# NORM-INFLATION RESULTS FOR THE BBM EQUATION 

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#### Abstract

Considered here is the periodic initial-value probem for the regularized long-wave (BBM) equation $$
u_{t}+u_{x}+u u_{x}-u_{x x t}=0
$$

Adding to previous work in the literature, it is shown here that for any $s<0$, there is smooth initial data that is small in the $L_{2}$-based Sobolev spaces $H^{s}$, but the solution emanating from it becomes arbitrarily large in arbitrarily small time. This so called norm inflation result has as a consequence the previously determined conclusion that this problem is ill-posed in these negative-norm spaces.


## 1. Introduction

This note derives from the paper [7] where it was shown that the initial-value problem

$$
\begin{gather*}
u_{t}+u_{x}+u u_{x}-u_{x x t}=0, \\
u(0, x)=u_{0}(x), \tag{1.1}
\end{gather*}
$$

for the regularized long-wave or BBM equation is globally well posed in the $L_{2^{-}}$ based Sobolev spaces $H^{r}(\mathbb{R})$ provided $r \geq 0$. In the same paper, it was shown that the map that takes initial data to solutions cannot be locally $C^{2}$ if $r<0$. This latter result suggests, but does not prove, that the problem (1.1) is not well posed in $H^{r}$ for negative values of $r$. Later, Panthee [15] showed that this solution map, were it to exist on all of $H^{r}(\mathbb{R})$, could not even be continuous, thus proving that the problem is ill posed in the $L_{2}$-based Sobolev spaces with negative index. Indeed, he showed that there is a sequence of smooth initial data $\left\{\phi_{n}\right\}_{n=1}^{\infty}$ such that $\phi_{n} \rightarrow 0$ in $H^{r}(\mathbb{R})$ but the associated solutions, $\left\{u_{n}\right\}_{n=1}^{\infty}$ have the property that $\|u(\cdot, t)\|_{H^{r}}$ is bounded away from zero for all small values of $t>0$ and all $n \geq 1$.

The BBM equation itself was initially put forward in 16 and 3] as an approximate description of long-crested, surface water waves. It is an alternative to the classical Korteweg-de Vries equation and has been shown to be equivalent in that, for physically relevant initial data, the solutions of the two models differ by higher order terms on a long time scale (see [6.) It predicts the propagation of surface water waves pretty well in its range of validity [5]. Finally, it is known rigorously to be a good approximation to solutions of the full, inviscid, water-wave problem by combining results in [1, [4] and [13] (see also [14]).

It is our purpose here to show that in fact, for $r<0$, the problem (1.1) is not only not well posed, but features blow-up in the $H^{r}$-norm in arbitrarily short time. This will be done in the context of the periodic initial-value problem wherein $u_{0}$ is

[^0]a periodic distribution lying in $H_{p e r}^{r}$ for some $r<0$. Similar results hold for $H^{r}(\mathbb{R})$, but are not explicated here.

More precisely, it will be shown that, for any given $r<0$, there is a sequence $\left\{u_{0}^{n}\right\}_{n=1}^{\infty}$ of smooth initial data such that $u_{0}^{n} \rightarrow 0$ in $H_{p e r}^{r}$ and a sequence $\left\{T_{n}\right\}_{n=1}^{\infty}$ of positive times tending to 0 as $n \rightarrow \infty$ such that the corresponding solutions $\left\{u_{n}\right\}_{n=1}^{\infty}$ emanating from this initial data, whose existence is guaranteed by the periodic version [9] of the theory for the initial-value problem, are such that for $n=1,2,3, \cdots$,

$$
\left\|u\left(\cdot, T_{n}\right)\right\|_{H_{p e r}^{s}} \geq n
$$

This insures in particular that the solution map $\mathcal{S}$ that associates solutions to initial data, which exists on $L_{2}$, cannot be extended continuously to all of $H_{p e r}^{s}$, thus reproducing Panthee's conclusion. Results of this sort go by the appellation norm inflation for obvious reasons. The idea originated in the work of Bourgain and Pavlović [8] for the three-dimensional Navier-Stokes equation. The method of construction there was applied to some other dissipative fluid equations by the second author and her collaborators, see [12, 11, 10. It suggests that the method is generic as well as sophisticated.

