# NORM-INFLATION RESULTS FOR THE BBM EQUATION

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ABSTRACT. Considered here is the periodic initial-value probem for the regularized long-wave (BBM) equation

 $u_t + u_x + uu_x - u_{xxt} = 0.$ 

Adding to previous work in the literature, it is shown here that for any s < 0, there is smooth initial data that is small in the  $L_2$ -based Sobolev spaces  $H^s$ , but the solution emanating from it becomes arbitrarily large in arbitrarily small time. This so called *norm inflation* result has as a consequence the previously determined conclusion that this problem is ill-posed in these negative-norm spaces.

# 1. INTRODUCTION

This note derives from the paper [7] where it was shown that the initial-value problem

(1.1) 
$$u_t + u_x + uu_x - u_{xxt} = 0, u(0, x) = u_0(x),$$

for the regularized long-wave or BBM equation is globally well posed in the  $L_{2^-}$ based Sobolev spaces  $H^r(\mathbb{R})$  provided  $r \geq 0$ . In the same paper, it was shown that the map that takes initial data to solutions cannot be locally  $C^2$  if r < 0. This latter result suggests, but does not prove, that the problem (1.1) is not well posed in  $H^r$  for negative values of r. Later, Panthee [15] showed that this solution map, were it to exist on all of  $H^r(\mathbb{R})$ , could not even be continuous, thus proving that the problem is ill posed in the  $L_2$ -based Sobolev spaces with negative index. Indeed, he showed that there is a sequence of smooth initial data  $\{\phi_n\}_{n=1}^{\infty}$  such that  $\phi_n \to 0$ in  $H^r(\mathbb{R})$  but the associated solutions,  $\{u_n\}_{n=1}^{\infty}$  have the property that  $||u(\cdot,t)||_{H^r}$ is bounded away from zero for all small values of t > 0 and all  $n \geq 1$ .

The BBM equation itself was initially put forward in [16] and [3] as an approximate description of long-crested, surface water waves. It is an alternative to the classical Korteweg-de Vries equation and has been shown to be equivalent in that, for physically relevant initial data, the solutions of the two models differ by higher order terms on a long time scale (see [6].) It predicts the propagation of surface water waves pretty well in its range of validity [5]. Finally, it is known rigorously to be a good approximation to solutions of the full, inviscid, water-wave problem by combining results in [1], [4] and [13] (see also [14]).

It is our purpose here to show that in fact, for r < 0, the problem (1.1) is not only not well posed, but features blow-up in the  $H^r$ -norm in arbitrarily short time. This will be done in the context of the periodic initial-value problem wherein  $u_0$  is

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a periodic distribution lying in  $H_{per}^r$  for some r < 0. Similar results hold for  $H^r(\mathbb{R})$ , but are not explicated here.

More precisely, it will be shown that, for any given r < 0, there is a sequence  $\{u_0^n\}_{n=1}^{\infty}$  of smooth initial data such that  $u_0^n \to 0$  in  $H_{per}^r$  and a sequence  $\{T_n\}_{n=1}^{\infty}$  of positive times tending to 0 as  $n \to \infty$  such that the corresponding solutions  $\{u_n\}_{n=1}^{\infty}$  emanating from this initial data, whose existence is guaranteed by the periodic version [9] of the theory for the initial-value problem, are such that for  $n = 1, 2, 3, \cdots$ ,

$$\|u(\cdot, T_n)\|_{H^s_{ner}} \ge n.$$

This insures in particular that the solution map S that associates solutions to initial data, which exists on  $L_2$ , cannot be extended continuously to all of  $H^s_{per}$ , thus reproducing Panthee's conclusion. Results of this sort go by the appellation norm inflation for obvious reasons. The idea originated in the work of Bourgain and Pavlović [8] for the three-dimensional Navier-Stokes equation. The method of construction there was applied to some other dissipative fluid equations by the second author and her collaborators, see [12, 11, 10]. It suggests that the method is generic as well as sophisticated.

