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Stability of Solitary-Wave Solutions of Systems of Dispersive Equations

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Abstract The present study is concerned with systems

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + \frac{\partial}{\partial x} P(u, v) = 0, \\ \frac{\partial v}{\partial t} + \frac{\partial^3 v}{\partial x^3} + \frac{\partial}{\partial x} Q(u, v) = 0, \end{cases}$$

of Korteweg–de Vries type, coupled through their nonlinear terms. Here, $u = u(x, t)$ and $v = v(x, t)$ are real-valued functions of a real spatial variable x and a real temporal variable t. The nonlinearities P and Q are homogeneous, quadratic polynomials with real coefficients $A, B, \ldots$, *viz.*

$$P(u, v) = Au^2 + Buv + Cv^2, \qquad Q(u, v) = Du^2 + Euv + Fv^2,$$

in the dependent variables u and v. A satisfactory theory of local well-posedness is in place for such systems. Here, attention is drawn to their solitary-wave solutions. Special traveling waves termed proportional solitary waves are introduced and

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determined. Under the same conditions developed earlier for global well-posedness, stability criteria are obtained for these special, traveling-wave solutions.

1 Introduction

Nonlinear, dispersive wave equations arise in a number of important application areas. Because of this, and because their mathematical properties are interesting and subtle, they have seen enormous development since the 1960s when they first came to the fore (see Miura [27] for a sketch of the early history of the subject). The theory for a single nonlinear, dispersive wave equation is well developed by now, though there are still interesting open issues. The theory for coupled systems of such equations is much less developed, though they, too, arise as models of a range of physical phenomena. Considered here is a paradigm class of such systems, namely coupled Korteweg–de Vries equations. The systems we have in mind take the form

$$\begin{cases} u_t + u_{xxx} + P(u,v)_x = 0, \\ v_t + v_{xxx} + Q(u,v)_x = 0, \end{cases} \tag{1.1}$$

which comprise two linear Korteweg–de Vries equations coupled through their nonlinearity. Here, the dependent variables $u = u(x,t)$ and $v = v(x,t)$ are real-valued functions defined on $\mathbb{R} \times \mathbb{R}^+$ and subscripts connote partial differentiation. The nonlinearities are taken to be homogeneous quadratic polynomials in u and v, *viz.*

$$P(u,v) = Au^2 + Buv + Cv^2, \qquad Q(u,v) = Du^2 + Euv + Fv^2,$$

with given real coefficient $A, B, \ldots, F$, though parts of the theory developed here hold for much more general nonlinearities. Such systems and their close relatives arise as models for waves in a number of situations. For example, the model for Madden–Julian atmospheric oscillations recently developed by Majda and Biello [26] fits exactly into this class of systems. The surface water wave models put forward in [9,10] have specializations with the same sort of coupled KdV structure as in (1.1) (see also [13,14]). The Gear–Grimshaw system [20] arising in internal wave propagation likewise has features similar to the simpler models in (1.1). A particular system of the type displayed above, but with BBM-type dispersion, was studied by Hakkaev [23].

Recently, theory for the pure initial-value problem posed on the entire real line $\mathbb{R}$ and for the periodic initial-value problem for such systems has been developed (see [2, 3,11,15,29], and with BBM-type dispersion, [22]). One of the hallmarks of equations featuring both nonlinear and dispersive effects, as these systems do, is the existence of solitary-wave solutions. It is typical of nonlinear dispersive wave equations that solitary waves not only exist, but that they play a distinguished role in the large-time asymptotics of general solutions to the initial-value problem. One of the telling precursors of the resolution into solitary waves seen so frequently in the evolution of solutions to nonlinear, dispersive wave equations is the stability of individual solitary waves to small perturbations.

It is our purpose here to examine both the question of existence of solitary-wave solutions and their stability for the systems displayed in (1.1). Naturally, the outcome of our analysis will depend upon the particular coefficients $A, B, \ldots, F$ appearing in the model. The study undertaken here features rigorous analysis. In a companion paper [7], numerical simulations using conservative, discontinuous Galerkin methods developed in [8] are presented that began to fill in gaps in the picture revealed by the theory put forward here. In the present essay, attention is restricted to those systems (1.1) that satisfy a condition arising in [11] that guarantees global existence of smooth solutions corresponding to suitable initial data. In fact, only a local well-posedness theory is needed to set the question of stability on firm ground (see, e.g. [16] where certain solitary waves are seen to be stable even in the absence of global well-posedness for general initial data). As mentioned already, discussion of the case wherein the system itself is not globally well posed will appear in [7].

The development proceeds as follows. Section 2 reviews the conditions for global existence appearing in [11] that will pervade the further discussion. In Sect. 3, the definition of solitary waves and their stability, which is in fact orbital stability, is reviewed. Explicit, exact solutions are then found. Of course, there is general theory pertaining to the question of existence of traveling-wave solutions, but it does not appear that such theory can pick out all the solitary waves found here. It transpires that for a given speed $\omega > 0$ of propagation, there can be one or more solitary-wave solutions. Once solitary waves are in hand, the results pertaining to their stability or instability can be explained. Section 4 features a preliminary reduction of the complexity of the system. The question of stability for the reduced system is addressed in Sect. 5.1, which is the heart of the paper. Our stability theory is informed by the original theory for a single equation developed in the works of Benjamin [4] and Bona [6] (and see also the later works [1,21,30]). While the method is not new, its application to a system of equations requires further elucidation. Section 5.2 brings the results from the reduced system back to the original system (1.1). We conclude with a direct application of our theory to the Majda–Biello system (see Remark 5.6).

Notation

The notation used is mostly standard. The norm of a function f in the Lebesgue spaces $L_p(\mathbb{R})$ is denoted $|f|_p$. The same notation is employed for the norm of a vector $\mathbf{u} \in L_p(\mathbb{R}) \times L_p(\mathbb{R})$. The norm of a function u in the L_2–based Sobolev space $H^s(\mathbb{R})$ is written $\|u\|_s$. Similarly, the norm of a two-vector $\mathbf{u} = (u, v) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ is written $\|\mathbf{u}\|_s$. For $T > 0$, the space $C(0, T : X)$ is the collection of all continuous maps from $[0, T]$ into the Banach space X with the norm induced by the norm $\|\cdot\|_X$ on X and the supremum-norm on $[0, T]$. The Bourgain space $X_{s,r} = X_{s,r}(\mathbb{R})$ is defined to be the Banach space of all tempered distributions u on $\mathbb{R}^2$ such that

$$\|u\|_{s,r}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+\xi^2)^s(1+|\xi^3-\tau|)^r|\widehat{u}(\xi,\tau)|^2\,d\xi d\tau$$

is finite, where $\hat{u}$ connotes the Fourier transform of u in both x and t. For $T > 0$ finite, the restrictions to $[0, T]$ of elements in $X_{s,r}$ are denoted $X_{s,r}^T$. The space $X_{s,r}^T$ is endowed with the quotient norm. It is well known that

$$X_{s,r}^T \subset C(0, T : H^s) \tag{1.2}$$

provided that $r > \frac{1}{2}$.

Throughout, an unadorned integral will always denote integration over the entire real line $\mathbb{R}$; thus,

$$\int f(x)dx = \int_{-\infty}^{\infty} f(x)dx.$$

2 Global Well-Posedness

The system (1.1) is always locally well posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for any $s > -\frac{3}{4}$. This is a straightforward consequence of the extant local well-posedness theory for the Korteweg–de Vries equation itself. In [11], it was shown that global well posedness of solutions corresponding to initial data in the just mentioned Sobolev spaces obtains as soon as the coeficients $A, B, \ldots, F$ satisfy the condition to be explained now.

Consider the pair of linear equations

$$\begin{cases} 2Ba + (E - 2A)b - 4Dc = 0, \\ 4Ca + (2F - B)b - 2Ec = 0, \end{cases} \tag{2.1}$$

for the unknowns a, b, c. This system comprises two equations in three unknowns and consequently always has at least a one-dimensional subspace of solutions. This system of equations arises from asking for values a, b, c for which the quadratic functional

$$\Omega(u, v) = \int \left(au^2 + buv + cv^2\right) dx \tag{2.2}$$

is time-independent whenever (u, v) is a sufficiently regular solution of the initial-value problem for the system (1.1).

The system (2.1) implies that

$$\frac{\partial}{\partial v}\Big\{2aP(u, v) + bQ(u, v)\Big\} = \frac{\partial}{\partial u}\Big\{bP(u, v) + 2cQ(u, v)\Big\}, \tag{2.3}$$

which in turn means that there is a cubic polynomial R of the form

$$R(u, v) = \frac{\alpha}{3}u^3 + \beta u^2 v + \gamma uv^2 + \frac{\delta}{3}v^3 \tag{2.4}$$

such that

$$\begin{aligned} &\frac{\partial}{\partial u}R(u, v) = 2aP(u, v) + bQ(u, v) \text{ and} \\ &\frac{\partial}{\partial v}R(u, v) = bP(u, v) + 2cQ(u, v). \end{aligned} \tag{2.5}$$

The coefficients $\alpha, \beta, \gamma, \delta$ can be written explicitly as

$$\alpha = 2aA + bD,\ \beta = bA + 2cD,\ \gamma = 2aC + bF,\ \delta = bC + 2cF, \tag{2.6}$$

or, because of the equations (2.1) satisfied by a, b, c,

$$\beta = aB + \frac{1}{2}bE \quad \text{and} \quad \gamma = \frac{1}{2}bB + cE. \tag{2.7}$$

It was shown in [11] that if a, b, c satisfies (2.1), then it is also the case that the functional

$$\Theta(u, v) = \int \Big(au_x^2 + bu_x v_x + cv_x^2 - R(u, v) \Big)\, dx \tag{2.8}$$

is time-independent when (u, v) is a sufficiently smooth solution of (1.1) that decays suitably to zero as $x \to \pm\infty$.

