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# TRAVELING FRONTS OF A CONSERVATION LAW WITH HYPER-DISSIPATION

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**Abstract.** Studied here are traveling-front solutions $\phi_\epsilon(x-ct)$ of a conservation law with hyper-dissipation appended. The evolution equation in question is a simple conservation law with a fourth-order dissipative term, namely

$$u_t + 2uu_x + \epsilon u_{xxxx} = 0,$$

where $\epsilon > 0$. The traveling front is restricted by the asymptotic conditions $\phi_\epsilon(x) \to L_\pm$ as $x \to \pm\infty$, where $L_+ < L_-$, and the symmetry condition $\phi_\epsilon(x)+\phi_\epsilon(-x) = L_- + L_+$ for all $x \in \mathbb{R}$. Such fronts are shown to exist and proven to be unique. Unlike the corresponding fronts for the Burgers' equation, they do not decay monotonically to their asymptotic states, but oscillate infinitely often around them. Despite this oscillation, it is also shown that $\phi_\epsilon(x) \to L_+$ as $\epsilon \to 0$, for all $x > 0$, and $\phi_\epsilon(x) \to L_-$ as $\epsilon \to 0$, for all $x < 0$.

## 1. Introduction

Investigated here is the existence and some detailed properties of traveling fronts of the perturbed conservation law

$$u_t + 2uu_x + \epsilon u_{xxxx} = 0, \tag{1.1}$$

where $\epsilon > 0$ is a fixed constant. In addition, attention is also given to the convergence of these fronts as the perturbation parameter $\epsilon > 0$ tends to

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zero. By a traveling front, we mean a solution of (1.1) of the form

$$u(t,x) = \phi(x - ct) \tag{1.2),}$$

satisfying the asymptotic conditions

$$\lim_{x\to\pm\infty} \phi(x) = L_\pm, \tag{1.3}$$

where it is expected that $L_+ < L_-$.

Such a program is elementary in case the dissipation is of the form $-\epsilon u_{xx}$, so that the problem is in fact Burgers' equation. In this case, there is an exact formula for a traveling front with the given asymptotic conditions, namely

$$u_\epsilon(t,x) = \phi_\epsilon(z) = \tfrac{1}{2}\gamma\tanh\Big(-\frac{\gamma}{\epsilon}z\Big) + \tfrac{1}{2}c$$

where $z = x - ct$, $c = L_- + L_+$, and $\gamma = L_- - L_+$. Moreover, the uniqueness of this front, modulo the translation group in the underlying spatial domain, is also easily established. Note that in this case, the front is strictly monotone decreasing over the entire real axis. For Burgers' equation, these traveling fronts play a distinguished role in the long-time asymptotics of solutions whose large-space asymptotes are as in (1.3). A similar result is expected for (1.1), which is one reason for interest in its traveling-wave solutions.

While the term $-\epsilon u_{xx}$ is often appended when dissipation needs to be inserted in time-dependent models arising in practice (see [1], [2] and [5] for recent examples), it is not always the correct asymptotic form for real dissipation in underlying physical systems (see, e.g., [3], [4] and the references contained therein, where other types of dissipation arise). One of the simplest generalization of the classical Burgers' equation is the model (1.1) above, as the dissipation is still localized in the spatial variable $x$. This evolution equation has been considered in some detail by Tadmor [9], who was interested in the non-dissipative limit of the time-dependent problem. As we will see, while some of the overall aspects of the traveling fronts of (1.1) are the same as for the Burgers' front shown above (e.g., the dependence of the speed of propagation $c$ on the asymptotic conditions (1.3)), other aspects differ markedly. And, in any case, so far as we are aware, there is no exact formula for these traveling fronts.

Substituting a function $u$ of the form (1.2) into (1.1) leads to

$$-c\phi' + 2\phi\phi' + \epsilon\phi'''' = 0,$$

or, what is the same,

$$(-c\phi + \phi^2 + \epsilon\phi''')' = 0.$$

Upon integrating once, there appears

$$-c\phi + \phi^2 + \epsilon\phi''' = k \tag{1.4}$$

for some constant $k \in \mathbb{R}$.

Applying the asymptotic conditions (1.3) to (1.4) formally yields the equation

$$L^2 - cL - k = 0,$$

which both $L_+$ and $L_-$ should satisfy. In consequence, it is expected that

$$L_\pm = \frac{c}{2} \mp b \tag{1.5}$$

where

$$b = \sqrt{k + \frac{c^2}{4}}. \tag{1.6}$$

Thus, just as for Burgers' equation, the speed $c$ of the traveling front is necessarily related to the spatial asymptotic limits of the front by the relation

$$c = L_+ + L_-, \tag{1.7}$$

and the constant $k$ in (1.4) must be

$$k = -L_+L_-. \tag{1.8}$$

Note that, as long as $L_+ \neq L_-$, then if $c$ and $k$ are given by (1.7) and (1.8), the discriminant $k + \frac{1}{4}c^2$ above is positive, and so (1.6) provides a well-defined, positive real number.

Stated now are the principal results of our study. The proof of these theorems is the focus of the remainder of the paper.

**Theorem 1.1.** *Fix $\epsilon > 0$. Let $L_+ < L_-$ and let $c$ be given by* (1.7). *There is a unique traveling-front solution $u_\epsilon(t,x) = \phi_\epsilon(x - ct)$ of equation* (1.1) *such that $\phi_\epsilon : \mathbb{R} \to \mathbb{R}$ satisfies*
*(i)* $\lim_{x\to\pm\infty} \phi_\epsilon(x) = L_\pm$ *and*
*(ii)* $\phi_\epsilon(x) + \phi_\epsilon(-x) = c$ *for all $x \in \mathbb{R}$.*
*The traveling front $\phi_\epsilon$ has the additional property that $\phi_\epsilon(x) < \phi_\epsilon(-x)$ for all $x > 0$. Furthermore, it oscillates infinitely often around $L_\pm$ as $x \to \pm\infty$ and satisfies the decay estimates*

$$|\phi_\epsilon(x) - L_+| \le Cb\,\exp\Big(\frac{-b^{\frac{1}{3}}x}{(4\epsilon)^{\frac{1}{3}}}\Big),$$

$$|\phi_\epsilon'(x)| \le Cb\Big(\frac{b}{\epsilon}\Big)^{\frac{1}{3}}\exp\Big(\frac{-b^{\frac{1}{3}}x}{(4\epsilon)^{\frac{1}{3}}}\Big),$$

$$|\phi_\epsilon''(x)| \le Cb\Big(\frac{b}{\epsilon}\Big)^{\frac{2}{3}}\exp\Big(\frac{-b^{\frac{1}{3}}x}{(4\epsilon)^{\frac{1}{3}}}\Big),$$

*for all $x \ge x_0$, where $x_0 > 0$ is any fixed positive value, $b$ is given by* (1.6) *and* (1.8)*, and the constants denoted by $C$ are independent of the parameters, $\epsilon$, $L_\pm$, and $c$. A similar estimate holds for negative values $x \le -x_0$, with $L_+$ replaced by $L_-$ of course.*

**Theorem 1.2.** *Let $L_+ < L_-$ and let $c$ be given as in* (1.7)*. Fix an $\epsilon > 0$ and let*

$$u_\epsilon(t, x) = \phi_\epsilon(x - ct)$$

*be the traveling-front solution of* (1.1) *with the properties described in Theorem* 1.1*. Then, for any $x_0 > 0$,*

$$\lim_{\epsilon\to 0} \phi_\epsilon(x) = L_+,$$

*uniformly for $x \ge x_0 > 0$, and*

$$\lim_{\epsilon\to 0} \phi_\epsilon(x) = L_-$$

*uniformly for $x \le -x_0 < 0$.*

The results in Theorem 1.2 are consistent with those obtained by Tadmor [9] for the limit $\epsilon \to 0$ of the time-dependent problem (1.1).