### Notation

The notation used throughout is standard. For  $r \in \mathbb{R}$ , the collection  $H_{per}^r$  is the homogeneous space of  $2\pi$ -periodic distributions whose norm

$$||f||_r^2 = \sum_{k=1}^\infty k^{2r} (|f_k|^2 + |g_k|^2)$$

is finite. Elements in  $H_{per}^r$  all have mean zero over the period domain  $[0, 2\pi]$ . Here, the  $\{f_k\}$  are the Fourier sine coefficients and the  $\{g_k\}$  are the Fourier cosine coefficients of f. Notice that  $\dot{H}_{per}^0$  may be viewed simply as the  $L_2$ -functions on the period domain  $[0, 2\pi]$  with mean zero. If X is any Banach space, the set C([0, T]; X) consists of the continuous functions from the real interval [0, T] into X with its usual norm.

# 2. NORM INFLATION

The principal result of our study is the following theorem.

**Theorem 2.1.** Let r < 0 by given. Then there is a sequence  $\{u_0^j\}_{j=1}^{\infty}$  of  $\mathcal{C}^{\infty}$ , periodic initial data such that

$$u_0^{(j)} \to 0 \quad \text{as} \quad j \to \infty$$

in  $\dot{H}_{per}^r$  and a sequence  $\{T_j\}_{j=1}^{\infty}$  of positive times tending to zero as  $j \to \infty$  such that if  $u_j(x,t)$  is the solution emanating from  $u_0^{(j)}$ , then

$$\|u(\cdot,T_j)\|_{\dot{H}^r_{per}} \ge j$$

for all  $j = 1, 2, \cdots$ .

**Proof:** Fix s > 0, let r = -s and consider a wavenumber  $k_1 \in \mathbb{N}$  which, in due course, will be taken to be large. Let  $k_2 = k_1 + 1$ , define  $\bar{u}$  by  $\bar{u} = \sin(k_1x) + \sin(k_2x)$  and consider the  $2\pi$ -periodic, men zero initial data  $u_0 = k_1^{\gamma} \bar{u}$  for (1.1) where  $\gamma > 0$  will be restricted presently. Of course,  $u_0$  is smooth, so the theory developed in [9] implies that a unique, global, smooth solution emanates from this initial data.

Notice also that the solution preserves the property of having zero mean, so it lies in  $C([0,T]; \dot{H}_{per}^{\rho})$  for all  $\rho \in \mathbb{R}$ .

Let  $\varphi(D_x)$  be the Fourier multiplier operator given in terms of its Fourier transform by  $\varphi(D_x)u(\xi) = \frac{\xi}{1+\xi^2}\hat{u}(\xi)$ . The equation (1.1) can be rewritten as

(2.2) 
$$iu_t = \varphi(D_x)u + \frac{1}{2}\varphi(D_x)\left(u^2\right),$$
$$u(0,x) = u_0(x).$$

Let  $S(t) = e^{-it\varphi(D_x)}$  be the unitary group defining the evolution of the linear BBM equation. Then, Duhamel's principle allows the solution of (1.1)-(2.2) to be written in the form

(2.3) 
$$u(x,t) = S(t)u_0(x) + u_1(s,t) + y(x,t)$$

where

$$u_1(x,t) = \frac{1}{2} \int_0^t S(t-\tau)\varphi(D_x) \left(S(\tau)u_0\right)^2 d\tau$$

is the first order approximation of the nonlinear term in the differential-integral equation in (2.2). The function y(x,t) is the remainder, which may be expressed implicitly in the sightly complicated, but useful form

(2.4) 
$$y(x,t) = \int_0^t S(t-\tau)\varphi(D_x) \left[ G_0(\tau) + G_1(\tau) + G_2(\tau) \right] d\tau$$

with

$$G_0(\tau) = \frac{1}{2}u_1^2(\tau) + u_1(\tau)S(\tau)u_0,$$
  

$$G_1(\tau) = u_1(\tau)y(\tau) + y(\tau)S(\tau)u_0,$$
  

$$G_2(\tau) = \frac{1}{2}y^2(\tau),$$

where the spatial dependence has been supressed for ease of reading. The strategy is to show that by choosing  $k_1$  sufficiently large,  $u_1$  becomes large in a short time in the space  $\dot{H}_{per}^r = \dot{H}_{per}^{-s}$ , while the error term y remains under control in the same space.