## Notation

The notation used throughout is standard. For $r \in \mathbb{R}$, the collection $\dot{H}_{p e r}^{r}$ is the homogeneous space of $2 \pi$-periodic distributions whose norm

$$
\|f\|_{r}^{2}=\sum_{k=1}^{\infty} k^{2 r}\left(\left|f_{k}\right|^{2}+\left|g_{k}\right|^{2}\right)
$$

is finite. Elements in $\dot{H}_{p e r}^{r}$ all have mean zero over the period domain $[0,2 \pi]$. Here, the $\left\{f_{k}\right\}$ are the Fourier sine coefficients and the $\left\{g_{k}\right\}$ are the Fourier cosine coefficients of $f$. Notice that $\dot{H}_{\text {per }}^{0}$ may be viewed simply as the $L_{2}$-functions on the period domain $[0,2 \pi]$ with mean zero. If $X$ is any Banach space, the set $C([0, T] ; X)$ consists of the continuous functions from the real interval $[0, T]$ into $X$ with its usual norm.

## 2. Norm inflation

The principal result of our study is the following theorem.
Theorem 2.1. Let $r<0$ by given. Then there is a sequence $\left\{u_{0}^{j}\right\}_{j=1}^{\infty}$ of $\mathcal{C}^{\infty}$, periodic initial data such that

$$
u_{0}^{(j)} \rightarrow 0 \quad \text { as } \quad j \rightarrow \infty
$$

in $\dot{H}_{p e r}^{r}$ and a sequence $\left\{T_{j}\right\}_{j=1}^{\infty}$ of positive times tending to zero as $j \rightarrow \infty$ such that if $u_{j}(x, t)$ is the solution emanating from $u_{0}^{(j)}$, then

$$
\left\|u\left(\cdot, T_{j}\right)\right\|_{\dot{H}_{p e r}^{r}} \geq j
$$

for all $j=1,2, \cdots$.
Proof: Fix $s>0$, let $r=-s$ and consider a wavenumber $k_{1} \in \mathbb{N}$ which, in due course, will be taken to be large. Let $k_{2}=k_{1}+1$, define $\bar{u}$ by $\bar{u}=\sin \left(k_{1} x\right)+\sin \left(k_{2} x\right)$ and consider the $2 \pi$-periodic, men zero initial data $u_{0}=k_{1}^{\gamma} \bar{u}$ for (1.1) where $\gamma>0$ will be restricted presently. Of course, $u_{0}$ is smooth, so the theory developed in [9] implies that a unique, global, smooth solution emanates from this initial data.

Notice also that the solution preserves the property of having zero mean, so it lies in $C\left([0, T] ; \dot{H}_{p e r}^{\rho}\right)$ for all $\rho \in \mathbb{R}$.

Let $\varphi\left(D_{x}\right)$ be the Fourier multiplier operator given in terms of its Fourier transform by $\widehat{\varphi\left(D_{x}\right)} u(\xi)=\frac{\xi}{1+\xi^{2}} \hat{u}(\xi)$. The equation (1.1) can be rewritten as

$$
\begin{gather*}
i u_{t}=\varphi\left(D_{x}\right) u+\frac{1}{2} \varphi\left(D_{x}\right)\left(u^{2}\right)  \tag{2.2}\\
u(0, x)=u_{0}(x)
\end{gather*}
$$

Let $S(t)=e^{-i t \varphi\left(D_{x}\right)}$ be the unitary group defining the evolution of the linear BBM equation. Then, Duhamel's principle allows the solution of (1.1)-(2.2) to be written in the form

$$
\begin{equation*}
u(x, t)=S(t) u_{0}(x)+u_{1}(s, t)+y(x, t) \tag{2.3}
\end{equation*}
$$

where

$$
\left.u_{1}(x, t)=\frac{1}{2} \int_{0}^{t} S(t-\tau) \varphi\left(D_{x}\right)\left(S(\tau) u_{0}\right)\right)^{2} d \tau
$$

