THE AMPLE CONE OF MODULI SPACES OF SHEAVES ON SURFACES AND THE BRILL-NOETHER PROBLEM

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Abstract. This survey discusses recent work on the Brill-Noether problem for rational surfaces and the problem of computing the ample cones of moduli spaces of sheaves. We discuss the relationship between the problem of computing the ample cone and the classification of Chern characters of stable sheaves. We also show that moduli spaces of rank 2 sheaves on very general hypersurfaces of degree d in $\mathbb{P}^3$ can have arbitrarily many irreducible components as d tends to infinity.

1. Introduction

In this paper, we discuss recent developments concerning the birational geometry of moduli spaces of Gieseker semistable sheaves on surfaces following [CH16b, CH17a] and [CH17b]. We will focus on the Brill-Noether problem and computing important birational invariants such as ample cones. The paper grew out of the first author's talk at the Abel Symposium in Svolvær, Norway in August 2017. While this paper is largely a survey, we will also present some new results and examples on surfaces of general type.

Let X be a smooth, complex projective surface and let H be an ample divisor. Given a Chern character $\mathbf{v}$, Gieseker [Gie77] and Maruyama [Mar78] construct a moduli space $M_{X,H}(\mathbf{v})$ that parameterizes S-equivalence classes of H-Gieseker semistable sheaves on X with Chern character $\mathbf{v}$. The moduli spaces $M_{X,H}(\mathbf{v})$ carry fundamental information on algebro-geometric invariants such as linear systems on X and they play a central role in Donaldson's theory of differentiable structures [Don90], in representation theory [Nak99] and mathematical physics [Wit95].

Rank one stable sheaves are of the form $L \otimes I_Z$, where L is a line bundle on X and I_Z is an ideal sheaf of points on X. Consequently, when the rank of $\mathbf{v}$ is one, the moduli space $M_{X,H}(\mathbf{v})$ fibers over $\mathrm{Pic}^{\mathrm{ch}_1(\mathbf{v})}(X)$ with fibers isomorphic to a Hilbert scheme of points on X. The Hilbert scheme $X^{[n]}$ of n points on X is a smooth, projective irreducible variety of dimension $2n$ [Fog68]. Hence, the basic geometric invariants of $M_{X,H}(\mathbf{v})$ such as dimension and irreducibility are well-understood. When the rank of $\mathbf{v}$ is higher, much less is known. The following questions are open in general.

(1) For which Chern characters $\mathbf{v}$ is the moduli space $M_{X,H}(\mathbf{v})$ non-empty?
(2) When is $M_{X,H}(\mathbf{v})$ irreducible and of the expected dimension?
(3) What are the singularities of $M_{X,H}(\mathbf{v})$? Is $M_{X,H}(\mathbf{v})$ reduced?

The currently known answers to these questions often have two flavors. There are results that hold on arbitrary surfaces under numerical restrictions on $\mathbf{v}$. For example, the Bogomolov inequality, which asserts that the discriminant Δ of a stable sheaf has to be nonnegative, imposes strong restrictions on the existence of stable sheaves. When $\Delta(\mathbf{v}) \gg 0$, then theorems of Donaldson [Don90], Li [LiJ93, LiJ94] and O'Grady [O'G96] show that the moduli spaces $M_{X,H}(\mathbf{v})$ behave well.

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They are nonempty, irreducible, of the expected dimension and generically smooth (see [HuL10]). Then there are results on specific surfaces. The question of when $M_{X,H}(\mathbf{v})$ is nonempty has been answered for surfaces such as K3 surfaces, Abelian surfaces and $\mathbb{P}^2$ (see [DLP85, HuL10, LeP97, Muk84, Yos01]). In these cases, the moduli spaces are irreducible and often have more structure. For example, when X is a $K3$ surface, $\mathbf{v}$ is a primitive character and H is sufficiently general, then $M_{X,H}(\mathbf{v})$ is a hyperkähler manifold. We will briefly recall a sampling of these results in §2.5.

When X is a surface of general type and Δ is positive but small, the moduli space $M_{X,H}(\mathbf{v})$ can exhibit pathological behavior. The moduli spaces can be reducible, non-reduced and can have components of different dimensions (see [Mes97, MS11, MS13a, MS13b]). The pathological behavior is already present in hypersurfaces in $\mathbb{P}^3$. In §3 we will show the following.

Theorem 1.1. *Given a positive integer k, there exists an integer d_k such that for all $d \geq d_k$, there exists a moduli space $M_{X_d,H}(\mathbf{v}_d)$ with at least k components, where X_d is a very general surface of degree d in $\mathbb{P}^3$, H is the hyperplane class and $\mathbf{v}_d$ is a Chern character of rank 2.*

When the moduli space $M_{X,H}(\mathbf{v})$ is irreducible and normal, one can ask for finer topological and birational invariants of $M_{X,H}(\mathbf{v})$.

(1) Compute the ample and effective cones of divisors of $M_{X,H}(\mathbf{v})$.
(2) Run the minimal model program for $M_{X,H}(\mathbf{v})$ and use wall-crossing to compute topological invariants of $M_{X,H}(\mathbf{v})$.
(3) Compute the cohomology of tautological sheaves on $M_{X,H}(\mathbf{v})$.

In recent years, Bridgeland stability has allowed many researchers to compute ample and effective cones of $M_{X,H}(\mathbf{v})$ and run the minimal model program on $M_{X,H}(\mathbf{v})$ [Abe17, ABCH13, BM14a, BM14b, BC13, BHL+15, CH16a, CH17a, CHW17, LZ16, MYY14, MYY15, Nue16a, Rya16, YY14, Yos12]. Again the results have two flavors. There are detailed answers for special surfaces such as K3 surfaces and $\mathbb{P}^2$ for all Chern characters. These results crucially depend on the classification of Chern characters of stable sheaves on these surfaces. There are also results on arbitrary surfaces for Chern characters with $\Delta \gg 0$. We will review some of these results in §5 and §6.

Assume that $M_{X,H}(\mathbf{v})$ is irreducible and normal. We will primarily focus on two main techniques for constructing nef and effective divisors on $M_{X,H}(\mathbf{v})$. Brill-Noether divisors provide a large class of natural divisors on $M_{X,H}(\mathbf{v})$. Let $\mathcal{W}$ be a sheaf with Chern character $\mathbf{w}$ such that $\chi(\mathbf{v}, \mathbf{w}) = 0$. Consider the locus

$$D_{\mathcal{W}} := \{\mathcal{V} \in M_{X,H}(\mathbf{v}) | h^1(X, \mathcal{W} \otimes \mathcal{V}) \neq 0\}.$$

When $D_{\mathcal{W}}$ is not the entire moduli space, it is an effective divisor. The Brill-Noether problem asks to determine the invariants $\mathbf{w}$ for which there exists a sheaf $\mathcal{W}$ with Chern character $\mathbf{w}$ such that $D_{\mathcal{W}}$ is an effective divisor. In particular, when $\chi(\mathbf{v}) = 0$, we can ask whether the cohomology of the general sheaf $\mathcal{V} \in M_{X,H}(\mathbf{v})$ vanishes. In §4, following [CH16b] we will recall the answer to this question for certain rational surfaces such as $\mathbb{P}^2$, Hirzebruch surfaces and del Pezzo surfaces. We will also discuss an application of these cohomology computations to classifying moduli spaces $M_{X,H}(\mathbf{v})$ whose general member is globally generated.

The other tool for constructing important divisor classes is the Positivity Lemma of Bayer and Macrì [BM14a]. Bayer and Macrì construct a nef divisor class on moduli spaces of Bridgeland stable objects under certain assumptions. Viewing $M_{X,H}(\mathbf{v})$ as a moduli space of Bridgeland stable objects for suitable stability conditions, they obtain nef divisors on $M_{X,H}(\mathbf{v})$. Assume X has Picard rank 1 and $\Delta \gg 0$. Following [CH17a], we will explain how to compute the nef cone of $M_{X,H}(\mathbf{v})$ using this machinery and show that the problem of computing the ample cones of $M_{X,H}(\mathbf{v})$ and the problem of characterizing Chern characters of stable bundles on X are intimately related.

