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A non-homogeneous boundary-value problem for the Korteweg–de Vries equation posed on a finite domain II

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ABSTRACT

Studied here is an initial- and boundary-value problem for the Korteweg–de Vries equation

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0,$$

posed on a bounded interval $I=\{x\colon\ a\leqslant x\leqslant b\}$. This problem features non-homogeneous boundary conditions applied at $x=a$ and $x=b$ and is known to be well-posed in the L_2-based Sobolev space $H^s(I)$ for any $s>-\frac{3}{4}$. It is shown here that this initial–boundary-value problem is in fact well-posed in $H^s(I)$ for any $s>-1$. Moreover, the solution map that associates the solution to the auxiliary data is not only continuous, but also analytic between the relevant function classes. The improvement on the previous theory comes about because of a more exacting appreciation of the damping that is inherent in the imposition of the boundary conditions.

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1. Introduction

In this paper, we continue the study of the Korteweg–de Vries equation (KdV-equation henceforth)

$$u_t+u_x+uu_x+u_{xxx}=0 \tag{1.1}$$

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posed on a finite interval (a,b), which without loss of generality is taken to be $(0,1)$, subject to an initial condition

$$u(x,0)=\phi(x), \quad \text{for } x\in(0,1), \tag{1.2}$$

and the non-homogeneous boundary conditions

$$u(0,t)=h_1(t), \qquad u(1,t)=h_2(t), \qquad u_x(1,t)=h_3(t), \quad \text{for } t\geqslant 0, \tag{1.3}$$

where the initial value ϕ and the boundary data h_j, $j=1,2,3$, are given functions. As is usual in studies of the KdV-equation, $u=u(x,t)$ is a real-valued function of the real variables x and t which often correspond in applications to space and time, respectively, and subscripts denote partial differentiation. The principal concern of the present essay is the *well-posedness* of the initial–boundary-value problem (IBVP henceforth) (1.1)–(1.3) in the classical Sobolev space $H^s(0,1)$ for negative values of s.

We remind the reader that while there is marvelous theory developed for the pure initial-value problem (IVP from now on) (1.1)–(1.2) set on the whole line $I=R$ (cf. [6,7,16,30–32,41,44] and the references therein), and for the initial–boundary-value problem of Eq. (1.1) posed on the half line $I=R^+$ (cf. [3,5,8,9,15,20,23–26] and the references therein), some of which will be discussed presently, when the equation is used in practical situations, one inevitably encounters a finite domain where lateral boundary conditions must be imposed. Hence, theory for such boundary-value problems, while more complicated and less elegant than the theory on the whole line, is important, as is discussed in some detail in the works [1,2,4] and [14] for example. And, though a theory related to very rough auxiliary data such as that which is the focus here is not relevant to real applications of equations like (1.1), the issue is mathematically challenging and, moreover, the representations derived in Section 2 do find use in the analysis of practically important issues (see e.g. [2]).

Recall that *the IBVP* (1.1)–(1.3) *is said to be locally well-posed in the space* $H^s(0,1)$ *for some* $s\in R$ *if for given* $T>0$ *and suitably compatible auxiliary data*[1]

$$\phi\in H^s(0,1) \quad \text{and} \quad \vec{h}=(h_1,h_2,h_3)\in H^{\mu_1}(0,T)\times H^{\mu_2}(0,T)\times H^{\mu_3}(0,T),$$

there exists a T^* *with* $0<T^*\leqslant T$ *depending only on*

$$r\equiv\|\phi\|_{H^s(0,1)}+\|\vec{h}\|_{H^{\mu_1}(0,T)\times H^{\mu_2}(0,T)\times H^{\mu_3}(0,T)}$$

such that (1.1)–(1.3) *admits a unique solution* $u\in C([0,T^*];H^s(0,1))$ *which depends continuously on* ϕ *and* $\vec{h}$ *in their respective spaces. If* $T^*=T$ *for any compatible* ϕ *and* $\vec{h}$, *the IBVP* (1.1)–(1.3) *is said to be globally well-posed in* $H^s(0,1)$.

The reader is referred to [10,17–19,27,28,37–40,42,43] and the references contained therein for various studies of the IBVP (1.1)–(1.3). In particular, we showed in [10] that (1.1)–(1.3) is locally well-posed in the space $H^s(0,1)$ with $\mu_1=\mu_2=\frac{s+1}{3}$ and $\mu_3=\frac{s}{3}$ for any $s\geqslant 0$ and is globally well-posed in the space $H^s(0,1)$ for any $s\geqslant 3$. In the case wherein $0\leqslant s<3$, we showed that (1.1)–(1.3) is globally well-posed with compatible

$$(\phi,\vec{h})\in H^s(0,1)\times H^{\mu_1(s)}(0,T)\times H^{\mu_1(s)}(0,T)\times H^{\mu_2(s)}(0,T),$$

[1] The reader is refereed to [10] for the definition of compatibility of the initial value ϕ and the boundary data $\vec{h}$. Compatibility does not arise as an issue in spaces where the relevant traces of the auxiliary data at $(x,t)=(0,0)$ and $(x,t)=(1,0)$ do not exist, such as those that are the principal focus here.

where $\mu_1(s) = \epsilon + (5s+9)/18$, $\mu_2(s) = \epsilon + (5s+3)/18$ and ϵ is an arbitrary small positive constant. This result has been improved recently by Faminskii [27]; he showed that (1.1)–(1.3) is globally well-posed in the space $H^s(0,1)$ for any s with $0 \leqslant s < 3$ for compatible data

$$(\phi, \vec{h}) \in H^s(0,1) \times H^{\frac{s+1}{3}+\epsilon}(0,T) \times H^{\frac{s+1}{3}+\epsilon}(0,T) \times H^{\frac{s}{3}+\epsilon}(0,T).$$

Even more recently, Holmer showed in [28] that (1.1)–(1.3) is locally well-posed in $H^s(0,1)$ for any $s > -\frac{3}{4}$. The following question then arises naturally.

Question 1.1. *Is the IBVP (1.1)–(1.3) well-posed in the space $H^s(0,1)$ for some values of $s \leqslant -\frac{3}{4}$?*

The same issue arises for the IVP for the KdV equation posed on the whole line R, *viz.*

$$\left.\begin{aligned} &u_t + uu_x + u_{xxx} = 0, \quad x, t \in R, \\ &u(x,0) = \phi(x), \end{aligned}\right\} \tag{1.4}$$

with Sobolev-class initial data ϕ, posed with periodic boundary conditions so that ϕ is periodic and solutions having the same period are sought, or posed in a quarter plane, *viz.*

$$\left.\begin{aligned} &u_t + u_x + uu_x + u_{xxx} = 0, \qquad x \in R^+,\ t \in R^+, \\ &u(x,0) = \phi(x), \quad u(0,t) = h(t), \quad x, t \in R^+. \end{aligned}\right\} \tag{1.5}$$

After considerable effort by a number of researchers, it has been understood that the IVP (1.4) is well-posed in the space $H^s(R)$ for $s > -\frac{3}{4}$, whereas the periodic IVP for (1.4) is well-posed in the space $H^s_{\mathrm{per}}((a,b))$ for $s \geqslant -\frac{1}{2}$ [33,34]. The IBVP (1.5) is also known to be well-posed in the space $H^s(R^+)$ for any $s > -\frac{3}{4}$ [12,28].

In the context of the pure IVP or the periodic IVP (1.4), or the quarter-plane problem (1.5), one can ask the same question as for the finite-interval problem.

Question 1.2. *Is the IVP (1.4) well-posed in $H^s(R)$ for some $s < -\frac{3}{4}$; is the periodic IVP (1.4) well-posed in $H^s_{\mathrm{per}}((a,b))$ for some $s < -\frac{1}{2}$; is the IBVP (1.5) well-posed in $H^s(R^+)$ for some $s \leqslant -\frac{3}{4}$?*

For the IVP (1.4), when $s < -\frac{3}{4}$, it has been shown to be ill-posed in $H^s(R)$ in the sense that the solution map, if it were to exist, cannot be locally uniformly continuous. The same can be said for the periodic IVP (1.4); it is ill-posed in $H^s(S)$ when $s < -\frac{1}{2}$ in the sense that the solution map cannot be locally uniformly continuous. When $s = -\frac{3}{4}$, a weaker form of local well-posedness was established for the IVP (1.4) in [21]. Thus, the indications were that the answer to Question 1.2 was almost certainly negative. However, Kappeler and Topalov [29] recently demonstrated that the IVP (1.4) is (globally) well-posed in the space $H^s(S)$ for $s \geqslant -1$. In addition, Molinet and Ribaud [35] showed that the pure initial-value problem

$$\left.\begin{aligned} &u_t + uu_x + u_{xxx} - u_{xx} = 0, \quad -\infty < x < \infty,\ t > 0, \\ &u(x,0) = \psi(x), \qquad\qquad\quad -\infty < x < \infty, \end{aligned}\right\} \tag{1.6}$$

for the KdV–Burgers equation is well-posed in the space $H^s(R)$ for $s > -1$ and is ill-posed when $s < -1$ in the sense that the corresponding solution map is not C^2. Both of these results are a little surprising. Molinet and Ribaud achieved their result by taking full advantage of the combination of the dispersion introduced through the term u_{xxx} and the dissipation introduced through the Burgers' term $-u_{xx}$. The corresponding solution map is real analytic when $s > -1$. In contrast, the approach of Kappeler and Topalov is based on the classical inverse scattering transform. The corresponding solution map associated with the periodic IVP (1.4) is continuous, but not locally uniformly continuous when $s < -\frac{1}{2}$. The interested reader is also refereed to [22,23] for similar discussions for the Burgers equation.

There is an interesting connection between the KdV equation and the KdV–Burgers equation. Let $\alpha, \beta \in R$ be given and consider the transformation

$$u(x,t) = e^{\alpha x+\beta t} v(x,t).$$

A direct calculation shows that u is a solution of the KdV equation

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1.7}$$

if and only if v is a solution of the equation

$$v_t + (\alpha + \alpha^3 + \beta)v + (3\alpha^2 + 1)v_x + v_{xxx} + 3\alpha v_{xx} + e^{\beta t+\alpha x}(\alpha v + v v_x) = 0. \tag{1.8}$$

This connection between the KdV equation and the KdV–Burgers equation led us to consider in [13] the following IBVP for the KdV–Burgers equation posed in a quarter plane:

$$\left.\begin{aligned} &u_t + a(x,t)uu_x + u_{xxx} - u_{xx} = 0, \quad x,t > 0,\\ &u(x,0) = \psi(x), \quad u(x,0) = h(t), \quad x,t > 0, \end{aligned}\right\} \tag{1.9}$$

where $a = a(x,t)$ is a given smooth function. It was shown in [13] that the IBVP (1.9) is locally well-posed in the space $H^s(R^+)$ for any $s > -1$. Consequently, there emerges the following well-posedness result for the IBVP (1.5), which provides a partial answer to Question 1.2 for the KdV equation posed in a quarter plane.

Theorem. *Let $\nu > 0$ be given. Then for any $s > -1$ with $s \neq 3m + \frac{1}{2}$, $m = 0, 1, \ldots,$ the IBVP (1.5) is locally well-posed in the weighted Sobolev space*

$$H^s_\nu(R^+) = \{f \in H^s(R^+);\ e^{\nu x} f \in H^s(R^+)\}.$$

Moreover, the correspondence $(\psi, h) \mapsto u$ of data with the associated solution is an analytic mapping between the relevant spaces.

In this paper, interest is focused on Question 1.1 for the IBVP (1.1)–(1.3). The following two theorems are the principal outcomes of the present study.

Theorem 1.1. *The IBVP (1.1)–(1.3) is locally well-posed in the space $H^s(0,1)$ for any $-1 < s \leqslant 0$. The correspondence $(\psi, h_1, h_2, h_3) \mapsto u$ is an analytic mapping from $H^s(0,1) \times H^{\frac{1}{3}}(0,T) \times H^{\frac{1}{3}}(0,T) \times H^0(0,T)$ to $C([0,T]; H^s(R^+))$, where $T > 0$ is any value less than the existence time provided by the local well-posedness.*

Theorem 1.2. *For any $\epsilon > 0$, the IBVP (1.1)–(1.3) is globally well-posed in the space $H^s(0,1)$ for any s in the range $-1 < s \leqslant 0$ and for auxiliary data $\phi \in H^s(0,1)$ and $\vec{h} \in H^{\frac{1}{3}+\epsilon}(0,T) \times H^{\frac{1}{3}+\epsilon}(0,T) \times H^{\epsilon}(0,T)$.*

In addition, it will be shown that the IBVP (1.1)–(1.3) possesses a strong smoothing property, similar to that of the heat equation.

Theorem 1.3. *Let $s > -1$ be given. If $\vec{h} \in H^\infty_{loc}(R^+) \times H^\infty_{loc}(R^+) \times H^\infty_{loc}(R^+)$, then for any $\phi \in H^s(0,1)$, the corresponding solution u of the IBVP (1.1)–(1.3) belongs to the space $C(R^+; H^\infty(0,1))$.*

Thus it is concluded that the IBVP (1.1)–(1.3) has the same local theory as that established by Kappeler and Topalov using inverse scattering theory for the periodic initial-value problem. An advantage of the present arguments is that they do not depend upon the rigid structure of inverse scattering theory, and so may be expected to apply to a range of nonlinear dispersive equations.

The rather strong damping property enunciated in Theorem 1.2, and which is a consequence of imposing boundary conditions, indicates why one might hope for a better local existence theory than obtained heretofore. Indeed, the improved theory developed here owes almost entirely to a better appreciation of the smoothing induced by the imposition of boundary values.

Observe that a function $u = u(x,t)$ solves the IBVP (1.1)–(1.3) if and only if $v = e^{-2t+x}u$ solves the IBVP

$$\left.\begin{aligned} &v_t + 4v_x + \frac{1}{2}\left(e^{4t-2x}v\right)_x + v_{xxx} - 3v_{xx} = 0, \quad x \in (0,1),\\ &v(x,0) = \phi^*(x),\\ &v(0,t) = h_1^*(t), \quad v(1,t) = h_2^*(t), \quad v_x(1,t) = h_3^*(t), \end{aligned}\right\} \tag{1.10}$$

for a KdV–Burgers-type equation posed on $(0,1)$, where $\phi^*(x) = e^x\phi(x)$ and $\vec{h}^* = (h_1^*, h_2^*, h_3^*)$ with

$$h_1^*(t) = e^{-2t}h_1(t), \qquad h_2^*(t) = e^{-2t+1}h_2(t), \qquad h_3^*(t) = e^{-2t+1}\big(h_2(t) + h_3(t)\big).$$

Thus, to prove Theorems 1.1–1.3 for the KdV equation, one needs only study the IBVP (1.10) and establish for it the same well-posedness results as those described in Theorems 1.1–1.3. More precisely, it suffices to prove the following well-posedness results for the IBVP (1.10).

Theorem 1.4.