is the first order approximation of the nonlinear term in the differential-integral equation in (2.2). The function $y(x, t)$ is the remainder, which may be expressed implicitly in the sightly complicated, but useful form

$$
\begin{equation*}
y(x, t)=\int_{0}^{t} S(t-\tau) \varphi\left(D_{x}\right)\left[G_{0}(\tau)+G_{1}(\tau)+G_{2}(\tau)\right] d \tau \tag{2.4}
\end{equation*}
$$

with

$$
\begin{aligned}
& G_{0}(\tau)=\frac{1}{2} u_{1}^{2}(\tau)+u_{1}(\tau) S(\tau) u_{0} \\
& G_{1}(\tau)=u_{1}(\tau) y(\tau)+y(\tau) S(\tau) u_{0} \\
& G_{2}(\tau)=\frac{1}{2} y^{2}(\tau)
\end{aligned}
$$

where the spatial dependence has been supressed for ease of reading. The strategy is to show that by choosing $k_{1}$ sufficiently large, $u_{1}$ becomes large in a short time in the space $\dot{H}_{p e r}^{r}=\dot{H}_{\text {per }}^{-s}$, while the error term $y$ remains under control in the same space.

In contrast to dissipative equations, the linear dispersion operator $S(t)$ only translates the wave, but does not change its magnitude; more precisely, for $k=$ $1,2, \cdots$,

$$
\begin{equation*}
S(t) \sin (k x)=\sin \left(k x-\frac{k}{1+k^{2}} t\right), \quad S(t) \cos (k x)=\cos \left(k x-\frac{k}{1+k^{2}} t\right) \tag{2.5}
\end{equation*}
$$

On the other hand, the operator $\varphi\left(D_{x}\right)$ both decreases the amplitude of its argument and adds rotation viz.

$$
\begin{equation*}
\varphi\left(D_{x}\right) \sin k x=-i \frac{k}{1+k^{2}} \cos k x, \quad \varphi\left(D_{x}\right) \cos k x=i \frac{k}{1+k^{2}} \sin k x \tag{2.6}
\end{equation*}
$$

It follows from this that $\varphi\left(D_{x}\right)$ vanishes on constant functions.
It is clear that if $s>0$, then

$$
\begin{array}{ll} 
& \left\|S(t) u_{0}\right\|_{-s}=\left\|u_{0}\right\|_{-s} \sim k_{1}^{\gamma-s}, \\
\text { while } \quad & \left\|S(t) u_{0}\right\|_{0}=\left\|u_{0}\right\|_{0} \sim k_{1}^{\gamma} . \tag{2.7}
\end{array}
$$

As we want the initial data to be small in $\dot{H}_{p e r}^{-s}, \gamma$ is restricted to the range $(0, s)$. The formulas in (2.5) imply

$$
S(\tau) \bar{u}=\sin \left(k_{1} x-\frac{k_{1}}{1+k_{1}^{2}} \tau\right)+\sin \left(k_{2} x-\frac{k_{2}}{1+k_{2}^{2}} \tau\right)
$$

so that

$$
\begin{aligned}
{[S(\tau) \bar{u}]^{2}=\frac{1}{2} } & {\left[1-\cos \left(2 k_{1} x-\frac{2 k_{1}}{1+k_{1}^{2}} \tau\right)\right]+\frac{1}{2}\left[1-\cos \left(2 k_{2} x-\frac{2 k_{2}}{1+k_{2}^{2}} \tau\right)\right] } \\
& +\cos \left(\left(k_{1}-k_{2}\right) x-\left(\frac{k_{1}}{1+k_{1}^{2}}-\frac{k_{2}}{1+k_{2}^{2}}\right) \tau\right) \\
& -\cos \left(\left(k_{1}+k_{2}\right) x-\left(\frac{k_{1}}{1+k_{1}^{2}}+\frac{k_{2}}{1+k_{2}^{2}}\right) \tau\right)
\end{aligned}
$$