The organization of the paper. In §2, we will recall definitions and results on Gieseker stability and review basic constructions such as elementary modifications and the Serre construction. In

§3, we will discuss unexpected behavior of moduli spaces of sheaves on general type surfaces when the discriminant is small. In particular, we will prove Theorem 1.1. In §4, we will review recent developments on the Brill-Noether Problem for rational surfaces following [CH16b] and [CH17b]. In §5, we will review basic facts on Bridgeland stability. Finally, in §6, we will discuss a method for computing ample cones of moduli spaces of sheaves following [CH17a].

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2. Preliminaries

In this section, we collect basic definitions and facts on Gieseker and twisted Gieseker semistability, prioritary sheaves, and classification of stable sheaves on surfaces such as $\mathbb{P}^2$ and K3 surfaces.

2.1. **Gieseker and μ-stability.** We refer the reader to [CH15, HuL10, Hui17] and [LeP97] for more detailed information on Gieseker (semi)stability and moduli spaces of stable sheaves. Let X be a smooth, complex projective surface and let H be an ample divisor on X. Let $\mathbf{v}$ denote a Chern character on X and define the *H-slope* $\mu_H(\mathbf{v})$, the *total slope* $\nu(\mathbf{v})$ and *discriminant* $\Delta(\mathbf{v})$ by the formulae

$$\mu_H(\mathbf{v}) = \frac{c_1(\mathbf{v})\cdot H}{r(\mathbf{v})\cdot H^2}, \quad \nu(\mathbf{v}) = \frac{c_1(\mathbf{v})}{r(\mathbf{v})}, \quad \Delta(\mathbf{v}) = \frac{1}{2}\nu(\mathbf{v})^2 - \frac{\mathrm{ch}_2(\mathbf{v})}{r(\mathbf{v})},$$

respectively. The H-slope, total slope and discriminant of a sheaf $\mathcal{V}$ of positive rank is defined to be the H-slope, total slope and discriminant of its Chern character. The Chern character $(r, \mathrm{ch}_1, \mathrm{ch}_2)$ of a positive rank sheaf can be recovered from (r, ν, Δ). The advantage is that the slope and the discriminant are additive on tensor products

$$\nu(\mathcal{V}\otimes\mathcal{W}) = \nu(\mathcal{V}) + \nu(\mathcal{W})$$
$$\Delta(\mathcal{V}\otimes\mathcal{W}) = \Delta(\mathcal{V}) + \Delta(\mathcal{W}).$$

If L is a line bundle on X, then $\Delta(L) = 0$. Consequently, tensoring a sheaf with a line bundle preserves the discriminant. Set

$$P(\nu) = \chi(\mathcal{O}_X) + \frac{1}{2}\nu\cdot(\nu - K_X).$$

The Riemann-Roch formula in terms of these invariants reads

$$\chi(\mathcal{V}) = r(\mathcal{V})(P(\nu(\mathcal{V})) - \Delta(\mathcal{V})).$$

Definition 2.1. A torsion-free coherent sheaf $\mathcal{V}$ is *μ_H-(semi)stable* if for every nonzero subsheaf $\mathcal{W}$ of smaller rank, we have

$$\mu_H(\mathcal{W}) \underset{(-)}{<} \mu_H(\mathcal{V}).$$

The Hilbert and reduced Hilbert polynomials $P_{H,\mathcal{V}}$ and $p_{H,\mathcal{V}}$ of a pure d-dimensional, coherent sheaf $\mathcal{V}$ with respect to H are defined by

$$P_{H,\mathcal{V}}(m) = \chi(\mathcal{V}(mH)) = a_d\frac{m^d}{d!} + \text{l.o.t}, \qquad p_{H,\mathcal{V}} = \frac{P_{H,\mathcal{V}}}{a_d}.$$

The sheaf $\mathcal{V}$ is *H-Gieseker (semi)stable* if for every proper subsheaf $\mathcal{W}$,

$$p_{H,\mathcal{W}}(m) \underset{(-)}{<} p_{H,\mathcal{V}}(m)$$

for $m \gg 0$.

Expressing the Hilbert polynomial in terms of μ_H and Δ, one obtains the following implications

$$\mu_H\text{-stability} \implies H\text{-Gieseker stability} \implies H\text{-Gieseker semistability} \implies \mu_H\text{-semistability}.$$

The reverse implications are false in general. However, when $c_1 \cdot H$ and rH^2 are relatively prime, then μ_H-stability and μ_H-semistability coincide and all 4 concepts agree. When the ample class H is fixed or understood from context, we will drop it from our notation. We will often refer to Gieseker (semi)stability simply as (semi)stability.

Two sheaves $\mathcal{V}$ and $\mathcal{W}$ are *S-equivalent* with respect to a notion of stability if they have the same Jordan-Hölder factors with respect to that notion of stability. Gieseker [Gie77] and Maruyama [Mar78] prove that there exists a (possibly empty) projective scheme parameterizing S-equivalence classes of H-Gieseker semistable sheaves (see [HuL10, Theorem 4.3.4]).

The Bogomolov inequality. The *Bogomolov inequality* asserts that a μ_H-semistable sheaf $\mathcal{V}$ satisfies $\Delta(\mathcal{V}) \geq 0$ and imposes a strong restriction on the existence of semistable sheaves. Since a line bundle L has $\Delta(L) = 0$, the Bogomolov inequality is sharp. However, the inequalities may be improved for (nonintegral) slopes s depending on the surface X. Given a rank r and a total slope ν, let $\Delta^H_{\min,\nu,r}$ denote the minimal discriminant of a μ_H-semistable bundle with total slope ν and rank at most r. By definition any μ_H-semistable sheaf with total slope ν and rank at most r satisfies the inequality $\Delta \geq \Delta^H_{\min,\nu,r}$. We will refer to such inequalities as *sharp Bogomolov inequalities*.

Remark 2.2. Determining the sharp Bogomolov inequalities on X is equivalent to classifying Chern characters of μ_H-semistable sheaves on X. Once there exists a μ_H-semistable sheaf $\mathcal{V}$ of rank r, total slope ν, and discriminant $\Delta^H_{\min,\nu,r}$, by performing elementary modifications (explained in detail below) we obtain μ_H-semistable sheaves for all integral Chern characters of rank r, total slope ν and $\Delta > \Delta^H_{\min,\nu,r}$. Similarly, if there exists a μ_H-stable sheaf of rank r, total slope ν and discriminant Δ_0, then there exists a μ_H-stable sheaf for every integral Chern character of rank r, total slope ν and discriminant $\Delta \geq \Delta_0$.

The existence of Gieseker semistable sheaves is more subtle. For example, on $\mathbb{P}^2$ with $H = \mathcal{O}_{\mathbb{P}^2}(1)$, $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$ is a Gieseker semistable sheaf with $(r, \mu_H, \Delta) = (2, 0, 0)$. However, any Gieseker semistable sheaf with $r = 2$, $\mu_H = 0$ and $\Delta > 0$, in fact has $\Delta \geq 1$. We will shortly see that there does not exist a Gieseker semistable sheaf with $\Delta = \frac{1}{2}$. Let I_p denote the ideal sheaf of a point $p \in \mathbb{P}^2$. The sheaf $\mathcal{O}_{\mathbb{P}^2} \oplus I_p$ is a μ_H-semistable sheaf with $\Delta = \frac{1}{2}$.