These two invariants play a central role in the global well-posedness theory developed in [11] (and see also the related theory of Oh [29]). They will also be crucial to understanding the stability of the solitary-wave solutions of (1.1).

Of course, for the invariant functional Ω to be useful in providing bounds on the solution (u, v), it must be the case that the quadratic form

$$q(u, v) = au^2 + buv + cv^2$$

appearing in the integrand of Ω is strictly positive (or strictly negative) definite. This is true if and only if

$$b^2 < 4ac. \tag{2.9}$$

which can be expressed in terms of the original coefficients $A, B, \ldots, F$. In the case when the linear system (2.1) has rank 2, the generic case, the condition (2.9) is equivalent to the inequality

$$\begin{aligned}(4CD - BE)^2 \ &< 2EC(E - 2A)^2 + 2BD(2F - B)^2 \\ &\quad - \big[4CD + BE\big](E - 2A)(2F - B).\end{aligned} \tag{2.10}$$

The main theorem that emerges from the analysis in [11] is the following:

Theorem 2.1 *Suppose that the constants a, b and c satisfy the linear system (2.1) and that the positive definiteness condition (2.9)–(2.10) holds. Then, for any $s > -\frac{3}{4}$ and any $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, there is a pair of functions $(u, v) \in C([0, \infty) : H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ starting at (u_0, v_0) when $t = 0$ that satisfies the system (1.1). Moreover, there is an $r > \frac{1}{2}$ such that for any $T > 0$, this solution pair (u, v) lies in $X^T_{s,r} \times X^T_{s,r}$ and is unique within this class. The solution (u, v) depends continuously in $X^T_{s,r} \times X^T_{s,r}$ on variations of (u_0, v_0) in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, and hence, for any $T > 0$, continuously in $C([0, T] : H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ on perturbations of (u_0, v_0).*

Two points are worth making in tandem with this theorem. First, the condition (2.9) is not a necessary condition for global well-posedness of the sysem (1.1). Indeed, the example

$$\begin{cases} u_t + u_{xxx} + uu_x = 0, \\ v_t + v_{xxx} + (Du^2 + Euv + Fv^2)_x = 0 \end{cases} \tag{2.11}$$

is easily seen to be globally well-posed. This system comprises a KdV-equation driving another KdV-equation with a variable coefficient linear term and a forcing term. Solving the first equation globally and then substituting the determined value of u into the equation for the evolution of v yields an equation that is clearly globally well-posed. This is a degenerate case where the positive definiteness criterion just misses, which is to say, $b^2 = 4ac$. Indeed, as will appear later, whenever $b^2 = 4ac$, the system (1.1) may be put into the form (2.11) by a change of the dependent variables. In light of this fact, the above theorem can in fact be strengthened, drawing the same conclusions with only the hypothesis $4ac \geq b^2$.

Second, there are choices of the coefficients $A, B, \ldots, F$ that yield systems that do not have a global well-posedness theory. Explicit examples are given in [18] and other indications of the existence of such systems are provided in our companion paper [7].

3 Solitary-Wave Solutions

Attention is now turned to the solitary-wave solutions of the system (1.1). A solitary wave traveling to the right with speed $\omega > 0$ has the form $(u_s(x,t), v_s(x,t)) = (\phi_\omega(x - \omega t), \psi_\omega(x - \omega t))$ where ϕ_ω and ψ_ω are smooth, real-valued functions of one real variable which decay to zero at $\pm\infty$. For the time being, view ω as fixed and ignore the subscripts on the two shape functions $\phi = \phi_\omega$ and $\psi = \psi_\omega$. Traveling-wave solutions of the system (1.1) of partial differential equations that decay to zero, along with their second derivatives, as $x \to \pm\infty$ satisfy the coupled system

$$\begin{cases} -\omega\phi + \phi'' + A\phi^2 + B\phi\psi + C\psi^2 = 0, \\ -\omega\psi + \psi'' + D\phi^2 + E\phi\psi + F\psi^2 = 0, \end{cases} \tag{3.1}$$

of ordinary differential equations. Of course, $(\phi, \psi) = (0, 0)$ is always a trivial solution which is of no interest here. In case the positive definiteness condition (2.9) holds, this trivial solution is always stable in the sense that if the initial data starts out small in H^1, it remains small there for all time.

Attention is now focused upon solutions (ϕ, ψ) for which at least one of the two components ϕ and ψ is a non-constant function.

There are general methods to attack the issue of existence of traveling-wave solutions of the system (3.1) (see, for example, [12,25,30]). In the present work, we search for solitary-wave solutions (ϕ, ψ) having one of two special forms, namely, $\psi(y) = \mu\phi(y)$ or $\phi(y) = \nu\psi(y)$ where μ and ν are real numbers. This includes the cases where $\mu = 0$ or $\nu = 0$ so that the traveling wave has the form $(\phi, 0)$ or $(0, \psi)$, respectively. We call traveling-wave solutions of the form $(\phi, \mu\phi)$ and $(\nu\psi, \psi)$

proportional solitary waves. We hasten to add that there are non-proportional solitary-wave solutions of such systems as well. Indeed, one of the interesting things about this simple class of systems is that they often have several, quite different solitary-wave solutions. This is true even of proportional solitary waves, as is shown in this section. The existence of non-proportional solitary waves will become clearer in the associated paper [7]. At the same time, the proportional solitary waves appear to often play a significant role in the longer-term asymptotics of solutions to the initial-value problem, and so are worth the extended study they are given here.

Without loss of generality, focus upon the case $\psi = \mu\phi$. Replacing ψ with $\mu\phi$ in (3.1) leads to the pair of equations

$$\begin{cases} -(\omega\phi - \phi'') + (A + B\mu + C\mu^2)\phi^2 = 0, \\ -\mu(\omega\phi - \phi'') + (D + E\mu + F\mu^2)\phi^2 = 0. \end{cases} \tag{3.2}$$

Demanding that $\phi \neq 0$ solves both these two equations implies that

$$\mu(A + B\mu + C\mu^2) = D + E\mu + F\mu^2, \tag{3.3}$$

or, what is the same,

$$C\mu^3 + (B - F)\mu^2 + (A - E)\mu - D = 0. \tag{3.4}$$

Notice that if μ is a solution of (3.3) and satisfies $A+B\mu+C\mu^2 = 0$, it follows from the first equation in (3.2) that necessarily $\phi = 0$, whence $\psi = 0$ and so corresponding to this particular μ, there is no non-trivial solution of the form $(\phi, \mu\phi)$. The existence of these special proportional solitary-wave solutions requires μ to be a real solution of (3.3) or (3.4) as well as having $A + B\mu + C\mu^2 \neq 0$.

Proposition 3.1 *If equation* (3.3) *or* (3.4) *has a solution* $\mu \in \mathbb{R}$ *for which the quantity* $A+B\mu+C\mu^2 \neq 0$, *then for any propagation speed* $\omega > 0$, (1.1) *has a solitary-wave solution* (u_s, v_s) *with*

$$u_s(x,t) = \phi(x - \omega t) = \frac{3\omega}{2(A + B\mu + C\mu^2)} \operatorname{sech}^2\left(\frac{\sqrt{\omega}}{2}(x - \omega t)\right). \tag{3.5}$$

and $v_s(x,t) = \mu u_s(x,t)$. *Similarly, if* $\nu \in \mathbb{R}$ *satisfies*

$$A\nu^2 + B\nu + C = \nu(D\nu^2 + E\nu + F) \tag{3.6}$$

and $D\nu^2 + E\nu + F \neq 0$, *then for any* $\omega > 0$, (1.1) *has a solitary-wave solution* (u_s, v_s) *where*

$$v_s(x,t) = \psi(x - \omega t) = \frac{3\omega}{2(D\nu^2 + E\nu + F)} \operatorname{sech}^2\left(\frac{\sqrt{\omega}}{2}(x - \omega t)\right) \tag{3.7}$$

and $u_s = \nu v_s$.

Remark 3.2 The proportional solitary-wave solutions (ϕ, ψ) of speed $\omega > 0$ are seen to be 2-vector multiples of the square of the hyperbolic secant, *viz.*

$$(\phi, \psi) = (\mu_1, \mu_2)\, 3\omega\, \mathrm{sech}^2\left(\frac{\sqrt{\omega}}{2}\,\cdot\right). \tag{3.8}$$

If neither of μ_1 and μ_2 is zero, then $\mu = \frac{\mu_2}{\mu_1} = \frac{1}{\nu}$,

$$\mu_1 = \frac{1}{2(A + B\mu + C\mu^2)} = \frac{\nu}{2(D\nu^2 + E\nu + F)}$$

and

$$\mu_2 = \frac{1}{2(D\nu^2 + E\nu + F)} = \frac{\mu}{2(A + B\mu + E\mu^2)}.$$

If one of μ_1 and μ_2 is zero, say $\mu_2 = 0$, then it must be the case that $\mu_1 \neq 0$. In this circumstance, $\mu = 0$ and $\nu \neq 0$, the proportional solitary-wave solution is of form $(\phi, 0)$. Conversely, the solitary-wave solution $(\phi, 0)$ corresponds to $\mu_2 = 0$ in (3.8). If a proportional solitary-wave solution (ϕ, ψ) is given, then μ_1 and μ_2 are uniquely determined.

Hence, a proportional solitary-wave solution (ϕ, ψ) may be delineated in the form $(\mu_1, \mu_2)\, 3\omega\, \mathrm{sech}^2(\frac{\sqrt{\omega}}{2}\,\cdot)$, or in one or both of the forms $(\phi, \mu\phi)$ and $(\nu\psi, \psi)$.