It then follows from (2.6) that

$$
\begin{aligned}
\frac{1}{2} \varphi\left(D_{x}\right)[S(\tau) \bar{u}]^{2}= & -\frac{i}{4} \frac{2 k_{1}}{1+4 k_{1}^{2}} \sin \left(2 k_{1} x-\frac{2 k_{1}}{1+k_{1}^{2}} \tau\right) \\
& -\frac{i}{4} \frac{2 k_{2}}{1+4 k_{2}^{2}} \sin \left(2 k_{2} x-\frac{2 k_{2}}{1+k_{2}^{2}} \tau\right) \\
& +\frac{i}{2} \frac{k_{1}-k_{2}}{1+\left(k_{1}-k_{2}\right)^{2}} \sin \left(\left(k_{1}-k_{2}\right) x-\left(\frac{k_{1}}{1+k_{1}^{2}}-\frac{k_{2}}{1+k_{2}^{2}}\right) \tau\right) \\
& -\frac{i}{2} \frac{k_{1}+k_{2}}{1+\left(k_{1}+k_{2}\right)^{2}} \sin \left(\left(k_{1}+k_{2}\right) x-\left(\frac{k_{1}}{1+k_{1}^{2}}+\frac{k_{2}}{1+k_{2}^{2}}\right) \tau\right) \\
\equiv & I_{1}+I_{2}+I_{3}+I_{4} .
\end{aligned}
$$

Consider now the function $\sin (k x-\omega t)$ and calculate as follows:

$$
\begin{align*}
\int_{0}^{t} S(t & -\tau) \sin (k x-\omega \tau) d \tau=\int_{0}^{t} \sin \left(k x-\frac{k}{1+k^{2}}(t-\tau)-\omega \tau\right) d \tau \\
& =\left(\frac{k}{1+k^{2}}-\omega\right)^{-1}\left(\cos \left(k x-\frac{k}{1+k^{2}} t\right)-\cos (k x-\omega t)\right) \tag{2.8}
\end{align*}
$$

where use has been made of (2.5).

The latter formula, applied four times, allows us to calculate $u_{1}$ explicitly, to wit,

$$
\begin{aligned}
u_{1}= & k_{1}^{2 \gamma} \int_{0}^{t} S(t-\tau)\left[I_{1}+I_{2}+I_{3}+I_{4}\right] d \tau \\
= & -\frac{i k_{1}^{2 \gamma}}{12} \frac{1+k_{1}^{2}}{k_{1}^{2}}\left[\cos \left(2 k_{1} x-\frac{2 k_{1}}{1+k_{1}^{2}} t\right)-\cos \left(2 k_{1} x-\frac{2 k_{1}}{1+4 k_{1}^{2}} t\right)\right] \\
& -\frac{i k_{1}^{2 \gamma}}{12} \frac{1+k_{2}^{2}}{k_{2}^{2}}\left[\cos \left(2 k_{2} x-\frac{2 k_{2}}{1+k_{2}^{2}} t\right)-\cos \left(2 k_{2} x-\frac{2 k_{2}}{1+4 k_{2}^{2}} t\right)\right] \\
& +\frac{i k_{1}^{2 \gamma}}{2} \frac{k_{1}-k_{2}}{1+\left(k_{1}-k_{2}\right)^{2}}\left[\frac{k_{1}}{1+k_{1}^{2}}-\frac{k_{2}}{1+k_{2}^{2}}-\frac{k_{1}-k_{2}}{1+\left(k_{1}-k_{2}\right)^{2}}\right]^{-1} . \\
& {\left[\cos \left(\left(k_{1}-k_{2}\right) x-\left(\frac{k_{1}}{1+k_{1}^{2}}-\frac{k_{2}}{1+k_{2}^{2}}\right) t\right)-\cos \left(\left(k_{1}-k_{2}\right) x-\frac{k_{1}-k_{2}}{1+\left(k_{1}-k_{2}\right)^{2}} t\right)\right] } \\
& -\frac{i k_{1}^{2 \gamma}}{2} \frac{k_{1}+k_{2}}{1+\left(k_{1}+k_{2}\right)^{2}}\left[\frac{k_{1}}{1+k_{1}^{2}}+\frac{k_{2}}{1+k_{2}^{2}}-\frac{k_{1}+k_{2}}{1+\left(k_{1}+k_{2}\right)^{2}}\right]^{-1} \cdot \\
& {\left[\cos \left(\left(k_{1}+k_{2}\right) x-\left(\frac{k_{1}}{1+k_{1}^{2}}+\frac{k_{2}}{1+k_{2}^{2}}\right) t\right)-\cos \left(\left(k_{1}+k_{2}\right) x-\frac{k_{1}+k_{2}}{1+\left(k_{1}+k_{2}\right)^{2}} t\right)\right] . }
\end{aligned}
$$