(a) *The IBVP (1.10) is locally well-posed in the space $H^s(0,1)$ for any $-1 < s \leqslant 0$. Moreover, the correspondence of auxiliary data to solutions $(\phi^*, \vec{h}^*) \mapsto u$ is an analytic mapping of $H^s(0,1) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$ to $C(0,T; H^s(0,1))$ for appropriate $T > 0$.*

(b) *For any $T > 0$, the IBVP (1.10) is well-posed in the space $H^s(0,1)$ for any $-1 < s \leqslant 0$ with $\phi^* \in H^s(0,1)$ and $\vec{h}^* \in H^{\frac{1}{3}+\epsilon}(0,T) \times H^{\frac{1}{3}+\epsilon}(0,T) \times H^{\epsilon}(0,T)$. (Thus the problem is globally well-posed in these spaces.)*

(c) *Let $s > -1$ be given. If $\vec{h}^* \in H^\infty_{loc}(R^+) \times H^\infty_{loc}(R^+) \times H^\infty_{loc}(R^+)$, then for any $\phi^* \in H^s(0,1)$, the corresponding solution v of the IBVP (1.1)–(1.3) belongs to the space $C(R^+; H^\infty(0,1))$.*

This result will be proved using the Laplace transform approach developed in our earlier papers [12,13].

The paper is organized as follows. In Section 2, explicit representation formulas are presented for solutions of initial-boundary-value problems for the linear KdV–Burgers equation. These are developed along the lines put forward in [12,13]. Various estimates will be established for the linear problems associated to (1.10). These will play a central role in the analysis of the nonlinear problems. In Section 3, the well-posedness results for the IBVP (1.10) as described in Theorem 1.4 are established. The technical Appendix A contains proofs of some of the lemmas that arise in the analysis underlying the principal results.

2. Linear problems

This section is divided into two subsections. In the first, consideration is given to linear problems associated to the KdV–Burgers equation. Explicit representation formulas for solutions of an initial-boundary-value problem for this equation will be derived. Then, the boundary integral operators that arise in the solution formulas will be extended from the domain $(0,1)\times R^+$ to the whole plane $R\times R$ using the approach developed in [13]. The extended boundary integral operator will play a crucial role in our analysis. The second subsection contains estimates of solutions of the linear problems and of the boundary integral operators.

2.1. Solution formulas

Consideration is first directed to the non-homogeneous boundary-value problem

$$\left.\begin{aligned} &u_t+4u_x+u_{xxx}-3u_{xx}=0, \quad u(x,0)=0,\\ &u(0,t)=h_1(t), \quad u(1,t)=h_2(t), \quad u_x(1,t)=h_3(t). \end{aligned}\right\} \tag{2.1}$$

Applying the Laplace transform with respect to t, (2.1) is converted to

$$\left.\begin{aligned} &s\hat{u}(x,s)+4\hat{u}_x(x,s)+\hat{u}_{xxx}(x,s)-3\hat{u}_{xx}(x,s)=0,\\ &\hat{u}(0,s)=\hat{h}_1(s), \quad \hat{u}(1,s)=\hat{h}_2(s), \quad \hat{u}_x(1,s)=\hat{h}_3(s) \end{aligned}\right\} \tag{2.2}$$

where

$$\hat{u}(x,s)=\int_0^{+\infty} e^{-st}u(x,t)\,dt$$

and

$$\hat{h}_j(s)=\int_0^{+\infty} e^{-st}h_j(t)\,dt, \quad j=1,2,3.$$

The solution $\hat{u}(x,s)$ of (2.2) can be written in the form

$$\hat{u}(x,s)=\sum_{j=1}^{3} c_j(s)e^{\lambda_j(s)x}$$

where the $\lambda_j(s)$, $j=1,2,3$, are the solutions of the characteristic equation

$$s+4\lambda+\lambda^3-3\lambda^2=0$$

and the $c_j=c_j(s)$, $j=1,2,3$, solve the linear system

$$\begin{cases} c_1+c_2+c_3=\hat{h}_1(s),\\ c_1e^{\lambda_1(s)}+c_2e^{\lambda_2(s)}+c_3e^{\lambda_3(s)}=\hat{h}_2(s),\\ c_1\lambda_1(s)e^{\lambda_1(s)}+c_2\lambda_2(s)e^{\lambda_2(s)}+c_3\lambda_3(s)e^{\lambda_3(s)}=\hat{h}_3(s). \end{cases}$$

Let $\Delta(s)$ be the determinant of the coefficient matrix of the left-hand side of this system, and $\Delta_i(s)$ the determinants of the matrices that are obtained by replacing the ith column of $\Delta(s)$ by the column vector $(\hat{h}_1(s),\hat{h}_2(s),\hat{h}_3(s))^T$, $i=1,2,3$. Cramer's rule implies that

$$c_j=\frac{\Delta_j(s)}{\Delta(s)}, \quad j=1,2,3.$$

Taking the inverse Laplace transform of $\hat{u}$ yields the representation

$$u(x,t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\hat{u}(x,s)\,ds = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_j(s)}{\Delta(s)}e^{\lambda_j(s)x}\,ds$$

which holds for any $r > 0$. The solution u of (2.14) may also be written in the form

$$u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t)$$

where $u_m(x,t)$ solves (2.14) with $h_j \equiv 0$ when $j \neq m$, $m, j = 1,2,3$; thus u_m has the representation

$$u_m(x,t) = \sum_{j=1}^{3} u_{j,m}(x,t)$$

with

$$u_{j,m}(x,t) = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_m(s)\,ds$$

for $m, j = 1,2,3$. Here $\Delta_{j,m}(s)$ is obtained from $\Delta_j(s)$ by letting $\hat{h}_m(t) = 1$ and $h_k(t) \equiv 0$ for $k \neq m$, $k, m = 1,2,3$. It is straightforward to determine that in the last two formulas, the right-hand sides are continuous with respect to r for $r \geqslant 0$. As the left-hand sides do not depend on r, it follows that we may take $r = 0$ in these formulas and in those appearing below. Write $u_{j,m}$ in the form

$$\begin{aligned} u_{j,m}(x,t) &= \frac{1}{2\pi i}\int_{0}^{+i\infty} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_m(s)\,ds + \frac{1}{2\pi i}\int_{-i\infty}^{0} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_m(s)\,ds \\ &\equiv u_{j,m}^{+}(x,t) + u_{j,m}^{-}(x,t), \end{aligned}$$

for $m, j = 1,2,3$. Making the substitution $s = i(\rho^3 - \rho)$ with $1 \leqslant \rho < +\infty$ in the characteristic equation

$$s + 4\lambda + \lambda^3 - 3\lambda^2 = 0, \tag{2.3}$$

the three roots λ_j^{+} may be written as a function of ρ rather than s, viz. $\lambda_j^{+}(\rho)$, $j = 1,2,3$, with

$$\operatorname{Re}\lambda_1^{+}(\rho) \geqslant 0, \qquad \operatorname{Re}\lambda_2^{+}(\rho) \geqslant 0, \qquad \operatorname{Re}\lambda_3^{+}(\rho) \leqslant 0$$

and, as $\rho \to \infty$,

$$\lambda_1^{+}(\rho) = i\rho + 1 + O\left(\frac{1}{\rho}\right), \qquad \lambda_2^{+}(\rho) = \frac{\sqrt{3}-i}{2}\rho + 1 + O\left(\frac{1}{\rho}\right),$$

$$\lambda_3^{+}(\rho) = \frac{-\sqrt{3}-i}{2}\rho + 1 + O\left(\frac{1}{\rho}\right).$$

Thus $u^+_{j,m}(x,t)$ and $u^-_{j,m}(x,t)$ have the form

$$u^+_{j,m}(x,t)=\frac{1}{2\pi}\int_1^{+\infty} e^{i(\rho^3-\rho)t}e^{\lambda_j^+(\rho)x}\frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)}(3\rho^2-1)\hat{h}^+_m(\rho)\,d\rho$$

and

$$u^-_{j,m}(x,t)=\frac{1}{2\pi}\int_1^{+\infty} e^{-i(\rho^3-\rho)t}e^{\lambda_j^-(\rho)x}\frac{\Delta^-_{j,m}(\rho)}{\Delta^-(\rho)}(3\rho^2-1)\hat{h}^-_m(\rho)\,d\rho$$

where $\hat{h}^+_m(\rho)=\hat{h}_m(i(\rho^3-\rho))$, $\Delta^+(\rho)$ and $\Delta^+_{j,m}(\rho)$ are obtained from $\Delta(s)$ and $\Delta_{j,m}(s)$, respectively, by replacing s with $i(\rho^3-\rho)$ and $\lambda_j(s)$ with $\lambda_j^+(\rho)$, for $j=1,2,3$. Notice that, with an obvious notation, $\Delta^-(\rho)=\overline{\Delta^+}(\rho)$ and $\Delta^-_{j,m}(\rho)=\overline{\Delta^+_{j,m}}(\rho)$ for $j=1,2,3$, and $\hat{h}^-_m(\rho)=\overline{\hat{h}^+_m}(\rho)$. In consequence, it is also the case that $u^-_{j,m}(x,t)=\overline{u^+_{j,m}(x,t)}$, $j,m=1,2,3$.

It will be helpful to know the large-ρ asymptotics of the ratios

$$\frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)} \quad\text{and}\quad \frac{\Delta^-_{j,m}(\rho)}{\Delta^-(\rho)}.$$

Since

$$\lambda_1(s)+\lambda_2(s)+\lambda_3(s)\equiv 3,$$

it is readily seen that

$$\Delta(s)=\big(\lambda_3(s)-\lambda_2(s)\big)e^{3-\lambda_1(s)}+\big(\lambda_1(s)-\lambda_3(s)\big)e^{3-\lambda_2(s)}+\big(\lambda_2(s)-\lambda_1(s)\big)e^{3-\lambda_3(s)},$$

$$\begin{cases}\Delta_{1,1}(s)=\big(\lambda_3(s)-\lambda_2(s)\big)e^{3-\lambda_1(s)},\\ \Delta_{2,1}(s)=\big(\lambda_1(s)-\lambda_3(s)\big)e^{3-\lambda_2(s)},\\ \Delta_{3,1}(s)=\big(\lambda_2(s)-\lambda_1(s)\big)e^{3-\lambda_3(s)},\end{cases}\qquad \begin{cases}\Delta_{1,2}(s)=\lambda_2(s)e^{\lambda_2(s)}-\lambda_3(s)e^{\lambda_3(s)},\\ \Delta_{2,2}(s)=\lambda_3(s)e^{\lambda_3(s)}-\lambda_1(s)e^{\lambda_1(s)},\\ \Delta_{3,2}(s)=\lambda_1(s)e^{\lambda_1(s)}-\lambda_3(s)e^{\lambda_2(s)},\end{cases}$$

and

$$\Delta_{1,3}(s)=e^{\lambda_3(s)}-e^{\lambda_2(s)},\qquad \Delta_{2,3}(s)=e^{\lambda_1(s)}-e^{\lambda_3(s)},\qquad \Delta_{3,3}(s)=e^{\lambda_2(s)}-e^{\lambda_1(s)}.$$

Therefore, it follows immediately that as a function of the variable ρ introduced above,

$$\frac{\Delta^+_{1,1}(\rho)}{\Delta^+(\rho)}\sim e^{\frac{-\sqrt{3}}{2}\rho},\qquad \frac{\Delta^+_{2,1}(\rho)}{\Delta^+(\rho)}\sim e^{-\sqrt{3}\rho},\qquad \frac{\Delta^+_{3,1}(\rho)}{\Delta^+(\rho)}\sim 1, \tag{2.4}$$

$$\frac{\Delta^+_{1,2}(\rho)}{\Delta^+(\rho)}\sim e^{-3},\qquad \frac{\Delta^+_{2,2}(\rho)}{\Delta^+(\rho)}\sim e^{-3},\qquad \frac{\Delta^+_{3,2}(\rho)}{\Delta^+(\rho)}\sim e^{-3}, \tag{2.5}$$

$$\frac{\Delta^+_{1,3}(\rho)}{\Delta^+(\rho)}\sim e^{-3}\rho^{-1},\qquad \frac{\Delta^+_{2,3}(\rho)}{\Delta^+(\rho)}\sim e^{-3}\rho^{-1},\qquad \frac{\Delta^+_{3,2}(\rho)}{\Delta^+(\rho)}\sim e^{-3}\rho^{-1} \tag{2.6}$$

as $\rho\to+\infty$.

As already noted,

$$u^-_{j,m}(x,t) = \overline{u^+_{j,m}(x,t)}, \quad j,m = 1,2,3,$$

and

$$u^+_{3,m}(x,t) = \frac{1}{2\pi}\int_1^{+\infty} e^{i(\rho^3-\rho)t} e^{\lambda_3^+(\rho)x}(3\rho^2-1)\hat{h}^{*+}_{3,m}(\rho)\,d\rho$$

for $m = 1,2,3$, whilst

$$u^+_{j,m}(x,t) = \frac{1}{2\pi}\int_1^{+\infty} e^{i(\rho^3-\rho)t} e^{-\lambda_j^+(\rho)(1-x)}(3\rho^2-1)\hat{h}^{*+}_{j,m}(\rho)\,d\rho \quad (\text{let } x' = 1-x)$$
$$= \frac{1}{2\pi}\int_1^{+\infty} e^{i(\rho^3-\rho)t} e^{-\lambda_j^+(\rho)x'}(3\rho^2-1)\hat{h}^{*+}_{j,m}(\rho)\,d\rho$$

for $m = 1,2,3$ and $j = 1,2$, with

$$\hat{h}^{*+}_{3,m}(\rho) = \frac{\Delta^+_{3,m}(\rho)}{\Delta^+(\rho)}\hat{h}^+_m(\rho), \qquad \hat{h}^{*+}_{j,m}(\rho) = \frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)}e^{\lambda_j^+(\rho)}\hat{h}^+_m(\rho)$$

for $j = 1,2$ and $m = 1,2,3$. Moreover, it follows straightforwardly from (2.4)–(2.6) that

$$h_1 \in H_0^{\frac{s+1}{3}}(R^+) \implies h^*_{j,1} \in H^\infty(R), \quad j=1,2, \quad h^*_{3,1} \in H_0^{\frac{s+1}{3}}(R),$$
$$h_2 \in H_0^{\frac{s+1}{3}}(R^+) \implies h^*_{j,2} \in H_0^{\frac{s+1}{3}}(R), \quad j=1,2,3,$$
$$h_3 \in H_0^{\frac{s}{3}}(R^+) \implies h^*_{j,3} \in H_0^{\frac{s+1}{3}}(R), \quad j=1,2,3.$$

For given $m, j = 1,2,3$, let $W_{j,m}$ be an operator on $H_0^s(R^+)$ defined as follows; for any $h \in H_0^s(R^+)$,

$$[W_{j,m}h](x,t) \equiv [U_{j,m}h](x,t) + \overline{[U_{j,m}h](x,t)} \tag{2.7}$$

with

$$[U_{j,m}h](x,t) \equiv \frac{1}{2\pi}\int_1^{+\infty} e^{i(\rho^3-\rho)t} e^{-\lambda_j^+(\rho)(1-x)}(3\rho^2-1)\hat{h}^{*+}_{j,m}(\rho)\,d\rho \tag{2.8}$$

for $j = 1,2$, $m = 1,2,3$ and

$$[U_{3,m}h](x,t) \equiv \frac{1}{2\pi}\int_1^{+\infty} e^{i(\rho^3-\rho)t} e^{\lambda_3^+(\rho)x}(3\rho^2-1)\hat{h}^{*+}_{3,m}(\rho)\,d\rho \tag{2.9}$$

for $m = 1, 2, 3$. Here, the functions $\hat{h}^{*+}_{j,m}$ are defined by

$$\hat{h}^{*+}_{3,m}(\rho) = \frac{\Delta^+_{3,m}(\rho)}{\Delta^+(\rho)}\hat{h}^+(\rho), \qquad \hat{h}^{*+}_{j,m}(\rho) = \frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)}e^{\lambda^+_j(\rho)}\hat{h}^+(\rho)$$

for $j = 1, 2$ and $m = 1, 2, 3$, where $\hat{h}^+(\rho) = \hat{h}(i(\rho^3 - \rho))$.

