integration with respect to t gives

$$\int_{-\infty}^{\infty}(1+\alpha(\xi))|\widehat{w}(\xi,t)|^2\,d\xi \leq \int_{-\infty}^{\infty}(1+\alpha(\xi))|\widehat{w}_0(\xi)|^2\,d\xi + c\int_0^t \|v(\cdot,\tau)\|_s\|w(\cdot,\tau)\|_{\frac{r}{2}}^2\,d\tau,$$

which may be extended to

$$\|w(\cdot,t)\|_{\frac{r}{2}}^2 \leq \frac{\tilde{\gamma}}{\gamma_0}\|w_0\|_{\frac{r}{2}}^2 + \frac{c}{\gamma_0}\int_0^t \|v(\cdot,\tau)\|_s\|w(\cdot,\tau)\|_{\frac{r}{2}}^2\,d\tau$$

on account of (A2). Since $v \in C(0,T;H^s)$, $|v(\cdot,t)|_2$ is well defined for $0 \leq t \leq T$. Gronwall's lemma thus indicates that

$$\|w(\cdot,t)\|_{\frac{r}{2}}^2 \leq \frac{\tilde{\gamma}}{\gamma_0}\|w_0\|_{\frac{r}{2}}^2 \exp\Big(\frac{c}{\gamma_0}\int_0^t \|v(\cdot,\tau)\|_s\,d\tau\Big).$$

The proof is complete. □

Combining the last two lemmas leads to the following conclusion.

Theorem 3.3. *Let $r > 1$ and s satisfy any one of the three conditions enunciated Theorem 3.1. In addition, suppose $g(u) = \frac{1}{2}u^2$ and $s+\frac{r}{2} \geq 1$. Then, the initial-value problem (0.1) is globally well-posed in H^s.*

Proof. If $s \geq \frac{r}{2}$, the global well-posedness is guaranteed by Theorem 3.2. It thus suffices to consider the case where $1 - \frac{r}{2} \leq s < \frac{r}{2}$.

Let $T > 0$ be arbitrary and let $\epsilon = \epsilon(T)$ be the small positive number whose existence is guaranteed by Lemma 3. Since $H^{\frac{r}{2}}$ is dense in H^s, there is $\phi_\epsilon \in H^{\frac{r}{2}}$ such that $\|\phi_\epsilon - u_0\|_s \leq \epsilon$. Then Lemma 3 guarantees that the initial-value problem

$$\left.\begin{aligned} &v_t + v_x + vv_x + Lv_t = 0, \quad x \in \mathbb{R}, \quad t > 0,\\ &v(x,0) = u_0 - \phi_\epsilon, \quad x \in \mathbb{R}\end{aligned}\right\} \tag{3.6}$$

has a unique solution $v \in C(0,T;H^s)$.

Now, consider the Cauchy problem

$$\left.\begin{aligned} &w_t + w_x + \frac{1}{2}(2vw + w^2)_x + Lw_t = 0, \quad x \in \mathbb{R}, \quad t > 0,\\ &w(x,0) = \phi_\epsilon(x), \quad x \in \mathbb{R},\end{aligned}\right\} \tag{3.7}$$

where v is the solution of (3.6). Note that since $\phi_\epsilon \in H^{\frac{r}{2}}$, Lemma 4 implies that (3.7) has a unique solution $w \in C(0;T;H^{\frac{r}{2}})$. It is straightforward to verify that $u = v + w \in C(0,T;H^s)$ solves the original problem,

$$\left.\begin{aligned} &u_t + u_x + uu_x + Lu_t = 0, \quad x \in \mathbb{R}, \quad t > 0,\\ &u(x,0) = u_0, \quad x \in \mathbb{R}.\end{aligned}\right\} \tag{3.8}$$

Since $T > 0$ was arbitrary, local well-posedness together with existence on $[0,T]$ for any T establishes the result. ◻

4. **Bore-like initial data.** The theory developed in Sections 2 and 3 concentrated on initial wave profiles that decay to zero at $\pm\infty$, at least in a weak sense. Attention is turned now to initial data that possesses different asymptotic states at $+\infty$ and $-\infty$. In the water wave context, this corresponds to bore propagation in field situations (see Peregrine 1966, 1967) and hydraulic surges in laboratory configurations. In other physical systems, such data is generated when a signal corresponding to a surge moves into an undisturbed stretch of the medium of propagation. Theoretical work on the bore problem in the context of the BBM-equation was initiated by Benjamin *et al.* (1972) (see also the papers of Bona and Schonbek 1985 and Bona, Rajopadhye and Schonbek 1994, where further theory was developed for both the BBM and the Korteweg-de Vries equations).