A study of the various constants appearing above reveals that, up to absolute constants,

$$
\begin{aligned}
u_{1} \sim & -i k_{1}^{2 \gamma}\left[\cos \left(2 k_{1} x-\frac{2 k_{1}}{1+k_{1}^{2}} t\right)-\cos \left(2 k_{1} x-\frac{2 k_{1}}{1+4 k_{1}^{2}} t\right)\right] \\
& -i k_{1}^{2 \gamma}\left[\cos \left(2 k_{2} x-\frac{2 k_{2}}{1+k_{2}^{2}} t\right)-\cos \left(2 k_{2} x-\frac{2 k_{2}}{1+4 k_{2}^{2}} t\right)\right] \\
& +i k_{1}^{2 \gamma}\left[\cos \left(x-\left(\frac{k_{1}}{1+k_{1}^{2}}-\frac{k_{2}}{1+k_{2}^{2}}\right) t\right)-\cos \left(x-\frac{t}{2}\right)\right] \\
& -i k_{1}^{2 \gamma}\left[\cos \left(\left(k_{1}+k_{2}\right) x-\left(\frac{k_{1}}{1+k_{1}^{2}}+\frac{k_{2}}{1+k_{2}^{2}}\right) t\right)\right. \\
& \left.-\cos \left(\left(k_{1}+k_{2}\right) x-\frac{k_{1}+k_{2}}{1+\left(k_{1}+k_{2}\right)^{2}} t\right)\right] .
\end{aligned}
$$

as $k_{1}$ becomes large. Since

$$
\left|\cos \left(k x-\omega_{1} t\right)-\cos \left(k x-\omega_{2} t\right)\right| \leq\left|\omega_{1}-\omega_{2}\right| t
$$

straightforward calculations show that the first, second and fourth terms above are uniformly small compared to the third term, for large values of $k_{1}$. Indeed, they are all of order $k_{1}^{2 \gamma-1} t$, whereas the third term is of order $k_{1}^{2 \gamma} t$.

It follows from this that for all $t \geq 0$,

$$
\begin{align*}
\left\|u_{1}(t, \cdot)\right\|_{-s} & \sim k_{1}^{2 \gamma} t \quad \text { and likewise }  \tag{2.9}\\
\left\|u_{1}(t, \cdot)\right\|_{0} & \sim k_{1}^{2 \gamma} t
\end{align*}
$$

Thus, by taking $k_{1}$ large, the $\dot{H}_{\text {per }}^{-s}$-norm of $u_{1}$ can be made as big as we like.
As mentioned earlier, an estimate of the error term $y$ is needed to complete the argument. It will in fact be shown that $y$ is even bounded in $L_{2}$, let along $\dot{H}_{p e r}^{-s}$.

To this end, use is made of one of a periodic version of one the bilinear estimates in 7.
Lemma 2.2. Let $u, v \in H_{p e r}^{q}$ with $q \geq 0$. Then

$$
\begin{equation*}
\left\|\varphi\left(D_{x}\right)(u v)\right\|_{q} \lesssim\|u\|_{q}\|v\|_{q} \tag{2.10}
\end{equation*}
$$

where the implied constant only depends upon $q$.
The proof of this result is the same as the proof of Lemma 1 in [7, with sums replacing integrals.

