PROBLEM SET 2

This homework is due Wednesday September 7 in the beginning of class. No late homework will be accepted. You may collaborate on the homework. However, the final write-up must be yours and should reflect your own understanding of the problem. Please be sure to properly cite any help you get. The following problems refer to problems in the book.

Do problems Do problems 1.8(a,d), 1.10(b,c), 1.12(a,b,d) on page 54 and the following two problems.

Problem 1. In this problem you will learn about fields. While reading the definition, it might help to keep the example of real numbers with ordinary addition and multiplication in mind. A field $(F, +, \cdot)$ is a set F together with two binary operations called addition $+ : F \times F \to F$ and multiplication $\cdot : F \times F \to F$ that satisfy the following axioms. (This is a long list of axioms, 4 properties for addition $+$ and 4 properties for multiplication $\cdot$ and 1 property for how the two interact.)

(1) Addition is commutative, that is for any two elements x, y of F,

$$x + y = y + x.$$

(2) Addition is associative, that is for any three elementsx, y, z of F,

$$(x + y) + z = x + (y + z).$$

(3) There is an element zero $0 \in F$ such that $0 + x = x$ for every element $x \in F$.

(4) For every element $x \in F$, there is an additive inverse $y \in F$ such that $x + y = 0$.

(5) Multiplication is commutative, that is for any two elements x, y of F,

$$x \cdot y = y \cdot x.$$

(6) Multiplication is associative, for any three elements x, y, z of F,

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

(7) There is an element one $1 \in F$ different from 0 such that $1 \cdot x = x$ for every element $x \in F$.

(8) Every non-zero element $x \in F$ has a multiplicative inverse $y \in F$ such that $x \cdot y = 1$.

(9) For any three elements x, y, z in F, $x \cdot (y + z) = x \cdot y + x \cdot z$.

1. Check that real numbers with the usual addition and multiplication satisfy all these axioms.

2. Check that complex numbers with the usual addition and multiplication satisfy all these axioms. Recall that a complex number can be written as $z = x + iy$ where x and y are real numbers and $i^2 = -1$. Addition and multiplication for complex numbers are defined by

$$(x + iy) + (u + iv) = x + u + i(y + v)$$

and

$$(x + iy) \cdot (u + iv) = xu - yv + i(xv + yu).$$

In this course, we will do linear algebra over real or complex numbers. However, it is important to do linear algebra over other fields. Many of the results we prove will be valid over arbitrary fields. Here are a few examples of more exotic fields. You might find some of these examples challenging, but I am including them to make you aware of the potential applications of the course to fields ranging from differential equations to number theory.

3. Show that the set of rational functions in one variable under the usual addition and multiplication is a field. (Recall that rational functions are functions that can be expressed as a ratio of two polynomials.)

4. (Challenge) Let p be a prime number. Let $\mathbb{Z}/p\mathbb{Z}$ be the set of integers $\{0, 1, 2, \ldots, p-1\}$. Define $i \oplus j$ to be the remainder when $i + j$ is divided by p. Similarly, define $i \otimes j$ as the remainder when $i \cdot j$ is divided by p. Show that$\mathbb{Z}/p\mathbb{Z}$ with the operations $\oplus$ and $\otimes$ is a field.

Problem 2. Prove the following properties of fields.

(1) If F is a field, then F has at least two elements.
(2) There is a unique zero 0 in a field.
(3) There is a unique one 1 in a field.
(4) Every element $x \in F$ has a unique additive inverse (this is usually denoted by $-x$).
(5) Every non-zero element $x \in F$ has a unique multiplicative inverse (this is usually denoted by x^{-1}).
(6) A subset G of a field F which is itself a field under the same addition and multiplication is called a subfield. Show that the set of rational numbers is a subfield of the real numbers (under the usual addition and multiplication).