The 2×2 matrix

$$\mathcal{M} = \mathcal{M}(\mu_1, \mu_2) = \begin{pmatrix} 2A\mu_1 + B\mu_2 & B\mu_1 + 2C\mu_2 \\ 2D\mu_1 + E\mu_2 & E\mu_1 + 2F\mu_2 \end{pmatrix} \tag{3.9}$$

associated with a proportional solitary-wave solution given in the form (3.8) will appear frequently. Notice that this matrix depends upon the coefficients $A, B, \ldots, F$, and so is not only a function of μ_1 and μ_2. This dependence is suppressed to keep the notation readable. Note also that

$$3\omega\mathcal{M}(\mu_1, \mu_2)\, \mathrm{sech}^2\left(\frac{\sqrt{\omega}}{2}\,\cdot\right) = \begin{pmatrix} 2A\phi + B\psi & B\phi + 2C\psi \\ 2D\phi + E\psi & E\phi + 2F\psi \end{pmatrix}, \tag{3.10}$$

where $(\phi(z), \psi(z)) = (\mu_1, \mu_2)\, 3\omega\, \mathrm{sech}^2\left(\frac{\sqrt{\omega}}{2}\, z\right)$.

A natural question following on Proposition 3.1 is, does each of the systems displayed in (1.1) have non-trivial proportional solitary-wave solutions? The answer is "no". For example, in (2.11) with $F = 0$, $E = \frac{1}{2}$, $D \neq 0$, the system has no such solitary-wave solutions. On the other hand, for the same system when $D = F = 0$ and $E = \frac{1}{2}$, there are an infinite number of such solutions, namely $(u_s, \mu u_s)$ where $u_s(x, t) = \phi(x - \omega t) = 3\omega\, \mathrm{sech}^2\left(\frac{\sqrt{\omega}}{2}\,(x - \omega t)\right)$ and μ is any real number. These systems do not satisfy the positive definiteness criterion (2.9) and so are not within the purview of the present study.

4 Reduction of the System and the Number of Solitary-Wave Solutions

4.1 Reduction

It is supposed from here on that the constants a, b, c satisfy the linear system (2.1) and that the matrix

$$\mathcal{N} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \tag{4.1}$$

is positive definite. As noted already, systems for which this is not the case will be discussed separately. (Most instances where such systems arise in practice fall into the category of having a positive definite matrix $\mathcal{N}$, though there is an interesting exception which will appear later.) Multiply the system (1.1) by the matrix $\mathcal{N}$. It follows from (2.3) that

$$(\partial_t + \partial_{xxx}) \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \begin{pmatrix} P(u,v) \\ Q(u,v) \end{pmatrix} = 0, \tag{4.2}$$

is the same as

$$(\partial_t + \partial_{xxx}) \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} \partial_u R(u,v) \\ \partial_v R(u,v) \end{pmatrix} = 0, \tag{4.3}$$

or, since R has the form displayed in (2.4),

$$(\partial_t + \partial_{xxx}) \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} \alpha u^2 + 2\beta uv + \gamma v^2 \\ \beta u^2 + 2\gamma uv + \delta v^2 \end{pmatrix} = 0, \tag{4.4}$$

where α, β, γ and δ are as in (2.6). Since the matrix $\mathcal{N}$ is positive definite, its square root $\mathcal{N}^{\frac{1}{2}}$ is well defined as is the inverse $\mathcal{N}^{-\frac{1}{2}}$. Introduce new dependent variables $\tilde{u}$ and $\tilde{v}$ by

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \mathcal{N}^{\frac{1}{2}} \begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{N}^{-\frac{1}{2}} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}. \tag{4.5}$$

In the new variables $\tilde{u}$ and $\tilde{v}$, $R(u,v) = R(u(\tilde{u},\tilde{v}), v(\tilde{u},\tilde{v})) = \tilde{R}(\tilde{u},\tilde{v})$ is still a homogeneous, cubic polynomial function of $\tilde{u}$ and $\tilde{v}$, say,

$$R(u,v) = \tilde{R}(\tilde{u},\tilde{v}) = \frac{1}{3}\tilde{\alpha}\tilde{u}^3 + \tilde{\beta}\tilde{u}^2\tilde{v} + \tilde{\gamma}\tilde{u}\tilde{v}^2 + \frac{1}{3}\tilde{\delta}\tilde{v}^3. \tag{4.6}$$

A calculation reveals that

$$\begin{aligned} \nabla_{(u,v)} R(u,v) &= \big(\partial_u R(u,v), \partial_v R(u,v)\big) = \big(\partial_u \tilde{R}(\tilde{u},\tilde{v}), \partial_v \tilde{R}(\tilde{u},\tilde{v})\big) \\ &= \big(\partial_{\tilde{u}} \tilde{R}(\tilde{u},\tilde{v})), \partial_{\tilde{v}} \tilde{R}(\tilde{u},\tilde{v})\big) \frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)} \end{aligned}$$

$$= \nabla_{(\tilde{u},\tilde{v})} \tilde{R}(\tilde{u}, \tilde{v}) \mathcal{N}^{\frac{1}{2}}.$$

Taking the transpose of the last formula and using the fact that the matrix $\mathcal{N}$ is symmetric, the system (4.3), rewritten in the variables $(\tilde{u}, \tilde{v})$, becomes

$$(\partial_t + \partial_{xxx}) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \partial_x \begin{pmatrix} \partial_{\tilde{u}} \tilde{R}(\tilde{u}, \tilde{v}) \\ \partial_{\tilde{v}} \tilde{R}(\tilde{u}, \tilde{v}) \end{pmatrix} = 0. \tag{4.7}$$

In more detail, suppose

$$\mathcal{N}^{-\frac{1}{2}} = \begin{pmatrix} n_{11} & n_{12} \\ n_{12} & n_{22} \end{pmatrix}.$$

Then the coeficients of $\tilde{R}(\tilde{u}, \tilde{v})$ are

$$\tilde{\alpha} = \alpha n_{11}^3 + 3\beta n_{11}^2 n_{12} + 3\gamma n_{11} n_{12}^2 + \delta n_{12}^3,$$

$$\tilde{\beta} = \alpha n_{11}^2 n_{12} + \beta (n_{11}^2 n_{22} + 2 n_{11} n_{12}^2) + \gamma (2 n_{11} n_{12} n_{22} + n_{12}^3) + \delta n_{12}^2 n_{22},$$

$$\tilde{\gamma} = \alpha n_{11} n_{12}^2 + \beta (2 n_{11} n_{12} n_{22} + n_{12}^3) + \gamma (n_{11} n_{22}^2 + 2 n_{12}^2 n_{22}) + \delta n_{12} n_{22}^2,$$

$$\tilde{\delta} = \alpha n_{12}^3 + 3\beta n_{12}^2 n_{22} + 3\gamma n_{12} n_{22}^2 + \delta n_{22}^3.$$

The system (4.7) thus has the form

$$\begin{cases} \tilde{u}_t + \tilde{u}_{xxx} + (\tilde{\alpha}\tilde{u}^2 + 2\tilde{\beta}\tilde{u}\tilde{v} + \tilde{\gamma}\tilde{v}^2)_x = 0, \\ \tilde{v}_t + \tilde{v}_{xxx} + (\tilde{\beta}\tilde{u}^2 + 2\tilde{\gamma}\tilde{u}\tilde{v} + \tilde{\delta}\tilde{v}^2)_x = 0, \end{cases} \tag{4.8}$$

where $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\delta}$ are as above.

It is worth note that the system (4.8) features only four distinct coefficients rather than the original six. The two invariants for (4.8) have the form

$$\Omega(\tilde{u}.\tilde{v}) = \int \left(\frac{1}{2}\tilde{u}^2 + \frac{1}{2}\tilde{v}^2 \right) dx = \int \left(au^2 + buv + cv^2 \right) dx = \Omega(u, v)$$

and

$$\begin{aligned} \Theta(\tilde{u}, \tilde{v}) &= \int \left(\frac{1}{2}\tilde{u}_x^2 + \frac{1}{2}\tilde{v}_x^2 - \frac{1}{3}\tilde{\alpha}\tilde{u}^3 - \tilde{\beta}\tilde{u}^2\tilde{v} - \tilde{\gamma}\tilde{u}\tilde{v}^2 - \frac{1}{3}\tilde{v}^3 \right) dx \\ &= \int \left(au_x^2 + bu_x v_x + cv_x^2 - R(u, v) \right) dx = \Theta(u, v), \end{aligned}$$

where R is as before. In other words, the change of variables (4.5) does not change the values of the two invariants.

It is straightforward to see that (ϕ, ψ) is a solitary-wave solution of (1.1) if and only if $(\tilde{\phi}, \tilde{\psi})$ is a solitary-wave solution of (4.8), where

$$\begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix} = \mathcal{N}^{\frac{1}{2}} \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \tag{4.9}$$

Furthermore, (ϕ, ψ) is a proportional solitary-wave solution of (1.1) if and only if $(\tilde{\phi}, \tilde{\psi})$ is a proportional solitary-wave solution of (4.8).