Thanks to the scaling properties of equation (1.1), which are inherited by equation (1.4), the proof of Theorem 1.1 can be reduced to the case $\epsilon = 1$, $c = 0$ and $k = 1$. Indeed, fix $\epsilon > 0$, let $L_+ < L_-$, and let $c$ and $k$ be given by (1.7) and (1.8). Put $b = \sqrt{k + \frac{c^2}{4}}$. It follows that a function $v : \mathbb{R} \to \mathbb{R}$ is a solution of

$$v''' = -v^2 + 1 \tag{1.9}$$

if and only if the function $\phi_\epsilon : \mathbb{R} \to \mathbb{R}$ given by

$$\phi_\epsilon(x) = bv\Big(\Big(\frac{b}{\epsilon}\Big)^{\frac{1}{3}}x\Big) + \frac{c}{2} \tag{1.10}$$

is a solution of equation (1.4). Moreover,

$$\lim_{x\to\pm\infty} \phi_\epsilon(x) = L_\pm \quad \text{if and only if} \quad \lim_{x\to\pm\infty} v(x) = \mp 1.$$

Theorems 1.1 and 1.2 are therefore immediate consequences of formula (1.10) and the following result.

**Theorem 1.3.** *There exists a unique, regular, odd solution* $v : \mathbb{R} \to \mathbb{R}$ *of the ordinary differential equation* (1.9) *such that* $v(x) \to \mp 1$ *as* $x \to \pm\infty$*. This solution has the additional property that* $v(x) < 0$ *for all* $x > 0$*. Furthermore, the solution oscillates infinitely often around* $-1$ *as* $x \to \infty$ *and satisfies the inequalities*

$$|v(x) + 1| \le C\exp\Big(-\frac{x}{2^{\frac{2}{3}}}\Big),$$
$$|v'(x)| \le C\exp\Big(-\frac{x}{2^{\frac{2}{3}}}\Big), \qquad |v''(x)| \le C\exp\Big(-\frac{x}{2^{\frac{2}{3}}}\Big),$$

*uniformly for* $x > x_0$*, where* $x_0 > 0$ *is fixed and the various constants denoted* $C$ *depend only upon* $x_0$*.*

Of course, it would be interesting to prove that any solution $v$ of (1.9) for which $v(x) \to \mp 1$ must in fact be odd (up to a spatial translation).

Equation (1.9) has been studied previously. Kopell and Howard [6] proved the existence part of Theorem 1.3, as well as “local” uniqueness by showing that the stable manifold of the stationary solution $v \equiv -1$ transversally intersects the manifold of solutions given by the initial conditions $v(0) = v''(0) = 0$. Bounded solutions to (1.9) can also be proved to exist using the Conley index. See Smoller’s book [8] for an exposition of Conley’s treatment of (1.9). This equation is also treated in the upcoming book by Hastings and McLeod [7], and we refer the reader to this work for more historical information. Thanks go to Stuart Hastings for providing these references.

Our proof uses an elementary shooting argument to show the existence part of Theorem 1.3. The exponential, oscillatory decay to the asymptotic boundary values follows from an auxiliary, dynamical systems argument. Uniqueness depends on some delicate calculations, not unlike some of those found in [6], but we obtain more than “local” uniqueness. We thank Bill Troy for sharing some numerical simulations which suggested the uniqueness proof.

## 2. Proof of Theorem 1.3

Theorem 1.3 will be established via a shooting argument. To this end, consider the initial-value problem

$$v''' = -v^2 + 1, \tag{2.1}$$
$$v(0) = v''(0) = 0, \tag{2.2}$$
$$v'(0) = \alpha \in \mathbb{R}. \tag{2.3}$$

Because $v(0) = 0$, (2.1) implies that $v'''(0) = 1$. Also, if $v$ is a solution of (2.1) with the above initial conditions, so is $w$, where $w(x) = -v(-x)$. Since the solution of (2.1) with specified initial conditions $v(0)$, $v'(0)$, and $v''(0)$ is unique, it follows that $v = w$, which is to say that $v$ is an odd function. The parameter $\alpha \in \mathbb{R}$ plays the role of the "shooting parameter" in the proof of existence of a solution to (2.1) with the desired properties. Since any solution of the initial-value problem (2.1)–(2.3) is odd, it suffices to consider the behavior of the solution for $x > 0$. In developing an argument for existence of a solution, three auxiliary functions will prove to be useful, *viz.*

$$E(x) = v'(x)v''(x) - v(x) + \tfrac{1}{3}v(x)^3,$$
$$F(x) = \big(v(x)^2 - 1\big)v'(x) + \tfrac{1}{2}v''(x)^2 = -v'(x)v'''(x) + \tfrac{1}{2}v''(x)^2,$$

and

$$G(x) = \tfrac{3}{2}E(x) - v'(x)v''(x).$$

Note that

$$E'(x) = v''(x)^2 \geq 0, \quad F'(x) = 2v(x)v'(x)^2, \quad \text{and} \quad G'(x) = F(x).$$

Of these three auxiliary functions, $E(x)$ plays the major role in our development, so the reader should especially take note of it. Observe that $E(x)$ is in fact an increasing function, not just nondecreasing. Indeed, if $E'(x) = v''(x)^2$ were to vanish on any interval, it would follow that $v'''(x)$ vanishes on that same interval, and so by (2.1), $v$ would be constant on that interval. By uniqueness of the initial-value problem associated to (2.1) at any point, it would follow that $v$ is either identically equal to 1 or to $-1$ for all $x \in \mathbb{R}$, which is excluded by (2.2). This is formalized in the following proposition.

**Proposition 2.1.** *Let $v$ be a non-constant solution of* (2.1). *Then the function $E$, given by*

$$E(x) = v'(x)v''(x) - v(x) + \frac{1}{3}v(x)^3,$$

*is strictly increasing.*

The two sets $A = \{\alpha \in \mathbb{R} : v(x_0) \geq 0 \text{ for some } x_0 > 0\}$ and $B = \{\alpha \in R : \text{for all } x > 0,\ v(x) < 0, \text{ and } F(x_0) < 0 \text{ for some } x_0 > 0\}$ are also central to the shooting argument. Clearly, they are disjoint.