Proposition 2.1. *For given h_1, h_2, and h_3, let $\vec{h} = (h_1, h_2, h_3)$. Then the solution of (2.14) may be written in the form*

$$u(x,t) = [\mathcal{W}_{bdr}\vec{h}](x,t) := \sum_{j,m=1}^{3} [\mathcal{W}_{j,m}h_m](x,t).$$

Next, consider the same problem posed with zero boundary condition, but non-trivial initial data, *viz.*

$$\left.\begin{array}{l} u_t + 4u_x + u_{xxx} - 3u_{xx} = 0, \quad u(x,0) = \phi(x), \\ u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0. \end{array}\right\} \tag{2.10}$$

By semigroup theory [36], its solution may be obtained in the form

$$u(t) = W_c(t)\phi \tag{2.11}$$

where the spatial variable is suppressed and $W_c(t)$ is the C_0-semigroup in the space $L^2(0,1)$ generated by the operator

$$Af = -f''' - 4f' + 3f''$$

with the domain

$$\mathcal{D}(A) = \{f \in H^3(0,1) \mid f(0) = 0,\ f(1) = f'(1) = 0\}.$$

By Duhamel's principle, one may use the semigroup $W_c(t)$ to formally write the solution of the forced linear problem

$$\left.\begin{array}{l} u_t + u_x + u_{xxx} = f(x,t), \quad u(x,0) = 0, \\ u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0 \end{array}\right\} \tag{2.12}$$

in the form

$$u(t) = \int_0^t W_c(t-\tau)f(\cdot,\tau)\,d\tau. \tag{2.13}$$

Recall the explicit solution formula

$$u(x,t) = W_R(t)\phi(x) = c\int_{-\infty}^{\infty} e^{i(\xi^3-4\xi)t-3\xi^2 t}e^{ix\xi}\int_{-\infty}^{\infty} e^{-iy\xi}\psi(y)\,dy\,d\xi \tag{2.14}$$

for the pure IVP

$$\left.\begin{aligned} &u_t + 4u_x - 3u_{xx} + u_{xxx} = 0, && x, t \in R, \\ &u(x,0) = \psi(x), && x \in R. \end{aligned}\right\} \tag{2.15}$$

The formula for $W_R(t)$ is explicit and simple. We take advantage of this simplicity to give a related representation of $W_c(t)$ in terms of $W_R(t)$ and $\mathcal{W}_{bdr}(t)$.

Let a function ϕ be defined on the interval $(0,1)$ and let ϕ^* be an extension of ϕ to the whole line R. The mapping $\phi \to \phi^*$ can be organized so that it defines a bounded linear operator B from $H^s(0,1)$ to $H^s(R)$. Henceforth, $\phi^* = B\phi$ will refer to the result of such an extension operator applied to $\phi \in H^s(0,1)$. Assume that $v = v(x,t)$ is the solution of

$$v_t + 4v_x - 3v_{xx} + v_{xxx} = 0, \qquad v(x,0) = \phi^*(x)$$

for $x \in R$, $t \geqslant 0$. If $g_1(t) = v(0,t)$, $g_2(t) = v(1,t)$, $g_3(t) = v_x(1,t)$ and $\vec{g} = (g_1, g_2, g_3)$, then $v_{\vec{g}} = v_{\vec{g}}(x,t) = [\mathcal{W}_{bdr}(t)\vec{g}](x)$ is the corresponding solution of the non-homogeneous boundary-value problem (2.10) with boundary condition $h_j(t) = g_j(t)$, $j = 1,2,3$, for $t \geqslant 0$. It is clear that for $x \in (0,1)$ the function $v(x,t) - v_{\vec{g}}(x,t)$ solves the IBVP (2.10), and this leads directly to a representation of the semigroup $W_c(t)$ in terms of $\mathcal{W}_{bdr}(t)$ and $W_R(t)$.

Proposition 2.2. *For a given s and $\phi \in H^s(0,1)$, if ϕ^* is its extension to R as described above, then $W_c(t)\phi$ may be written in the form*

$$W_c(t)\phi = W_R(t)\phi^* - \mathcal{W}_{bdr}(t)\vec{g} \tag{2.16}$$

for any $x \in (0,1)$ and $t > 0$, where $\vec{g}$ is obtained from the trace of $W_R(t)\phi^*$ at $x = 0$ and at $x = 1$ as indicated above.

In a similar manner, one may derive an alternative representation for solutions of the inhomogeneous initial-boundary-value problem (2.12).

Proposition 2.3. *If $f^*(\cdot,t) = Bf(\cdot,t)$ is an extension of f from $[0,1] \times R^+$ to $R \times R^+$, say, then the solution u of (2.12) may be written in the form*

$$u(\cdot,t) = \int_0^t W_R(t-\tau) f^*(\cdot,\tau)\,d\tau - \mathcal{W}_{bdr}(t)\vec{v}$$

for any $x, t \geqslant 0$ where $\vec{v} \equiv \vec{v}(t) = (v_1(t), v_2(t), v_3(t))$ is the appropriate boundary traces of

$$q(x,t) = \int_0^t W_R(t-\tau) f^*(\tau)\,d\tau$$

at $x = 0$ and $x = 1$, which is to say,

$$v_1(t) = q(0,t), \qquad v_2(t) = q(1,t), \qquad v_3(t) = q_x(1,t).$$

Of course, the formulas in Propositions 2.2 and 2.3 only provide solutions of the partial differential equation for $0 < x < 1$ and $t > 0$. Indeed, $W_c(t)\phi(x)$, the left-hand of (2.16), is only defined for this range of x and t. However, it will be convenient to extend the terms on the right-hand side of (2.16) in such a way that they are defined for all $x, t \in R$. This will provide a context in which to establish the well-posedness of the nonlinear problem in the framework of Bourgain spaces. Note that the term $W_R(t)$ can be redefined as

$$W_R(t)\phi = c\int_{-\infty}^{\infty} e^{i(\xi^3-4\xi)t-3\xi^2|t|}e^{ix\xi}\int_{-\infty}^{\infty} e^{-iy\xi}\phi(y)\,dy\,d\xi$$

for all $x, t \in R$ and thus only a suitable extension of the second term in both formulas is needed to extend the entire formula from $(0,1)\times R^+$ to $R \times R$. Because of the structure of the boundary integral operators $[W_{jm}h](x,t)$, $j, m = 1, 2, 3$ (see (2.7)–(2.9)), it suffices to consider extending an integral operator $\mathcal{U}_{bdr}(t)$ of the form

$$\big[\mathcal{U}_{bdr}(t)h\big](x) = \frac{1}{2\pi}\operatorname{Re}\int_{1}^{\infty} e^{it(\mu^3-\mu)}e^{(\alpha(\mu)+i\beta(\mu))x}\big(3\mu^2-1\big)\hat{h}^+(\mu)\,d\mu \tag{2.17}$$

with $\hat{h}^+(\mu) = \hat{h}(i(\mu^3-\mu))$, where both $\alpha(\mu)$ and $\beta(\mu)$ are real-valued functions and $\alpha(\mu) \leqslant 0$ for all μ.

Attention is thus turned to providing an extension of the boundary integral operator $\mathcal{U}_{bdr}(t)$. Rewrite $\mathcal{U}_{bdr}(t)$ as

$$\begin{aligned}
\big[\mathcal{U}_{bdr}(t)h\big](x) &= \frac{1}{2\pi}\operatorname{Re}\int_{1}^{\infty} e^{it(\mu^3-\mu)}e^{(\alpha(\mu)+i\beta(\mu))x}\big(3\mu^2-1\big)\hat{h}^+(\mu)\,d\mu \\
&= \frac{1}{2\pi}\operatorname{Re}\int_{1}^{4} e^{i\mu^3t-i\mu t}e^{(\alpha(\mu)+i\beta(\mu))\phi_3(x)}\big(3\mu^2-1\big)\phi_1(\mu)\hat{h}^+(\mu)\,d\mu \\
&\quad + \frac{1}{2\pi}\operatorname{Re}\int_{\frac{2}{\sqrt{3}}}^{\infty} e^{i\mu^3t-i\mu t}e^{(\alpha(\mu)+i\beta(\mu))x}\big(3\mu^2-1\big)\phi_2(\mu)\hat{h}^+(\mu)\,d\mu \\
&:= \frac{1}{2\pi}\big\{I_1(x,t)+I_2(x,t)\big\}
\end{aligned}$$

where $\phi_1(\mu)$ and $\phi_2(\mu)$ are nonnegative cut-off functions satisfying

$$\phi_1(\mu)+\phi_2(\mu) = 1 \quad \text{for all } \mu \in R^+$$

with $\operatorname{supp}\phi_1 \subset (-1,4)$, $\operatorname{supp}\phi_2 \subset (3,\infty)$ and $\phi_3(x)$ is a smooth function on R such that

$$\phi_3(x) = \begin{cases} x & \text{for } x \geqslant 0, \\ 0 & \text{for } x \leqslant -1. \end{cases}$$

The integral $I_1(x,t)$ is naturally defined for all values of x and t and, viewed as a function defined on $R \times R$, is in fact C^∞-smooth there, with all its derivatives decreasing rapidly as $x \to \pm\infty$. Thus no complicated extension of I_1 is required as the obvious one suffices. It is otherwise for I_2. To discuss $I_2(x,t)$, it is convenient to let $\mu(\lambda)$ denote the positive solution of

$$\mu^3 - \mu = \lambda$$

for $\lambda \geqslant 0$ and $\mu \geqslant 1$, while $\mu(\lambda) = -\mu(-\lambda)$ for $\lambda < 0$. By a change of variables, the integral I_2 can be rewritten in the form

$$\begin{aligned} I_2(x,t) &= \operatorname{Re} \int_{\frac{2}{3\sqrt{3}}}^{\infty} e^{i\lambda t} e^{(\alpha_\mu(\lambda)+i\beta_\mu(\lambda))x} e^{-i\lambda s} \phi_2\big(\mu(\lambda)\big)\hat{h}(\lambda)\,d\lambda \\ &:= E(x,t) \end{aligned}$$

for $x \geqslant 0$ with $\alpha_\mu(\lambda) := \alpha(\mu(\lambda))$ and $\beta_\mu(\lambda) := \beta(\mu(\lambda))$. Let the extension of $E(x,t)$ to $x<0$ be $g(x,t)$ and write

$$I_2(x,t) = \begin{cases} E(x,t), & x \geqslant 0, \\ g(x,t), & x < 0, \end{cases}$$

where $g(x,t)$ is to be defined.

Using the argument appearing in [13] (see Section 2), one may rewrite $\mathcal{F}_{x,t}[I_2](\xi,t)$ as

$$\begin{aligned} \mathcal{F}_{x,t}[I_2] = {}& \mathcal{F}_t\left[\int_0^\infty \big(E(x,t)\cos(x\xi) + g(-x,t)\cos(x\xi)\big)\,dx\right] \\ & + \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{1}{\xi-\eta}\mathcal{F}_t\left[\int_0^\infty \cos(\eta x)E(x,t)\,dx - \int_0^\infty \cos(\eta x)g(-x,t)\,dx\right]d\eta. \end{aligned}$$

For $x > 0$, choose $g(-x,t)$ such that

$$\begin{aligned} \mathcal{F}_t\left[\int_0^\infty g(-x,t)\cos(x\xi)\,dx\right](\tau) = {}& -\mathcal{F}_t\left[\int_0^\infty E(x,t)\cos(x\xi)\,dx\right](\tau)\Theta(\xi,\tau) \\ & + \mathcal{F}_t\left[\int_0^\infty E(x,t)\cos(x\xi)\,dx\right](\tau)\big(1-\Theta(\xi,\tau)\big)\nu(\xi)\omega(\tau) \end{aligned} \quad (2.18)$$

where $\Theta(\xi,\tau) = \chi(|\xi| - \delta|\tau|^{1/3})$ with $\delta > 0$ fixed, $0 \leqslant \chi(\xi) \leqslant 1$ everywhere, and

$$\chi(\xi) = \begin{cases} 1, & \xi < 0, \\ 0, & \xi > 0, \end{cases}$$

whilst

$$\nu(\xi) = \begin{cases} 1 & \text{if } |\xi| \geqslant 1, \\ 0 & \text{if } |\xi| < 1, \end{cases}$$

and $\omega(\tau)$ is a smooth and bounded function to be specified momentarily. It is easy to see that such a g is a combination of even and odd extensions, *viz.*

$$\mathcal{F}_{x,t}[I_2] := \hat{I}_{21}(\xi,\tau) + \hat{I}_{22}(\xi,\tau)$$

where

$$\hat{I}_{21}(\xi,\tau) = \mathcal{F}_t\left[\int_0^\infty E(x,t)\cos(x\xi)\,dx\right](\tau)\big(1-\Theta(\xi,\tau)\big)\big(1+\nu(\xi)\omega(\tau)\big)$$

and

$$\begin{aligned}\hat{I}_{22}(\xi,\tau) &= \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{1}{\xi-\eta}\mathcal{F}_t\left[\int_0^\infty E(x,t)\cos(x\eta)\,dx\right](\tau)\\ &\quad\times\big(2\Theta(\eta,\tau)+\big(1-\Theta(\eta,\tau)\big)\big(1+\nu(\xi)\omega(\tau)\big)\big)\,d\eta\\ &= \frac{i}{\pi}\int_0^\infty\left(\frac{1}{\xi-\eta}+\frac{1}{\xi+\eta}\right)\mathcal{F}_t\left[\int_0^\infty E(x,t)\cos(x\eta)\,dx\right](\tau)\\ &\quad\times\big(2\Theta(\eta,\tau)+\big(1-\Theta(\eta,\tau)\big)\big(1+\nu(\xi)\omega(\tau)\big)\big)\,d\eta.\end{aligned}$$

Because of the algebraic identity

$$\frac{1}{\xi-\eta}+\frac{1}{\xi+\eta} = \frac{2}{\xi}\left(1+\frac{\eta^2}{\xi^2-\eta^2}\right),$$