Lemma 4.1 *Suppose* $(\phi, \psi) = (\mu_1, \mu_2)3\omega \operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x)$ *is a proportional solitary-wave solution of* (1.1) *and* $\mathcal{N}$ *defined in* (4.1) *is positive definite. Define the vector*

$$\begin{pmatrix} \tilde{\mu}_1 \\ \tilde{\mu}_2 \end{pmatrix} = \mathcal{N}^{\frac{1}{2}} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.$$

If $(\tilde{\phi}, \tilde{\psi})$ *is defined as in* (4.9)*, so that* $(\tilde{\phi}, \tilde{\psi}) = (\tilde{\mu}_1, \tilde{\mu}_2)\, 3\omega \operatorname{sech}^2(\frac{\sqrt{\omega}}{2}\, x)$*, then,*

$$\tilde{\mathcal{M}}(\tilde{\mu}_1, \tilde{\mu}_2) = \mathcal{N}^{\frac{1}{2}} \mathcal{M}(\mu_1, \mu_2) \mathcal{N}^{-\frac{1}{2}},$$

where $\mathcal{M}(\mu_1, \mu_2)$ *is as in* (3.9) *corresponding to system* (1.1) *and* $\tilde{\mathcal{M}}(\tilde{\mu}_1, \tilde{\mu}_2)$ *is defined in the same way but corresponding to the reduced system* (4.8)*.*

Proof By definition,

$$\begin{aligned} 3\omega \tilde{\mathcal{M}}(\tilde{\mu}_1, \tilde{\mu}_2) \operatorname{sech}^2\left(\frac{\sqrt{\omega}}{2}\cdot\right) &= \begin{pmatrix} 2\tilde{\alpha}\tilde{\phi} + 2\tilde{\beta}\tilde{\psi} & 2\tilde{\beta}\tilde{\phi} + 2\tilde{\gamma}\tilde{\psi} \\ 2\tilde{\beta}\tilde{\phi} + 2\tilde{\gamma}\tilde{\psi} & 2\tilde{\gamma}\tilde{\phi} + 2\tilde{\delta}\tilde{\psi} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{R}_{\tilde{\phi}\tilde{\phi}} & \tilde{R}_{\tilde{\phi}\tilde{\psi}} \\ \tilde{R}_{\tilde{\phi}\tilde{\psi}} & \tilde{R}_{\tilde{\psi}\tilde{\psi}} \end{pmatrix} \end{aligned} \tag{4.10}$$

where $\tilde{R}$ is defined in (4.6). The chain rule implies that

$$\begin{aligned} 3\omega \tilde{\mathcal{M}}(\tilde{\mu}_1, \tilde{\mu}_2) \operatorname{sech}^2(\frac{\sqrt{\omega}}{2}\cdot) &= \frac{\partial(\phi, \psi)}{\partial(\tilde{\phi}, \tilde{\psi})} \begin{pmatrix} R_{\tilde{\phi}\phi} & R_{\tilde{\psi}\phi} \\ R_{\tilde{\phi}\psi} & R_{\tilde{\psi}\psi} \end{pmatrix} \\ &= \frac{\partial(\phi, \psi)}{\partial(\tilde{\phi}, \tilde{\psi})} \begin{pmatrix} R_{\phi\phi} & R_{\phi\psi} \\ R_{\psi\phi} & R_{\psi\psi} \end{pmatrix} \frac{\partial(\phi, \psi)}{\partial(\tilde{\phi}, \tilde{\psi})} \\ &= \mathcal{N}^{-\frac{1}{2}} \begin{pmatrix} 2\alpha\phi + 2\beta\psi & 2\beta\phi + 2\gamma\psi \\ 2\beta\phi + 2\gamma\psi & 2\gamma\phi + 2\delta\psi \end{pmatrix} \mathcal{N}^{-\frac{1}{2}} \\ &= \mathcal{N}^{-\frac{1}{2}} \mathcal{N} \begin{pmatrix} 2A\phi + B\psi & B\phi + 2C\psi \\ 2D\phi + E\psi & E\phi + 2F\psi \end{pmatrix} \mathcal{N}^{-\frac{1}{2}} \\ &= 3\omega \mathcal{N}^{\frac{1}{2}} \mathcal{M}(\mu_1, \mu_2) \mathcal{N}^{-\frac{1}{2}} \operatorname{sech}^2(\frac{\sqrt{\omega}}{2}\cdot). \end{aligned}$$

The result follows immediately. □

Thus, we see that when $\mathcal{N}$ is positive definite, the system (1.1) is equivalent to a system of form

$$\begin{cases} u_t + u_{xxx} + \left(Au^2 + Buv + \frac{1}{2}Ev^2\right)_x = 0, \\ v_t + v_{xxx} + \left(\frac{1}{2}Bu^2 + Euv + Fv^2\right)_x = 0, \end{cases} \tag{4.11}$$

under a linear change of the dependent variables. In the positive-definite case, the ostensibly six-parameter system (1.1) is seen to be equivalent to the four-parameter family (4.11). This reduction is not especially helpful as far as local and global well-posedness is concerned. However, it greatly simplifies the classification of the existence and stability properties of proportional solitary-wave solutions.

4.2 Explicit Solitary-Wave Solutions

In this section, a classification of proportional solitary-wave solutions is undertaken. For the reduced system (4.11), here is the overarching result.

Theorem 4.1 *The system* (4.11) *has one, two or three proportional solitary-wave solutions if A, B, E and F do not all vanish.*

The proof of this theorem is broken into several lemmas. More detail about which of the three cases obtains for a particular system is provided in these lemmas.

Lemma 4.2 *The system of equations*

$$\begin{cases} u_t + u_{xxx} + (Au^2)_x = 0, \\ v_t + v_{xxx} + (Fv^2)_x = 0, \end{cases} \tag{4.12}$$

has 1 *or* 3 *proportional solitary-wave solutions provided at least one of A and F is not equal to zero.*

Proof If both A and F are non-zero, then corresponding to any $\omega > 0$, the decoupled system (4.12) has three solitary wave-solutions, $(\phi, 0)$, $(0, \psi)$ and (ϕ, ψ) where $\phi(x) = \frac{3\omega}{2A}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x)$ and $\psi(x) = \frac{3\omega}{2F}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x)$. If one of A and F is zero, say $F = 0$, then the system has one such solitary-wave solution, namely $(\phi, 0)$ where $\phi(x) = \frac{3\omega}{2A}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x)$. □

Consider next the system

$$\begin{cases} u_t + u_{xxx} + (Au^2 + \frac{1}{2}Ev^2)_x = 0, \\ v_t + v_{xxx} + (Euv + Fv^2)_x = 0, \end{cases} \tag{4.13}$$

where $E \neq 0$. It is clear that it has no solitary-wave solutions of form $(0, \psi)$. All proportional solitary-wave solutions must therefore be of the form $(\phi, \mu\phi)$ where

$\phi \neq 0$. A necessary condition for there to be a proportional solitary wave is that μ is a real root of the cubic polynomial

$$f(\mu) = E\mu^3 - 2F\mu^2 + 2(A - E)\mu. \tag{4.14}$$

Hence, the maximum number of proportional solitary-wave solutions is the number of distinct real roots of $f(\mu)$. A natural question is, does every real root μ of (4.14) have a corresponding non-zero solitary-wave solution $(\phi, \mu\phi)$? Of course, in this case, 0 is always a real root.

The possibilities for real roots of the polynomial f above are (I) $\mu = 0$ is the only real root and it is simple, (II) $\mu = 0$ is a root of multiplicity 2 and hence there is another, non-zero real root, (III) $\mu = 0$ is a root of multiplicity 3, or (IV) in addition to $\mu = 0$, $f(\mu)$ has two more non-zero real roots.

These four possibilities are now investigated.

Lemma 4.3 *If* $\mu = 0$ *is the only real root of the cubic polynomial defined in* (4.14) *and it is simple, then the system* (4.13) *has only one proportional solitary-wave solution.*

Proof $\mu = 0$ being the only real root of f means that the maximum number of proportional solitary-wave solutions is 1. Moreover, since zero is simple, the quadratic equation $Ex^2 - 2Fx + 2(A - E) = 0$ has no real root. In consequence, it must have a negative discriminant, which is to say,

$$F^2 + 2E^2 - 2AE < 0,$$

whence $A \neq 0$. By Proposition 3.1, the system (4.13) has a solitary-wave solution $(\phi, 0)$ where $\phi(x) = \frac{3\omega}{2A}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x)$. □

Lemma 4.4 *If* $\mu = 0$ *is a real root of* f *of multiplicity 2, then the system* (4.13) *has two solitary-wave solutions of the form* $(\phi, \mu\phi)$.

Proof It must be the case that $A = E$ and $E, F \neq 0$. Then, the system has two solitary-wave solutions of the form $(\phi, \mu\phi)$. One is $(\phi, 0)$ and the other is $(\phi_1, \tilde{\mu}\phi_1)$ where $\tilde{\mu} = \frac{2F}{A}$ with

$$\phi(x) = \frac{3\omega}{2A}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x) \quad \text{and} \quad \phi_1(x) = \frac{3\omega}{A(2 + \tilde{\mu}^2)}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x).$$

□

Lemma 4.5 *If* $\mu = 0$ *is a real root of* f *of multiplicity 3, then the system* (4.13) *has only one solitary-wave solution of the special form* $(\phi, \mu\phi)$.

Proof In this case, $A = E \neq 0$ and $F = 0$. Clearly, (4.13) will then have only one proportional solitary-wave solution which is $(\phi, 0)$ where $\phi(x) = \frac{3\omega}{2A}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x)$. □

In case (IV), the cubic function f has three real roots, $\mu = 0$, μ_+ and μ_- where

$$\mu_\pm = \frac{F \pm \sqrt{F^2 + 2E^2 - 2AE}}{E}. \tag{4.15}$$

Does each of these roots necessarily correspond to a non-trivial solitary-wave solution $(\phi, \mu\phi)$ for system (4.13)?

Lemma 4.6 *In case (IV), if $A = 0$, or $A \neq 0$ and $\frac{E}{2A} = -\frac{F^2}{E^2}$, then the system (4.13) has only two proportional solitary-wave solutions.*

Otherwise, each of the three real roots of f has a corresponding proportional solitary-wave solution of the system (4.13) .