Twisted Gieseker semistability. We will need a further variant of Gieseker semistability. Let D be an arbitrary $\mathbb{Q}$-divisor on X. Formally define $\operatorname{ch}^D = e^{-D}\operatorname{ch}$, which explicitly gives

$$\operatorname{ch}_0^D = \operatorname{ch}_0, \quad \operatorname{ch}_1^D = \operatorname{ch}_1 - D\operatorname{ch}_0, \quad \operatorname{ch}_2^D = \operatorname{ch}_2 - D \cdot \operatorname{ch}_1 + \frac{D^2}{2}\operatorname{ch}_0.$$

For a sheaf $\mathcal{V}$, define the (H, D)-*slope* $\mu_{H,D}$ and the (H, D)-*discriminant* $\Delta_{H,D}$ by

$$\mu_{H,D} = \frac{H \cdot \operatorname{ch}_1^D}{H^2 \operatorname{ch}_0^D}, \quad \Delta_{H,D} = \frac{1}{2}\mu_{H,D}^2 - \frac{\operatorname{ch}_2^D}{H^2 \operatorname{ch}_0^D}.$$

A torsion-free coherent sheaf $\mathcal{V}$ is $\mu_{H,D}$-(semi)stable if for every nonzero, proper subsheaf $\mathcal{W} \subset \mathcal{V}$ of smaller rank

$$\mu_{H,D}(\mathcal{W}) \leq \mu_{H,D}(\mathcal{V}).$$

Since μ_H and $\mu_{H,D}$ only differ by the constant $\frac{D \cdot H}{H^2}$, μ_H-(semi)stability and $\mu_{H,D}$-(semi)stability coincide.

We need to introduce one final change of coordinates that take into account the Riemann-Roch formula. Define $\overline{\operatorname{ch}}^D = \operatorname{ch}^{D + \frac{1}{2}K_X}$ and define the modified invariants

$$\overline{\mu}_{H,D} = \mu_{H,D+\frac{1}{2}K_X}, \quad \overline{\Delta}_{H,D} = \Delta_{H,D+\frac{1}{2}K_X}.$$

The *reduced D-twisted Hilbert polynomial* of a torsion free sheaf $\mathcal{V}$ is defined by the formal Euler characteristic

$$p_{H,D}^{\mathcal{V}}(m) = \frac{\chi(\mathcal{V}(mH - D))}{\mathrm{rk}(\mathcal{V})}.$$

The sheaf $\mathcal{V}$ is (H,D)-*twisted Gieseker (semi)stable* if for every nonzero proper subsheaf $\mathcal{W} \subset \mathcal{V}$ of smaller rank, we have $p_{H,D}^{\mathcal{W}}(m) \le p_{H,D}^{\mathcal{V}}(m)$ for $m \gg 0$. When $D = 0$, this notion coincides with usual H-Gieseker (semi)stability. The notion of (H,D)-twisted Gieseker semistability can be reformulated in terms of the invariants μ_H and $\overline{\Delta}_{H,D}$. The sheaf $\mathcal{V}$ is (H,D)-twisted Gieseker (semi)stable if and only if

(1) $\mathcal{V}$ is μ_H-semistable; and
(2) if $\mathcal{W} \subset \mathcal{V}$ is a proper nonzero subsheaf of smaller rank, then

$$\overline{\Delta}_{H,D}(\mathcal{W}) \ge \overline{\Delta}_{H,D}(\mathcal{V}).$$

Matsuki and Wentworth [MW97] prove that there exists a projective moduli space $M_{X,H,D}(\mathbf{v})$ parameterizing S-equivalence classes of (H,D)-twisted Gieseker semistable sheaves with Chern character $\mathbf{v}$. These moduli spaces were initially constructed to study the birational geometry of $M_{X,H}(\mathbf{v})$. Under certain assumptions, one obtains different birational models of $M_{X,H}(\mathbf{v})$ by varying the divisor D.

2.2. **Prioritary sheaves.** It is often difficult to construct semistable bundles or check that a given bundle is semistable. When K_X is negative, there is a weaker notion which is easier to work with.

Definition 2.3. Let D be an effective divisor on X. A torsion-free coherent sheaf $\mathcal{V}$ is D-*prioritary* on X if $\mathrm{Ext}^2(\mathcal{V}, \mathcal{V}(-D)) = 0$. Let $\mathcal{P}_{X,D}(\mathbf{v})$ denote the stack of D-prioritary sheaves on X with Chern character $\mathbf{v}$.

If $H \cdot (K_X + D) < 0$, then the stack $\mathcal{M}_{X,H}(\mathbf{v})$ of H-Gieseker semistable sheaves is a (possibly empty) open substack of $\mathcal{P}_{X,D}(\mathbf{v})$. If $\mathcal{V}$ is μ_H-semistable, then by Serre duality

$$\mathrm{Ext}^2(\mathcal{V}, \mathcal{V}(-D)) = \mathrm{Hom}(\mathcal{V}, \mathcal{V}(K_X + D))^* = 0,$$

where the last equality follows because $\mu_H(\mathcal{V}) < \mu_H(\mathcal{V}(K_X + D))$ by assumption. Hence, every μ_H-semistable sheaf is D-prioritary.

This concept is especially useful when X is $\mathbb{P}^2$ and D is the hyperplane class L or X is a birationally ruled surface and D is the fiber class F. We will use the following fundamental theorem of Walter numerous times.

Theorem 2.4 (Walter [Wal98])**.** *Let X be a birationally ruled surface with fiber class F and let $\mathbf{v}$ be a Chern character with positive rank. Then the stack $\mathcal{P}_{X,F}(\mathbf{v})$ is irreducible whenever it is nonempty. Moreover, if $\mathrm{rk}(\mathbf{v}) \ge 2$, then the general element of $\mathcal{P}_{X,F}(\mathbf{v})$ is a vector bundle. In particular, if H is a polarization such that $H \cdot (K_X + F) < 0$ and $M_{X,H}(\mathbf{v})$ is nonempty, then $M_{X,H}(\mathbf{v})$ is irreducible.*

Remark 2.5. Walter's theorem generalizes an earlier theorem of Hirschowitz and Laszlo [HiL93] which asserts that if $\mathbf{v}$ is a positive rank Chern character, then $\mathcal{P}_{\mathbb{P}^2,L}(\mathbf{v})$ is irreducible whenever it is nonempty.

Remark 2.6. When X is a Hirzebruch surface, $K_X + F$ is anti-effective. Hence, the condition $H \cdot (K_X + F) < 0$ holds for every polarization H. On an arbitrary birationally ruled surface, F is nef and $K_X \cdot F < 0$ by adjunction. Since ampleness is an open condition and nef divisors are in the closure of the ample cone, there exist polarizations H sufficiently close to F such that $H \cdot (K_X + F) < 0$. However, this inequality in general imposes conditions on the polarization H.

Checking a sheaf is prioritary is much easier than checking it is stable. Prioritary sheaves are also easier to construct.

Example 2.7. Let L be the hyperplane class on $\mathbb{P}^2$ and let a be an integer. Then vector bundles of the form $\mathcal{O}_{\mathbb{P}^2}(a)^m \oplus \mathcal{O}_{\mathbb{P}^2}(a+1)^{r-m}$ are L-prioritary even though they are not μ_L-semistable.

Every vector bundle of rank r on a rational curve is a direct sum of line bundles $\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$. The vector bundle is called *balanced* if $|a_i - a_j| \leq 1$ for every $1 \leq i \leq j \leq r$.

Let D be a smooth curve on X. The condition of being D-prioritary is useful for understanding the restriction of bundles from X to D and especially useful when D is a rational curve. Let $\mathcal{F}_s/S$ be a family of D-prioritary sheaves on X which are locally free on D. Then the condition of being D-prioritary implies that the natural map

$$\mathrm{Ext}^1_X(\mathcal{F}_s, \mathcal{F}_s) \to \mathrm{Ext}^1_D(\mathcal{F}_s|_D, \mathcal{F}_s|_D)$$

is surjective. Consequently, we obtain the following.

