Proofs versus Counterexamples

Some useful tips for students of Math 320. March 8, 2009

My friends, you have been writing proofs in order to disprove things. That's bad. Generally, a single counterexample is more appropriate for a disproof. The error persists throughout much of your work, but I'll give you a particular instance. Consider this recent quiz question:

Is $f : M_{2\times 2} \to \mathbb{R}$**,** $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ **an isomorphism?**

Most of you realized the answer was "NO"; one popular justification was the fact that f does not preserve scaling. A good way to write this would be:

The map f does not preserve scaling. Here is a counterexample:

$$f\left(2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) = 2\cdot 2 - 0\cdot 0 = 4$$

$$\neq 2f\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 2(1\cdot 1 - 0\cdot 0) = 2.$$ Q.E.D.

That's a good answer because it shows that the general statement "f preserves scaling" is not always true. Most people didn't do it like that, though. Here's an inferior and much more typical solution from you kids:

The map f does not preserve scaling. Here is a proof. On the one hand we have:

$$f\left(r\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f\left(\begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}\right) = ra\cdot rd - rb\cdot rc = r^2(ad - bc).$$

However on the other hand we have:

$$\neq rf\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = r(a\cdot d - b\cdot c).$$

The two expressions are only equal when $r = 1$ or 0, or $a = b = c = d = 0$. In all other cases the expressions are not equal, so scaling is not preserved. Q.E.D.

Both solutions are entirely correct. However, the second one goes rather above and beyond what is needed, badly obfuscating the essential point.

ADVICE: When you want to show something is **not true in general**, you find a single, specific, concrete counterexample, and you avoid using general symbols like a, b, c, and d to represent arbitrary cases. When you want to prove something is **always true**, you start with a general form of the relevant objects, using symbols like a, b, c, and d, and then show that the statement holds no matter what their values are. **DO NOT CONFUSE THESE TWO METHODS.**

I find the following analogy helps: Imagine the statement is a balloon. To prove the balloon holds air, you must check that it is solid everywhere, that it doesn't have any holes anywhere on its surface. To burst the balloon, you merely need to poke the balloon with a pin at **one** point.