In the present contribution, the assumptions on the initial data are weakened and the theory is extended to the broader class of models featured in (0.1).

The mathematical problem amounts to being confronted with the prospect of solutions $u = u(x,t)$ satisfying the boundary conditions

$$\lim_{x\to-\infty} u(x,t) = 1, \qquad \lim_{x\to+\infty} u(x,t) = 0, \tag{4.1}$$

say. The question is, if the initial disturbance is bore-shaped, will the wave evolve in a bore-like pattern? If so, how long will this pattern last? Bona, Rajopadhye and Schonbek (1994) showed that the BBM-equation with bore-like initial data as in (4.1) is globally well posed and that the solution maintains the boundary behavior

(4.1) for all time. In this section, the generalized BBM-type model equations (0.1) will be discussed in the bore context.

Consider the initial-value problem

$$\left.\begin{aligned} &u_t + u_x + g(u)_x + Lu_t = 0, \\ &u(x,0) = u_0(x), \end{aligned}\right\} \tag{4.2}$$

where the operator L and nonlinear function g are as described in Section 2 and the initial data u_0 satisfies the bore condition (4.1). Following the technique used by Bona, Rajopadhye and Schonbek (1994), u_0 can be decomposed into the sum of two parts v_0 and ϕ, say, where $\phi \in C^\infty(\mathbb{R})$ satisfies the bore condition (4.1) and its derivative ϕ' lies in H^∞, and v_0 is a measurable function on $\mathbb{R}$ whose smoothness is determined by the smoothness of u_0.

Introduce a new variable $v = v(x,t)$ by $u(x,t) = v(x,t)+\phi(x)$. Upon substitution of this form into (4.2), there emerges the initial-value problem

$$\left.\begin{aligned} &(I+L)v_t + v_x + \big(g(v+\phi) - g(\phi)\big)_x = -\big(1+g'(\phi)\big)\phi' \\ &v(x,0) = v_0 \end{aligned}\right\} \tag{4.3}$$

for v. Inverting the operator $I+L$ and then integrating with respect to t over $[0,t]$ leads to the integral equation

$$v = v_0 + \int_0^t K * \{v + g(v+\phi) - g(\phi)\}(\cdot,\tau)\, d\tau + tM * \big(1+g'(\phi)\big)\phi' \tag{4.4}$$

where the integral kernels K and M are determined via their Fourier symbols, *viz.*

$$\widehat{K}(\xi) = \frac{-i\xi}{\sqrt{2\pi}\,(1+\alpha(\xi))} \quad \text{and} \quad \widehat{M}(\xi) = \frac{-1}{\sqrt{2\pi}\,(1+\alpha(\xi))},$$

respectively. The following result is the analog in the bore context of Theorem 2.1.

Theorem 4.1. *Suppose the nonlinear function g and the integral kernel K satisfy hypotheses (H1) and (H2) in Section 2. Moreover, suppose that*

$$\inf_{\xi\in\mathbb{R}} \alpha(\xi) > -1 \quad \textit{and} \quad \liminf_{|\xi|\to\infty} \frac{\alpha(\xi)}{1+|\xi|} > 0.$$

Then, for any q such that

$$q \geq \max\Big\{p,\ \frac{r(p-1)}{r-1}\Big\},$$

if $v_0 \in L_q$, then there is a positive number $T = T(|\phi|_\infty, |\phi'|_q) > 0$ such that the integral equation (4.4) has an unique solution $v \in C(0,T;L_q)$ and, moreover, the mapping $v_0 \mapsto v$ is continuous from L_q to $C(0,T;L_q)$.

Proof. For any $v \in C(0,\infty;L_q)$, modify the definition of the operator $\mathcal{A}$ in Section 2 as follows:

$$\mathcal{A}v = v_0 + tM * \Big(\big(1+g'(\phi)\big)\phi'\Big) + \int_0^t K * \Big\{v + g(v+\phi) - g(\phi)\Big\}\, d\tau. \tag{4.5}$$