Proposition 2.8 ([CH17b], Proposition 2.6)**.** *Let D be a smooth curve on X and let $\mathcal{F}_s/S$ be a complete family of D-prioritary sheaves on X which are locally free on D. Then the restricted family $\mathcal{F}_s|_D/S$ is also a complete family. In particular, if D is a rational curve, then $\mathcal{F}_s|_D$ is balanced for $s \in U$, where U is a nonempty dense open subset of S.*

2.3. **Elementary modifications.** An *elementary modification* of a torsion-free sheaf $\mathcal{V}$ on X is any sheaf given by an exact sequence

$$0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{O}_p \to 0,$$

where $p \in X$ is a point. Using the defining exact sequence, the following are immediate:

$$\mathrm{rk}(\mathcal{V}') = \mathrm{rk}(\mathcal{V}), \quad c_1(\mathcal{V}') = c_1(\mathcal{V}), \quad \mathrm{ch}_2(\mathcal{V}') = \mathrm{ch}_2(\mathcal{V}) - 1.$$

In particular,

$$\chi(\mathcal{V}') = \chi(\mathcal{V}) - 1, \quad \Delta(\mathcal{V}') = \Delta(\mathcal{V}) + \frac{1}{r}.$$

Assume $\phi : \mathcal{F} \to \mathcal{V}'$ is an injective sheaf homomorphism. Composing ϕ with the inclusion of $\mathcal{V}'$ into $\mathcal{V}$, we can view $\mathcal{F}$ as a subsheaf of $\mathcal{V}$. Consequently, an elementary modification of a μ_H-(semi)stable sheaf is again μ_H-(semi)stable. As discussed in Remark 2.2, elementary modifications of Gieseker (semi)stable sheaves do not need to be Gieseker (semi)stable.

For future reference, we observe the following easy lemma.

Lemma 2.9 ([CH17b], Lemma 2.7)**.** *Let $\mathcal{V}'$ be a general elementary modification of $\mathcal{V}$ at a general point $p \in X$. Then:*

(1) If $\mathcal{V}$ is D-prioritary, then $\mathcal{V}'$ is D-prioritary.
(2) $H^2(X, \mathcal{V}) = H^2(X, \mathcal{V}')$.
(3) If $h^0(X, \mathcal{V}) > 0$, then $h^0(X, \mathcal{V}') = h^0(X, \mathcal{V}) - 1$ and $h^1(X, \mathcal{V}') = h^1(X, \mathcal{V})$. If $h^0(X, \mathcal{V}) = 0$, then $h^1(X, \mathcal{V}') = h^1(X, \mathcal{V}) + 1$. In particular, if at most one of h^0 or h^1 is nonzero for $\mathcal{V}$, then at most one of h^0 or h^1 is nonzero for $\mathcal{V}'$.

2.4. **The Serre construction.** The Serre construction provides a method for constructing locally free sheaves on any variety, but it takes a particularly simple form on surfaces. Let Z be collection of n distinct points on X. We say that Z satisfies the *Cayley-Bacharach property* with respect to a line bundle L if any section of L vanishing on any subscheme $Z' \subset Z$ of length $n-1$ vanishes on Z.

Theorem 2.10 ([HuL10], Theorem 5.1.1)**.** *There exists a locally free extension $\mathcal{V}$ of the form*

$$0 \to L_1 \to \mathcal{V} \to L_2 \otimes I_Z \to 0$$

if and only if Z satisfies the Cayley-Bacharach property with respect to the line bundle $L_1^{-1} \otimes L_2 \otimes K_X$.

Below we will use the Serre construction to construct vector bundles on hypersurfaces in $\mathbb{P}^3$.

2.5. **Non-emptiness and irreducibility of the moduli spaces of sheaves.** In this subsection, we collect some facts concerning non-emptiness and irreducibility of the moduli spaces $M_{X,H}(\mathbf{v})$.

The Hilbert scheme. When $\mathrm{rk}(\mathbf{v}) = 1$, the basic geometric invariants of the moduli spaces $M_{X,H}(\mathbf{v})$ are well-understood. A sheaf of rank 1 is isomorphic to $L \otimes I_Z$, where L is a line bundle on X and I_Z is an ideal sheaf of points on X. The Hilbert scheme $X^{[n]}$ parameterizes ideal sheaves of length n. By Fogarty's theorem, $X^{[n]}$ is smooth and irreducible.

Theorem 2.11 (Fogarty [Fog68])**.** *Let X be a smooth, irreducible projective surface. Then the Hilbert scheme $X^{[n]}$ of n points on X is a smooth, irreducible projective variety of dimension $2n$.*

Furthermore, Fogarty computes the Picard group of $X^{[n]}$. When the irregularity $h^{1,0}(X) = q(X)$ is not 0, self-correspondences of X can complicate the answer. For simplicity, we will assume that $q(X) = 0$. Let $X^{(n)}$ denote the nth symmetric product of X. The Hilbert-Chow morphism $h : X^{[n]} \to X^{(n)}$ associates to a scheme of length n its support counted with multiplicity. The exceptional locus of h is an irreducible divisor B parameterizing the locus of nonreduced schemes in $X^{[n]}$. A line bundle L on X induces a symmetric line bundle $L \boxtimes \cdots \boxtimes L$ on X^n, which descends to $X^{(n)}$. Hence, it induces a line bundle $L^{[n]}$ on $X^{[n]}$ by pullback. We can thus embed $\mathrm{Pic}(X)$ in $\mathrm{Pic}(X^{[n]})$.

Theorem 2.12 (Fogarty [Fog73])**.** *Let X be a smooth, complex projective surface with $q(X) = 0$. Then $\mathrm{Pic}(X^{[n]}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}\frac{B}{2}$.*

Higher rank moduli spaces. When the rank of $\mathbf{v}$ is at least 2, much less is known about the geometry of $M_{X,H}(\mathbf{v})$. The moduli spaces $M_{X,H}(\mathbf{v})$ are well behaved as $\Delta(\mathbf{v})$ gets large. The results of Donaldson, Li, culminating in O'Grady's provide good examples of this general principle.

Theorem 2.13 (O'Grady [O'G96])**.** *There is an explicit constant $A = A(r, H, X)$ depending on the rank r, the ample H and the surface X, such that for $\Delta(\mathbf{v}) \geq A$, the moduli space $M_{X,H}(\mathbf{v})$ is normal, irreducible and of the expected dimension. Furthermore, if $r \geq 2$, the general point of $M_{X,H}(\mathbf{v})$ parameterizes a μ_H-stable vector bundle.*

In the next section, we will see that $M_{X,H}(\mathbf{v})$ can be reducible with arbitrarily many components of different dimensions when Δ is small. On the other hand, for special surfaces $M_{X,H}(\mathbf{v})$ may have very nice structure. The K-trivial surfaces often exhibit the simplest behavior.

Moduli spaces of K-trivial surfaces. The moduli spaces of K3 surfaces have been studied thoroughly by many authors including Mukai, Markman, Huybrechts and Yoshioka. We refer the reader to [HuL10] for a list of references. Define the Mukai pairing on $K_{num}(X)$ by

$$\langle \mathbf{v}, \mathbf{w} \rangle = -\chi(\mathbf{v}, \mathbf{w}).$$

Since K_X is trivial, Serre duality implies that the Mukai pairing is symmetric. By deformation theory, the moduli space $M_{X,H}(\mathbf{v})$ has expected dimension $\langle \mathbf{v}, \mathbf{v} \rangle + 2$. A class $\mathbf{v}$ is *primitive* if it is not the multiple of another class. The class $\mathbf{v}$ is called *positive* if $\langle \mathbf{v}, \mathbf{v} \rangle \geq -2$. If the polarization H is generic and $\mathbf{v}$ is primitive, then there are no strictly semistable sheaves. The main existence theorem on K3 surfaces is the following (see [HuL10, Chapter 6] for references).