In contrast to dissipative equations, the linear dispersion operator S(t) only translates the wave, but does not change its magnitude; more precisely, for  $k = 1, 2, \cdots$ ,

(2.5) 
$$S(t)\sin(kx) = \sin\left(kx - \frac{k}{1+k^2}t\right), \ S(t)\cos(kx) = \cos\left(kx - \frac{k}{1+k^2}t\right).$$

On the other hand, the operator  $\varphi(D_x)$  both decreases the amplitude of its argument and adds rotation *viz*.

(2.6) 
$$\varphi(D_x)\sin kx = -i\frac{k}{1+k^2}\cos kx, \qquad \varphi(D_x)\cos kx = i\frac{k}{1+k^2}\sin kx.$$

It follows from this that  $\varphi(D_x)$  vanishes on constant functions.

It is clear that if s > 0, then

(2.7) 
$$\|S(t)u_0\|_{-s} = \|u_0\|_{-s} \sim k_1^{\gamma-s}, \\ \|S(t)u_0\|_0 = \|u_0\|_0 \sim k_1^{\gamma}.$$

As we want the initial data to be small in  $\dot{H}^{-s}_{per}$ ,  $\gamma$  is restricted to the range (0, s). The formulas in (2.5) imply

$$S(\tau)\bar{u} = \sin\left(k_1x - \frac{k_1}{1+k_1^2}\tau\right) + \sin\left(k_2x - \frac{k_2}{1+k_2^2}\tau\right),\,$$

so that

$$\begin{bmatrix} S(\tau)\bar{u} \end{bmatrix}^2 = \frac{1}{2} \left[ 1 - \cos\left(2k_1x - \frac{2k_1}{1+k_1^2}\tau\right) \right] + \frac{1}{2} \left[ 1 - \cos\left(2k_2x - \frac{2k_2}{1+k_2^2}\tau\right) \right] + \cos\left((k_1 - k_2)x - \left(\frac{k_1}{1+k_1^2} - \frac{k_2}{1+k_2^2}\right)\tau\right) - \cos\left((k_1 + k_2)x - \left(\frac{k_1}{1+k_1^2} + \frac{k_2}{1+k_2^2}\right)\tau\right).$$

It then follows from (2.6) that

$$\begin{split} \frac{1}{2}\varphi(D_x) \left[S(\tau)\bar{u}\right]^2 &= -\frac{i}{4} \frac{2k_1}{1+4k_1^2} \sin\left(2k_1x - \frac{2k_1}{1+k_1^2}\tau\right) \\ &\quad -\frac{i}{4} \frac{2k_2}{1+4k_2^2} \sin\left(2k_2x - \frac{2k_2}{1+k_2^2}\tau\right) \\ &\quad +\frac{i}{2} \frac{k_1 - k_2}{1+(k_1 - k_2)^2} \sin\left((k_1 - k_2)x - \left(\frac{k_1}{1+k_1^2} - \frac{k_2}{1+k_2^2}\right)\tau\right) \\ &\quad -\frac{i}{2} \frac{k_1 + k_2}{1+(k_1 + k_2)^2} \sin\left((k_1 + k_2)x - \left(\frac{k_1}{1+k_1^2} + \frac{k_2}{1+k_2^2}\right)\tau\right) \\ &\equiv I_1 + I_2 + I_3 + I_4. \end{split}$$

Consider now the function  $\sin\left(kx-\omega t\right)$  and calculate as follows:

(2.8) 
$$\int_0^t S(t-\tau)\sin(kx-\omega\tau)d\tau = \int_0^t \sin\left(kx-\frac{k}{1+k^2}(t-\tau)-\omega\tau\right)d\tau$$
$$= \left(\frac{k}{1+k^2}-\omega\right)^{-1}\left(\cos\left(kx-\frac{k}{1+k^2}t\right)-\cos\left(kx-\omega t\right)\right)$$

where use has been made of (2.5).