$\hat{I}_{22}(\xi,\tau)$ may be written as

$$\begin{aligned}\hat{I}_{22}(\xi,\tau) &= \frac{2i}{\pi\xi}\int_0^\infty\mathcal{F}_t\left[\int_0^\infty E(x,t)\cos(x\eta)\,dx\right](\tau)\\ &\quad\times\big(2\Theta(\eta,\tau)+\big(1-\Theta(\eta,\tau)\big)\big(1+\nu(\xi)\omega(\tau)\big)\big)\,d\eta\\ &\quad+\frac{2i}{\pi\xi}\int_0^\infty\frac{(\eta/\xi)^2}{1-(\eta/\xi)^2}\mathcal{F}_t\left[\int_0^\infty E(x,t)\cos(x\eta)\,dx\right](\tau)\\ &\quad\times\big(2\Theta(\eta,\tau)+\big(1-\Theta(\eta,\tau)\big)\big(1+\nu(\xi)\omega(\tau)\big)\big)\,d\eta\\ &:= Q_1(\xi,\tau)+Q_2(\xi,\tau).\end{aligned}\tag{2.19}$$

Choose a C^∞-smooth function $\omega(\tau)$ (cf. [13]) such that for all τ,

$$Q_1(\xi,\tau)\equiv 0,\quad \text{for } |\xi|\geqslant 1.$$

Hence, for $|\xi|>1$,

$$\hat{I}_{22}(\xi,\tau) = Q_2(\xi,\tau).$$

Thus, when $|\xi| > 1$ and $\tau \geqslant 0$,

$$\hat{I}_{22}(\xi,\tau) = \frac{2iC_2}{\xi}\int_0^{\infty} \frac{\eta^2}{\xi^2-\eta^2}\Bigg[\sum_{m=1}^{4} K_{m1}(\eta,\tau)\phi_2\big(\mu(\tau)\big)\hat{h}(\tau)\Bigg] \times \big(2\Theta(\eta,\tau) + \big(1-\Theta(\eta,\tau)\big)\big(1+\omega(\tau)\big)\big)\,d\eta \tag{2.20}$$

whereas

$$\hat{I}_{22}(\xi,\tau) = \frac{2iC_2}{\xi}\int_0^{\infty} \frac{\eta^2}{\xi^2-\eta^2}\Bigg[\sum_{m=1}^{4} K_{m2}(\eta,-\tau)\phi_2\big(\mu(-\tau)\big)\hat{h}(\tau)\Bigg] \times \big(2\Theta(\eta,\tau) + \big(1-\Theta(\eta,\tau)\big)\big(1+\omega(\tau)\big)\big)\,d\eta$$

when $|\xi| \geqslant 1$ and $\tau < 0$. Here, the functions K_{ij} are

$$\begin{aligned}
K_{11}(\eta,\lambda) &= \frac{-\alpha_\mu(\lambda)}{2(\alpha_\mu^2(\lambda) + (\eta+\beta_\mu(\lambda))^2)},\\
K_{21}(\eta,\lambda) &= \frac{-\alpha_\mu(\lambda)}{2(\alpha_\mu^2(\lambda) + (\eta-\beta_\mu(\lambda))^2)},\\
K_{31}(\eta,\lambda) &= \frac{\alpha_\mu^2(\lambda)\beta_\mu(\lambda)i}{(\alpha_\mu^2(\lambda) + (\eta+\beta_\mu(\lambda))^2)(\alpha_\mu^2(\lambda) + (\eta-\beta_\mu(\lambda))^2)},\\
K_{41}(\eta,\lambda) &= \frac{(\beta_\mu^2(\lambda)-\eta^2)\beta_\mu(\lambda)i}{(\alpha_\mu^2(\lambda) + (\eta+\beta_\mu(\lambda))^2)(\alpha_\mu^2(\lambda) + (\eta-\beta_\mu(\lambda))^2)}
\end{aligned} \tag{2.21}$$

and

$$\begin{cases} K_{12}(\eta,\lambda) = K_{11}(\eta,\lambda), & K_{22}(\eta,\lambda) = K_{21}(\eta,\lambda),\\ K_{32}(\eta,\lambda) = -K_{31}(\eta,\lambda), & K_{42}(\eta,\lambda) = -K_{41}(\eta,\lambda).\end{cases}$$

The extension of the operator $W_{j,m}(t)$ as just outlined will be denoted by $\mathcal{BI}^{(j,m)}(t)$ for $j,m = 1,2,3$. The boundary integral operator corresponding to this extension of $\mathcal{W}_{bdr}(t) = \sum_{j,m=1}^{3} W_{j,m}(t)$ is denoted by $\mathcal{BI}(t)$.

2.2. Linear estimates

In this subsection, estimates for solutions of associated linear problems for the KdV–Burgers equation are provided. These are used in establishing the well-posedness of the nonlinear problems in the next section.

For given $s \in R$, $b \in [0,1]$ and any function $w \equiv w(x,t) : R \times R \to R$, define

$$\Lambda_{s,b}(w) = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big\langle i\big(\tau - (\xi^3 - 4\xi)\big) + 3\xi^2\big\rangle^{2b} \langle\xi\rangle^{2s} \big|\hat{w}(\xi,\tau)\big|^2\,d\xi\,d\tau\right)^{1/2}$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$. Let $X_{s,b}$ be the space of all functions w satisfying

$$\|w\|_{X_{s,b}} := \Lambda_{s,b}(w) < \infty \tag{2.22}$$

and

$$\mathcal{X}_{s,b} \equiv C\big(R; H^s(R)\big) \cap X_{s,b} \tag{2.23}$$

with the norm

$$\|w\|_{\mathcal{X}_{s,b}} = \Big(\sup_{t\in R}\|w(\cdot,t)\|^2_{H^s(R)} + \|w\|^2_{X_{s,b}}\Big)^{1/2}. \tag{2.24}$$

Consider first the semigroup $\{W_R(t)\}_0^\infty$ associated to the linear KdV–Burgers equation posed on the whole line R. Recall that for any $\phi \in \mathcal{S}'$,

$$\mathcal{F}_x\big(W_R(t)\phi\big)(\xi) = \exp\big[-3\xi^2 t + i\big(\xi^3 - 4\xi\big)t\big]\hat{\phi}(\xi)$$

for all $t \geqslant 0$, and we extend W_R to a linear operator defined on the whole real axis by setting

$$\mathcal{F}_x\big(W_R(t)\phi\big)(\xi) = \exp\big[-3\xi^2|t| + i\big(\xi^3 - 4\xi\big)t\big]\hat{\phi}(\xi)$$

for $t \in R$. The proof of the following proposition regarding $\{W_R(t)\}_0^\infty$ follows the argument in [35, Section 3], with some minor modifications.

Proposition 2.4. *Let $-\infty < s < \infty$, $0 < b \leqslant 1$, $0 < \delta < \frac{1}{2}$ and $\delta' > 0$ be given.*

(i) *There exists a constant C depending only on s and b such that*

$$\big\|\psi(t)W_R(t)\phi\big\|_{\mathcal{X}_{s,b}} \leqslant C\|\phi\|_{H^s(R)}. \tag{2.25}$$

(ii) *There exists $C_\delta > 0$ such that for all $u \in X_{s,-1/2+\delta}$,*

$$\left\|\psi(t)\int_0^t W_R(t-t')f(t')\,dt'\right\|_{\mathcal{X}_{s,\frac{1}{2}}} \leqslant C_\delta\|f\|_{X_{s,-\frac{1}{2}+\delta}}. \tag{2.26}$$

(iii) *For all $f \in X_{s,-\frac{1}{2}+\delta}$, the mapping*

$$t \mapsto \int_0^t W_R(t-t')f(t')\,dt'$$

lies in $C(R^+, H^{s+2\delta}(R))$. In addition, if $\{f_n\}$ is a sequence with $f_n \to 0$ in $X_{s,-\frac{1}{2}+\delta}$, then

$$\left\|\int_0^t W_R(t-t')f_n(t')\,dt'\right\|_{L^\infty(R^+,H^{s+2\delta}(R))} \to 0 \quad \text{as } n \to \infty.$$

The next two propositions establish an estimate for the spatial traces of $W_R(t)\phi$ and the integral $\int_0^t W_R(t-t')f(\cdot,t')\,dt$. The proof of these inequalities can be found in [13].

Proposition 2.5. *Let $s \in [-1, 2]$ be given. There exists a constant C depending only on s such that*

$$\sup_{x\in R}\left\|W_R(t)\phi\right\|_{H_t^{\frac{s+1}{3}}(R)} \leqslant C\|\phi\|_{H^s(R)} \tag{2.27}$$

and

$$\sup_{x\in R}\left\|\partial_x W_R(t)\phi\right\|_{H_t^{\frac{s}{3}}(R)} \leqslant C\|\phi\|_{H^s(R)} \tag{2.28}$$

for any $\phi \in H^s(R)$.

Proposition 2.6. *Let $0 \leqslant b < 1/2$, $-1 \leqslant s \leqslant 2-3b$, $\psi \in C_0^\infty(R)$ and*

$$w(x,t) = \int_0^t W_R(t-t')f(\cdot,t')\,dt'.$$

There exists C depending only on b, s and ψ such that

$$\sup_{x\in R}\left\|\psi(\cdot)w(x,\cdot)\right\|_{H_t^{\frac{s+1}{3}}(R)} \leqslant C\|f\|_{X_{s,-b}} \tag{2.29}$$

and

$$\sup_{x\in R}\left\|\psi(t)w_x(x,t)\right\|_{H_t^{s/3}(R)} \leqslant C\|f\|_{X_{s,-b}}. \tag{2.30}$$

Finally, attention is turned to the boundary integral operators $\mathcal{BI}^{(j,m)}(t)$, $j,m = 1,2,3$.

Proposition 2.7. *Let $\psi \in C_0^\infty(R)$ be given and assume that $0 \leqslant b < 1/2 - s/3$ with $s \leqslant 0$ and $b \neq \frac{1}{2}$. Then, there exists a constant C such that for any $h \in H_0^{\frac{2b+s}{3}}(R^+)$*

$$\left\|\psi\mathcal{BI}^{(j,m)}(h)\right\|_{X_{s,b}} \leqslant C\|h\|_{H^{\frac{2b+s}{3}}(R^+)}, \quad j,m = 1,2,3. \tag{2.31}$$

Proposition 2.8. *Let $-\frac{3}{2} < \alpha < \frac{1}{2}$ and $-\frac{1}{2} < \beta < 1$ be given. There exist constants C_α and C_β such that*

$$\sup_{t\in R}\left\|\mathcal{BI}^{(j,m)}h\right\|_{H^\alpha(R)} \leqslant C\|h\|_{H^{(\alpha+1)/3}(R^+)}, \quad j,m = 1,2,3. \tag{2.32}$$

The proofs of Propositions 2.7 and 2.8 are similar to those in Section 3 of [13]. A sketch of the proof for Proposition 2.7 will be given in Appendix A.

Observe that

$$\|w\|_{L^2(0,T;H^s(R))} \leqslant C\Lambda_{s,b}(\psi w)$$

for any $s \in R$ and $b \geqslant 0$, where $\psi \in C_0^\infty(R)$ and $\psi(t) = 1$ when $t \in (0,T)$. The following result, which follows from Propositions 2.7 and 2.8, presents a boundary smoothing property of the linear KdV–Burgers equation, which is the same as that which holds for the linear KdV equation (see again [13]).

Corollary 2.9. *For any given $T > 0$ and $s \geqslant -\frac{3}{2}$, there exists a constant C such that*

$$\|\mathcal{W}_{bdr}\vec{h}\|_{L^2(0,T;H^{s+1}(R^+))} \leqslant C\|\vec{h}\|_{H^{\frac{1+s}{3}}(R^+)\times H^{\frac{1+s}{3}}(R^+)\times H^{\frac{s}{3}}(R^+)} \tag{2.33}$$

for any $\vec{h} \in H_0^{\frac{1+s}{3}}(R^+) \times H_0^{\frac{1+s}{3}}(R^+) \times H_0^{\frac{s}{3}}(R^+)$.

The boundary integral operator $\mathcal{W}_{bdr}$ also possesses the sharp Kato smoothing property as described below.

Proposition 2.10. *For any given $T > 0$ and $s \geqslant -\frac{3}{2}$, there exists a constant C such that*

$$\sup_{x\in R^+} \|\partial_x \mathcal{W}_{bdr}\vec{h}\|_{H_t^{\frac{s}{3}}(R^+)} \leqslant C\|\vec{h}\|_{H^{\frac{1+s}{3}}(R^+)\times H^{\frac{1+s}{3}}(R^+)\times H^{\frac{s}{3}}(R^+)} \tag{2.34}$$

for any $\vec{h} \in H_0^{\frac{1+s}{3}}(R^+) \times H_0^{\frac{1+s}{3}}(R^+) \times H_0^{\frac{s}{3}}(R^+)$.

3. The nonlinear problem

In this section, we study the well-posedness of the nonlinear IBVP

$$\left.\begin{aligned}
&u_t + 4u_x + \frac{1}{2}\left(e^{4t-2x}u^2\right)_x + u_{xxx} - 3u_{xx} = 0, \quad \text{for } x \in (0,1),\ t > 0,\\
&u(x,0) = \phi(x), \quad \text{for } x \in (0,1),\\
&u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u_x(1,t) = h_3(t), \quad \text{for } t \geqslant 0.
\end{aligned}\right\} \tag{3.1}$$

Let $\mathcal{Y}_{s,b}$ be the space of all functions w in $\mathcal{X}_{s,b}$ (see (2.22)–(2.24)) satisfying

$$\sup_{x\in R}\left\|w_x(x,\cdot)\right\|_{H_t^{\frac{s}{3}}(R)} < +\infty.$$

For any $w \in \mathcal{Y}_{s,b}$, define

$$\|w\|_{\mathcal{Y}_{s,b}} = \left(\|w\|^2_{\mathcal{X}_{s,b}} + \sup_{x\in R}\left\|w_x(x,\cdot)\right\|^2_{H_t^{\frac{s}{3}}(R)}\right)^{\frac{1}{2}}.$$

The above Bourgain-type spaces are defined for functions whose domain is the whole plane $R \times R$. However, the IBVP (3.1) is posed on the domain $(0,1) \times R^+$ and we are seeking its solution in the space $C(R^+; H^s(0,1))$ corresponding to a given initial value in the space $H^s(0,1)$ and boundary data in the space $H_{\text{loc}}^{\frac{s+1}{3}}(R^+) \times H_{\text{loc}}^{\frac{s+1}{3}}(R^+) \times H_{\text{loc}}^{\frac{s}{3}}(R^+)$. It is thus natural to consider restricted versions of these Bourgain-type spaces to the strip $(0,1) \times R^+$. Let Ω denote a subinterval of R; define a restricted version of the Bourgain space $X_{s,b}$ to the domain $(0,1) \times \Omega$ as follows:

$$X_{s,b}\big((0,1)\times\Omega\big) = X_{s,b}|_{(0,1)\times\Omega}$$

with the quotient norm

$$\|u\|_{X_{s,b}((0,1)\times\Omega)} \equiv \inf_{w\in X_{s,b}}\left\{\|w\|_{X_{s,b}}:\ w(x,t) = u(x,t) \text{ on } (0,1)\times\Omega\right\}.$$

The spaces $\mathcal{X}_{s,b}((0,1)\times\Omega)$ and $\mathcal{Y}_{s,b}((0,1)\times\Omega)$ are defined similarly. In addition, define

$$\mathcal{H}^s_{\mathrm{loc}}(R^+)=H^{\frac{s+1}{3}}_{\mathrm{loc}}(R^+)\times H^{\frac{s+1}{3}}_{\mathrm{loc}}(R^+)\times H^{\frac{s}{3}}_{\mathrm{loc}}(R^+)$$

and

$$\mathcal{H}^s(0,T)=H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s}{3}}(0,T)$$

for $0<T\leqslant\infty$.