Proof If $A = 0$, then corresponding to the root $\mu = 0$, there is no non-trivial solitary-wave solution $(\phi, 0)$ since the second component being zero implies the first to be zero as well. Consider the other two roots μ_+ and μ_- of f. By Proposition 3.1, it is sufficient to check whether either of $2A + E\mu_\pm^2$ can vanish. Apparently,

$$2A + E\mu_\pm^2 = E\mu_\pm^2 \neq 0,$$

whence, $(\phi_+, \mu_+\phi_+)$ and $(\phi_-, \mu_-\phi_-)$ are indeed solitary-wave solutions, with

$$\phi_+(x) = \frac{3\omega}{E\mu_+^2} \operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x) \text{ and } \phi_-(x) = \frac{3\omega}{E\mu_-^2} \operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x). \tag{4.16}$$

If $A \neq 0$ and $\frac{E}{2A} = -\frac{F^2}{E^2}$, then corresponding to the root $\mu = 0$, there is a solitary-wave solution $(\phi, 0)$ where $\phi(x) = \frac{3\omega}{2A} \operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x)$. By Proposition 3.1, whether or not $(\phi_+, \mu_+\phi_+)$ or $(\phi_-, \mu_-\phi_-)$ is a solitary-wave solution depends on if the quantities $2A + E\mu_+^2$ and $2A + E\mu_-^2$ are zero. A simple calculation reveals that when $\frac{E}{2A} = -\frac{F^2}{E^2}$, or, what is the same, $E^3 + 2AF^2 = 0$ occurs, then

$$2A + E\mu_\pm^2 = 2E + 2F\mu_\pm = \frac{2E^2 + 2F^2 \pm 2\,\mathrm{sgn}\,(F)(E^2 + F^2)}{E}.$$

One of these two values is indeed zero; more precisely,

$$2A + E\mu_+^2 \neq 0 \quad \text{but} \quad 2A + E\mu_-^2 = 0 \quad \text{if } F > 0$$

and

$$2A + E\mu_+^2 = 0 \quad \text{but} \quad 2A + E\mu_-^2 \neq 0 \quad \text{if } F < 0.$$

This shows that the system (4.13) has two proportional solitary waves $(\phi, 0)$ and $(\phi_+, \mu_+\phi_+)$ when $E^3 + 2AF^2 = 0$ and $F > 0$ and two proportional solitary-waves $(\phi, 0)$ and $(\phi_-, \mu_-\phi_-)$ when $E^3 + 2AF^2 = 0$ and $F < 0$.

For the remaining case where $A \neq 0$ and $E^3 + 2AF^2 \neq 0$, so that $2A + E\mu_\pm^2 \neq 0$, Proposition 3.1 yields three solitary-wave solutions $(\phi_0, 0)$, $(\phi_+, \mu_+\phi_+)$ and $(\phi_-, \mu_-\phi_-)$, where

$$\phi_0(x) = \frac{3\omega}{2A} \operatorname{sech}^2\left(\frac{\sqrt{\omega}}{2}x\right), \quad \phi_\pm(x) = \frac{3\omega}{2A + E\mu_\pm^2} \operatorname{sech}^2\left(\frac{\sqrt{\omega}}{2}x\right). \tag{4.17}$$

□

To summarize, system (4.13) has one, two or three proportional solitary-wave solutions if at least one of A, E and F is not zero.

We are ready to prove Theorem 4.1.

Proof (of Theorem 4.1) If one of B and E is zero, the conclusion of the theorem is a straightforward consequence of the last five lemmas. So, assume that $B, E \neq 0$. The cubic polynomial equation

$$E\mu^3 + 2(B - F)\mu^2 + 2(A - E)\mu - B = 0$$

or

$$\mu(E\mu^2 + 2B\mu + 2A) = 2F\mu^2 + 2E\mu + B$$

always has at least one real root, μ_* say. Under the change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{1 + \mu_*^2}} \begin{pmatrix} 1 & -\mu_* \\ \mu_* & 1 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

the system (4.11) reduces to

$$\begin{cases} \tilde{u}_t + \tilde{u}_{xxx} + (\tilde{A}\tilde{u}^2 + \frac{1}{2}\tilde{E}\tilde{v}^2)_x = 0, \\ \tilde{v}_t + \tilde{v}_{xxx} + (\tilde{E}\tilde{u}\tilde{v} + \tilde{F}\tilde{v}^2)_x = 0. \end{cases} \tag{4.18}$$

where

$$\tilde{A} = \frac{1}{2}\left(1 + \mu_*^2\right)^{-\frac{1}{2}}\left(2A + 2B\mu_* + E\mu_*^2\right),$$

$$\begin{aligned} \tilde{E} &= \left(1 + \mu_*^2\right)^{-\frac{3}{2}}\left(E + 2(F - B)\mu_* + 2(A - E)\mu_*^2 + B\mu_*^3\right) \\ &= \left(1 + \mu_*^2\right)^{-\frac{1}{2}}\left((2F - B)\mu_* + E(1 - \mu_*^2)\right) \end{aligned}$$

and

$$\begin{aligned} \tilde{F} &= \frac{1}{2}\left(1 + \mu_*^2\right)^{-\frac{3}{2}}\left(2F - 3E\mu_* + 3B\mu_*^2 - 2A\mu_*^3\right) \\ &= \frac{1}{2}\left(1 + \mu_*^2\right)^{-\frac{1}{2}}\left(B + 2F - (2A + E)\mu_*\right). \end{aligned}$$

One checks immediately that $A = E = B = F = 0$ if and only if $\tilde{A} = \tilde{E} = \tilde{F} = 0$. Moreover, (u_s, v_s) is a solitary-wave solution of (4.11) if and only $(\tilde{u}_s, \tilde{v}_s)$ is a solitary-wave solution of (4.18). By the last five lemmas, (4.18) has one, two or three solitary-wave solutions of the form $(\phi, \mu\phi)$ or $(\nu\psi, \psi)$, hence, so does (4.11). The theorem is proved. □

5 Stability of Solitary-Wave Solutions

Some preliminary discussion about stability will be useful. These remarks are motivated by the earlier works [4–6,17,30].

On the Sobolev space $H^1 \times H^1 = H^1(\mathbb{R}) \times H^1(\mathbb{R})$, define a pseudo-metric d by

$$d\big((f_1, g_1), (f_2, g_2)\big) = \inf_{\tau \in \mathbb{R}} \Big\{ \| f_1(\cdot + \tau) - f_2(\cdot)\|_1 + \| f_2(\cdot + \tau) - f_2(\cdot)\|_1 \Big\}$$

where $\|\cdot\|_1$ is the standard H^1-norm. For any $(f_0, g_0) \in H^1 \times H^1$, $U_r(f_0, g_0)$ is the ball of radius r about (f_0, g_0) in $H^1 \times H^1$ equipped with the pseudo-metric d, *viz.*

$$U_r(f_0, g_0) = \big\{(f, g) \in H^1 \times H^1 : d\big((f, g), (f_0, g_0)\big) < r\big\}.$$

Definition 1 A traveling-wave solution (ϕ, ψ) of (4.11) is said to be stable in $H^1 \times H^1$ if for any $\epsilon > 0$, there is a $\delta > 0$ such that whenever the initial data (u_0, v_0) of (1.1) lies in $U_\delta(\phi, \psi)$, then the solution $\big(u(\cdot, t), v(\cdot, t)\big) \in U_\epsilon(\phi, \psi)$ for all $t \geq 0$. If it is not stable, then it is said to be unstable.

Remark : 1. Since the orbit $\mathcal{O} \subset H^1 \times H^1$ of a solitary-wave solution (ϕ, ψ) of the system (4.11) is

$$\mathcal{O} = \big\{(f, g) : \exists y \in \mathbb{R},\ f(x) = \phi(x + y),\ g(x) = \psi(x + y)\ \ \forall x \in \mathbb{R}\big\},$$

it follows that the definition of stability offered above is just the usual notion of orbital stability.

2. A decoupled KdV-KdV system

$$\begin{cases} u_t + u_{xxx} + 2Auu_x = 0, \\ v_t + v_{xxx} + 2Fvv_x = 0, \end{cases}$$

where $A, F \neq 0$, has, up to translations, exactly the three solitary-wave solutions $(\phi, 0)$, $(0, \psi)$ and (ϕ, ψ) of speed ω. Here, the shape functions are $\phi(z) = \frac{3\omega}{2A}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}z)$ and $\psi(z) = \frac{3\omega}{2F}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}z)$. The first two solitary-wave pairs are clearly stable according to Definition 5.1 because the zero-solution of the Korteweg–de Vries equation is stable. However, the third one is not. Consider perturbing the solitary-wave solution $(\phi, \psi) = (\phi_\omega, \psi_\omega)$ with the pair $(\phi_{\omega+\epsilon} - \phi_\omega, \psi_{\omega-\epsilon} - \psi_\omega)$.

For ϵ small, this comprises a small perturbation in $H^1 \times H^1$. After time $t > 0$, the solution (u, v) of the system with the perturbed initial data is exactly

$$\big(u(x,t), v(x,t)\big) = \big(\phi_{\omega+\epsilon}(x - (\omega+\epsilon)t), \psi_{\omega-\epsilon}(x - (\omega-\epsilon)t)\big).$$

Thus, it transpires that $d\big((u, v), (\phi_\omega, \psi_\omega)\big)$ grows with time like $C\epsilon(1+t)$ for some time-independent constant C, at least on a time interval of length $O(\frac{1}{\epsilon})$, and so does not remain small for all time.

5.1 Analysis of the Reduced System

We now embark upon preliminary analysis that will lead to criteria for stability of the proportional solitary-wave solutions of the reduced, coupled KdV-systems (4.11). Note that when the system is written in the form (4.11), the two invariants are

$$\Omega(u, v) = \int \Big(\frac{1}{2}u^2 + \frac{1}{2}v^2\Big)\,dx$$

and

$$\Theta(u, v) = \int \Big(\frac{1}{2}u_x^2 + \frac{1}{2}v_x^2 - R(u, v)\Big)\,dx$$

where

$$R(u, v) = \frac{1}{3}Au^3 + \frac{1}{2}Bu^2v + \frac{1}{2}Euv^2 + \frac{1}{3}Fv^3.$$

As seen in Sect. 4, the system (4.11) has at least one, explicit, proportional solitary-wave solution, say (ϕ, ψ). As in Remark 3.2, there are two numbers μ_1, μ_2 such that

$$\big(\phi(x), \psi(x)\big) = (\mu_1, \mu_2)\,3\omega\,\mathrm{sech}^2\left(\frac{\sqrt{\omega}}{2}x\right) \tag{5.1}$$

for $x \in \mathbb{R}$.