Theorem 2.14 (Mukai, Yoshioka [Muk84], [Yos99])**.** *Let (X, H) be a polarized smooth K3 surface. Let $\mathbf{v} \in K_{num}(X)$ be the class of a sheaf and write $\mathbf{v} = m\mathbf{v}_0$, where $\mathbf{v}_0$ is primitive and m is a positive integer. If $\mathbf{v}_0$ is positive, then $M_{X,H}(\mathbf{v})$ is nonempty. Furthermore, if $m = 1$ and H is generic, then $M_{X,H}(\mathbf{v})$ is a smooth, irreducible, holomorphic symplectic variety. Conversely, if $M_{X,H}(\mathbf{v})$ is nonempty and the polarization is sufficiently generic, then $\mathbf{v}_0$ is positive.*

In fact, the topology of the moduli spaces $M_{X,H}(\mathbf{v})$ is well-understood thanks to results of Huybrechts and Yoshioka.

Theorem 2.15 (Huybrechts, Yoshioka, [Huy99, Yos99]). *A smooth moduli space of sheaves on a K3 surface is deformation equivalent to a Hilbert scheme of points on a K3 surface.*

There is a similarly detailed study of the moduli spaces of sheaves on abelian surfaces due to Yoshioka.

Theorem 2.16 (Yoshioka, [Yos01]). *Let (X, H) be a polarized abelian surface. Let $\mathbf{v}$ be a primitive, positive Mukai vector with $c_1(\mathbf{v}) \in \mathrm{NS}(X)$ and $v^2 \geq 2$. If H is a polarization such that there are no strictly semistable sheaves, then $M_{X,H}(\mathbf{v})$ is deformation equivalent to $\hat{X} \times X^{[\frac{v^2}{2}]}$.*

In particular, the dimension and irreducibility of the moduli spaces are known for abelian surfaces as well. The story for Enriques surfaces is more recent and has been worked out in detail by Nuer and Yoshioka. The story is slightly more complicated since K_X in that case is not trivial but torsion of order 2. Recall that an Enriques surface is called *unnodal* if it does not contain any smooth rational curves.

Theorem 2.17 (Nuer, [Nue16b], Theorem 1.1). *Let $\mathbf{v}$ be a primitive Mukai vector of positive rank with $\mathbf{v}^2 \geq -1$. For a polarization H generic with respect to v on an unnodal Enriques surface X, the moduli space $M_{X,H}(\mathbf{v}, L)$ of semistable sheaves with determinant L is nonempty unless $\mathbf{v}^2 = 0$, $2|c_1(L)$ and $2 \nmid L + \frac{\mathrm{rk}(\mathbf{v})}{2} K_X$. When nonempty, the moduli space $M_{X,H}(\mathbf{v}, L)$ is irreducible.*

There is a nice description of the non-primitive case as well (see [Nue16b, Theorem 1.4]).

Moduli spaces of K-negative surfaces. Already for $\mathbb{P}^2$ the classification of stable Chern characters is more complicated. For example, the moduli space may be empty even though the expected dimension is positive. In the case of $\mathbb{P}^2$ a complete answer is known thanks to the work of Drézet and Le Potier (see [CH15, CHW17, DLP85, Hui17, LeP97] for more detailed expositions).

A stable bundle $\mathcal{E}$ on $\mathbb{P}^2$ is called *exceptional* if $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) = 0$. Drézet has completely classified these bundles [Dre87]. Define an *exceptional slope* to be the slope of an exceptional bundle. Then there is a unique exceptional bundle for each exceptional slope. All exceptional bundles are obtained via a sequence of mutations starting with line bundles. Furthermore, the set of exceptional slopes is known via an explicit bijection to the dyadic integers (see [DLP85]) and there is an easy algorithm for computing the continued fraction expansions of exceptional slopes (see [CHW17, §4], [Hui16]).

If $\mathcal{V}$ and $\mathcal{W}$ are semistable sheaves with $\mu_H(\mathcal{V}) < \mu_H(\mathcal{W}) < \mu_H(\mathcal{V}) + 3$, then $\chi(\mathcal{W}, \mathcal{V}) \leq 0$. By stability, $\hom(\mathcal{W}, \mathcal{V}) = 0$. By Serre duality and stability, $\mathrm{ext}^2(\mathcal{W}, \mathcal{V}) = \hom(\mathcal{V}, \mathcal{W}(-3))^* = 0$. Given an exceptional bundle $\mathcal{E}_\alpha$ of slope α, any stable sheaf $\mathcal{V}$ with slope $\alpha - 3 < \mu \leq \alpha$ must satisfy $\chi(\mathcal{E}_\alpha, \mathcal{V}) \leq 0$. Similarly, any stable sheaf $\mathcal{V}$ with slope μ such that $\alpha < \mu < \alpha + 3$ must satisfy $\chi(\mathcal{V}, \mathcal{E}_\alpha) \leq 0$. The conditions $\chi(\mathcal{E}_\alpha, \cdot) = 0$ and $\chi(\cdot, \mathcal{E}_\alpha) = 0$ define parabolas in the (μ, Δ)-plane. Around each exceptional slope, there is an interval I_α where E_α gives the best inequality. The union of the parabolas giving the best inequalities is a fractal curve $\Delta = \delta(\mu)$ called the *Drézet-Le Potier curve*. The main classification theorem of Drézet and Le Potier is then the following.

Theorem 2.18 (Drézet-Le Potier [DLP85]). *The moduli space $M_{\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)}(\mathbf{v})$ is positive dimensional if and only if $\Delta(\mathbf{v}) \geq \delta(\mu(\mathbf{v}))$. In that case, the moduli space is irreducible, normal, factorial and of the expected dimension. Furthermore, if the rank is at least 2, the general point of the moduli space parameterizes a μ-stable vector bundle.*

Rudakov [Rud94] has worked out a similar classification for $\mathbb{P}^1 \times \mathbb{P}^1$ for the polarization $(1, 1)$. Although there are many special results for small rank or special Chern characters, in general, even for del Pezzo or Hirzebruch surfaces a detailed classification of stable Chern characters similar to the Drézet -Le Potier classification is not known.

Surfaces of general type. We are not aware of any complete classification of stable Chern characters on a surface of general type. There have been extensive studies of small rank cases on special surfaces such as rank two bundles on quintic hypersurfaces in $\mathbb{P}^3$ (see [MS11, MS13a, MS13b]). As the next section indicates, the geometry of $M_{X,H}(\mathbf{v})$ may exhibit very complicated behavior especially when $\Delta(\mathbf{v})$ is small. It is likely very challenging to obtain a complete classification of stable Chern characters and the components of the moduli spaces $M_{X,H}(\mathbf{v})$ for general type surfaces.

3. Pathological behavior for small Δ

In this section, following ideas of Mestrano and Simpson [MS13b], we show how to use Hilbert schemes of space curves to detect components of moduli spaces $M_X(2,1,n)$ of sheaves on very general surfaces $X \subset \mathbb{P}^3$ of sufficiently large degree d with c_1 equal to the hyperplane class and $c_2 = n$. As an application, we show that for any number $k > 0$, there is a number d_k such that if $d \geq d_k$, then there are moduli spaces $M_X(2,1,n)$ with at least k irreducible components.

Throughout this section, we let $X \subset \mathbb{P}^3$ be a very general surface of degree $d \geq 5$. Then the Noether-Lefschetz theorem guarantees that $\operatorname{Pic} X \cong \mathbb{Z}$, generated by $\mathcal{O}_X(1)$. We have $K_X = \mathcal{O}_X(d-4)$. By the restriction sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(k-d) \to \mathcal{O}_{\mathbb{P}^3}(k) \to \mathcal{O}_X(k) \to 0,$$

the line bundles $\mathcal{O}_X(k)$ all have $H^1(\mathcal{O}_X(k)) = 0$, and if $k < d$ then $H^0(\mathcal{O}_X(k)) \cong H^0(\mathcal{O}_{\mathbb{P}^3}(k))$. Thus for $k < d$ the sections of $\mathcal{O}_X(k)$ can be interpreted as surfaces in $\mathbb{P}^3$.