The proof of Theorem 1.3 is made in several stages. The strategy is to show that $A$ and $B$ are both non-empty open subsets of $\mathbb{R}$. It follows immediately from this that there exists $\alpha \notin A \cup B$. In fact, it will transpire that there is exactly one such value of $\alpha$. One then needs to establish that if $\alpha \notin A \cup B$, the resulting solution of (2.1) has the desired properties. The uniqueness part of Theorem 1.3 is proven by showing that if $\alpha \in A$ or if $\alpha \in B$, then the associated solution $v$ of (2.1)–(2.3) cannot converge to $-1$ as $x \to \infty$.

The argument in favor of Theorem 1.3 proceeds as follows. Existence and uniqueness of $\alpha \notin A \cup B$ is the object of Corollary 2.12 below. The fact that if $\alpha \in B$, then $v(x)$ cannot converge to $-1$ as $x \to \infty$ is proved in Proposition 2.4. The asymptotic behavior of $v(x)$ in case $\alpha \notin A \cup B$ is proved in Propositions 2.13–2.16. Finally, the somewhat technical proposition that if $\alpha \in A$, then $v(x)$ can not converge to $-1$ as $x \to \infty$ is proved in Section 3.

**Proposition 2.2.** *There exists $\alpha_0 < 0$ such that $(\alpha_0, \infty) \subset A$.*

**Proof.** If $\alpha > 0$, then obviously $\alpha \in A$. Consider $\alpha = 0$ and let $v_0$ be the solution of (2.1) with

$$v_0(0) = v_0'(0) = v_0''(0) = 0.$$

It follows that $v_0'''(x) > 0$ in a neighborhood of 0, and hence that $v_0'$ is strictly convex in a neighborhood of 0. Thus $v_0'(0) = 0$ is a strict local minimum for $v_0'$. This means that $v_0'(x) > 0$, and thus $v_0(x) > 0$ for $x > 0$ near 0. In particular, $0 \in A$.

Fix $x_1 > 0$ such that $v_0(x_1) > 0$. By continuous dependence, it follows that if $\alpha < 0$ is sufficiently close to 0, then the resulting solution $v_\alpha$ of the initial-value problem (2.1)–(2.3) is positive at $x_1$. The proposition is thereby established.

**Proposition 2.3.** *The set $A$ is open.*

**Proof.** Let $\alpha \in A$, $\alpha < 0$, and let $v$ be the resulting solution of the initial-value problem (2.1)–(2.3). Let $x_0 > 0$ be the smallest positive value such that $v(x_0) = 0$. By continuous dependence on initial data, it is enough to prove that $v'(x_0) > 0$. Since necessarily $v'(x_0) \geq 0$, it suffices to prove $v'(x_0) \neq 0$. To see this is the case, note first that $E(0) = 0$ since $v(0) = v''(0) = 0$. By Proposition 2.1, it is known that $E(x_0) = v'(x_0)v''(x_0) > 0$, so of course, $v'(x_0) \neq 0$.

**Proposition 2.4.** *Let $\alpha < 0$ and let $v$ be the resulting solution of the initial-value problem* (2.1)–(2.3). *The following are equivalent.*

(i) *For all $x > 0$, $v(x) < 0$ and there exists an $x_0 > 0$ with $F(x_0) < 0$, i.e., $\alpha \in B$.*
(ii) *For all $x > 0$, $v(x) < 0$ and there exists an $x_0 > 0$ such that $F(x) < 0$, for all $x \geq x_0$.*

(iii) *There exists an* $x_0 > 0$ *such that* $v(x) < 0$ *for all* $x \in (0, x_0]$, $v(x_0) < -1$, *and* $F(x_0) < 0$.

*Furthermore, when any of the above conditions hold, it cannot happen that* $v(x)$ *converges to* $-1$ *as* $x \to \infty$.

**Proof.** That (i) and (ii) are equivalent is an immediate consequence of the fact that $F'(x) = 2v(x)v'(x)^2$.

To prove that (ii) implies (iii), it suffices to show that if $x_0$ is as in the statement (ii), then there exists $x_1 \geq x_0$ such that $v(x_1) < -1$. Suppose to the contrary that $v(x) \geq -1$ for all $x \geq x_0$. Since $F(x) < 0$ for all $x \geq x_0$ it follows that $(v(x)^2 - 1)v'(x) < 0$ for all $x \geq x_0$. Since $-1 \leq v(x) < 0$ for all $x \geq x_0$, it follows that $v(x)^2 - 1 < 0$ and so $v'(x) > 0$ for all $x \geq x_0$. Hence $v$ is an increasing function for $x \geq x_0$, which implies that $v(x) \to L$ as $x \to \infty$, where necessarily $-1 < L \leq 0$, i.e., $L^2 \neq 1$. This implies, by (2.1), that $v'''(x)$ has a nonzero limit as $x \to \infty$, which is clearly impossible (since then $v$ would then behave asymptotically as a cubic polynomial).

Finally, attention is turned to the assertion that (iii) implies (i) and to the last statement of the proposition. Let $x_0$ be as in statement (iii). Then it must be the case that $v'(x) < 0$ for all $x \geq x_0$. To see this, first observe that since $v(x_0) < -1$ and $F(x_0) < 0$, it follows that $v'(x_0) < 0$. Suppose it is false that $v'(x) < 0$ for all $x \geq x_0$, and let $x_1 > x_0$ be the first point where $v'(x_1) = 0$. Then $F'(x) \leq 0$ on $[x_0, x_1]$ (since $v(x) < 0$ on $[x_0, x_1]$), and so $F(x_1) \leq F(x_0) < 0$. On the other hand, $v'(x_1) = 0$ implies that $F(x_1) \geq 0$. This establishes the claim. The advertised conclusions follow since $v(x) < v(x_0) < -1$ for all $x > x_0$.

**Corollary 2.5.** *The set* $B$ *is open in* $(-\infty, 0)$.

**Proof.** Let $\alpha_0 < 0$ and let $v_0$ be the resulting solution of the initial-value problem (2.1)–(2.3) with $\alpha = \alpha_0$ . Suppose that $v_0$ satisfies statement (iii) in the previous proposition. By continuous dependence of both $v$ and $v'$ on the initial data, it is clear that the statement (iii) still holds for the solution $v$ of (2.1)–(2.3) with $\alpha$ close enough to $\alpha_0$.

**Proposition 2.6.** *Let* $\alpha \in \mathbb{R}$ *and let* $v$ *be the resulting solution of the initial-value problem* (2.1)–(2.3). *Suppose there exists an* $x_0 > 0$ *such that* $v(x_0) = -\sqrt{3}$. *Then* $v(x) < -\sqrt{3}$ *for all* $x > x_0$. *In particular,* $v(x)$ *cannot converge to* $-1$ *as* $x \to \infty$.