The latter formula, applied four times, allows us to calculate  $u_1$  explicitly, to wit,

$$\begin{split} u_1 =& k_1^{2\gamma} \int_0^t S(t-\tau) \left[ I_1 + I_2 + I_3 + I_4 \right] d\tau \\ = & -\frac{ik_1^{2\gamma}}{12} \frac{1+k_1^2}{k_1^2} \left[ \cos\left(2k_1x - \frac{2k_1}{1+k_1^2}t\right) - \cos\left(2k_1x - \frac{2k_1}{1+4k_1^2}t\right) \right] \\ & -\frac{ik_1^{2\gamma}}{12} \frac{1+k_2^2}{k_2^2} \left[ \cos\left(2k_2x - \frac{2k_2}{1+k_2^2}t\right) - \cos\left(2k_2x - \frac{2k_2}{1+4k_2^2}t\right) \right] \\ & +\frac{ik_1^{2\gamma}}{2} \frac{k_1 - k_2}{1+(k_1 - k_2)^2} \left[ \frac{k_1}{1+k_1^2} - \frac{k_2}{1+k_2^2} - \frac{k_1 - k_2}{1+(k_1 - k_2)^2} \right]^{-1} \cdot \\ & \left[ \cos\left((k_1 - k_2)x - \left(\frac{k_1}{1+k_1^2} - \frac{k_2}{1+k_2^2}\right)t\right) - \cos\left((k_1 - k_2)x - \frac{k_1 - k_2}{1+(k_1 - k_2)^2}t\right) \right] \\ & -\frac{ik_1^{2\gamma}}{2} \frac{k_1 + k_2}{1+(k_1 + k_2)^2} \left[ \frac{k_1}{1+k_1^2} + \frac{k_2}{1+k_2^2} - \frac{k_1 + k_2}{1+(k_1 + k_2)^2} \right]^{-1} \cdot \\ & \left[ \cos\left((k_1 + k_2)x - \left(\frac{k_1}{1+k_1^2} + \frac{k_2}{1+k_2^2}\right)t\right) - \cos\left((k_1 + k_2)x - \frac{k_1 + k_2}{1+(k_1 + k_2)^2}t\right) \right] \end{split}$$

A study of the various constants appearing above reveals that, up to absolute constants,

$$u_{1} \sim -ik_{1}^{2\gamma} \left[ \cos\left(2k_{1}x - \frac{2k_{1}}{1+k_{1}^{2}}t\right) - \cos\left(2k_{1}x - \frac{2k_{1}}{1+4k_{1}^{2}}t\right) \right] -ik_{1}^{2\gamma} \left[ \cos\left(2k_{2}x - \frac{2k_{2}}{1+k_{2}^{2}}t\right) - \cos\left(2k_{2}x - \frac{2k_{2}}{1+4k_{2}^{2}}t\right) \right] +ik_{1}^{2\gamma} \left[ \cos\left(x - \left(\frac{k_{1}}{1+k_{1}^{2}} - \frac{k_{2}}{1+k_{2}^{2}}\right)t\right) - \cos\left(x - \frac{t}{2}\right) \right] -ik_{1}^{2\gamma} \left[ \cos\left((k_{1}+k_{2})x - \left(\frac{k_{1}}{1+k_{1}^{2}} + \frac{k_{2}}{1+k_{2}^{2}}\right)t\right) - \cos\left((k_{1}+k_{2})x - \left(\frac{k_{1}+k_{2}}{1+k_{2}^{2}}t\right) \right] \right]$$

as  $k_1$  becomes large. Since

$$\left|\cos(kx-\omega_1t)-\cos(kx-\omega_2t)\right| \leq |\omega_1-\omega_2|t,$$

straightforward calculations show that the first, second and fourth terms above are uniformly small compared to the third term, for large values of  $k_1$ . Indeed, they are all of order  $k_1^{2\gamma-1}t$ , whereas the third term is of order  $k_1^{2\gamma}t$ . It follows from this that for all  $t \geq 0$ ,

(2.9) 
$$\begin{aligned} \|u_1(t,\cdot)\|_{-s} \sim k_1^{2\gamma}t \quad \text{and likewise} \\ \|u_1(t,\cdot)\|_0 \sim k_1^{2\gamma}t. \end{aligned}$$

Thus, by taking  $k_1$  large, the  $\dot{H}_{per}^{-s}$ -norm of  $u_1$  can be made as big as we like.

As mentioned earlier, an estimate of the error term y is needed to complete the argument. It will in fact be shown that y is even bounded in  $L_2$ , let along  $\dot{H}_{per}^{-s}$ . To this end, use is made of one of a periodic version of one the bilinear estimates in [7].