For the IBVP (3.1), the following well-posedness result obtains.

Theorem 3.1. *Let $-1<s\leqslant 0$, $T>0$ and $r>0$ be given. There exist $T^*\in(0,T]$ and $b\in(0,\frac{1}{2})$ such that for a given pair $(\phi,\vec{h})\in H^s(0,1)\times\mathcal{H}^s_{\mathrm{loc}}(R^+)$ satisfying*

$$\|\phi\|_{H^s(0,1)}+\|\vec{h}\|_{\mathcal{H}^s(0,T)}\leqslant r,$$

the IBVP (3.1) admits a unique solution $u\in\mathcal{Y}_{s,b}((0,1)\times(0,T^*))$. Moreover, the solution u depends continuously on ϕ and h in the corresponding spaces.

The proof of Theorem 3.1 is based on the results expounded in Section 2 and the following lemmas. The solution of the non-homogeneous linear problem

$$\left.\begin{aligned}&u_t+4u_x-3u_{xx}+u_{xxx}=0,\quad \text{for } x\in(0,1),\ t\geqslant 0,\\&u(x,0)=0,\\&u(0,t)=h_1(t),\quad u(1,t)=h_2(t),\quad u_x(1,t)=h_3(t)\end{aligned}\right\}\tag{3.2}$$

may be written in the form

$$u(x,t)=\big[\mathcal{W}_{bdr}(t)\vec{h}\big](x)$$

for $x\in(0,1)$, $t\geqslant 0$ as expounded in Section 2.

Lemma 3.2. *For a given pair (b,s) satisfying*

$$0\leqslant b<\frac{1}{2}-\frac{s}{3}\quad \text{with } s\leqslant 0 \text{ and } b<\frac{1}{2},\tag{3.3}$$

there exists a constant C such that for any $T>0$ and any $\vec{h}\in\mathcal{H}^s(0,T)$, the corresponding solution u of (3.2) belongs to the restricted Bourgain space $\mathcal{Y}_{s,b}((0,1)\times(0,T))$ and satisfies

$$\|u\|_{\mathcal{Y}_{s,b}((0,1)\times(0,T))}\leqslant C\|\vec{h}\|_{\mathcal{H}^s(0,T)}.\tag{3.4}$$

Proof. For $T>0$, let $\vec{h}_1\in\mathcal{H}^s(R^+)$ be such that $\vec{h}_1\equiv\vec{h}$ in the space $\mathcal{H}^s(0,T)$ and

$$\|\vec{h}_1\|_{\mathcal{H}^s(R^+)}\leqslant C\|\vec{h}\|_{\mathcal{H}^s(0,T)}.$$

Let $\psi_1\in C_0^\infty(R)$ be so that $\psi_1(t)=1$ for all $t\in[0,T]$. Define

$$u_1(x,t)=\big[\mathcal{BI}(t)\vec{h}_1\big](x).$$

Observing that

$$u(x,t) = u_1(x,t) \quad \text{for } (x,t) \in R^+ \times [0,T],$$

and using Propositions 2.7 and 2.8, one arrives at the inequalities

$$\|u\|_{\mathcal{Y}_{s,b}((0,1)\times(0,T))} \leqslant \|\psi_1 u_1\|_{\mathcal{Y}_{s,b}} \leqslant C\|\vec{h}_1\|_{\mathcal{H}^s(R^+)},$$

from which (3.4) follows. The proof is complete. □

Consider the same linear equation posed with zero boundary conditions, but non-trivial initial data, *viz.*

$$\left.\begin{aligned} &u_t + 4u_x - 3u_{xx} + u_{xxx} = 0, \quad \text{for } x \in (0,1),\ t \geqslant 0,\\ &u(x,0) = \phi(x),\\ &u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0. \end{aligned}\right\} \tag{3.5}$$

Its solution can be written as

$$u(x,t) = \big[\mathcal{W}_c(t)\phi\big](x)$$

for $x, t \geqslant 0$.

Lemma 3.3. *For a given pair (b,s) satisfying (3.3), there exists a constant C such that for any $T > 0$ and any $\phi \in H_0^s(R^+)$, the corresponding solution u of (3.5) belongs to the restricted Bourgain space $\mathcal{Y}_{s,b}(R^+ \times (0,T))$ and satisfies the inequality*

$$\|u\|_{\mathcal{Y}_{s,b}(R^+\times(0,T))} \leqslant C\|\phi\|_{H^s(R^+)}. \tag{3.6}$$

Proof. According to Proposition 2.2, one may write $W_c(t)\phi$ as

$$\mathcal{W}_c(t)\phi = W_R(t)\phi^* - \mathcal{W}_{bdr}(t)\vec{g}$$

for any $x, t > 0$, where $\phi^* \in H^s(R)$, ϕ^* equals ϕ when restricted on R^+, and $\vec{g} = (g_1, g_2, g_3)$ is the associated boundary trace values of $v = W_R(t)\phi^*$, which is to say,

$$g_1(t) = v(0,t), \qquad g_2(t) = v(1,t), \qquad g_3(t) = v_x(1,t).$$

The inequality (3.6) follows from Propositions 2.4, 2.5 and Lemma 3.2. □

We turn consideration to the forced linear problem

$$\left.\begin{aligned} &u_t + 4u_x - 3u_{xx} + u_{xxx} = f, \quad \text{for } x \in (0,1),\ t \geqslant 0,\\ &u(x,0) = 0,\\ &u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0. \end{aligned}\right\} \tag{3.7}$$

Its solution can be written in the form

$$u(\cdot,t)=\int_0^t \mathcal{W}_c(t-\tau)f(\cdot,\tau)\,d\tau.$$

Lemma 3.4. *Assume that $-1<s\leqslant 0$ and $0<b,b'<\frac{1}{2}$. There exists a constant C such that for any $T>0$ and any $f\in X_{s,-b}(R^+\times(0,T))$, the corresponding solution u of (3.7) belongs to the space $\mathcal{X}_{s,b'}(R^+\times(0,T))$ and satisfies the estimate*

$$\|u\|_{\mathcal{Y}_{s,b'}(R^+\times(0,T))}\leqslant C\|f\|_{X_{s,-b}(R^+\times(0,T))}. \tag{3.8}$$

Proof. By Proposition 2.3,

$$u(\cdot,t)=\int_0^t W_R(t-\tau)f(\cdot,\tau)\,d\tau-\mathcal{W}_{bdr}(t)\vec{v}$$

for any $x,t>0$ where $\vec{v}\equiv\vec{v}(t)$ is the boundary trace values of $\int_0^t W_R(t-\tau)f(\cdot,\tau)\,d\tau$. The estimate (3.8) then follows from Propositions 2.4, 2.7 and Lemma 3.2. □

The next lemma presents a version of so-called bilinear estimates in the restricted Bourgain space $X_{s,b}((0,1)\times(0,T))$ which follows from minor modifications of the proof of Lemma 3.1 in [35].

Lemma 3.5. *Given $s>-1$ and $T>0$, there exist positive constants C, μ, δ and $b\in(0,\frac{1}{2})$ such that*

$$\left\|\partial_x(uv)\right\|_{X_{s,-1/2+\delta}((0,1)\times(0,T))}\leqslant CT^{\mu}\|u\|_{X_{s,b}((0,1)\times(0,T))}\|v\|_{X_{s,b}((0,1)\times(0,T))} \tag{3.9}$$

for any $u,v\in X_{s,b}((0,1)\times(0,T))$.

The way is now prepared for the proof of Theorem 3.1.

Proof of Theorem 3.1. By applying Lemmas 3.2–3.5, Theorem 3.1 can be established by the standard contraction mapping principle.

In more detail, let $\phi\in H^s(0,1)$ and $\vec{h}\in\mathcal{H}^s_{loc}(R^+)$ be given with $s\in(-1,0]$. For given θ with $0<\theta\leqslant 1$ (to be chosen precisely presently) and $v,w\in\mathcal{Y}_{s,b}((0,1)\times(0,\theta))$, define

$$\mathbf{F}(w)=W_c(t)\phi+\mathcal{W}_{bdr}(t)\vec{h}-\int_0^t \mathcal{W}_c(t-\tau)\left(e^{4t-2x}w^2\right)_x(\tau)\,d\tau.$$

Using Lemmas 3.2–3.5, it is seen that

$$\left\|\mathbf{F}(w)\right\|_{\mathcal{Y}_{s,b}((0,1)\times(0,\theta))}\leqslant C_1\left(\|\phi\|_{H^s(0,1)}+\|\vec{h}\|_{\mathcal{H}^s(0,T)}\right)+C_2\theta^{\mu}\|w\|^2_{\mathcal{Y}_{s,b}((0,1)\times(0,\theta))}$$

and

$$\left\|\mathbf{F}(v)-\mathbf{F}(w)\right\|_{\mathcal{Y}_{s,b}((0,1)\times(0,\theta))}\leqslant C_2\theta^{\mu}\|v-w\|_{\mathcal{Y}_{s,b}((0,1)\times(0,\theta))}\|v+w\|_{\mathcal{Y}_{s,b}((0,1)\times(0,\theta))}$$

where the constants C_1 and C_2 are independent of θ, v and w. Let B_r be the ball of radius r in the space $\mathcal{Y}_{s,b}(R^+ \times (0,\theta))$ where

$$r = 2C_1\big(\|\phi\|_{H^s(R^+)} + \|\vec{h}\|_{\mathcal{H}^s(0,T)}\big),$$

and choose $\theta = T^*$ small enough that

$$2C_2\big(T^*\big)^{\mu} r \equiv \beta < 1.$$

It then follows readily that $\mathbf{F}$ maps B_r into itself and that for $w, v \in B_r$,

$$\big\|\mathbf{F}(w) - \mathbf{F}(v)\big\|_{\mathcal{Y}_{s,b}((0,1)\times(0,T^*))} \leqslant \beta \|w - v\|_{\mathcal{Y}_{s,b}((0,1)\times(0,T^*))}.$$

Thus, the function $\mathbf{F}$ is a contraction mapping of the ball B_r. The fixed point u of this map $\mathbf{F}$ in B_r is the advertised solution. □

The well-posedness result presented in Theorem 3.1 is conditional since uniqueness is established in the space $\mathcal{Y}_{s,b}((0,1)\times(0,T))$ rather than in the space $C([0,T]; H^s(0,1))$. However, following the procedure developed in [11], one can show that in fact uniqueness holds in the space $C([0,T]; H^s(0,1))$.

Proposition 3.6. *Let $s \in (-1,0]$ and $r > 0$ be given. There exists a $T > 0$ depending only on s and r such that for given $(\phi, \vec{h}) \in H^s(0,1) \times \mathcal{H}_{loc}(R^+)$ satisfying*

$$\|\phi\|_{H^s(0,1)} + \|\vec{h}\|_{\mathcal{H}^s(0,T)} \leqslant r,$$

the IBVP (3.1) admits a unique solution $u \in C([0,T]; H^s(0,1))$. Moreover, the solution u depends continuously on ϕ and h in the respective spaces.

This well-posedness result is, in fact, valid for any $s > -1$.

Theorem 3.7. *Let $s > -1$ and $r > 0$ be given. There exists a $T > 0$ depending only on s and r such that for a given s-compatible (see [10] for the definition) $(\phi, \vec{h}) \in H^s(0,1) \times \mathcal{H}_{loc}(R^+)$ satisfying*

$$\|\phi\|_{H^s(0,1)} + \|\vec{h}\|_{\mathcal{H}^s(0,T)} \leqslant r,$$

the IBVP (3.1) admits a unique solution $u \in C([0,T]; H^s(0,1))$. Moreover, the solution u depends continuously on ϕ and h in the respective spaces.

Proof. When $s \geqslant 0$, we refer to [10] for the proof. □

Next, consider the issue of global well-posedness of the IBVP (3.1). The proof of the following theorem may be found in [13].

Theorem 3.8. *Let $s \geqslant 0$ and $T > 0$ be given. For any compatible*

$$(\phi, \vec{h}) \in H^s\big(R^+\big) \times \mathcal{H}^{s+}_{loc}\big(R^+\big),$$

the IBVP (3.1) admits a unique solution $u \in C([0,T]; H^s(R^+))$, where $s^+ = s$ when $s \geqslant 3$ and $s^+ = s + \epsilon$ when $0 \leqslant s < 3$ with arbitrarily small $\epsilon > 0$. Moreover, the solution u depends analytically on ϕ and h in the respective spaces.

When $-1 < s < 0$, Molinet and Ribaud [35] showed that the IVP for the KdV–Burgers equation is also globally well-posed in the space $H^s(R)$ by taking advantage of the dissipative smoothing property of the KdV–Burgers equation. We have a similar global well-posedness result for the IBVP (3.1).

Theorem 3.9. *Let $-1 < s < 0$, $\epsilon' > 0$ and $T > 0$ be given. For any given*

$$(\phi, \vec{h}) \in H^s(R^+) \times \mathcal{H}^{\epsilon'}_{\text{loc}}(R^+),$$

the IBVP (3.1) admits a unique solution $u \in C([0,T]; H^s(R^+))$. Moreover, the solution u has the property that $u \in L^2([\epsilon, T]; H^1(R^+))$ for any $\epsilon > 0$.

The following lemma is needed in the proof of the last theorem.

Lemma 3.10. *Given $s > -1$ and $T > 0$, there exist $\delta > 0$ and b with $0 < b < \frac{1}{2}$ such that for all $u \in \mathcal{X}_{s,b}(R^+ \times (0,T))$,*

$$w = \int_0^t \mathcal{W}_c(t-\tau)(uu_x)(\tau)\,d\tau$$

lies in $L^2(0,T; H^{s+\delta}(R^+))$.