3.1. **The construction.** Let $\mathcal{H}_{e,g}$ be the Hilbert scheme of curves of degree $e \geq 3$ and genus g in $\mathbb{P}^3$, and let $\mathcal{R} = \mathcal{R}_{e,g} \subset \mathcal{H}_{e,g}$ be an open subset of an irreducible component of $\mathcal{H}_{e,g}$ parameterizing nondegenerate smooth irreducible curves which are transverse to X. Let $C \subset \mathbb{P}^3$ be a general curve parameterized by $\mathcal{R}$. By Riemann-Roch, the Hilbert polynomial of $\mathcal{O}_C$ is

$$\chi(\mathcal{O}_C(m)) = em - g + 1.$$

We further assume that $d \geq 5$ is large enough that the following two properties hold.

(1) We have vanishings

$$h^1(\mathcal{O}_C(d-4)) = 0, \qquad h^1(I_{C\subset\mathbb{P}^3}(d-4)) = 0, \qquad \text{and} \qquad h^1(I_{C\subset\mathbb{P}^3}(d-3)) = 0.$$

(2) The curve C can be cut out by homogeneous forms of degree $d-3$.

By passing to an open subset of $\mathcal{R}$, we can without loss of generality assume the above properties hold for *every* curve C parameterized by $\mathcal{R}$. These properties imply many additional vanishings.

Lemma 3.1. *If $k \geq d-4$, then the sheaves $\mathcal{O}_C(k)$ and $I_{C\subset\mathbb{P}^3}(k)$ have no higher cohomology.*

Proof. Exact sequences of the form

$$0 \to \mathcal{O}_C(d-4) \to \mathcal{O}_C(k) \to \mathcal{O}_Z \to 0$$

for Z zero-dimensional show that $H^1(\mathcal{O}_C(d-4)) = 0$ implies $H^1(\mathcal{O}_C(k)) = 0$. The sequences

$$0 \to I_{C\subset\mathbb{P}^3}(k) \to \mathcal{O}_{\mathbb{P}^3}(k) \to \mathcal{O}_C(k) \to 0$$

then show that $H^2(I_{C\subset\mathbb{P}^3}(k)) = H^3(I_{C\subset\mathbb{P}^3}(k)) = 0$. Recall that a smooth irreducible curve $C \subset \mathbb{P}^3$ is called *k-normal* if $H^1(I_{C\subset\mathbb{P}^3}(k)) = 0$. Since C is $(d-3)$-normal and $\mathcal{O}_C(d-4)$ is nonspecial, we also have that C is k-normal for all $k \geq d-3$ by [ACGH85, Exercise III.D-5]. $\square$

We put

$$n := n(d,e,g) = h^0(\mathcal{O}_C(d-3)) + 1 = e(d-3) - g + 2.$$

Then $C \cap X$ consists of $de > n$ points. Let $Z \subset C \cap X$ be a collection of n points. We study rank 2 bundles $\mathcal{E}$ on X which fit as extensions

$$0 \to \mathcal{O}_X \to \mathcal{E} \to I_{Z\subset X}(1) \to 0.$$

Proposition 3.2. *We have $\mathrm{ext}^1(I_{Z\subset X}(1), \mathcal{O}_X) = 1$. Let $\mathcal{E}$ be the sheaf given by a nontrivial extension class. Then $\mathcal{E}$ is a μ-stable vector bundle in $M_X(2,1,n)$.*

Let $U_{\mathcal{R}}(n) \subset M_X(2,1,n)$ be the locus parameterizing the sheaves $\mathcal{E}$ which can be constructed by varying C in $\mathcal{R}$ and choosing the scheme $Z \subset C \cap X$ arbitrarily. Then $U_{\mathcal{R}}(n)$ is irreducible and $\dim U_{\mathcal{R}}(n) = \dim \mathcal{R} \geq 4e$.

Proof. First view the curve C and a collection $Z' \subset C$ of $n-1$ points as fixed; we claim that there is a surface $X \subset \mathbb{P}^3$ of degree d which contains Z' but does not contain C. Consider the restriction sequence

$$0 \to I_{C\subset \mathbb{P}^3}(d) \to I_{Z'\subset \mathbb{P}^3}(d) \to I_{Z'\subset C}(d) \to 0.$$

Then

$$\chi(I_{Z'\subset C}(d)) = ed - g + 1 - (n-1) = 3e > 0,$$

so $h^0(I_{Z'\subset C}(d)) > 0$. Also $h^1(I_{C\subset\mathbb{P}^3}(d)) = 0$ by our choice of d, so there is a surface X of degree d which vanishes on Z' and does not contain C.

Let $W \subset \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ be the subset of surfaces which do not contain C, and consider the correspondence

$$\Sigma = \{(X, Z') : Z' \subset C \cap X\} \subset W \times \mathrm{Sym}^{n-1} C.$$

Then Σ is irreducible by a standard monodromy argument, and dominates the second factor. Therefore the locus of (X, Z') such that Z' imposes $n-1 = h^0(\mathcal{O}_C(d-3))$ conditions on sections of $\mathcal{O}_C(d-3)$ is a dense open subset. Hence if X is very general and $Z \subset C \cap X$ is any collection of n points, then $h^0(I_{Z\subset C}(d-3)) = 0$ and $h^1(I_{Z\subset C}(d-3)) = 1$. Since $H^0(\mathcal{O}_{\mathbb{P}^3}(d-3)) \to H^0(\mathcal{O}_C(d-3))$ is surjective, we see that Z imposes $n-1$ conditions on surfaces of degree $d-3$. Then by Serre duality we have

$$\mathrm{ext}^1(I_{Z\subset X}(1), \mathcal{O}_X) = \mathrm{ext}^1(\mathcal{O}_X, I_{Z\subset X}(d-3)) = h^1(I_{Z\subset X}(d-3)) = h^1(I_{Z\subset\mathbb{P}^3}(d-3)) = 1.$$

The sheaf $\mathcal{E}$ is a vector bundle since the scheme Z satisfies the Cayley-Bacharach property for the line bundle $K_X(1) = \mathcal{O}_X(d-3)$: every section of $\mathcal{O}_X(d-3)$ which vanishes at some $n-1$ points $Z' \subset Z$ vanishes on C and so vanishes at all of Z. The Chern class computation is elementary. The defining sequence for $\mathcal{E}$ shows that no line bundle $\mathcal{O}_X(k)$, $k \geq 1$, admits a nonzero map to $\mathcal{E}$. Since $c_1(\mathcal{E})$ is odd, the μ-stability of $\mathcal{E}$ follows.

Given a bundle $\mathcal{E}$ constructed by this method, observe that every section of $\mathcal{O}_{\mathbb{P}^3}(d-3)$ which vanishes along Z also has to contain the nondegenerate irreducible curve C. This implies that the collection of points Z is not coplanar, and therefore $h^0(\mathcal{E}) = 1$. This unique section allows us to uniquely recover the sheaf $I_Z(1)$ as the cokernel of the inclusion $\mathcal{O}_X \to \mathcal{E}$. Since the ideal of C is generated in degree $d-3$ or less, we can recover C as the common zero locus of all the forms in $H^0(I_Z(d-3))$. Thus there is a finite mapping $U_{\mathcal{R}}(n) \to \mathcal{R}$ given by sending $\mathcal{E}$ to C, and $\dim U_{\mathcal{R}}(n) = \dim \mathcal{R}$. The dimension estimate $\dim \mathcal{R} \geq 4e$ for components of the Hilbert scheme parameterizing smooth curves is well-known. The irreducibility of $U_{\mathcal{R}}(n)$ is immediate from the irreducibility of Σ. $\square$

Remark 3.3. The previous discussion can be modified to allow $\mathcal{R}$ to be either the Hilbert scheme of lines or conics which intersect X transversely. We still have $\mathrm{ext}^1(I_{Z\subset X}(1), \mathcal{O}_X) = 1$, and $\mathcal{E}$ is a μ-stable vector bundle in $M_X(2,1,n)$. Varying C and Z still sweeps out a locus $U_{\mathcal{R}}(n) \subset M_X(2,1,n)$. However, since C is degenerate the dimension estimate for $U_{\mathcal{R}}(n)$ changes as follows:

(1) If $\mathcal{R}$ parameterizes lines, we have $n = d-1$. Since $h^0(I_Z(1)) = 2$ we get $h^0(\mathcal{E}) = 3$, and $\mathcal{E}$ no longer determines the line C. Any sheaf $\mathcal{E} \in U_{\mathcal{R}}(n)$ arises from up to a 2-dimensional family of schemes Z, so $\dim U_{\mathcal{R}}(n) \geq 4 - 2 = 2$. Equality holds; see Remark 3.7.