**Proof.** Let $x_0 > 0$ be the smallest positive value where $v(x_0) = -\sqrt{3}$. Then, naturally, $v'(x_0) \leq 0$ and $v(x_0)^2 = 3$. Since $E$ is increasing (Proposition 2.1), it follows that

$$E(x_0) = v'(x_0)v''(x_0) > E(0) = 0.$$

Since $v'(x_0) \le 0$, the latter inequality implies that $v'(x_0) < 0$, $v''(x_0) < 0$. We claim that $v'(x) < 0$ and $v''(x) < 0$ for all $x > x_0$. To prove this, suppose the contrary, and let $x_1 > x_0$ be the first point where either $v'(x_1) = 0$ or $v''(x_1) = 0$. Since $v'(x) < 0$ on $(x_0, x_1)$, it follows that $v(x_1) < v(x_0) = -\sqrt{3}$, so that $v(x_1)^2 > 3$. As a consequence, the inequality

$$E(x_1) = v'(x_1)v''(x_1) - v(x_1) + \tfrac{1}{3}v(x_1)^3 = 0 + v(x_1)\big[-1 + \tfrac{1}{3}v(x_1)^2\big] < 0$$

must hold. On the other hand, this is impossible since $E(0) = 0$ and $E$ is increasing. The claim follows and thereby the proposition.

**Proposition 2.7.** *Let $\alpha < 0$ and let $v$ be the resulting solution of the initial-value problem* (2.1)–(2.3). *Suppose there exists $x_0 > 0$ such that $v(x) < 0$ for all $x \in (0, x_0]$ and $v(x_0) = -\sqrt{3}$. Then, the parameter $\alpha$ lies in $B$.*

**Proof.** Repeating the proof of the previous proposition, it transpires that $v(x) < 0$ for all $x > 0$ and that $v'(x) < 0$ and $v''(x) < 0$ for all $x > x_0$. Furthermore, for all $x \ge x_0$,

$$F''(x) = 2v'(x)^3 + 4v(x)v'(x)v''(x) < 0.$$

Since also $F'(x) = 2v(x)v'(x)^2 < 0$ for all $x > 0$, it follows that $F$ is decreasing and (strictly) concave for $x > x_0$. Hence, it must ultimately become negative, which proves that $\alpha \in B$.

**Corollary 2.8.** *If $\alpha \notin A \cup B$ and if $v$ is the resulting solution of the initial-value problem* (2.1)–(2.3), *then $-\sqrt{3} < v(x) < 0$ for all $x > 0$.*

**Proposition 2.9.** *Let $\alpha < 0$. Suppose there is an $x_1 > 0$ such that*

$$\alpha = -\frac{x_1^2}{3} - \frac{\sqrt{3}}{x_1}.$$

*Then $\alpha \in B$. In particular, $B$ is nonempty.*

**Proof.** Let $\alpha < 0$ and let $v$ be the resulting solution of the initial-value problem (2.1)–(2.3). Let $x_0 > 0$ be such that $v(x)^2 \le 3$ on $[0, x_0]$. It follows that $v(x)^2 - 1 \le 2$, and in fact that $|v(x)^2 - 1| \le 2$ on $[0, x_0]$. In other words, $|v'''(x)| \le 2$ on $[0, x_0]$. Integrating repeatedly, using the initial conditions (2.2) and (2.3), there obtains the inequalities

(a) $|v''(x)| \le 2x$,
(b) $|v'(x) - \alpha| \le x^2$,
(c) $\alpha - x^2 \le v'(x) \le \alpha + x^2$,

(d) $\alpha x - x^3/3 \le v(x) \le \alpha x + x^3/3$,

all of which are valid on the interval $[0, x_0]$. In particular, if $\alpha + \frac{1}{3}x_0^2 < 0$, it follows that $v(x) < 0$ on the interval $(0, x_0]$.

Let $x_1$ be as given. If it happens that $v(x)^2 \le 3$ on $[0, x_1]$, then the above analysis shows that $v(x) < 0$ on $(0, x_1]$ and that

$$v(x_1) \le \alpha x_1 + \tfrac{1}{3}x_1^3 = -\sqrt{3}.$$

Thus, in fact, $v(x_1) = -\sqrt{3}$ and so $\alpha \in B$. On the other hand, if $v(x)^2$ is not bounded by 3 on $[0, x_1]$, let $x_0 < x_1$ be the first value where $v(x_0)^2 = 3$. Since $\alpha + \frac{1}{3}x_0^2 < \alpha + \frac{1}{3}x_1^2 < 0$, it follows from the above that $v(x) < 0$ on $(0, x_0]$, which again implies that $\alpha \in B$.

The proposition is thus established.

**Corollary 2.10.** *The set* $\{\alpha < 0 : \alpha \notin A \cup B\}$ *is not empty.*

**Proposition 2.11.** *The set* $\{\alpha \in \mathbb{R} : \alpha \notin A \cup B\}$ *contains at most* (*and therefore precisely*) *one element.*

**Proof.** Suppose $\alpha_1, \alpha_2 \notin A \cup B$. Then, of course both are negative and we suppose that $\alpha_2 > \alpha_1$, say. Let $v_1$ and $v_2$ be the solutions of the initial-value problem (2.1)–(2.3) with $\alpha = \alpha_1$ and $\alpha = \alpha_2$, respectively. Setting $w = v_2 - v_1$, it transpires that

$$w''' = -v_2^2 + v_1^2 = -(v_2 + v_1)w,$$

and $w(0) = 0$ with $w'(0) = \alpha_2 - \alpha_1 > 0$. It follows that $w'(x) > 0$ and $w(x) > 0$ for small $x > 0$. Since $-(v_2(x) + v_1(x)) > 0$ for $x > 0$, it follows that $w'''(x) > 0$ for small $x > 0$, and since $w''(0) = 0$, it also follows that $w''(x) > 0$ for small $x > 0$.

We claim that $w(x) > 0$ for all $x > 0$. Suppose not, and let $x_0$ be the first positive zero of $w$. Then, it must be the case that $w'''(x) > 0$ on $(0, x_0)$, whence $w''(x) > 0$ and so $w'(x) > 0$ on $(0, x_0)$. This of course makes it impossible to have $w(x_0) = 0$, and proves the claim.

It is thus seen that $w(x) > 0$, $w'(x) > 0$, $w''(x) > 0$, and $w'''(x) > 0$ for all $x > 0$. This certainly implies that $w(x) \to \infty$ as $x \to \infty$, which is impossible since $w = v_2 - v_1$ and both $v_1$ and $v_2$ are bounded functions (by Corollary 2.8).

This proposition is proved.