It is sufficient to prove that $\mathcal{A}$ has a fixed point in $C(0,T;L_q)$ for some $T>0$. Note as before that for any $v \in L_q$,

$$v + g(v+\phi) - g(\phi) = \int_0^1 \Big(1 + g'(\phi + sv)\Big)\, ds\, v.$$

In consequence, it follows that

$$\begin{aligned}|K*\big(v+g(v+\phi)-g(\phi)\big)|_q \le & C_0\Big||K|*\Big(\big(1+\big(|\phi|+|v|\big)^{p-1}\big)|v|\Big)\Big|_q\\ \le & C_1\Big||K|*\Big(\big(1+|\phi|^{p-1}+|v|^{p-1}\big)|v|\Big)\Big|_q\end{aligned}$$

where C_1 is a constant only dependent on p. Applying Young's inequality yields

$$|K*\big(v+g(v+\phi)-g(\phi)\big)|_q \le C_1(1+|\phi|_\infty^{p-1})|K|_1|v|_q + C_1|K|_{q/(q+1-p)}|v|_q^p.$$

Hence, it is seen that

$$K*\big(v+g(v+\phi)-g(\phi)\big)\in L_q.$$

Since $\phi\in C_b^\infty$, $\phi'\in H^\infty$, g is a C^1-function and the operator M is defined by its Fourier symbol $-1/\big(\sqrt{2\pi}(1+\alpha(\xi))\big)$ where α has the growth property just described, it follows that

$$M*\Big(\big(1+g'(\phi)\big)\phi'\Big)\in H^1\subset L_q$$

because

$$\begin{aligned}&\int_{-\infty}^{\infty}(1+|\xi|)^2\Big|\mathcal{F}\Big(M*\big((1+g'(\phi))\phi'\big)\Big)(\xi)\Big|^2\,d\xi\\ =&\int_{-\infty}^{\infty}\frac{(1+|\xi|)^2}{(1+\alpha(\xi))^2}\Big|\mathcal{F}\big((1+g'(\phi))\phi'\big)(\xi)\Big|^2\,d\xi<\infty.\end{aligned}$$

So, the operator $\mathcal{A}$ maps $C(0,\infty;L_q)$ to itself. Let B_β be, as before, the closed ball of radius $\beta>0$ centered at the origin in L_q. For any $v,w\in C(0,\infty;L_q)$,

$$\mathcal{A}v(\cdot,t)-\mathcal{A}w(\cdot,t)=\int_0^t K*\{v-w+g(v+\phi)-g(w+\phi)\}(\cdot,\tau)\,d\tau.$$

Hence, if $v,w\in C(0,\infty;B_\beta)$, then Young's inequality yields

$$|\mathcal{A}v(\cdot,t)-\mathcal{A}w(\cdot,t)|_q\le C\int_0^t\big(1+(|\phi|_\infty+\beta)^{p-1}\big)|v(\cdot,\tau)-w(\cdot,\tau)|_q\,d\tau \tag{4.6}$$

where the constant C may be taken to be

$$C=C_0\max_{0\le j\le p-1}\{|K|_{q/(q-j)}\}.$$

Following the line of argument laid down in the proof of Theorem 2.1, choose

$$\beta=2|v_0|_q+2\Big|M*\Big(\big(1+g'(\phi)\big)\phi'\Big)\Big|_q$$

and

$$T=\min\big\{1,1/\big(2C(|\phi|_\infty+\beta)^{p-1}\big)\big\}.$$

The operator $\mathcal{A}$ is then contractive on $C(0,T;B_\beta)$ and the stated results follow directly. □

Theorem 4.2. *(Regularity) Let $v\in C(0,T;L_q)$ be the solution in Theorem 4.1 In addition, suppose for some $k\ge1$, the nonlinear function $g\in C^k$ and $g^{(k)}$ is bounded by a polynomial of degree less than or equal to $p-k$. Then for $j=1,\cdots,k$,*

$$\frac{\partial^j v}{\partial t^j}\in C(0,T;L_q).$$

Proof. The proof is virtually the same as the proof of Proposition 1 in Section 2, and so is omitted. □

The following further regularity result is the analog of Proposition 2. As the proof is entirely similar, it is also omitted.

Theorem 4.3. *(Regularity) Let $v \in C(0,T;L_q)$ be the solution of (4.4) obtained in Theorem 4.1. Suppose in addition that for some $k \geq 1$, $v_0 \in C_b^{k-1}$ and $g \in C^k$ and its j^{th} derivative $g^{(j)}$ is bounded by a polynomial of degree less than or equal to $p-j$ for $j = 1, 2, \cdots, k$. Then $v \in C(0,T;C_b^{k-1} \cap W_q^{k-1})$.*

Proposition 3. *In the above Theorem, if $p = 2n-1 > 1$ is an odd integer and there are two positive numbers γ_1 and γ_2 such that the nonlinear function $g(z) \geq (\gamma_1 - 1)z + 2n\gamma_2 z^{2n-1}$ for all $z \geq 0$, then the equation (4.4) is well-posed in $L_2 \cap L_{2n}$ globally in time, in the sense that for any initial data $v_0 \in L_2 \cap L_{2n}$ and $\bar{T} > 0$, there is a unique solution v lying in $C(0,\bar{T};L_2 \cap L_{2n})$.*

Proof. Theorem 4.1 guarantees that there is $T > 0$ such that (4.4) has a unique solution $v \in C(0,T;L_{2n})$. As in the proof of Theorem 2.4, it can be shown that v also lies in $C(0,T;L_2)$. It is sufficient to show that the solution can be extended to times that are arbitrarily large.