(2) If $\mathcal{R}$ parameterizes conics, we have $n = 2d-4$. Here $h^0(I_Z(1)) = 1$ and $h^0(\mathcal{E}) = 2$, and $\mathcal{E}$ no longer determines the conic C. By the same considerations as above, $\dim U_{\mathcal{R}}(n) \geq 8-1 = 7$, and again equality holds by Remark 3.7.

3.2. **Tangent space.** Let $\mathcal{S}_{\mathcal{R}}(n)$ be an irreducible component of $M_X(2,1,n)$ which contains $U_{\mathcal{R}}(n)$. In this section we study the tangent space to $M_X(2,1,n)$ at points of $U_{\mathcal{R}}(n)$ to find an upper bound on the dimension of $\mathcal{S}_{\mathcal{R}}(n)$. This computation generalizes a computation from [MS13b] in the case where $d = 6$ and $\mathcal{R}$ parameterizes twisted cubics.

Let $\mathcal{E} \in U_{\mathcal{R}}(n)$. Since $\mathcal{E}$ is locally free, we have $\operatorname{Ext}^1(\mathcal{E},\mathcal{E}) \cong H^1(\mathcal{E}^* \otimes \mathcal{E})$. Since $\mathcal{E}$ has rank 2, we have $\mathcal{E}^* \cong \mathcal{E}(-1)$. Also, $\mathcal{E}^* \otimes \mathcal{E}$ splits as a direct sum

$$\mathcal{E}^* \otimes \mathcal{E} \cong (\mathcal{E} \otimes \mathcal{E})(-1) \cong (\operatorname{Sym}^2 \mathcal{E} \oplus \bigwedge^2 \mathcal{E})(-1) \cong \mathcal{V} \oplus \mathcal{O}_X$$

where $\mathcal{V} := (\operatorname{Sym}^2 \mathcal{E})(-1)$. Since $h^1(\mathcal{O}_X) = 0$, we find that $h^1(\mathcal{V})$ is the dimension of the tangent space to the moduli space at $\mathcal{E}$.

The bundle $\mathcal{V}$ fits in a convenient exact sequence, namely

$$0 \to \mathcal{E}(-1) \to \mathcal{V} \to I_{2Z\subset X}(1) \to 0.$$

Here, $2Z \subset X$ is the subscheme defined by the symbolic square of the ideal $I_{Z\subset X}$ of Z, so it consists of a union of n planar double points and has length $3n$. We write $2C$ for the "rope" in $\mathbb{P}^3$ defined by the symbolic square of the ideal $I_{C\subset\mathbb{P}^3}$. Thus a surface contains $2C$ if and only if it is singular at every point of C.

Lemma 3.4. *We have $H^0(I_{2Z\subset X}(d-3)) \cong H^0(I_{2C\subset\mathbb{P}^3}(d-3))$. That is, every degree $d-3$ form tangent to X at each point of Z is singular along the entire curve C.*

Proof. Let $F \in H^0(I_{2Z\subset X}(d-3)) \subset H^0(\mathcal{O}_{\mathbb{P}^3}(d-3))$ and let $Y : F = 0$. Then Y is a surface of degree $d-3$ containing Z, so Y contains C. Since C intersects X transversely and Y is tangent to X at the points of Z, we find that Y is singular at each point of Z. Then the partial derivatives $\partial F/\partial X_i$ each vanish at every point of Z. Since the partials have degree $d-4$, they must then also contain C, and therefore the partials of F vanish identically along C. Therefore $F \in H^0(I_{2C\subset\mathbb{P}^3}(d-3))$. The opposite containment is obvious. □

Slavov computes the Hilbert polynomial of $\mathcal{O}_{2C}$ as follows.

Lemma 3.5 ([Sla16])**.** *The Hilbert polynomial of $\mathcal{O}_{2C}$ is*

$$\chi(\mathcal{O}_{2C}(m)) = 3em - 4e - 5g + 5.$$

It is convenient to define

$$\alpha = h^0(I_{2C\subset\mathbb{P}^3}(d-3)) - \chi(I_{2C\subset\mathbb{P}^3}(d-3)),$$

so that in particular $\alpha = 0$ if d is sufficiently large. We now estimate the dimension of the tangent space.

Proposition 3.6. *The dimension $h^1(\mathcal{V})$ of the tangent space to $M_X(2,1,n)$ at $\mathcal{E} \in U_{\mathcal{R}}(n)$ satisfies the inequalities*

$$4e + 2g + \alpha \leq h^1(\mathcal{V}) \leq 5e + 2g - 3 + \alpha.$$

Proof. Since $\mathcal{E}$ is stable we have $h^0(\mathcal{V}) = 0$. The Euler characteristic $\chi(\mathcal{V})$ is computed using the exact sequences as follows.

$$\begin{aligned}\chi(\mathcal{V}) &= \chi(\mathcal{E}(-1)) + \chi(I_{2Z\subset X}(1))\\ &= \chi(\mathcal{O}_X(-1)) + \chi(\mathcal{O}_X) + \chi(\mathcal{O}_X(1)) - 4n\\ &= \binom{d}{3} + 1 + \binom{d-1}{3} + 4 + \binom{d-2}{3} - 4(e(d-3) - g + 2)\end{aligned}$$

We compute $h^2(\mathcal{V})$ by noting $\mathcal{V}$ is self-dual so $h^2(\mathcal{V}) = h^0(\mathcal{V}(d-4))$. Then we have an exact sequence

$$0 \to H^0(\mathcal{E}(d-5)) \to H^0(\mathcal{V}(d-4)) \to H^0(I_{2Z\subset X}(d-3)) \xrightarrow{\delta} H^1(\mathcal{E}(d-5)).$$

Now we compute the cohomology of $\mathcal{E}(d-5)$. We have

$$\begin{aligned}h^0(\mathcal{E}(d-5)) &= h^0(\mathcal{O}_X(d-5)) + h^0(I_{Z\subset X}(d-4))\\ &= h^0(\mathcal{O}_{\mathbb{P}^3}(d-5)) + h^0(I_{C\subset \mathbb{P}^3}(d-4))\\ &= \binom{d-2}{3} + h^0(\mathcal{O}_{\mathbb{P}^3}(d-4)) - h^0(\mathcal{O}_C(d-4))\\ &= \binom{d-2}{3} + \binom{d-1}{3} - (e(d-4) - g + 1).\end{aligned}$$

Note that $\mathcal{E}(d-5)$ is Serre dual to $\mathcal{E}$. So,

$$\begin{aligned}\chi(\mathcal{E}(d-5)) = \chi(\mathcal{E}) &= \chi(\mathcal{O}_X) + \chi(\mathcal{O}_X(1)) - n\\ &= 1 + \binom{d-1}{3} + 4 + \binom{d-2}{3} - (e(d-3) - g + 2).\end{aligned}$$

Also $h^2(\mathcal{E}(d-5)) = h^0(\mathcal{E}) = 1$, and therefore

$$h^1(\mathcal{E}(d-5)) = h^0(\mathcal{E}(d-5)) + h^2(\mathcal{E}(d-5)) - \chi(\mathcal{E}(d-5)) = e - 3.$$