**Corollary 2.12.** *There exists* $\alpha_0 < 0$ *such that* $A = (\alpha_0, \infty)$ *and* $B = (-\infty, \alpha_0)$.

**Remark.** Numerical simulations made by Bill Troy indicate that $\alpha_0 \approx -1.06076533$.

The properties of the solution of the initial-value problem (2.1)–(2.3) with $\alpha = \alpha_0 \notin A \cup B$ are now examined.

**Proposition 2.13.** *Let $\alpha = \alpha_0 < 0$ be the unique real number not lying in $A \cup B$. If $v$ is the resulting solution of the initial-value problem* (2.1)–(2.3), *then*

$$\lim_{x\to\infty} v(x) = -1, \qquad \lim_{x\to\infty} v'(x) = \lim_{x\to\infty} v''(x) = \lim_{x\to\infty} v'''(x) = 0.$$

**Proof.** Since $\alpha \notin A \cup B$, we know that $-\sqrt{3} < v(x) < 0$ and $F(x) \geq 0$ for all $x > 0$. Consequently, it must be the case that $G'(x) = F(x) \geq 0$ and so $G(x)$ has a limit (positive or infinite) as $x \to \infty$. Also $E'(x) \geq 0$, and so $E(x)$ has a limit (positive or infinite) as $x \to \infty$. In fact, both these limits have to be finite. Suppose, for example, that $E(x) \to \infty$ as $x \to \infty$. Since $v(x)$ is bounded, it follows that the only remaining term in the definition of $E(x)$, i.e., $v'(x)v''(x)$, must go to $\infty$ as $x \to \infty$. But this implies that $\frac{d}{dx}v'(x)^2 \to \infty$ as $x \to \infty$, which contradicts the fact that $v$ is bounded. Thus $E(x)$ has a finite limit as $x \to \infty$, and likewise $G(x)$.

Because $F$ and $G$ both have finite limits as $x \to \infty$, the quantity $v'(x)v''(x) = \frac{3}{2}E(x) - G(x)$ also has a finite limit as $x \to \infty$. In other words, $\frac{d}{dx}v'(x)^2$ has a limit as $x \to \infty$. This limit has to be zero since $v$ is bounded. It next follows from the definition of $E$ that $-v(x) + \frac{1}{3}v(x)^3$ has a limit as $x \to \infty$. This clearly implies that $v(x)$ has a limit as $x \to \infty$ (contained in the interval $[-\sqrt{3}, 0]$). Equation (2.1) now implies that $v'''(x)$ has a (finite) limit as $x \to \infty$, which must be zero since $v$ is bounded. That $v(x) \to -1$ is again a consequence of (2.1).

The final statement of Proposition 2.13 follows from the next result.

**Proposition 2.14.** *Let $v : [0, \infty) \to \mathbb{R}$ be any solution of* (2.1) *such that $v(x) \to -1$ as $x \to \infty$. It follows that $E(x) \to \frac{2}{3}$ as $x \to \infty$. Furthermore,*

$$\lim_{x\to\infty} v'(x) = \lim_{x\to\infty} v''(x) = \lim_{x\to\infty} v'''(x) = 0.$$

**Proof.** Since $E(x)$ is nondecreasing, it must have a limit, finite or infinite, as $x \to \infty$. Since $v(x) \to -1$ as $x \to \infty$, it follows from the definition of $E(x)$ that $v'(x)v''(x)$ has a limit, finite or infinite, as $x \to \infty$. In other words, the derivative $\frac{d}{dx}v'(x)^2$ has a limit as $x \to \infty$. This limit must be zero since $v$ is bounded. That $E(x) \to \frac{2}{3}$ as $x \to \infty$ now follows immediately.

Since $E(x)$ has a finite limit as $x \to \infty$ and $E'(x) \geq 0$, it follows that $E' \in L^1(0,\infty)$. In consequence, it must be the case that $v'' \in L^2(0,\infty)$. Furthermore, since $F'(x) = 2v(x)v'(x)^2 \leq 0$ for $x > 0$ sufficiently large, it follows that $F(x)$ must have a limit (finite or negative infinity) as $x \to \infty$. In fact, this limit is also finite. Indeed, if the limit is negative infinity, then $(v(x)^2 - 1)v'(x) \to -\infty$ as $x \to \infty$. Since $v(x)^2 - 1 \to 0$ it must be that $|v'(x)| \to \infty$, which contradicts $v$ being bounded. Hence, $F' \in L^1(0,\infty)$, and so $v' \in L^2(0,\infty)$. Putting this together, since $v'$ and $v''$ are both in $L^2(0,\infty)$, the product $v'v''$ is in $L^1(0,\infty)$, which implies that $v'(x)^2$ has a limit as $x \to \infty$. Since $v$ is bounded, this limit must be 0. Finally, since $F$, $v$ and $v'$ all have limits as $x \to \infty$, so must $v''(x)^2$, and this limit must again be 0.

**Proposition 2.15.** *If $\alpha = \alpha_0$ does not lie in $A \cup B$ and if $v$ is the resulting solution of the initial-value problem* (2.1)–(2.3), *then there exists an infinite sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \to \infty$ such that $v(x_n) = -1$ for all $n = 1, 2, 3, \cdots$.*

**Proof.** Suppose not. Then, either $v(x) > -1$ for all sufficiently large $x > 0$ or $v(x) < -1$ for all sufficiently large $x > 0$.

Consider first the possibility that $v(x) > -1$ for all sufficiently large $x > 0$. Since $v(x) < 0$, for all $x > 0$, it follows from (2.1) that $v'''(x) > 0$ for all sufficiently large $x > 0$, and so $v''$ is increasing near infinity. Since $v''(x) \to 0$ as $x \to \infty$, it follows that $v''(x) < 0$ for sufficiently large $x$. Thus, $v$ is strictly concave for large $x$. However, since $v(x) > -1$ for large $x$ and $v(x) \to -1$, there must be points $x_1$, arbitrarily large, where $v'(x_1) < 0$. By concavity, $v'(x) \leq v'(x_1) < 0$ for $x > x_1$, which contradicts the fact that $v(x) \to -1$.

Similarly, if $v(x) < -1$ for all sufficiently large $x > 0$, then $v'''(x) < 0$ for sufficiently large $x > 0$, which means that $v''(x)$ is decreasing for large $x$ and tends to 0. Thus, $v''(x) > 0$ for large $x$, which means that $v$ is strictly convex. This is impossible since $v(x) < -1$ for large $x$ and $v(x) \to -1$.