Proof. By Proposition 2.3, the distribution w may be rewritten $w = w_1 + w_2$ with

$$w_1 = \int_0^t \mathcal{W}_R(t-\tau)(uu_x)(\tau)\,d\tau \quad \text{and} \quad w_2 = \mathcal{W}_{bdr}(t)\vec{g}$$

where $\vec{g}$ is comprised of the appropriate boundary traces of w_1. According to Lemma 3.5, there exists a $\delta > 0$ such that $uu_x \in \mathcal{X}_{s,-\frac{1}{2}+\delta}(R^+ \times (0,T))$. Thus, $w_1 \in L^2(0,T; H^{s+\delta}(R^+))$ by Proposition 2.4. In addition, it follows from Proposition 2.6 that $\vec{g} \in \mathcal{H}^s(R^+)$ which yields that $w_2 \in L^2(0,T; H^{s+\delta}(R^+))$ by Corollary 2.9. The proof is complete. □

Proof of Theorem 3.9. By Theorem 3.1, the IBVP (3.1) admits a unique solution

$$u \in C([0,T^*]; H^s(R^+))$$

for some $T^* \leqslant T$. Moreover, u can be decomposed in the form

$$u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t)$$

with

$$u_1(x,t) = \mathcal{W}_c(t)\phi, \qquad u_2(x,t) = \mathcal{W}_{brd}(t)\vec{h}, \qquad u_3(x,t) = -\int_0^t \mathcal{W}_c(t-\tau)\left(au^2\right)_x(\tau)\,d\tau$$

where $a(x,t) = \frac{1}{2}e^{4t-2x}$. According to Corollary 2.9, $u_2 \in L^2(0,T; H^1(R^+))$. By Proposition 2.2,

$$u_1 = W_R(t)\phi^* - \mathcal{W}_{bdr}(t)\vec{h}_1$$

with the vector $\vec{h}_1$ having as components the relevant boundary traces of $W_R(t)\phi^*$. It therefore transpires that for any $\epsilon > 0$, $u_1 \in C([\epsilon, T]; H^\infty(R^+))$. As for u_3, it follows from Lemma 3.10 that $u_3 \in L^2(0, T^*; H^{s+\delta}(R^+))$ for some $\delta > 0$. Consequently, for any ϵ with $0 < \epsilon \leqslant T^*$, there is a $t_1 \in (0, \epsilon)$ such that $u(\cdot, t_1) \in H^{s+\delta}(R^+)$. Taking $\psi(x) = u(x, t_1)$ as a new initial value for the IBVP (3.1) and using the same argument, one arrives at $u(\cdot, t_2) \in H^{s+2\delta}(R^+)$ for some $t_1 < t_2 < \epsilon$. Repeating this procedure, one eventually arrives at the conclusion that $u(\cdot, t') \in H^{\frac{3}{2}}(R^+)$ for some $0 < t' < \epsilon$. The proof is completed by invoking Theorem 3.8. □

For the pure initial-value problem (1.6), if $\psi \in H^s(R)$ for some $s > -1$, then the corresponding solution u lies in $C([\epsilon, T]; H^\infty(R))$. We have a similar result in the present context, which follows directly from Theorem 3.9.

Corollary 3.11. *Let $s > -1$ and $T > 0$ be given. Assume that $\phi \in H^s(R^+)$ and $\vec{h} \in \mathcal{H}^\infty(0, T)$. Then the corresponding solution u of the IBVP (3.1) belongs to the space $C([\epsilon, T]; H^\infty(R^+))$ for any ϵ with $0 < \epsilon < T$.*

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Appendix A

In this appendix, a proof of Proposition 2.7 is presented. Since the proof is similar to that appearing in [13], a sketch suffices. The reader is referred to [13] for details.

Proof of Proposition 2.7. Recall that

$$\big[\mathcal{BI}^{(1,m)}(t)h\big](x) = I_1(x,t) + I_2(x,t)$$

where $I_1(x,t)$ is a function defined on the whole plane $R \times R$ and is, in fact, a C^∞-smooth function of x and t. For any $t \in R$,

$$\begin{aligned}\|I_1(x,t)\|_{L^2_x(R)} &\leqslant C\left\|(3\mu^2-1)\phi_1(\mu)\int_0^\infty e^{-i(\mu^3-\mu)\xi}h(\xi)\,d\xi\right\|_{L^2_\mu(R)} \\ &\leqslant C\|h\|_{L^2(R^+)}.\end{aligned}$$

This type of inequality is also valid for $\partial_x^j\partial_t^l I_1$ for any $j, l \geqslant 0$. Thus, it is straightforward to see that if $h \in L^2(R^+)$, then

$$\Lambda_{s,b}(\psi I_1) \leqslant C\|h\|_{L^2(R^+)} \tag{A.1}$$

for any given $b \geqslant 0$ and $s \in R$ where the constant C depends only on ψ, b and s.

To analyze $I_2(x,t)$, remember that

$$\mathcal{F}_{x,t}[I_2](\xi,\tau) = \hat{I}_{21}(\xi,\tau) + \hat{I}_{22}(\xi,\tau)$$

where, for $|\xi| > 1$,

$$\hat{I}_{21}(\xi,\tau)=\mathcal{F}_t\left[\int_0^\infty E(x,t)\cos(x\xi)\,dx\right]\big(1-\Theta(\xi,\tau)\big)\big(1+\omega(\tau)\big),$$

$$\hat{I}_{22}(\xi,\tau)=iC_2\int_0^\infty\left(\frac{1}{\xi-\eta}+\frac{1}{\xi+\eta}\right)\mathcal{F}_t\left[\int_0^\infty E(x,t)\cos(x\eta)\,dx\right]$$
$$\times\big(2\Theta(\eta,\tau)+\big(1-\Theta(\eta,\tau)\big)\big(1+\omega(\tau)\big)\big)\,d\eta.$$

Since the relevant estimates in the regions $|\xi|<1$ are straightforward, in what follows it is always assumed that $|\xi|\geqslant 1$. First, consider the term

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big\langle i\big(\tau-(\xi^3-4\xi)\big)+3\xi^2\big\rangle^{2b}\langle\xi\rangle^{2s}\big|\hat{I}_{21}(\xi,\tau)\big|^2\,d\xi\,d\tau. \qquad \square$$

Proposition A.1. *Let $s\leqslant 0$ and b with $0<b<\min\{\frac{1}{2}-\frac{s}{3},\frac{3}{2}\}$ be given. There exists a constant C such that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big\langle i\big(\tau-(\xi^3-4\xi)\big)+3\xi^2\big\rangle^{2b}\langle\xi\rangle^{2s}\big|\hat{I}_{21}(\xi,\tau)\big|^2\,d\xi\,d\tau<C\|h\|^2_{H^{\frac{2b+s}{3}}(R^+)} \tag{A.2}$$

for any $h\in H^{\frac{2b+s}{3}}(R^+)$.

Proof. According to (2.19)–(2.20),

$$\mathcal{F}_t\left[\int_0^\infty E(x,t)\cos(x,\xi)\,dx\right]=\sum_{m=1}^4 K_{m1}(\xi,\lambda)\phi_2\big(\mu(\lambda)\big)\hat{h}(\lambda)+\sum_{m=1}^4 K_{m2}(\xi,-\lambda)\phi_2\big(\mu(-\lambda)\big)\hat{h}(-\lambda).$$

In the following, detailed analysis is given for terms containing K_{21}; the estimates for the other terms follow similar lines. Suppose $\xi\geqslant 0$ in what follows. The case $\xi<0$ is entirely analogous. Use the notation

$$A_{m1}(\xi,\tau)=K_{m1}(\xi,\tau)\phi_2\big(\mu(\tau)\big)\hat{h}(\tau),\quad m=1,2,3.$$

For given $s\leqslant 0$ and $b>0$, we have

$$\int_{-\infty}^{\infty}\int_0^{\infty}\big\langle i\big(\tau-(\xi^3-4\xi)\big)+3\xi^2\big\rangle^{2b}\langle\xi\rangle^{2s}\big|A_{21}(\xi,\tau)\big(1+\omega(\tau)\big)\big(1-\Theta(\xi,\tau)\big)\big|^2\,d\xi\,d\tau$$
$$\leqslant C\int_{-\infty}^{\infty}\phi_2\big(\mu(\tau)\big)\big|\hat{h}(\tau)\big|^2B_{21}(\tau)\,d\tau$$

with

$$B_{21}(\tau)=\int_0^\infty \langle i(\tau-(\xi^3-4\xi))+3\xi^2\rangle^{2b}\langle\xi\rangle^{2s}\big|(1+\omega(\tau))(1-\Theta(\xi,\tau))\big|^2$$
$$\times\frac{\alpha_\mu^2(\tau)\phi_2(\mu(\tau))}{(\alpha_\mu^2(\tau)+(\xi-\beta_\mu(\tau))^2)^2}\,d\xi.$$

Claim. *If $b<\min\{\frac{1}{2}-\frac{s}{3},\frac{3}{2}\}$, then as $\tau\to\infty$,*

$$B_{21}(\tau)\sim\tau^{(4b+2s)/3}.$$

To see this claim is valid, note that in fact

$$B_{21}(\tau)=\int_{\delta|\tau|^{1/3}}^{\infty}\langle i(\tau-(\xi^3-4\xi))+3\xi^2\rangle^{2b}\langle\xi\rangle^{2s}\frac{\alpha_\mu^2(\tau)\phi_2(\mu(\tau))}{(\alpha_\mu^2(\tau)+(\xi-\beta_\mu(\tau))^2)^2}$$
$$\times\big(1+\omega(\tau)\big)^2\big(1-\Theta(\xi,\tau)\big)^2\,d\xi$$

since $\Theta(\xi,\tau)=1$ when $\xi<\delta|\tau|^{\frac{1}{3}}$, where $\delta>0$ is fixed, but arbitrary for the nonce. Let $\xi=\eta(\zeta)$ be the real solution of the equation

$$\xi^3-4\xi=\zeta,\quad 0\leqslant\zeta<\infty,\ 2\leqslant\xi<\infty,$$

that connects continuously to the unique real root as ζ becomes large. It is clear that

$$\eta(\zeta)\sim\zeta^{1/3}\quad\text{as }\zeta\to\infty.$$

For large τ, it is also the case that

$$\mu(\tau)\sim\tau^{1/3},\qquad|\alpha_\mu(\tau)|\sim1,\qquad|\beta_\mu(\tau)|\sim\tau^{1/3}.$$

Thus, for $\tau>0$ large enough,

$$B_{21}(\tau)\leqslant C\int_{\delta^3\tau}^{\infty}\frac{(1+|\tau-\zeta|+\zeta^{\frac{2}{3}})^{2b}}{(1+(\eta(\zeta)-\tau^{1/3})^2)^2}\langle\zeta\rangle^{2s/3}\frac{1}{3\eta^2(\zeta)-4}\,d\zeta$$
$$=C\int_{\delta^3\tau}^{2\tau}\frac{(1+|\tau-\zeta|+\zeta^{\frac{2}{3}})^{2b}}{(1+(\eta(\zeta)-\tau^{1/3})^2)^2}\langle\zeta\rangle^{2s/3}\frac{1}{3\eta^2(\zeta)-4}\,d\zeta$$
$$+C\int_{2\tau}^{\infty}\frac{(1+|\tau-\zeta|+\zeta^{\frac{2}{3}})^{2b}}{(1+(\eta(\zeta)-\tau^{1/3})^2)^2}\langle\zeta\rangle^{2s/3}\frac{1}{3\eta^2(\zeta)-4}\,d\zeta$$
$$:=G_{21-1}(\tau)+G_{21-2}(\tau).$$

Continuing this sequence of inequalities, it is seen that

$$
\begin{aligned}
G_{21-1}(\tau) &\leqslant C\big(1+|\tau|\big)^{2s/3} \int_{\delta^3\tau}^{2\tau} \frac{(1+|\tau-\zeta|+\zeta^{\frac{2}{3}})^{2b}}{(1+(\eta(\zeta)-\tau^{1/3})^2)^2} \frac{1}{3\eta^2(\zeta)-4}\, d\zeta \\
&\leqslant C\big(1+|\tau|\big)^{\frac{2s}{3}} \int_{\delta^3\tau}^{2\tau} \frac{(1+|\tau-\zeta|)^{2b}+\zeta^{\frac{4b}{3}}}{(1+(\eta(\zeta)-\tau^{\frac{1}{3}})^2)^2} \frac{1}{3\eta^2(\zeta)-4}\, d\zeta \\
&\leqslant C(1+\tau)^{\frac{4b}{3}+\frac{2s}{3}} \int_{\delta^3\tau^{\frac{1}{3}}}^{2\tau^{\frac{1}{3}}} \frac{1}{(1+(\xi-\tau^{\frac{1}{3}})^2)^2}\, d\zeta + C(1+\tau)^{\frac{2s}{3}} \int_{\delta\tau^{\frac{1}{3}}}^{2\tau^{\frac{1}{3}}} \frac{|\tau-\xi^3+4\xi|^{2b}}{(1+(\xi-\tau^{\frac{1}{3}})^2)^2}\, d\xi \\
&\leqslant C(1+\tau)^{\frac{4b}{3}+\frac{2s}{3}} + C(1+\tau)^{\frac{2s}{3}} \int_{\delta\tau^{\frac{1}{3}}}^{2\tau^{\frac{1}{3}}} \frac{|\tau|^{\frac{4b}{3}}|\xi-\tau^{\frac{1}{3}}|^{2b}}{(1+(\xi-\tau^{\frac{1}{3}})^2)^2}\, d\xi \\
&\leqslant C(1+\tau)^{\frac{4b}{3}+\frac{2s}{3}}
\end{aligned}
$$

if $4-2b>1$ or $b<\frac{3}{2}$, whereas

$$
\begin{aligned}
G_{21-2}(\tau) &\leqslant C \int_{2\tau}^{\infty} \frac{(1+|\tau-\zeta|+\zeta^{\frac{2}{3}})^{2b}}{(1+\zeta^{2/3})^2(1+\zeta)^{-2s/3}\zeta^{2/3}}\, d\zeta \\
&\leqslant C \int_{2\tau}^{\infty} \frac{\zeta^{2b}}{\zeta^{2(1-s/3)}}\, d\zeta \\
&\leqslant C\tau^{\frac{6b+2s-3}{3}}
\end{aligned}
$$

if $b<1/2-s/3$. The claim is thereby established.

As a consequence, the following estimate emerges. *For given $s \leqslant 0$ and $b<\min\{\frac{1}{2}-\frac{s}{3}, \frac{3}{2}\}$, there exists a constant C such that*

$$
\begin{aligned}
\int_{\frac{2}{3\sqrt{3}}}^{\infty} \phi_2(\mu(\tau))\big|\hat{h}(\tau)\big|^2 B_{21}(\tau)\, d\tau &\leqslant C \int_{\frac{2}{3\sqrt{3}}}^{\infty} \phi_2^2(\mu(\tau))\tau^{2(2b+s)/3}\big|\hat{h}(\tau)\big|^2\, d\tau \\
&\leqslant C\|h\|^2_{H^{\frac{2b+s}{3}}(R^+)} \qquad \text{(A.3)}
\end{aligned}
$$

for any $h \in H^{\frac{2b+s}{3}}(R^+)$.

The proof of Proposition A.1 is complete. □

Next, attention is given to the term

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle i\left(\tau-\left(\xi^3-4\xi\right)\right)+3\xi^2\right\rangle^{2b}\langle\xi\rangle^{2s}\left|\hat{I}_{22}(\xi,\tau)\right|^2 d\xi\,d\tau.$$

Proposition A.2. *Let s and b be given satisfying $0\geqslant s>-\frac{3}{2}$, $0\leqslant b<\frac{1}{2}-\frac{s}{3}$ and $b\neq\frac{1}{2}$. Then, there exists a constant C such that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle i\left(\tau-\left(\xi^3-4\xi\right)\right)+3\xi^2\right\rangle^{2b}\langle\xi\rangle^{2s}\left|\hat{I}_{22}(\xi,\tau)\right|^2 d\xi\,d\tau\leqslant C\|h\|^2_{H^{(3b+s-\frac{1}{2})/3}(R^+)} \tag{A.4}$$

for any $h\in H^{(3b+s-\frac{1}{2})/3}(R^+)$.