Introduce a Lyapunov functional

$$\Lambda(u, v) = \Theta(u, v) + \omega\Omega(u, v),$$

where (u, v) is the solution of (4.11) corresponding to the initial data (u_0, v_0) which lies in the small neighborhood of the solitary-wave solution (ϕ, ψ) of (4.11). The quantities

$$\Delta\Omega = \Omega(u, v) - \Omega(\phi, \psi) = \Omega(u_0, v_0) - \Omega(\phi, \psi)$$

and

$$\Delta\Theta = \Theta(u, v) - \Theta(\phi, \psi) = \Theta(u_0, v_0) - \Theta(\phi, \psi)$$

are independent of time t since both Ω and Θ are constants of the motion generated by the system (1.1). Define

$$h(x,t) = u(x,t) - \phi(x + a(t)),\ \ k(x,t) = v(x,t) - \psi(x + a(t))$$

where $a(t)$ is chosen so that

$$\|h(\cdot,t)\|^2 + \|k(\cdot,t)\|^2 = \min_{y\in\mathbb{R}} \int \Big(\big(u(x,t) - \phi(x+y)\big)^2 + \big(v(x,t) - \psi(x+y)\big)^2\Big) dx. \tag{5.2}$$

That such a finite value $a(t)$ can be inferred to exist, at least when h and k are relatively small compared to ϕ and ψ, follows from Lemma 1 in [6]. A calculation reveals the structure of the functional $\Lambda = \Theta + \omega\Omega$:

$$\begin{aligned}
\Delta\Lambda &= \Lambda(\phi + h, \psi + k) - \Lambda(\phi, \psi) = \Delta\Theta + \omega\Delta\Omega \\
&= \int \left\{ \frac{1}{2}(\phi_x + h_x)^2 + \frac{1}{2}(\psi_x + k_x)^2 - R((\phi + h), (\phi + h) \right\} dx \\
&\quad + \omega \int \left(\frac{1}{2}(\phi + h)^2 + \frac{1}{2}(\psi + k)^2 \right) dx - \Lambda(\phi, \psi) \\
&= \Lambda'_{(\phi,\psi)} \begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2}\Lambda''_{(\phi,\psi)}(h,k) - \int R(h,k)\, dx,
\end{aligned} \tag{5.3}$$

where

$$\begin{aligned}
\Lambda'_{(\phi,\psi)} \begin{pmatrix} h \\ k \end{pmatrix} &= \int \Big(-\big(\phi_{xx} + A\phi^2 + B\phi\psi + \frac{1}{2}E\psi^2\big) + \omega\phi \Big) h\, dx \\
&\quad + \int \Big(-\big(\psi_{xx} + \frac{1}{2}B\phi^2 + E\phi\psi + F\psi^2\big) + \omega\psi \Big) k\, dx.
\end{aligned} \tag{5.4}$$

Since (ϕ, ψ) is a solitary-wave solution with propagation speed ω, formula (5.4) together with the equations (3.1) defining the solitary wave imply that

$$\Lambda'_{(\phi,\psi)} \equiv 0.$$

That is to say, (ϕ, ψ) is a critical point of the functional Λ. The Hessian, or second Fréchet derivative of Λ, evaluated at (ϕ, ψ), is

$$\begin{aligned}\Lambda''_{(\phi,\psi)}(h,k) &= \int \Big\{\Big((h_x^2+k_x^2) - (2A\phi + B\psi)h^2 - (2B\phi + 2E\psi)hk \\ &\quad -(E\phi + 2F\psi)k^2\Big) + \omega(h^2+k^2)\Big\}\,dx \\ &= \left\langle \begin{pmatrix} h \\ k \end{pmatrix}, \mathcal{L}\begin{pmatrix} h \\ k \end{pmatrix} \right\rangle\end{aligned} \tag{5.5}$$

where $\langle \cdot, \cdot \rangle$ is the $L_2 \times L_2$ inner product,

$$\begin{aligned}\mathcal{L} &= \omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \partial_x^2 & 0 \\ 0 & \partial_x^2 \end{pmatrix} - \begin{pmatrix} 2A\phi + B\psi & B\phi + E\psi \\ B\phi + E\psi & E\phi + 2F\psi \end{pmatrix} \\ &= (\omega - \partial_x^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 3\,\omega\,\mathrm{sech}^2\left(\frac{\sqrt{\omega}}{2}x\right)\mathcal{M}(\mu_1,\mu_2)\end{aligned} \tag{5.6}$$

and the matrix $\mathcal{M}(\mu_1, \mu_2)$ is defined in (3.9).

This matrix will be denoted simply $\mathcal{M}$ when (μ_1, μ_2) and the coefficients A, B, E, F, and hence (ϕ, ψ), are understood from context.

It is straightforward to see that $\Delta\Lambda$ is bounded above in terms of the H^1–norms of h and k, for

$$\Delta\Lambda \le \gamma_1(\|h\|_1^2 + \|k\|_1^2) + \gamma_2(\|h\|_1^3 + \|k\|_1^3)$$

where γ_1 and γ_2 are positive numbers which are independent of h and k. If it can be demonstrated that $\Delta\Lambda$ has a lower bound of the form

$$\Delta\Lambda \ge \gamma_3(\|h\|_1^2 + \|k\|_1^2) - \gamma_4(\|h\|_1^3 + \|k\|_1^3), \tag{5.7}$$

where γ_3 and γ_4 are positive and independent of h and k, then following the arguments first laid out by Benjamin [4] and Bona [6], the orbital stability of the solitary-wave can be established. Of course, if the operator $\mathcal{L}$ that determines the Hessian is positive definite, (5.7) clearly holds since $R(h,k)$ is a homogeneous cubic polynomial whose integral $\int R(h,k)\,dx$ is composed of terms that can be bounded above by terms of order $O(\|h\|_1^3, \|k\|_1^3)$. If $\mathcal{L}$ is not positive definite, however, the quadratic term on the right-hand side of (5.7) is not always positive and the argument in favor of stability fails. Hence, to prove (5.7), a spectral analysis of the operator $\mathcal{L}$ is helpful.

Lemma 5.1 *The eigenvalues of the matrix* $\mathcal{M} = \mathcal{M}(\mu_1, \mu_2)$ *are* $\lambda_1 = 1$ *and* $\lambda_2 = \det \mathcal{M}$ *with corresponding eigenvectors*

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\mu_2 \\ \mu_1 \end{pmatrix},$$

respectively.

Proof Without loss of generality, assume $\mu_1 \neq 0$. Hence $(\phi, \psi) = (\phi, \mu\phi)$ where $\mu = \frac{\mu_2}{\mu_1}$ is a solution of (3.3) and $2A + 2B\mu + E\mu^2 = \frac{1}{\mu_1} \neq 0$. Denote the non-zero quantity $2A + 2B\mu + E\mu^2$ by κ. Then

$$\mathcal{M} = \frac{1}{\kappa}\begin{pmatrix} 2A + B\mu & B + E\mu \\ B + E\mu & E + 2F\mu \end{pmatrix}.$$

In terms of κ, the characteristic equation $\det(\lambda I - \mathcal{M}) = 0$ is equivalent to

$$0 = \det\begin{pmatrix} 2A + B\mu - \kappa\lambda & B + E\mu \\ B + E\mu & E + 2F\mu - \kappa\lambda \end{pmatrix}.$$

Multiply the second column by μ, add the result to the first column and notice that $\mu\kappa = \mu(2A + 2B\mu + E\mu^2) = B + 2E\mu + 2F\mu^2$. It follows immediately that

$$0 = \det\begin{pmatrix} \kappa - \kappa\lambda & B + E\mu \\ \mu\kappa - \mu\kappa\lambda & E + 2F\mu - \kappa\lambda \end{pmatrix} = \kappa(1-\lambda)\det\begin{pmatrix} 1 & B + E\mu \\ \mu & E + 2F\mu - \kappa\lambda \end{pmatrix}.$$

Thus, $\lambda = \lambda_1 = 1$ is a solution, and hence $\lambda_2 = \det\mathcal{M}$. Determining the eigenvectors is now straightforward. □

Define the unitary matrix U by

$$U = \frac{1}{\sqrt{\mu_1^2 + \mu_2^2}}\begin{pmatrix} \mu_1 & -\mu_2 \\ \mu_2 & \mu_1 \end{pmatrix}. \tag{5.8}$$

The matrix U diagonalizes $\mathcal{M}$, which is to say,

$$\mathcal{M} = U\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}U^t = U\begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix}U^t$$

where U^t is the transpose of U. In consequence,

$$U^t\mathcal{L}U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(\omega - \partial_{xx}) - 3\,\omega\,\mathrm{sech}^2\left(\frac{\sqrt{\omega}}{2}x\right)\begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{5.9}$$

and so it transpires that the spectrum of $\mathcal{L}$ is completely determined by the value of $\lambda_2 = \det\mathcal{M}$.

For real values of α, the complete spectrum of the self-adjoint operator $\mathcal{Q} = -\partial_{xx} - \alpha\,\mathrm{sech}^2(x)$ is known (see e.g. Landau and Lifschitz [24] or Morse and Feshbach [28]). The following results follow readily from the spectral properties of $\mathcal{Q}$.