**Proposition 2.16.** *If $\alpha = \alpha_0$ does not lie in $A \cup B$ and if $v$ is the resulting solution of the initial-value problem* (2.1)–(2.3), *then for any $x_0 > 0$, there exists $C = C(x_0)$ such that*

$$|v(x) + 1| \leq C\exp\Big(-\frac{x}{2^{\frac{2}{3}}}\Big), \quad |v'(x)| \leq C\exp\Big(-\frac{x}{2^{\frac{2}{3}}}\Big),$$
$$|v''(x)| \leq C\exp\Big(-\frac{x}{2^{\frac{2}{3}}}\Big),$$

*for all $x \geq x_0$.*

**Proof.** If we set $w = v'$ and $z = v''$, then equation (2.1) is the same as the system

$$\begin{aligned} v' &= w, \\ w' &= z, \\ z' &= 1 - v^2. \end{aligned}$$

This system, when linearized about the stationary point $(-1, 0, 0)$, becomes the constant coefficient system characterized by the matrix $M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$. The three eigenvalues of $M$ are $2^{\frac{1}{3}}$ and $2^{\frac{1}{3}}(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}) = -\frac{1}{2^{\frac{2}{3}}} \pm i\frac{\sqrt{3}}{2^{\frac{2}{3}}}$. The result follows since if $\alpha = \alpha_0 \notin A \cup B$, then $(v, w, z)$ is on the stable manifold of $(-1, 0, 0)$.

## 3. Completion of the Proof of Theorem 3.1

The goal of this section is to prove that if $\alpha \in A$, then the solution does not converge to $-1$. This will complete the proof of uniqueness and with it the proof of Theorem 1.3.

We already know, by Proposition 2.6, that if the solution $v(x) \to -1$ as $x \to \infty$, then $v(x) > -\sqrt{3}$ for all $x > 0$. Proposition 3.2 below shows, in addition, that in this situation, $v(x) < 2$ for all $x > 0$.

**Lemma 3.1.** *Let $v : [0, \infty) \to \mathbb{R}$ be a solution of* (2.1) *such that $v(0) = 0$ and $E(0) \geq 0$. Let $x_1 > 0$ be such that $v(x_1) > 0$. Then there exists $x_2 > x_1$ such that $v(x_2) < 0$.*

**Proof.** Suppose not. Then $v(x) > 0$ for all $x > x_1$. Let $x_0 \geq 0$ be such that $v(x_0) = 0$ and $v(x) > 0$ for all $x$ with $x_0 < x < x_1$. (It may be that $x_0 = 0$.)

Consider $x > x_0$. Observe that if $0 < v(x) < \sqrt{3}$, then neither $v'(x)$ nor $v''(x)$ can equal 0. Indeed, that would imply $E(x) < 0$ (impossible by Proposition 2.1). Thus $v'(x) > 0$ and $v''(x) > 0$ as long as $0 < v(x) < \sqrt{3}$. This implies that $v$ must attain the value of $\sqrt{3}$, and so we let $x_3 > x_0$ be the first point where $v(x_3) = \sqrt{3}$. Clearly, then, $v'(x_3) > 0$. Since $E(x_3) > 0$, it follows that $v''(x_3) > 0$. Suppose $v'(x) \geq 0$ for all $x > x_3$. It follows that $v$ is nondecreasing for $x > x_3$, and so has a limit $L > \sqrt{3}$ as $x \to \infty$. Thus $v'''(x) \to -L^2 + 1 < 0$ as $x \to \infty$, which implies, after integrating three times, that $v(x)$ ultimately becomes negative. Thus, there must exist $x_4 > x_3$ such that $v'(x_4) = 0$ and $v''(x_4) \leq 0$. Since $v(x_4) > \sqrt{3}$, it follows from (2.1) that $v'''(x_4) < 0$, and so $v''(x) < 0$, $v'(x) < 0$, and $v(x) < v(x_4)$ for $x > x_4$ and close to $x_4$. But now, neither $v'(x)$ nor $v''(x)$ can equal zero as long as $x > x_4$ and $0 \leq v(x) < v(x_4)$ since at such a point one would have $E(x) < E(x_4)$. This concludes the proof.

**Proposition 3.2.** *Let* $v : [0, \infty) \to \mathbb{R}$ *be a solution of* (2.1) *such that* $v(0) = 0$ *and* $E(0) \geq 0$. *Suppose there exists* $x_1 > 0$ *such that* $v(x_1) \geq 2$. *Then there exists* $x_2$ *such that* $E(x_2) \geq \frac{2}{3}$ *and so* $v(x)$ *can not converge to* $-1$ *as* $x \to \infty$.

**Proof.** By Lemma 3.1, there exists $x_2 > 0$ such that $v(x_2) \geq v(x_1) \geq 2$ and $v'(x_2) = 0$. For example, take $x_2$ such that $v(x)$ assumes its maximum on the bounded open interval containing $x_1$ where $v(x) > 0$. Since $E(x_2) \geq \frac{2}{3}$ and since $E(x)$ is increasing, the conclusion follows from Proposition 2.14.

Suppose now that $\alpha \in A$ and let $v$ be the resulting solution of (2.1) with initial conditions (2.2) and (2.3). We need to show that $v(x) \not\to -1$ as $x \to \infty$. This conclusion will be obtained by contradiction.

If $v$ is any solution of (2.1) with initial conditions (2.2) and (2.3) such that $v(x) \to -1$ as $x \to \infty$, then Propositions 2.6 and 3.2 imply

$$-\sqrt{3} < v(x) < 2, \tag{3.1}$$

for all $x > 0$. Moreover, it follows from (3.1) and (2.1) that, for all $x > 0$

$$-3 \leq v'''(x) \leq 1. \tag{3.2}$$

It turns out to be convenient to replace $v(x)$ by $v(x+x_0)$ for an appropriate value $x_0 \geq 0$. The choice of $x_0$ is made as follows. If $\alpha \geq 0$, then $x_0 = 0$. If $\alpha \in A$ and $\alpha < 0$ so that $v(x) < 0$ for small $x > 0$, then let $x_0 > 0$ be the smallest positive zero of $v$. In other words, in this latter case, $v(x) < 0$ on $(0, x_0)$ and $v(x_0) = 0$. It is clear that $v'(x_0) \geq 0$. Since $E(x_0) > 0$ by Proposition 2.1, it follows that $v'(x_0)v''(x_0) > 0$ and thus that both $v'(x_0) > 0$ and $v''(x_0) > 0$.

Thus, to establish that $v(x) \not\to -1$ as $x \to \infty$ if $\alpha \in A$, it suffices to prove the following result.

**Proposition 3.3.** *There does not exist a solution* $v : [0, \infty) \to \mathbb{R}$ *of* (2.1), *with initial conditions*

$$v(0) = 0, \tag{3.3}$$

$$v'(0) = \alpha \geq 0, \tag{3.4}$$

$$v''(0) = \beta \geq 0, \tag{3.5}$$

*which verifies* (3.1), *and thus* (3.2), *for all* $x > 0$.

Note that if $v$ satisfies the hypotheses of Proposition 3.3, then $E(0) = \alpha\beta \geq 0$ and $v(x) > 0$, $v'(x) > 0$, and $v''(x) > 0$ for small $x > 0$.

Proposition 3.3 is proved by contradiction. Let $v$ be a solution whose existence is denied by Proposition 3.3. The idea is to show that the estimate (3.2), along with the equation (2.1), renders (3.1) impossible.