Proof. As before, we only estimate one term in $\mathcal{F}_t[\int_0^\infty E(x,t)\cos(x\eta)\,dx]$, say

$$A_{21}(\xi,\tau)=K_{21}(\xi,\tau)\phi_2\left(\mu(\tau)\right)\hat{h}(\tau).$$

Notice that $K_{21}(-\xi,\tau)=K_{11}(\xi,\tau)$. Hence, attention may be restricted to the case wherein $\xi\geqslant 0$. Denote by q_2 the function

$$q_2(\xi,\tau)=\frac{1}{\alpha_\mu^2(\tau)+(\xi-\beta_\mu(\tau))^2}$$

for $\xi\geqslant 1$. Let D_2 be given by

$$D_2(\xi,\tau)=2\int_{-\infty}^{\infty}\frac{\eta^2\Theta(\eta,\tau)}{\xi(\xi^2-\eta^2)}q_2(\eta,\tau)\,d\eta+\left(1+\omega(\tau)\right)\int_{-\infty}^{\infty}\frac{\eta^2(1-\Theta(\eta,\tau))}{\xi(\xi^2-\eta^2)}q_2(\eta,\tau)\,d\eta$$

for $\xi\geqslant\delta_1\mu(\tau)$ and

$$D_2(\xi,\tau)=2\int_{-\infty}^{\infty}\frac{\xi\Theta(\eta,\tau)}{\xi^2-\eta^2}q_2(\eta,\tau)\,d\eta+\left(1+\omega(\tau)\right)\int_{-\infty}^{\infty}\frac{\xi(1-\Theta(\eta,\tau))}{\xi(\xi^2-\eta^2)}q_2(\eta,\tau)\,d\eta$$

for $0\leqslant\xi\leqslant\delta_1\mu(\tau)$, where $\delta_1>0$ is a small constant. The relevance of these functions will become clear presently. First, note that

$$A_{21}(\xi,\tau)=q_2(\xi,\tau)\phi_2\left(\mu(\tau)\right)\hat{h}(\tau)\left|\alpha\left(\mu(\tau)\right)\right|.$$

As for D_2, changing variables in the integrals of its definition shows it to have the form

$$
\begin{aligned}
D_2(\xi,\tau) &= 2\int_0^\infty \frac{\eta^2}{\xi(\xi^2-\eta^2)}\Theta(\eta,\tau)q_2(\eta,\tau)\,d\eta \\
&\quad + \big(1+\omega(\tau)\big)\int_0^\infty \frac{\eta^2}{\xi(\xi^2-\eta^2)}\big(1-\Theta(\eta,\tau)\big)q_2(\eta,\tau)\,d\eta \\
&= \frac{2}{\mu^2(\tau)}\int_0^{a_0} \frac{\eta^2}{y(y^2-\eta^2)}\Theta\big(\mu(\tau)\eta,\tau\big)p_2(\eta,\tau)\,d\eta \\
&\quad + \frac{1+\omega(\tau)}{\mu^2(\tau)}\int_{a_1}^\infty \frac{\eta^2}{y(y^2-\eta^2)}\big(1-\Theta\big(\mu(\tau)\eta,\tau\big)\big)p_2(\eta,\tau)\,d\eta \\
&:= D_{21}(y,\tau)+D_{22}(y,\tau)
\end{aligned}
$$

where

$$
a_0 = \frac{\delta|\tau|^{1/3}+1}{\mu(\tau)}, \qquad a_1 = \frac{\delta|\tau|^{1/3}}{\mu(\tau)}, \qquad y=\xi/\mu(\tau),
$$
$$
p_2(\eta,\tau) = \left(\frac{\alpha_\mu^2(\tau)}{\mu(\tau)^2} + \left(\eta - \frac{\beta_\mu(\tau)}{\mu(\tau)}\right)^2\right)^{-1}.
$$

We have similar definitions for $0 \leqslant y \leqslant \delta_1$. Remark that a_0 is bounded independent of τ and so for y large enough, $y^2-\eta^2$ is bounded below for $\eta \in [0,a_0]$. Thus,

$$
\begin{aligned}
D_{21}(y,\tau) &= \frac{2}{y^3\mu^2(\tau)}\int_0^{a_0} \frac{\eta^2}{1-(\eta/y)^2}\Theta\big(\mu(\tau)\eta,\tau\big)p_2(\eta,\tau)\,d\eta \\
&:= \frac{1}{y^3\mu^2(\tau)}D_{21,2}(\tau,y)
\end{aligned}
$$

with

$$
\big|D_{21,2}(\tau,y)\big| < C \quad \text{for any } \tau \text{ and } y.
$$

Turning to D_{22}, note that $\Theta(\mu(\tau)\eta,\tau)=0$ for $\eta \geqslant a_1$, so

$$
\begin{aligned}
&\int_{a_1}^\infty \frac{\eta^2}{y(y^2-\eta^2)}\big(1-\Theta\big(\mu(\tau)\eta,\tau\big)\big)p_2(\eta,\tau)\,d\eta \\
&\quad = \int_{a_1}^\infty \frac{\eta^2}{y(y^2-\eta^2)}p_2(\eta,\tau)\,d\eta \\
&\quad = \frac{1}{y^2}\int_{\frac{a_1}{y}}^\infty \frac{1}{1-z^2}z^2y^2p_2(zy,\tau)\,dz
\end{aligned}
$$

$$= \frac{1}{y^2}\int_{\frac{a_1}{y}}^{\infty} \frac{1}{1-z^2}\left(z^2y^2p_2(zy,\tau)-1\right)dz + \frac{1}{y^2}\int_{\frac{a_1}{y}}^{\infty}\frac{1}{1-z^2}\,dz$$
$$:= \frac{1}{y^2}\left(D_{22,1}(y,\tau)+D_{22,2}(y,\tau)\right).$$

Of course,

$$D_{22,2}(y,\tau) = \int_{\frac{a_1}{y}}^{\infty}\frac{1}{1-z^2}\,dz = -\int_0^{\frac{a_1}{y}}\frac{1}{1-\eta^2}\,d\eta$$

since

$$\int_0^{\infty}\frac{1}{1-\eta^2}\,d\eta = 0$$

as a principal-value integral. It is therefore clear that

$$\left|D_{22,2}(y,\tau)\right| \leqslant \frac{C}{y}$$

for some constant C independent of τ when y is large. As for $D_{22,1}(y,\tau)$, note that

$$\eta^2y^2p(\eta y,\tau)-1 = \frac{1}{y}\left(2\eta\frac{\beta_\mu(\tau)}{\mu(\tau)} - \frac{\alpha_\mu^2(\tau)+\beta_\mu^2(\tau)}{\mu(\tau)^2}\frac{1}{y}\right)\left(\left(\frac{\alpha_\mu(\tau)}{\mu(\tau)y}\right)^2 + \left(\eta-\frac{\beta_\mu(\tau)}{\mu(\tau)y}\right)^2\right)^{-1}$$
$$:= \frac{1}{y}p^*(\eta,y,\tau).$$

Rewrite $D_{22,1}(y,\tau)$ as

$$D_{22,1}(y,\tau) = \frac{1}{y}\left(\int_{a_1/y}^{1/2} + \int_{1/2}^{2} + \int_2^{\infty}\right)\frac{p^*(\eta,y,\tau)}{1-\eta^2}\,d\eta$$

to obtain

$$\left|\left(\int_{1/2}^{2} + \int_2^{\infty}\right)\frac{p^*(\eta,y,\tau)}{1-\eta^2}\,d\eta\right| \leqslant C$$

and

$$\left|\int_{a_1/y}^{1/2}\frac{p^*(\eta,y,\tau)}{1-\eta^2}\,d\eta\right| \leqslant Cy\mu(\tau)$$

where C is independent of τ and y for $\mu(\tau) \geqslant 3$ and y large. Thus, if $y > y_0$, then

$$\left|D_2\big(\mu(\tau)y,\tau\big)\right| \leqslant \frac{C}{y^2\mu(\tau)}$$

where C is independent of τ and y. The following calculation shows the relevance of D_2:

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \big\langle i\big(\tau-(\xi^3-4\xi)\big)+3\xi^2\big\rangle^{2b}\langle\xi\rangle^{2s} \\
&\qquad\times\left|\int_{-\infty}^{\infty}\frac{1}{\xi-\eta}A_{21}(\eta,\tau)\big(2\Theta(\eta,\tau)+\big(1-\Theta(\eta,\tau)\big)\big(1+\omega(\tau)\big)\big)\,d\eta\right|^2 d\xi\,d\tau \\
&\quad=\int_0^\infty \frac{1}{\pi^2}\phi_2^2\big(\mu(\tau)\big)|\hat{h}|^2(\tau)\alpha_\mu^2(\tau)\int_0^\infty \big\langle i\big(\tau-(\xi^3-4\xi)\big)+3\xi^2\big\rangle^{2b}\langle\xi\rangle^{2s} \\
&\qquad\times\left|\int_{-\infty}^{\infty}\frac{1}{\xi-\eta}q_2(\eta,\tau)\big(2\Theta(\eta,\tau)+\big(1-\Theta(\eta,\tau)\big)\big(1+\omega(\tau)\big)\big)\,d\eta\right|^2 d\xi\,d\tau \\
&\quad=\int_0^\infty \frac{1}{\pi^2}\phi_2^2\big(\mu(\tau)\big)|\hat{h}|^2(\tau)\alpha_\mu^2(\tau)\int_0^\infty \big\langle i\big(\tau-(\xi^3-4\xi)\big)+3\xi^2\big\rangle^{2b}\langle\xi\rangle^{2s}\left|D_2(\xi,\tau)\right|^2 d\xi\,d\tau.
\end{aligned}$$

Thus, appropriate bounds on D_2 yield bounds on the left-hand side of the last formula.

Consider the quantity

$$\begin{aligned}
E_2(\tau) &:= \alpha_\mu^2(\tau)\int_0^\infty \big\langle i\big(\tau-(\xi^3-4\xi)\big)+3\xi^2\big\rangle^{2b}\langle\xi\rangle^{2s}\left|D_2(\xi,\tau)\right|^2 d\xi \\
&= \alpha_\mu^2(\tau)\left(\int_0^{\delta_1\mu(\tau)}+\int_{\delta_1\mu(\tau)}^{y_0\mu(\tau)}+\int_{y_0\mu(\tau)}^{\infty}\right)\big\langle i\big(\tau-(\xi^3-4\xi)\big)+3\xi^2\big\rangle^{2b}\langle\xi\rangle^{2s}D_2^2(\xi,\tau)\,d\xi \\
&:= E_{21}(\tau)+E_{22}(\tau)+E_{23}(\tau)
\end{aligned}$$

where δ_1 is again a small positive constant. Then by the choice of $\omega(\tau)$,

$$\begin{aligned}
\left|E_{23}(\tau)\right| &\leqslant C\tau^{2/3}\int_{y_0\mu(\tau)}^{\infty}\xi^{6b+2s-4}\,d\xi \\
&\leqslant C\tau^{2/3}\mu(\tau)^{6b+2s-3}\int_{y_0}^{\infty}\xi^{6b+2s-4}\,d\xi \\
&\leqslant C\tau^{2b+\frac{2s}{3}-\frac{1}{3}}
\end{aligned}$$

for large τ if $6b+2s-4<-1$, which is to say $b<\frac{1}{2}-\frac{s}{3}$.

For $\delta_1 \leqslant y \leqslant y_0$,

$$|D_{21}| \leqslant \frac{C}{\mu^2(\tau)}\left(1 + \left|\int_0^{a_0} \frac{1}{y-\eta}\Theta\big(\mu(\tau)\eta, \tau\big)p_2(\eta, \tau)\, d\eta\right|\right) \leqslant \frac{C}{\mu^2(\tau)}$$

since

$$\left|\int_0^{a_0} \frac{1}{y-\eta}\Theta\big(\mu(\tau)\eta, \tau\big)p_2(\eta, \tau)\, d\eta\right| \leqslant \left|\int_0^{a_0} \frac{\Theta(\mu(\tau)\eta, \tau)}{y-\eta}\big(p_2(\eta, \tau) - p_2(y, \tau)\big)\, d\eta\right| + \left|p_2(y, \tau)\int_0^{a_0} \frac{\Theta(\mu(\tau)\eta, \tau)}{y-\eta}\, d\eta\right| \leqslant C.$$

For $D_{22}(\xi, \tau)$, note that if $\eta \geqslant \frac{3}{2}$, or $a_1 \leqslant \eta \leqslant \frac{1}{2}$, the term $p_2(\eta, \tau)$ is uniformly bounded. Thus, when $y \neq \frac{1}{2}$, $y \neq \frac{3}{2}$, and $\delta_1 < y < \frac{2}{3}$, or $\frac{5}{4} \leqslant y \leqslant y_0$,

$$\begin{aligned} \Delta(y, \tau) &:= \left|\left(\int_{a_1}^{\frac{1}{2}} + \int_{\frac{3}{2}}^{\infty}\right) \frac{\eta^2}{y^2(y^2-\eta^2)}\big(1 - \Theta\big(\mu(\tau)\eta, \tau\big)\big)p_2(\eta, \tau)\, d\eta\right| \\ &\leqslant \left|\left(\int_{a_1}^{\frac{1}{2}} + \int_{\frac{3}{2}}^{\infty}\right) \frac{\eta^2}{y^2(y^2-\eta^2)}\big(1 - \Theta\big(\mu(\tau)\eta, \tau\big)\big)\big(p_2(\eta, \tau) - p_2(y, \tau)\big)\, d\eta \right. \\ &\quad \left. + p_2(y, \tau)\left(\int_{a_1}^{\frac{1}{2}} + \int_{\frac{3}{2}}^{\infty}\right) \frac{\eta^2}{y^2(y^2-\eta^2)}\big(1 - \Theta\big(\mu(\tau)\eta, \tau\big)\big)\, d\eta\right| \leqslant C. \end{aligned}$$

It is also true that for $\frac{2}{3} \leqslant y \leqslant \frac{5}{4}$,

$$\Delta(y, \tau) \leqslant \left|\left(\int_{a_1}^{\frac{1}{2}} + \int_{\frac{3}{2}}^{\infty}\right) \frac{\eta^2}{(\eta^2+1)^2}\, d\eta\right| \leqslant C.$$

When $y = \frac{1}{2}$, or $\frac{3}{2}$, or $y \in [\frac{3}{8}, \frac{5}{8}]$ or $y \in (\frac{11}{8}, \frac{13}{8})$,

$$\left|\int_{a_1}^{\infty} \frac{\eta^2}{y^2(y^2-\eta^2)}\big(1 - \Theta\big(\mu(\tau)\eta, \tau\big)\big)p_2(\eta, \tau)\, d\eta\right|$$

$$= \left| \int_{a_1}^{2} \frac{\eta^2}{y^2(y^2-\eta^2)} \bigl(1-\Theta\bigl(\mu(\tau)\eta,\tau\bigr)\bigr)\bigl(p_2(\eta,\tau)-p_2(y,\tau)\bigr)\,d\eta \right.$$

$$\left. + \int_{a_1}^{\infty} \frac{\eta^2}{y^2(y^2-\eta^2)} \bigl(1-\Theta\bigl(\mu(\tau)\eta,\tau\bigr)\bigr)p_2(\eta,\tau)\,d\eta + p_2(y,\tau)\int_{a_1}^{2} \frac{\eta^2}{y^2(y^2-\eta^2)}\,d\eta \right|$$

$$\leqslant C\left(1+\int_{a_0}^{2} p_2(\eta,\tau)\,d\eta\right) \leqslant C\mu(\tau).$$