Lemma 5.2 *The operator $\mathcal{L}$ defined by (5.6) is a closed, unbounded, self-adjoint operator on $L_2 \times L_2$. The spectrum of $\mathcal{L}$ consists of a finite number of discrete eigenvalues (with finite-dimensional eigenspaces) and the continuous spectrum $[\omega, \infty)$. The values -5 and 0 are always eigenvalues of $\mathcal{L}$ with corresponding eigenvectors*

$$\chi_-(x) = U\begin{pmatrix} \mathrm{sech}^3(\frac{\sqrt{\omega}}{2}x) \\ 0 \end{pmatrix} = \frac{1}{\sqrt{\mu_1^2 + \mu_2^2}}\begin{pmatrix} \mu_1\,\mathrm{sech}^3(\frac{\sqrt{\omega}}{2}x) \\ \mu_2\,\mathrm{sech}^3(\frac{\sqrt{\omega}}{2}x) \end{pmatrix} \tag{5.10}$$

and

$$\chi_0(x) = U\begin{pmatrix} \frac{d}{dx}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x) \\ 0 \end{pmatrix} = \frac{1}{\sqrt{\mu_1^2+\mu_2^2}}\begin{pmatrix} \mu_1 \frac{d}{dx}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x) \\ \mu_2 \frac{d}{dx}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x) \end{pmatrix}, \quad (5.11)$$

respectively. Furthermore,

(a) *if* $\lambda_2 = \det \mathcal{M} < \frac{1}{2}$, *then* -5 *is the unique negative eigenvalue of* $\mathcal{L}$ *and both the eigenvalues* -5 *and* 0 *are simple, whereas*

(b) *if* $\frac{1}{2} < \lambda_2 = \det \mathcal{M} < 1$, *then* $\mathcal{L}$ *has exactly two negative eigenvalues and these are both simple. In addition to the eigenvalue* -5, *the other negative eigenvalue is* $-\beta_0^2 + 4$, *where*

$$\beta_0 = -\frac{1}{2} + \frac{1}{2}\sqrt{1+48\lambda_2}\,. \quad (5.12)$$

The corresponding eigenvector is

$$\chi_{-\beta_0^2+4} = U\begin{pmatrix} 0 \\ \operatorname{sech}^{\beta_0}(\frac{\sqrt{\omega}}{2}x) \end{pmatrix} = \frac{1}{\sqrt{\mu_1^2+\mu_2^2}}\begin{pmatrix} -\mu_2 \operatorname{sech}^{\beta_0}(\frac{\sqrt{\omega}}{2}x) \\ \mu_1 \operatorname{sech}^{\beta_0}(\frac{\sqrt{\omega}}{2}x) \end{pmatrix}. \quad (5.13)$$

(c) *If* $\lambda_2 = \det \mathcal{M} > 1$, *then the value of the quantity* β_0 *defined in part (b) above is greater than 3 so that* $-\beta_0^2 + 4 < -5$. *In case* β_0 *is not an integer,* $\mathcal{L}$ *has at least three different negative eigenvalues. In addition to the negative eigenvalue* -5, *the other negative eigenvalues are*

$$-\beta_0^2+4,\ -(\beta_0-1)^2+4, \cdots,\ -(\beta_0-\lfloor\beta_0\rfloor+2)^2+4. \quad (5.14)$$

In case β_0 *is an integer, then the negative eigenvalues of* $\mathcal{L}$ *are*

$$-\beta_0^2+4,\ -(\beta_0-1)^2+4, \cdots,\ -5,$$

all of which are integers. Notice that the last one is -5. *However,* -5 *and* 0 *are no longer simple. Besides* χ_- *and* χ_0, *the eigenvalues* -5 *and* 0 *also possess two more linearly independent eigenvectors, namely*

$$\chi^-(x) = U\begin{pmatrix} 0 \\ \operatorname{sech}^3(\frac{\sqrt{\omega}}{2}x) \end{pmatrix} = \frac{1}{\sqrt{\mu_1^2+\mu_2^2}}\begin{pmatrix} -\mu_2 \operatorname{sech}^3(\frac{\sqrt{\omega}}{2}x) \\ \mu_1 \operatorname{sech}^3(\frac{\sqrt{\omega}}{2}x) \end{pmatrix} \quad (5.15)$$

and

$$\chi^0(x) = U\begin{pmatrix} 0 \\ \frac{d}{dx}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x) \end{pmatrix} \quad (5.16)$$

$$= \frac{1}{\sqrt{\mu_1^2+\mu_2^2}} \begin{pmatrix} -\mu_2 \frac{d}{dx}\,\mathrm{sech}^2(\frac{\sqrt{\omega}}{2}x) \\ \mu_1 \frac{d}{dx}\,\mathrm{sech}^2(\frac{\sqrt{\omega}}{2}x) \end{pmatrix}, \tag{5.17}$$

respectively. Whether or not β_0 is an integer, $\chi_{-\beta_0^2+4}$ given in (5.13) *is the eigenvector corresponding to the bottom $-\beta_0^2+4$ of the spectrum.*

(d) *If* $\det \mathcal{M} = \frac{1}{2}$, *then $\mathcal{L}$ has a unique negative eigenvalue -5 which is simple. However, the eigenvalue* 0 *has two linearly independent eigenvectors, namely χ_0 given in* (5.11) *and χ^0 defined in* (5.16).

(e) *If* $\det \mathcal{M} = 1$, *then $\mathcal{L}$ has a unique negative eigenvalue -5 with multiplicity 2. The eigenvectors $\chi_-(x)$ given in* (5.10) *and χ^- in* (5.15) *span the eigenspace.*

The eigenvalue 0 *is not simple either; it has eigenvectors χ_0 given in* (5.11) *as well as χ^0 written in* (5.16).

Thus, $\mathcal{L}$ always has at least one negative eigenvalue and so is never positive definite. This problem can be circumvented. Here is the result in view.

Theorem 5.1 *Let $(\phi, \psi) = (\phi_\omega, \psi_\omega) = (\mu_1, \mu_2)\, 3\omega\, \mathrm{sech}^2(\frac{\sqrt{\omega}}{2}(x - \omega t))$ be a solitary-wave solution of* (4.11). *If* $\det \mathcal{M}(\mu_1, \mu_2) < \frac{1}{2}$, *then (ϕ, ψ) is stable.*

This result will be deduced making use of the following general result.

Theorem 5.2 *Let (ϕ, ψ) be a solitary-wave solution of a system of the form* (1.1) *for which the matrix $\mathcal{N}$ is positive definite. Suppose that (ϕ, ψ) is a local minimizer of the variational problem*

$$\min_{u,v\in H^1} \{\Theta(u, v) : \Omega(u, v) = \Omega(\phi, \psi)\}. \tag{5.18}$$

Then, this solitary wave is orbitally stable.

Results of this sort go back to the original paper of Benjamin [4], who credits Boussinesq [19] with already having the idea in the 19th century. The concentrated compactness method of Lions [25] provides a technical tool that allows such theorems to be established, though in the absence of a uniqueness result as is the case here, one only infers stability of the class of minimizers and not necessarily stability of the individual solitary-wave solutions. Theory that gives conditions under which a particular solitary-wave solution is a minimizer of (5.18) is couched in terms of properties of the operator $\mathcal{L}$. For example, [17], Lemma 5.2 (and see also [30]) slightly generalized to account for systems rather than single equations, implies the following.

Lemma 5.3 *Suppose the system* (1.1) *has a positive definite matrix $\mathcal{N}$ and a proportional solitary-wave solution (ϕ, ψ). If the associated linear operator $\mathcal{L}$ has a unique negative eigenvalue which is simple, zero is a simple eigenvalue and the rest of the spectrum is positive and bounded away from zero, then $(\phi, \psi)$is a local minimizer of the variational problem* (5.18) *provided that*

$$\langle \mathcal{L}\mathcal{Y}, \mathcal{Y}\rangle > 0 \tag{5.19}$$

for any non-zero vector $\mathcal{Y}$ in the subspace

$$\mathcal{S} = \left\{ \mathcal{Z} \in H^1 \times H^1 : \left\langle \mathcal{Z}, \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle = 0, \left\langle \mathcal{Z}, \begin{pmatrix} \phi_x \\ \psi_x \end{pmatrix} \right\rangle = 0 \right\}, \tag{5.20}$$

where $\langle \cdot, \cdot \rangle$ is again the $L_2 \times L_2$ inner product.

We do not enter into the details of the proof of this lemma. It consists of simply following the steps carried out in Sect. 5 of [17]. Notice that the hypothesis $d''(\omega) > 0$, to be explained presently, that appears in Lemma 5.2 of [17] is only there to guarantee the conclusion (5.19) holds in the subspace $\mathcal{S}$.

Remark 5.1 On the other hand, if there is a non-zero $\mathcal{Y} \in \mathcal{S}$ for which $\langle \mathcal{L}\mathcal{Y}, \mathcal{Y} \rangle < 0$, then (ϕ, ψ) is not a minimizer of (5.18). This case will be discussed in our companion paper [7] concerned with instability and singularity formation.

In light of Theorem 5.2, it becomes of interest to know when the hypotheses of Lemma 5.3 are satisfied.

Remark 5.2 We continue to assume that the matrix $\mathcal{N}$ is positive definite. Consulting Lemma 5.2 reveals that $\mathcal{L}$, when computed for one of the proportional solitary waves exposed in Sect. 4, has the spectral properties that are assumed in Lemma 5.3 exactly when $\det \mathcal{M} = \det \mathcal{M}(\mu_1, \mu_2) < \frac{1}{2}$. This property, that $\det \mathcal{M} = \det \mathcal{M}(\mu_1, \mu_2) < \frac{1}{2}$, is relatively straightforward to check in concrete situations.

Suppose $\{(\phi_\omega, \psi_\omega)\}_{\omega>0}$ to be any one of the branches of proportional solitary-wave solutions from the previous discussion. For a given speed of propagation ω, define the function $d(\omega)$ by

$$d(\omega) = \Theta(\phi_\omega, \psi_\omega) + \omega\Omega(\phi_\omega, \psi_\omega). \tag{5.21}$$

Because the branch $\{(\phi_\omega, \psi_\omega)\}_{\omega>0}$ depends smoothly upon ω, the function $d(\omega)$ is also smooth.

A checkable criterion that implies condition (5.19)–(5.20) holds is provided by a convexity condition on the function $d(\omega)$. This is the content of the following straightforward generalization of Lemma 5.1 in [17].