The first step is to integrate (3.2) three times, taking into account (3.3)–(3.5). This yields

$$\alpha x + \frac{\beta x^2}{2} - \frac{x^3}{2} \leq v(x) \leq \alpha x + \frac{\beta x^2}{2} + \frac{x^3}{6}, \tag{3.6}$$

for all $x > 0$. By assumption, the solution respects the bound $v(x) \leq 2$ for all $x > 0$, and so it must be that

$$\alpha x + \frac{\beta x^2}{2} - \frac{x^3}{2} \leq 2, \tag{3.7}$$

for all $x > 0$. In particular, since for $x > 0$,

$$\alpha x - \frac{x^3}{2} \leq \alpha x + \frac{\beta x^2}{2} - \frac{x^3}{2} \leq 2,$$

and

$$\frac{\beta x^2}{2} - \frac{x^3}{2} \leq \alpha x + \frac{\beta x^2}{2} - \frac{x^3}{2} \leq 2,$$

there obtains the restrictions

$$\alpha \leq \Big(\frac{3}{2^{\frac{1}{3}}}\Big), \quad \beta \leq 3 \tag{3.8}$$

on $\alpha$ and $\beta$. Admittedly, (3.8) is not the sharp consequence of (3.7), but it suffices for the present purposes. Thus, in the argument by contradiction, we may restrict ourselves to $\alpha \geq 0$ and $\beta \geq 0$ which satisfy (3.8). In fact, to further simplify matters, we consider $\alpha$ and $\beta$ such that

$$0 \leq \alpha \leq 3, \quad 0 \leq \beta \leq 3. \tag{3.9}$$

The next step is to substitute the right-hand side of estimate (3.6) back into (2.1), thereby obtaining

$$v'''(x) = 1 - v(x)^2 \geq 1 - \Big(\alpha x + \frac{\beta}{2}x^2 + \frac{1}{6}x^3\Big)^2$$

or, what is the same,

$$v'''(x) \geq 1 - \Big(\alpha^2 x^2 + \alpha\beta x^3 + (\frac{\beta^2}{4} + \frac{\alpha}{3})x^4 + \frac{\beta}{6}x^5 + \frac{1}{36}x^6\Big), \tag{3.10}$$

for all $x > 0$. If the inequality (3.10) is integrated three times, taking into account (3.3)–(3.5), there obtains the lower bound

$$v(x) \geq Q(\alpha, \beta, x), \tag{3.11}$$

on $v$(x), valid for all $x > 0$, where

$$Q(\alpha,\beta,x) = \alpha x + \frac{\beta}{2}x^2 + \frac{1}{6}x^3 - \Big[\frac{\alpha^2}{60}x^5 + \frac{\alpha\beta}{120}x^6 + \frac{1}{210}\Big(\frac{\beta^2}{4} + \frac{\alpha}{3}\Big)x^7 + \frac{\beta}{2^5 3^2 (7)}x^8 + \frac{1}{2^5 3^4 (7)}x^9\Big].$$

To complete the proof by contradiction, it is enough to show that for all $(\alpha, \beta)$ in the square $[0, 3] \times [0, 3]$, there exists some $x > 0$ such that $Q(\alpha, \beta, x) \geq 2$; i.e.,

$$\max_{x>0} Q(\alpha, \beta, x) \geq 2.$$

Since the smaller powers of $x$ in $Q(\alpha, \beta, x)$ have positive coefficients and the larger powers of $x$ have negative coefficients, this maximum is attained at a unique value $x_0 = x_0(\alpha, \beta) > 0$, i.e.,

$$Q(\alpha, \beta, x_0(\alpha, \beta)) = \max_{x>0} Q(\alpha, \beta, x).$$

Moreover, $x_0(\alpha, \beta)$ is the unique zero, as a function of $x > 0$, of $\frac{\partial}{\partial x}Q(\alpha, \beta, x)$, and by the implicit-function theorem, $x_0(\alpha, \beta)$ depends smoothly on $\alpha \geq 0$ and $\beta \geq 0$. There are no double roots since successive derivatives with respect to $x$ of $Q(\alpha, \beta, x)$ are given by the sum of an increasing (or at least a non-decreasing) function in $x > 0$ and a decreasing function of $x > 0$. It follows that the the solution to the minimax problem

$$\min_{0\leq\alpha\leq 3, 0\leq\beta\leq 3} \max_{x>0} Q(\alpha, \beta, x) \tag{3.13}$$

is realized at some point $(\overline{\alpha}, \overline{\beta}, \overline{x})$, where $\overline{x} = x_0(\overline{\alpha}, \overline{\beta})$. The issue under consideration is thus reduced to showing that $Q(\overline{\alpha}, \overline{\beta}, \overline{x}) \geq 2$.

To study the minimax problem (3.13), observe that the function $Q$ can also be expressed as a quadratic polynomial in $\alpha$ and $\beta$, with coefficients depending on $x$, *viz.*

$$Q(\alpha,\beta,x) = -\frac{x^5}{60}\Big(\alpha^2 + \frac{\alpha\beta x}{2} + \frac{\beta^2 x^2}{14}\Big) + \alpha x\Big(1 - \frac{x^6}{630}\Big) + \frac{\beta x^2}{2}\Big(1 - \frac{x^6}{2^4 3^2 (7)}\Big) + \frac{x^3}{6} - \frac{x^9}{2^5 3^4 (7)}.$$

For every $x > 0$, if $Q$ is viewed as a function of $(\alpha, \beta) \in \mathbb{R}^2$, then it is strictly concave and so admits a unique global maximum. Moreover, it is straightforward to determine that if $Q$ is considered as a function of all three variables, then its Hessian matrix, which we denote by $H_Q$, has at least two negative eigenvalues at every point.

Suppose first that $0 < \overline{\alpha} < 3$ and $0 < \overline{\beta} < 3$, i.e., that the solution to (3.13) is realized on the interior of the region defined by (3.9). It follows that $(\overline{\alpha}, \overline{\beta}, \overline{x})$ is a critical point of $Q$. Indeed, if we set

$$K(\alpha, \beta) = Q(\alpha, \beta, x_0(\alpha, \beta)),$$

then $K(\overline{\alpha}, \overline{\beta}) = Q(\overline{\alpha}, \overline{\beta}, x_0(\overline{\alpha}, \overline{\beta}))$ is a local minimum (in an open set) for $K$. Hence $(\overline{\alpha}, \overline{\beta})$ is a critical point of $K$. By the chain rule, since $\partial Q/\partial x = 0$ at all points $(\alpha, \beta, x_0(\alpha, \beta))$, it follows that $(\overline{\alpha}, \overline{\beta}, x_0(\overline{\alpha}, \overline{\beta}))$ is a critical point of $Q$. Since $K$ has a local minimum at the critical point $(\overline{\alpha}, \overline{\beta})$, it follows that the Hessian matrix of $K$ at $(\overline{\alpha}, \overline{\beta})$, which we denote $H_K = H_K(\overline{\alpha}, \overline{\beta})$, must be positive semi-definite. On the other hand, we have the relation