Introduce the abbreviation $X_{T}$ for $C\left([0, T] ; L^{2}\right)$ for ease of reading. The value of $T>0$ will be specified momentarily. It follows from (2.10) and the implicit relationship (2.4) for the remainder $y$ that

$$
\begin{align*}
\|y\|_{X_{T}} \lesssim & \lesssim\left\|u_{1}\right\|_{X_{T}}^{2}+T\left\|S(t) u_{0}\right\|_{X_{T}}\left\|u_{1}\right\|_{X_{T}}+T\left\|u_{1}\right\|_{X_{T}}\|y\|_{X_{T}} \\
& +T\left\|S(t) u_{0}\right\|_{X_{T}}\|y\|_{X_{T}}+T\|y\|_{X_{T}}^{2} \\
\lesssim & T^{3} k_{1}^{4 \gamma}+T^{2} k_{1}^{3 \gamma}+\left(k_{1}^{2 \gamma} T^{2}+k_{1}^{\gamma} T\right)\|y\|_{X_{T}}+T\|y\|_{X_{T}}^{2}  \tag{2.11}\\
= & \mathcal{A}+\mathcal{B} \mathcal{Y}+T \mathcal{Y}^{2}
\end{align*}
$$

where $\mathcal{Y}=\mathcal{Y}(T)=\|y\|_{X_{T}}$. As $y \in C\left([0, M] ; L_{2}\right)$ for all $M>0$, it follows that $\mathcal{Y}(T)$ is a continuous function of $T$. Moreover, $\mathcal{Y}(0)=0$.

Choose $T_{0}=k_{1}^{-\mu \gamma}$, where $\mu>\frac{3}{2}$. With this choice, we see that for $T \leq T_{0}$,

$$
\mathcal{A}=O\left(k_{1}^{\gamma(4-3 \mu)}+k_{1}^{\gamma(3-2 \mu)}\right) \quad \text { and } \quad \mathcal{B}=O\left(k_{1}^{2 \gamma(1-\mu)}+k_{1}^{\gamma(1-\mu)}\right)
$$

as $k_{1} \rightarrow \infty$ and all the exponents are negative.
Choose $k_{1}$ large enough that $\mathcal{B}<\frac{1}{2}$ and $T$ and $\mathcal{A}$ are both small. It follows in this circumstance that the quadratic polynomial

$$
p(z)=\mathcal{A}+(\mathcal{B}-1) z+T z^{2}
$$

has two positive roots, the smaller of which is denoted $\underline{z}$ and the larger $\bar{z}$. Of course, $p(z)<0$ for $z \in(\underline{\mathbf{z}}, \bar{z})$.

The inquality (2.11) may be expressed as

$$
p(\mathcal{Y}(T)) \geq 0
$$

As $\mathcal{Y}(T)$ is continuous and $\mathcal{Y}(0)=0$, it follows that $\mathcal{Y}(T) \leq \underline{\mathrm{z}}$ for all $T \in\left[0, T_{0}\right]$. For $k_{1}$ large, $T_{0}<1$. When combined with the fact that $\mathcal{B}<\frac{1}{2}$, it is readily deduced that

$$
\underline{\mathrm{z}} \leq 4 \mathcal{A}, \quad \text { whence } \quad \mathcal{Y}(T) \leq 4 \mathcal{A}
$$

thus assuring that the remainder $y(\cdot, t)$ is indeed uniformly bounded in $\dot{H}_{\text {per }}^{-s}$ for $t \leq T_{0}$ and large choices of $k_{1}$.

Taking a suitably chosen, increasing sequence $\left\{k_{1}^{(j)}\right\}_{j=1}^{\infty}$ of wavenumbers for which

$$
\lim _{j \rightarrow \infty} k_{1}^{(j)}=+\infty
$$

and with the indicated choices of $\gamma$ and $\mu$, (2.7) assures the initial data tends to zero in $\dot{H}_{\text {per }}^{-s}$. The decomposition (2.3) together with (2.7), (2.9) and the bound just obtained on $y$ then implies that the solutions $u_{j}$ blow up at times $T_{j}=\left(k_{1}^{(j)}\right)^{-\mu \gamma}$. The latter tend to zero as $j \rightarrow \infty$ since $\mu$ and $\gamma$ are both positive. This completes the proof of the theorem.

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[^0]:    The author M. Dai was partially supported by NSF grant DMS-1517583.