Theorem 5.3 *Suppose the system (1.1) has a positive definite matrix $\mathcal{N}$ and has a branch of proportional solitary-wave solutions $(\phi, \psi) = \{(\phi_\omega, \psi_\omega)\}_{\omega>0}$. Suppose that the associated linear operators $\mathcal{L} = \mathcal{L}_\omega$ have a unique negative eigenvalue which is simple, zero is a simple eigenvalue and the rest of the spectrum is positive and bounded away from zero. If $d''(\omega) > 0$ for all $\omega > 0$, then condition (5.19)–(5.20) holds.*

Remark 5.3 A scaling argument reveals that all the hypotheses and conclusions in this Theorem hold for all $\omega > 0$ if and only if they hold for one value of $\omega > 0$.

Finally, interest is directed toward the function $d(\omega)$ corresponding to a branch of proportional solitary waves of the system (4.11). In detail,

$$\begin{aligned} d(\omega) = \Lambda(\phi,\psi) &= \int_{-\infty}^{\infty} \left\{ \frac{1}{2}\omega(\phi^2+\psi^2) + \frac{1}{2}(\phi_x^2+\psi_x^2) - R(\phi,\psi) \right\} dx \\ &= \int_{-\infty}^{\infty} \left\{ \frac{1}{2}(\omega\phi - \phi_{xx})\phi + \frac{1}{2}(\omega\psi - \psi_{xx})\psi - R(\phi,\psi) \right\} dx. \end{aligned}$$

Let the quantity $A + B\mu + \frac{1}{2}E\mu^2$ be denoted again by κ. It follows that $\frac{1}{2}B + E\mu + F\mu^2 = \mu\kappa$, $\phi(x) = \phi_\omega(x) = \frac{3\omega}{2\kappa}\,\mathrm{sech}^2(\frac{1}{2}\sqrt{\omega}x)$ and thus

$$\begin{aligned} d(\omega) &= \int_{-\infty}^{\infty} \left\{ \frac{1}{2}\Big(A\phi^2 + B\phi\psi + \frac{1}{2}E\psi^2\Big)\phi + \frac{1}{2}\Big(\frac{1}{2}B\phi^2 + E\phi\psi + F\psi^2\Big)\psi \right. \\ &\quad \left. - \Big(\frac{1}{3}A\phi^3 + \frac{1}{2}B\phi^2\psi + \frac{1}{2}E\phi\psi^2 + \frac{1}{3}F\psi^3\Big) \right\} dx \\ &= \int_{-\infty}^{\infty} \Big(\frac{1}{6}A\phi^3 + \frac{1}{4}B\phi^2\psi + \frac{1}{4}E\phi\psi^2 + \frac{1}{6}F\psi^3\Big)\, dx. \end{aligned}$$

Taking out the common factor ϕ^3 in the integrand yields

$$\begin{aligned} d(\omega) &= \int_{-\infty}^{\infty} \Big(\frac{1}{6}A + \frac{1}{4}B\mu + \frac{1}{4}E\mu^2 + \frac{1}{6}F\mu^3\Big)\phi^3\, dx. \\ &= \int_{-\infty}^{\infty} \frac{1}{6}(1+\mu^2)\kappa\phi^3(x)\, dx \;=\; \frac{9(1+\mu^2)\omega^{\frac{5}{2}}}{2(2A+2B\mu+E\mu^2)^2}\int_{-\infty}^{\infty} \mathrm{sech}^6(x)\, dx \\ &= \frac{24(1+\mu^2)\omega^{\frac{5}{2}}}{5(2A+2B\mu+F\mu^2)^2}. \end{aligned}$$

Hence, $d''(\omega) > 0$ for *all* of the proportional solitary waves found here.

Combining Theorem 5.2, Lemma 5.3, Remark 5.2 and Theorem 5.3 completes the proof of Theorem 5.1. □

Remark 5.4 It is worth pointing out how different the situation is for the systems (1.1) as compared to the single, generalized KdV equations. There, the stability revolves exactly around whether or not the relevant function $d(\omega)$ is convex or not (see [17]). Here, while all the proportional solitary waves have stability functions $d(\omega)$ that are convex, this does not determine their stability (see [7]).

The results for the reduced system can be applied immediately to the decoupled KdV–KdV system

$$\begin{cases} u_t + u_{xxx} + (Au^2)_x = 0, \\ v_t + v_{xxx} + (Fv^2)_x = 0. \end{cases} \tag{5.22}$$

Assuming that $A, F \neq 0$, (5.22) has three proportional solitary-wave solutions, namely $(\phi, 0), (\psi, 0)$ and (ϕ, ψ) where $\phi(x) = \frac{3\omega}{2A}\,\mathrm{sech}^2(\frac{\sqrt{\omega}}{2}x)$ and $\psi(x) =$

$\frac{3\omega}{2F}\operatorname{sech}^2(\frac{\sqrt{\omega}}{2}x)$. As remarked earlier, the first two are stable and the third unstable. In these cases, it is straightforward to calculate that

$$det\,\mathcal{M}\left(\frac{1}{2A},0\right) = det\,\mathcal{M}\left(0,\frac{1}{2F}\right) = 0 < \frac{1}{2}$$

whilst the third has $det\,\mathcal{M}(\frac{1}{2A},\frac{1}{2F}) = 1 > \frac{1}{2}$.

5.2 Stability for the Full System

In this section, the results derived for the reduced system (4.11), or equivalently (4.8), are reinterpreted for the variables in the original pair (1.1) of equations. As has been the case throughout, we continue to presume the coefficients $A, B, \ldots, F$ are such that the linear equations (2.1) have a solution (a, b, c) for which the matrix $\mathcal{N}$ in (4.1) is positive definite.

As a reminder, through the variable change (4.5), system (1.1) is transformed to (4.8). Since the transformation between the two systems is invertible, a solitary-wave solution (ϕ, ψ) of the full system (1.1) exists and is stable if and only if $(\tilde{\phi}, \tilde{\psi})$ given in (4.9) exists and is stable for the system (4.8). Moreover, (ϕ, ψ) is a proportional solitary wave if and only if $(\tilde{\phi}, \tilde{\psi})$ is.

Let $(\phi, \psi) = (\mu_1, \mu_2)\, 3\omega \operatorname{sech}^2\left(\frac{\sqrt{\omega}}{2}x\right)$ be a non-trivial proportional solitary-wave solution of (1.1) so that $(\tilde{\phi}, \tilde{\psi}) = (\tilde{\mu}_1, \tilde{\mu}_2)\, 3\omega \operatorname{sech}^2\left(\frac{\sqrt{\omega}}{2}x\right)$ defined by (4.9) is the corresponding solitary-wave solution of (4.8). The matrix $\mathcal{M}$ introduced in (3.9) associated with (ϕ, ψ) and the matrix $\tilde{\mathcal{M}}$ for $(\tilde{\phi}, \tilde{\psi})$ are related by

$$\tilde{\mathcal{M}}(\tilde{\mu}_1, \tilde{\mu}_2) = \mathcal{N}^{\frac{1}{2}}\mathcal{M}(\mu_1, \mu_2)\mathcal{N}^{-\frac{1}{2}}$$

as shown in Lemma 4.1.

As these two matrices are similar, they have the same determinant. In consequence, the following stability result emerges.

Theorem 5.4 *Consider the general system* (1.1)*, and assume that there are real numbers* a, b *and* c *satisfying the system of linear algebraic equations* (2.1) *such that the matrix* $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ *is positive definite. Then a proportional solitary-wave solution* $(\phi, \psi) = (\mu_1, \mu_2)\, 3\omega \operatorname{sech}^2\left(\frac{\sqrt{\omega}}{2}x\right)$ *is orbitally stable if*

$$\det \mathcal{M}(\mu_1, \mu_2) < \frac{1}{2},$$

where $\mathcal{M}$ *is defined in* (3.9)*.*

Remark 5.5 Notice that a necessary condition for a proportional solitary-wave solution $(\mu_1, \mu_2)\, 3\omega \operatorname{sech}^2\left(\frac{\sqrt{\omega}}{2}x\right)$ to be orbitally unstable is that the determinant of the associated matrix $\mathcal{M}$ be greater than or equal to $\frac{1}{2}$. This point will be important in our companion paper [7] concerned with instablity and singularity formation.

Remark 5.6 The Majda–Biello system can be put in the form

$$u_t + \alpha u_{xxx} - (uv)_x = 0, \tag{5.23}$$

$$v_t + v_{xxx} - uu_x = 0. \tag{5.24}$$

In the situation arising in the derivation of this model, namely the interaction of equatorial baroclinic Rossby waves and mid-latitude barotropic waves, the parameter α takes values in the range between about .8 and 1. When $\alpha = 1$, this is a particular example of the class of systems (1.1) studied here wherein $A = 0$, $B = -1$, $C = 0$, $D = -\frac{1}{2}$ and $E = F = 0$. As is known already, this system is globally well posed since, up to multiples, the unique values a, b, c for which the functional Ω in (2.2) is time-independent are $a = c = 1$ and $b = 0$. For any propagation speed ω, our theory then provides two proportional solitary waves, namely

$$\left(\frac{\sqrt{2}}{2}, -\frac{1}{2}\right)\phi \quad \text{and} \quad \left(-\frac{\sqrt{2}}{2}, -\frac{1}{2}\right)\phi$$

$$\text{where} \quad \phi(\mathrm{x}) = 3\,\omega\,\mathrm{sech}^2\left(\frac{\sqrt{\omega}}{2}(\mathrm{x} - \omega \mathrm{t})\right).$$

The matrices $\mathcal{M}$ associated with these two solitary waves are

$$\mathcal{M} = \mathcal{M}\left(\pm\frac{\sqrt{2}}{2}, -\frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & \mp\frac{\sqrt{2}}{2} \\ \mp\frac{\sqrt{2}}{2} & 0 \end{pmatrix},$$

respectively. Both of these matrices have determinant $-\frac{1}{2}$ and hence it is concluded that both these solitary-wave solutions of the Majda–Biello system are stable.

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