$$H_K = M^T H_Q M, \tag{3.14}$$

where $H_Q = H_Q(\overline{\alpha}, \overline{\beta}, x_0(\overline{\alpha}, \overline{\beta}))$ and $M$ is a 3-by-2 matrix of rank 2 whose columns are independent tangent vectors to the surface defined by the graph of $x_0(\alpha, \beta)$ at the point $(\overline{\alpha}, \overline{\beta}, x_0(\overline{\alpha}, \overline{\beta}))$. This can be seen by comparing the second-order Taylor expansion of $K(\alpha, \beta)$ around the point $(\overline{\alpha}, \overline{\beta})$ with the second-order Taylor expansion of $Q(\alpha, \beta, x)$ around the point $(\overline{\alpha}, \overline{\beta}, \overline{x})$ with $\overline{x}$ replaced by $x_0(\alpha, \beta)$, and with $x_0(\alpha, \beta)$ expressed via its first-order Taylor expansion around $(\overline{\alpha}, \overline{\beta})$. Since $H_Q$ has at least two negative eigenvalues, (3.14) implies that $H_K(\overline{\alpha}, \overline{\beta})$ has at least one negative eigenvalue, which is impossible since it is positive semi-definite. Thus, the solution of the minimax problem (3.13) must be realized on the boundary of the square given by (3.9), i.e., with $\alpha = 0$ or 3 or $\beta = 0$ or 3.

We therefore consider $Q(\alpha, \beta, x)$ where $\alpha$ and $\beta$ are restricted to the boundary of the square given by (3.9). A somewhat tedious hand calculation reveals the following facts.

Consider first $Q(3, \beta, x)$, and in fact choose $x = 1$. In this case, observe that

$$Q(3, 0, 1) = -\frac{9}{60} + 3\Big(1 - \frac{1}{630}\Big) + \frac{1}{6} - \frac{1}{2^5 3^4 (7)} > 2,$$

and

$$Q(3, 3, 1) = -\frac{1}{60}\Big(9 + \frac{9}{2} + \frac{9}{14}\Big) + 3\Big(1 - \frac{1}{630}\Big) + \frac{3}{2}\Big(1 - \frac{1}{2^4 3^2 (7)}\Big) + \frac{1}{6} - \frac{1}{2^5 3^4 (7)} > 2.$$

Since $Q(3,\beta,1)$ is a concave function of $\beta$, it follows that $Q(3,\beta,1) > 2$ for all $\beta$ with $0 \le \beta \le 3$; i.e.,

$$\min_{0\le\beta\le3} \max_{x>0} Q(3,\beta,x) \ge \min_{0\le\beta\le3} Q(3,\beta,1) > 2.$$

This yields the desired result for the side of the square with $\alpha = 3$.

Attention is next turned to the side of the square with $\beta = 3$. Observe that

$$Q(0,3,2) = -\frac{8}{15}\Big(\frac{18}{7}\Big) + 6\Big(1 - \frac{4}{63}\Big) + \frac{8}{6} - \frac{16}{81(7)} > 2,$$

and

$$Q(2,3,2) = -\frac{8}{15}\Big(\frac{88}{7}\Big) + 4\Big(1 - \frac{64}{630}\Big) + 6\Big(1 - \frac{4}{63}\Big) + \frac{8}{6} - \frac{16}{81(7)} > 2.$$

Since $Q(\alpha,3,2)$ is a concave function of $\alpha$, it follows that $Q(\alpha,3,2) > 2$ for all $0 \le \alpha \le 2$, which is to say,

$$\min_{0\le\alpha\le2} \max_{x>0} Q(\alpha,3,x) > 2.$$

Furthermore,

$$Q(2,3,1) = -\frac{1}{60}\Big(7+\frac{9}{14}\Big)+2\Big(1-\frac{1}{630}\Big)+\frac{3}{2}\Big(1-\frac{1}{2^4 3^2(7)}\Big)+\frac{1}{6}-\frac{1}{2^5 3^4(7)} > 2.$$

Since $Q(3,3,1) > 2$ it follows in the same way that

$$\min_{2\le\alpha\le3} \max_{x>0} Q(\alpha,3,x) > 2.$$

Thus, the side of the square with $\beta = 3$ does not present an obstacle to the general line of argument being pursued.

Wishing to spare the reader the painful details (which are straightforwardly carried out), we note that the side of the square with $\alpha = 0$ can be handled using the fact that $Q(0,\beta,x)$ is a concave function of $\beta$, for any fixed $x > 0$, coupled with the observations

$$Q(0,0,3) > 2, \quad Q(0,1/2,3) > 2, \quad Q(0,1/2,2) > 2, \quad Q(0,3,2) > 2,$$

obtained by direct estimation. Similarly, the side with $\beta = 0$ follows via the inequalities

$$Q(0,0,3) > 2, \quad Q(1/2,0,3) > 2, \quad Q(1/2,0,2) > 2,$$
$$Q(2,0,2) > 2, \quad Q(2,0,1) > 2, \quad Q(3,0,1) > 2.$$

Putting all this together, the proof that the solution of the minimax problem (3.13) gives a value bigger than 2 is in hand, thus yielding the desired contradiction.

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## References

[1] D.M. Ambrose, J.L. Bona, and D.P. Nicholls, *Well-posedness of a model for water waves with viscosity*, submitted.

[2] F. Dias, A.I. Dyanchenko, and V.E. Zakharov, *Theory of weakly damped free-surface flows*, Phys. Lett. A, 372 (2008), 1297–1302.

[3] J.L. Bona, W.G. Pritchard, and L.R. Scott, *An evaluation of a model equation for water waves*, Philos. Trans. Royal Soc. London Series A, 302 (1981), 457–502.

[4] M. Chen, S. Dumont, L. Dupaigne, and O. Goubet, *Decay of solutions to a water wave model with non-local viscous dispersive term*, Discrete Cont. Dynamical Sys., Serirs S, 27 (2010), 1473-1492.

[5] M. Chen and O. Goubet, *Long-time asymptotic behavior of a dissipative Boussinesq system*, Discrete Cont. Dynamical Sys. Series A, 17 (2007), 61–80.

[6] S. Hasting and J.B. McLeod, "Classical Methods in Ordinary Differential Equations," in preparation.

[7] N. Kopell and L.N. Howard, *Bifurcations and trajectories joining critical points*, Advances in Math., 18 (1975), 308–358.

[8] J. Smoller, "Shock Waves and Reaction-Diffusion Equations," A Series of Comprehensive Studies in Mathematics, Vol. 258, Springer-Verlag, New York, 1983.

[9] E. Tadmor, *Burgers' equations with vanishing hyper-viscosity*, Commun. Math. Sci., 2 (2004), 317–324.