In consequence, we are left with considering the term

$$\Upsilon(y,\tau) = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\eta^2}{y^2(y^2-\eta^2)} \bigl(1-\Theta\bigl(\mu(\tau)\eta,\tau\bigr)\bigr)p_2(\eta,\tau)\,d\eta$$

for $\delta_1 \leqslant y \leqslant y_0$ and $y \notin (\frac{3}{8},\frac{5}{8})$, $y \notin (\frac{11}{8},\frac{13}{8})$. If $y \in (\delta_1,\frac{3}{8}]$ or $[\frac{13}{8},y_0]$, then the integral in Υ has no singularity, and

$$\bigl|\Upsilon(y,\tau)\bigr| \leqslant C\int_{\frac{1}{2}}^{\frac{3}{2}} p_2(\eta,\tau)\,d\eta \leqslant C\mu(\tau).$$

If instead $y \in [\frac{5}{8},\frac{11}{8}]$, then

$$\bigl|\Upsilon(y,\tau)\bigr| \leqslant C\left(\mu(\tau)+\left|\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{y-\eta}\bigl(1-\Theta\bigl(\mu(\tau)\eta,\tau\bigr)\bigr)p_2(\eta,\tau)\,d\eta\right|\right)$$

$$\leqslant C\left(\mu(\tau)+\left|\int_{y-\frac{1}{2}}^{y-\frac{3}{2}} \frac{1}{w}\bigl(1-\Theta\bigl(\mu(\tau)(y-w),\tau\bigr)\bigr)p_2(y-w,\tau)\,dw\right|\right)$$

$$\leqslant C\left(\mu(\tau)+\left|\int_{y-\frac{1}{2}}^{y-\frac{3}{2}} \frac{1}{w}p_2(y-w,\tau)\,dw\right|\right).$$

If $m = \min\{|y-\frac{1}{2}|, |y-\frac{3}{2}|\}$, then $m \geqslant \frac{1}{8}$ and so

$$\bigl|\Upsilon(y,\tau)\bigr| \leqslant C\left(\mu(\tau)+\left|\int_{-m}^{m} \frac{1}{w}p_2(y-w,\tau)\,dw\right|\right)$$

$$= C\left(\mu(\tau) + \left|\int_0^m \frac{1}{w}\big(p_2(y-w,\tau) - p_2(y+w,\tau)\big)\,dw\right|\right)$$

$$\leqslant C\left(\mu(\tau) + \int_0^m \frac{|y - \frac{\beta_\mu(\tau)}{\mu(\tau)}|}{w} p_2^{-1}(y-w,\tau) p_2^{-1}(y+w,\tau)\,dw\right).$$

Without loss of generality, assume $y > 1$. Let $v = (w - (y - \frac{\beta_\mu(\tau)}{\mu(\tau)}))\mu(\tau)$. Then

$$\big|\Upsilon(y,\tau)\big| \leqslant C\left(\mu(\tau) + \int_{-(y-\frac{\beta_\mu(\tau)}{\mu(\tau)})\mu(\tau)}^{(m-(y-\frac{\beta_\mu(\tau)}{\mu(\tau)}))\mu(\tau)} \frac{\mu^3(\tau)|y - \frac{\beta_\mu(\tau)}{\mu(\tau)}|}{(\alpha_\mu^2(\tau) + v^2)(\alpha_\mu^2(\tau) + (v + 2\mu(\tau)(y - \frac{\beta_\mu(\tau)}{\mu(\tau)})^2))}\,dv\right)$$

$$\leqslant C\left(\mu(\tau) + \frac{\mu^3(\tau)|y-1|}{1 + |\mu(\tau)(y-1)|^2}\right)$$

$$\leqslant C\left(\mu(\tau) + \frac{\mu(\tau)}{1 + |\mu(\tau)(y-1)|}\right).$$

Therefore, if $b \neq \frac{1}{2}$,

$$\big|E_{22}(\tau)\big| \leqslant C\left(\int_{\delta_1\mu(\tau)}^{y_0\mu(\tau)} \big\langle i\big(\tau - (\xi^3 - 4\xi)\big) + 3\xi^2\big\rangle^{2b} \langle\xi\rangle^{2s} \mu^{-2}(\tau)\,d\xi\right.$$

$$\left. + \int_{\frac{5}{8}\mu(\tau)}^{\frac{11}{8}\mu(\tau)} \big\langle i\big(\tau - (\xi^3 - 4\xi)\big) + 3\xi^2\big\rangle^{2b} \langle\xi\rangle^{2s} \frac{1}{(1 + |\xi - \mu(\tau)|)^2}\,d\xi\right)$$

$$\leqslant C\left(\tau^{\frac{6b+2s-1}{3}} + \big(1 + |\tau|^{\frac{1}{3}}\big)^{4b+2s} \int_{\frac{5}{8}\mu(\tau)}^{\frac{11}{8}\mu(\tau)} \frac{|\xi - \mu(\tau)|^{2b}}{(1 + |\xi - \mu(\tau)|)^2}\,d\xi\right)$$

$$\leqslant C\big(\tau^{\frac{6b+2s-1}{3}} + \tau^{\frac{6b+2s}{3}} \mu(\tau)^{2b-1}\big)$$

$$\leqslant C\tau^{(6b+2s-1)/3}.$$

When $b = \frac{1}{2}$,

$$\big|E_{22}(\tau)\big| \leqslant C\big(\tau^{\frac{6b+2s-1}{3}} + \tau^{\frac{4b+2s}{3}} \ln(\tau)\big) \leqslant C\tau^{\frac{2+2s}{3}} \ln(\tau).$$

If $0 \leqslant y \leqslant \frac{\xi}{\mu(\tau)} \leqslant \delta_1$, in $D_2 = D_{21} + D_{22}$, then

$$|D_{22}| \leqslant \frac{C}{\mu^2(\tau)}\left|\int_{a_1}^{\infty} \frac{2y}{y^2 - \eta^2}\big(1 - \Theta\big(\mu(\tau)\eta,\tau\big)\big) p_2(\eta,\tau)\,d\eta\right| \leqslant C\frac{|y|}{\mu(\tau)}$$

and

$$
\begin{aligned}
D_{21} &= \frac{1}{\mu^2(\tau)}\Bigg(\int_0^{a_0}\frac{1}{y-\eta}\big(p_2(\eta,\tau)-p_2(y,\tau)\big)\Theta\big(\mu(\tau)\eta,\tau\big)\,d\eta \\
&\quad + \int_0^{a_0}\frac{1}{y+\eta}\big(p_2(\eta,\tau)-p_2(-y,\tau)\big)\Theta\big(\mu(\tau)\eta,\tau\big)\,d\eta \\
&\quad + \int_0^{a_0}\left(\frac{1}{y-\eta}p_2(y,\tau)+\frac{1}{y+\eta}p_2(-y,\tau)\right)\Theta\big(\mu(\tau)\eta,\tau\big)\,d\eta\Bigg) \\
&:= \frac{1}{\mu^2(\tau)}(D_{21-1}+D_{21-2}+D_{21-3}).
\end{aligned}
$$

Recall that $p_2(\eta,\tau)=(v^2(\tau)+(\eta+w(\tau))^2)^{-1}$ with

$$
v(\tau)=\alpha_\mu(\tau)/\mu(\tau), \qquad w(\tau)=\beta_\mu(\tau)/\mu(\tau),
$$

so that

$$
\begin{aligned}
&D_{21-1}(y,\tau)+D_{21-2}(y,\tau) \\
&\quad = \int_0^{a_0}\frac{\Theta(\mu(\tau)\eta,\tau)}{v^2(\tau)+(\eta+w(\tau))^2}\left(\frac{y+\eta+2w(\tau)}{v^2(\tau)+(y+w(\tau))^2}-\frac{\eta-y+2w(\tau)}{v^2(\tau)+(-y+w(\tau))^2}\right)d\eta \\
&\quad = \int_0^{a_0}\frac{\Theta(\mu(\tau)\eta,\tau)}{v^2(\tau)+(\eta-w(\tau))^2}\Bigg[\frac{-4yw(\tau)(\eta+2w(\tau))}{v^2(\tau)+(y-w(\tau))^2)(v^2(\tau)+(y+w(\tau))^2)} \\
&\qquad + y\left(\frac{1}{v^2(\tau)+(y-w(\tau))^2}+\frac{1}{v^2(\tau)+(y+w(\tau))^2}\right)\Bigg]\,d\eta.
\end{aligned}
$$

It thus transpires that

$$
|D_{21-1}+D_{21-2}|\leqslant C|y|.
$$

Also, we see that

$$
\begin{aligned}
D_{21-3} &= p_2(y,\tau)\int_0^{a_0}\frac{1}{y-\eta}\Theta\big(\mu(\tau)\eta,\tau\big)\,d\eta + p_2(-y,\tau)\int_0^{a_0}\frac{1}{y+\eta}\Theta\big(\mu(\tau)\eta,\tau\big)\,d\eta \\
&= p_2(y,\tau)\left(\int_0^{a_1}\frac{1}{y-\eta}\,d\eta+\int_{a_1}^{a_0}\frac{1}{y-\eta}\Theta\big(\mu(\tau)\eta,\tau\big)\,d\eta\right) \\
&\quad + p_2(-y,\tau)\left(\int_0^{a_1}\frac{1}{y+\eta}\,d\eta+\int_{a_1}^{a_0}\frac{1}{y+\eta}\Theta\big(\mu(\tau)\eta,\tau\big)\,d\eta\right)
\end{aligned}
$$

$$
\begin{aligned}
&= p_2(y,\tau)\big(-\ln|a_1-y|+\ln|y|\big)+p_2(-y,\tau)\big(\ln|a_1+y|-\ln|y|\big)\\
&\quad + p_2(y,\tau)\int_{a_1}^{a_0}\frac{1}{\eta}\frac{\Theta(\mu(\tau)\eta,\tau)\,d\eta}{\frac{y}{\eta}-1}+p_2(-y,\tau)\int_{a_1}^{a_0}\frac{1}{\eta}\frac{\Theta(\mu(\tau)\eta,\tau)\,d\eta}{\frac{y}{\eta}+1}\\
&= \big(-p_2(y,\tau)+p_2(-y,\tau)\big)\big(\ln|a_1|-\ln|y|\big)+p_2(y,\tau)\left(-\ln\left|1-\frac{y}{a_1}\right|\right)\\
&\quad + p_2(-y,\tau)\ln\left(1+\frac{y}{a_1}\right)+p_2(y,\tau)\int_{a_1}^{a_0}\frac{1}{\eta}\left(-1+\frac{y/\eta}{y/\eta-1}\right)\Theta\big(\mu(\tau)\eta,\tau\big)\,d\eta\\
&\quad + p_2(-y,\tau)\int_{a_1}^{a_0}\frac{1}{\eta}\left(1-\frac{y/\eta}{y/\eta+1}\right)\Theta\big(\mu(\tau)\eta,\tau\big)\,d\eta\\
&= \big(-p_2(y,\tau)+p_2(-y,\tau)\big)\left(\ln|a_1|-\ln|y|+\int_{a_1}^{a_0}\frac{1}{\eta}\Theta\big(\mu(\tau)\eta,\tau\big)\,d\eta\right)\\
&\quad + p_2(y,\tau)\left(-\ln\left(\left|1-\frac{y}{a_1}\right|\right)+\int_{a_1}^{a_0}\frac{y}{(y-\eta)\eta}\Theta\big(\mu(\tau)\eta,\tau\big)\,d\eta\right)\\
&\quad + p_2(-y,\tau)\left(\ln\left(1+\frac{y}{a_1}\right)-\int_{a_1}^{a_0}\frac{y}{(y+\eta)\eta}\Theta\big(\mu(\tau)\eta,\tau\big)\,d\eta\right).
\end{aligned}
$$

It follows that

$$|D_{21-3}|\leqslant C|y|\big(\ln|y|+1\big)$$

and

$$|D_{21}|\leqslant \frac{C|y|(\ln|y|+1)}{\mu^2(\tau)},$$

which implies that

$$|D_2|\leqslant \frac{C|y|(\ln|y|+1)}{\mu^2(\tau)}.$$

Thus, it is apparent that

$$
\begin{aligned}
|E_{21}| &\leqslant C\int_0^{\delta_1\mu(\tau)}\big(1+|\tau|\big)^{2b}\langle\xi\rangle^{2s}\frac{\xi^2}{\tau^2}\left(1+\ln\left|\frac{\xi}{\mu(\tau)}\right|\right)d\xi\\
&\leqslant C\tau^{2b-2}\int_0^{\delta_1\mu(\tau)}\langle\xi\rangle^{2s}\xi^2\left(1+\ln\left|\frac{\xi}{\mu(\tau)}\right|\right)d\xi
\end{aligned}
$$

$$\leqslant C\tau^{2b-2}\int_0^{\delta_1}\left(1+|\mu(\tau)||\xi|\right)^{2s}\mu^3(\tau)\left(1+\ln|\xi|\right)\xi^2\,d\xi$$

$$\leqslant C\tau^{2b-1+\frac{2s}{3}}\int_0^{\delta_1}\xi^{2+2s}\left(1+\ln|\xi|\right)d\xi$$

$$\leqslant C\tau^{\frac{6b+2s-3}{3}}$$

if $2+2s>-1$. Combining these estimates, there obtains

$$\left|E_2(\tau)\right|\leqslant C\tau^{\frac{6b+2s-1}{3}}$$

if $s>-3/2$ and $0<b<\frac{1}{2}-\frac{s}{3}$, $b\neq\frac{1}{2}$. This in turn implies that

$$\int_{-\infty}^{\infty}\int_0^{\infty}\left\langle i\left(\tau-\left(\xi^3-4\xi\right)\right)+3\xi^2\right\rangle^{2b}\langle\xi\rangle^{2s}$$

$$\times\left|\int_{-\infty}^{\infty}\frac{1}{\xi-\eta}A_{21}(\eta,\tau)\left(2\Theta(\eta,\tau)+\left(1-\Theta(\eta,\tau)\right)\left(1+\omega(\tau)\right)\right)d\eta\right|^2 d\xi\,d\tau$$

$$\leqslant C\int_0^{\infty}\tau^{2b-(1-2s)/3}\left|\int_0^{\infty}h(s)e^{-is\tau}\,ds\right|^2 d\tau$$

$$\leqslant C\|h\|^2_{H^{b+\frac{s}{3}-\frac{1}{6}}}.$$

Similar estimates for other terms yield, in sum,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle i\left(\tau-\left(\xi^3-4\xi\right)\right)+3\xi^2\right\rangle^{2b}\langle\xi\rangle^{2s}\left|\hat{I}_{22}(\xi,\tau)\right|^2 d\xi\,d\tau\leqslant C\|h\|^2_{H^{b+\frac{s}{3}-\frac{1}{6}}}$$

if $0\geqslant s>-\frac{3}{2}$ and $0\leqslant b<\frac{1}{2}-\frac{s}{3}$ and $b\neq\frac{1}{2}$. This completes the proof of Proposition A.2. □

By combining the above two propositions, we complete the proof of Proposition 2.7. □

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