**Lemma 2.2.** Let  $u, v \in H_{per}^q$  with  $q \ge 0$ . Then

(2.10) 
$$\|\varphi(D_x)(uv)\|_q \lesssim \|u\|_q \|v\|_q$$

where the implied constant only depends upon q.

The proof of this result is the same as the proof of Lemma 1 in [7], with sums replacing integrals.

Introduce the abbreviation  $X_T$  for  $C([0,T]; L^2)$  for ease of reading. The value of T > 0 will be specified momentarily. It follows from (2.10) and the implicit relationship (2.4) for the remainder y that

(2.11)  
$$\begin{aligned} \|y\|_{X_{T}} \lesssim T \|u_{1}\|_{X_{T}}^{2} + T \|S(t)u_{0}\|_{X_{T}} \|u_{1}\|_{X_{T}} + T \|u_{1}\|_{X_{T}} \|y\|_{X_{T}} \\ &+ T \|S(t)u_{0}\|_{X_{T}} \|y\|_{X_{T}} + T \|y\|_{X_{T}}^{2} \\ \lesssim T^{3}k_{1}^{4\gamma} + T^{2}k_{1}^{3\gamma} + \left(k_{1}^{2\gamma}T^{2} + k_{1}^{\gamma}T\right) \|y\|_{X_{T}} + T \|y\|_{X_{T}}^{2} \\ &= \mathcal{A} + \mathcal{B}\mathcal{Y} + T\mathcal{Y}^{2}, \end{aligned}$$

where  $\mathcal{Y} = \mathcal{Y}(T) = ||y||_{X_T}$ . As  $y \in C([0, M]; L_2)$  for all M > 0, it follows that  $\mathcal{Y}(T)$  is a continuous function of T. Moreover,  $\mathcal{Y}(0) = 0$ .

Choose  $T_0 = k_1^{-\mu\gamma}$ , where  $\mu > \frac{3}{2}$ . With this choice, we see that for  $T \leq T_0$ ,

$$\mathcal{A} = O(k_1^{\gamma(4-3\mu)} + k_1^{\gamma(3-2\mu)}) \quad \text{and} \quad \mathcal{B} = O(k_1^{2\gamma(1-\mu)} + k_1^{\gamma(1-\mu)}),$$

as  $k_1 \to \infty$  and all the exponents are negative.

Choose  $k_1$  large enough that  $\mathcal{B} < \frac{1}{2}$  and T and  $\mathcal{A}$  are both small. It follows in this circumstance that the quadratic polynomial

$$p(z) = \mathcal{A} + (\mathcal{B} - 1)z + Tz^2$$

has two positive roots, the smaller of which is denoted  $\underline{z}$  and the larger  $\overline{z}$ . Of course, p(z) < 0 for  $z \in (\underline{z}, \overline{z})$ .

The inquality (2.11) may be expressed as

$$p(\mathcal{Y}(T)) \ge 0.$$

As  $\mathcal{Y}(T)$  is continuous and  $\mathcal{Y}(0) = 0$ , it follows that  $\mathcal{Y}(T) \leq \underline{z}$  for all  $T \in [0, T_0]$ . For  $k_1$  large,  $T_0 < 1$ . When combined with the fact that  $\mathcal{B} < \frac{1}{2}$ , it is readily deduced that

$$\underline{z} \leq 4\mathcal{A}$$
, whence  $\mathcal{Y}(T) \leq 4\mathcal{A}$ ,

thus assuring that the remainder  $y(\cdot, t)$  is indeed uniformly bounded in  $\dot{H}_{per}^{-s}$  for  $t \leq T_0$  and large choices of  $k_1$ .

Taking a suitably chosen, increasing sequence  $\{k_1^{(j)}\}_{j=1}^\infty$  of wavenumbers for which

$$\lim_{j \to \infty} k_1^{(j)} = +\infty.$$

and with the indicated choices of  $\gamma$  and  $\mu$ , (2.7) assures the initial data tends to zero in  $\dot{H}_{per}^{-s}$ . The decomposition (2.3) together with (2.7), (2.9) and the bound just obtained on y then implies that the solutions  $u_j$  blow up at times  $T_j = (k_1^{(j)})^{-\mu\gamma}$ . The latter tend to zero as  $j \to \infty$  since  $\mu$  and  $\gamma$  are both positive. This completes the proof of the theorem.

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