To this end, let $F(z) = \int_0^z g(z)\,dz$ be the primitive of g as before. Because of hypothesis (H2) and the restriction on g, it is easily deduced that for some positive constants γ_1 and γ_2, $F(z) \geq \frac{1}{2}(\gamma_1 - 1)z^2 + \gamma_2 z^{2n}$. Define a functional I by

$$I(v) = \int_{-\infty}^{\infty} \Big(\frac{1}{2}v^2 + F(v)\Big)\,dx.$$

If v is a solution of (4.4), then formally,

$$\begin{aligned}
\frac{d}{dt}I(v) &= \int_{-\infty}^{\infty} (v + g(v))\,v_t\,dx \\
&= \int_{-\infty}^{\infty} \big(v + g(v)\big)\Big(K * \big(v + g(v)\big) + M * (1 + g'(\phi))\phi'\Big)\,dx \\
&= \int_{-\infty}^{\infty} \big(v + g(v)\big)\Big(M * (1 + g'(\phi))\phi'\Big)\,dx \\
&\leq C_0\big|M * (1 + g'(\phi))\phi'\big|_2 |v|_2 + C_0\big|M * (1 + g'(\phi))\phi'\big|_{p+1} |v|_{p+1}^p \\
&\leq C_1(|v|_2^2 + |v|_{2n}^p) + C_2 \\
&\leq \bar{\gamma} I(v) + C_2,
\end{aligned}$$

where C_1 and C_2 are positive constants dependent only on the quantities $|\phi|_\infty$, $\|\phi'\|_1$ and $\bar{\gamma} = C_1/\min\{\frac{1}{2}\gamma_1, \gamma_2\}$. As before, this formal calculation is justified by the regularity theory combined with the continuous dependence result. A Gronwall-type argument then shows that for any $t > 0$,

$$I(v(\cdot,t)) \leq I(u_0)e^{\gamma_1 t} + \frac{C_2}{\gamma_1}(e^{\gamma_1 t} - 1).$$

This means that on any time interval $[0,\bar{T}]$, the L_2- and L_{2n}-norm of the solution v is finite. The standard extension argument then completes the proof. $\square$

Corollary 4. *For the generalized BBM-equation*

$$u_t + u_x + u^{2n-2}u_x - u_{xxt} = 0,$$

where $n \geq 2$, if the initial data $u_0 = v_0 + \phi$ where ϕ is an infinitely smooth bore and $v_0 \in L_2 \cap L_{2n}$, then there is a unique solution $u = v + \phi$ where $v \in C(0,\infty;L_2 \cap L_{2n})$ which depends continuously on v_0.

5. **Conclusion.** A satisfactory theory of local and global well-posedness has been put forward for the initial-value problems

$$\left.\begin{aligned} &u_t + u_x + g(u)_x + Lu_t = 0,\\ &u(x,0) = u_0(x) \end{aligned}\right\} \tag{5.1}$$

for generalized BBM-equations. Similar issues for KdV-type equations

$$\left.\begin{aligned} &v_t + v_x + g(u)_x - Lv_x = 0\\ &v(x,0) = v_0(x) \end{aligned}\right\} \tag{5.2}$$

have also attracted attentions (see Saut 1979). Results have been obtained both for initial data $u_0(x)$ that evanesces as $x \to \pm\infty$ and bore-like data.

An interesting issue not covered by the present developments is the so-called 'wavemaker', 'quarter-plane' or 'half-line' problem,

$$\left.\begin{aligned} &u_t + u_x + g(u)_x + Lu_t = 0, \quad x,\, t \geq 0,\\ &u(x,0) = u_0(x), \quad x \geq 0,\\ &u(0,t) = h(t), \quad t \geq 0. \end{aligned}\right\} \tag{5.3}$$

Theory for this problem for the BBM-equation itself was initiated by Bona and Bryant (1973), and has seen further development for more general nonlinearities (see Bona and Luo 1995). However, it remains an interesting question to provide theory for the initial-boundary-value problem (5.3) in case L is a non-local operator.

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