By Lemma 3.4 and 3.5 we have

$$\begin{aligned}h^0(I_{2Z\subset X}(d-3)) &= h^0(I_{2C\subset\mathbb{P}^3}(d-3))\\ &= \alpha + \chi(I_{2C\subset\mathbb{P}^3}(d-3))\\ &= \alpha + \chi(\mathcal{O}_{\mathbb{P}^3}(d-3)) - \chi(\mathcal{O}_{2C}(d-3))\\ &= \alpha + \binom{d}{3} - (3e(d-3) - 4e - 5g + 5).\end{aligned}$$

Combining these results, we conclude

$$\begin{aligned}h^1(\mathcal{V}) &= h^2(\mathcal{V}) - \chi(\mathcal{V})\\ &\le h^0(\mathcal{E}(d-5)) + h^0(I_{2Z\subset X}(d-3)) - \chi(\mathcal{V})\\ &= 5e + 2g - 3 + \alpha\end{aligned}$$

with equality if δ is 0 and

$$\begin{aligned}h^1(\mathcal{V}) &= h^2(\mathcal{V}) - \chi(\mathcal{V})\\ &\ge h^0(\mathcal{E}(d-5)) + h^0(I_{2Z\subset X}(d-3)) - h^1(\mathcal{E}(d-5)) - \chi(\mathcal{V})\\ &= 4e + 2g + \alpha\end{aligned}$$

with equality if δ is surjective. □

Remark 3.7. As in Remark 3.3, a similar result holds if $\mathcal{R}$ parameterizes lines or conics transverse to X, but adjustments need to be made since the curve C is degenerate:

(1) if $\mathcal{R}$ parameterizes lines then $h^0(\mathcal{E}) = 3$, so $h^1(\mathcal{E}(d-5)) = e-1$ and the bounds become the equality $h^1(\mathcal{V}) = 2+\alpha$. Furthermore, $\alpha = 0$ so long as $d \geq 3$, so $U_{\mathcal{R}}(n)$ is smooth and dense in $\mathcal{S}_{\mathcal{R}}(n)$.
(2) If $\mathcal{R}$ parameterizes conics then $h^0(\mathcal{E}) = 2$, so $h^1(\mathcal{E}(d-5)) = e-2$ and the bounds become the equality $h^1(\mathcal{V}) = 7+\alpha$. We have $\alpha = 0$ for $d \geq 5$, so $U_{\mathcal{R}}(n)$ is smooth and dense in $\mathcal{S}_{\mathcal{R}}(n)$.

Combining the results in this section yields the following dimension estimates.

Corollary 3.8. *With the assumptions above, the irreducible component $\mathcal{S}_{\mathcal{R}}(n)$ of $M_X(2,1,n)$ which contains $U_{\mathcal{R}}(n)$ has dimension satisfying*

$$4e \leq \dim \mathcal{S}_{\mathcal{R}}(n) \leq 5e + 2g - 3 + \alpha.$$

It is typically challenging to compute the dimension of $\mathcal{S}_{\mathcal{R}}(n)$ exactly. For example, if $g > 0$ then the expected dimension $4e$ of $U_{\mathcal{R}}(n)$ is strictly smaller than the lower bound $4e + 2g$ on the dimension of the tangent space. It is not clear whether sheaves in $U_{\mathcal{R}}(n)$ can be deformed outside this locus, or if $U_{\mathcal{R}}(n)$ is dense in $\mathcal{S}_{\mathcal{R}}(n)$ but $\mathcal{S}_{\mathcal{R}}(n)$ is everywhere nonreduced.

The one case where it is particularly easy to analyze things is when $\mathcal{R}$ parameterizes twisted cubic curves.

Corollary 3.9. *Suppose $d \geq 6$ and $\mathcal{R}$ parameterizes twisted cubic curves which are transverse to X. Then the closure of $U_{\mathcal{R}}(n)$ in $M_X(2,1,n)$ is an irreducible component of dimension 12 which is smooth at all points of $U_{\mathcal{R}}(n)$.*

Proof. The inequality $d \geq 5$ is sufficient to ensure that the assumptions on d in this section are satisfied. On the other hand, $d \geq 6$ is needed to give $\alpha = 0$; we have $\alpha = 1$ if $d = 5$. Then by Corollary 3.8, both $U_{\mathcal{R}}(n)$ and $\mathcal{S}_{\mathcal{R}}(n)$ have dimension 12 and the tangent space at any point of $U_{\mathcal{R}}(n)$ has dimension 12. ☐

3.3. **Elementary modifications.** In the previous subsection we used an open irreducible subset $\mathcal{R} \subset \mathcal{H}_{e,g}$ to construct a locus $U_{\mathcal{R}}(n)$ in $M_X(2,1,n)$ if $d = \deg X$ is sufficiently large. Here we have $n = n(d,e,g) = e(d-3) - g + 2$. We now use elementary modifications to construct additional loci in $M_X(2,1,s)$ for every $s \geq n$.

Definition 3.10. Let $s \geq n$. The locus $U_{\mathcal{R}}(s) \subset M_X(2,1,s)$ is the set of all sheaves which can be obtained from sheaves in $U_{\mathcal{R}}(n)$ by a sequence of $s-n$ elementary modifications at distinct points of X.

Given a sheaf $\mathcal{E} \in U_{\mathcal{R}}(n)$, a sheaf in $U_{\mathcal{R}}(s)$ is constructed by choosing $s-n$ points $p_1, \ldots, p_{s-n}$ of X and a hyperplane in the fiber $\mathcal{E}_{p_i}$ for each i. Since $U_{\mathcal{R}}(n)$ is irreducible, it follows that $U_{\mathcal{R}}(s)$ is irreducible of dimension $\dim(\mathcal{R}) + 3(s-n)$. Let $\mathcal{S}_{\mathcal{R}}(s) \subset M_X(2,1,s)$ be an irreducible component containing $U_{\mathcal{R}}(s)$. Our main result in this section bounds the dimension of $\mathcal{S}_{\mathcal{R}}(s)$.

Proposition 3.11. *We have*

$$4e + 3(s-n) \leq \dim \mathcal{S}_{\mathcal{R}}(s) \leq 5e + 2g - 3 + \alpha + 4(s-n).$$

Proof. The lower bound follows from the previous paragraph. Repeated application of the next lemma and Proposition 3.6 gives the upper bound. ☐

Lemma 3.12. *Suppose $\mathcal{E}$ is a stable rank r torsion-free sheaf on a surface X, let $p \in X$ be a point where $\mathcal{E}$ is locally free, and let $\mathcal{E}'$ be an elementary modification of $\mathcal{E}$ at p:*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{O}_p \to 0. \tag{1}$$

Then

$$\operatorname{ext}^1(\mathcal{E}', \mathcal{E}') \leq \operatorname{ext}^1(\mathcal{E}, \mathcal{E}) + 2r.$$

Proof. We first apply $\mathrm{Ext}(\mathcal{E},-)$ to the Sequence (1), and obtain the long exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E},\mathcal{E}') \to \mathrm{Hom}(\mathcal{E},\mathcal{E}) \to \mathrm{Hom}(\mathcal{E},\mathcal{O}_p) \to \mathrm{Ext}^1(\mathcal{E},\mathcal{E}') \to \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^1(\mathcal{E},\mathcal{O}_p).$$

We have $\mathrm{Hom}(\mathcal{E},\mathcal{E}) = \mathbb{C}\cdot \mathrm{id}$ by stability, and the map $\mathrm{Hom}(\mathcal{E},\mathcal{E}) \to \mathrm{Hom}(\mathcal{E},\mathcal{O}_p)$ carries id to the nonzero map $\mathcal{E} \to \mathcal{O}_p$ defining $\mathcal{E}'$. Therefore $\mathrm{Hom}(\mathcal{E},\mathcal{E}) \to \mathrm{Hom}(\mathcal{E},\mathcal{O}_p)$ is injective and $\mathrm{Hom}(\mathcal{E},\mathcal{E}') = 0$. Also $\hom(\mathcal{E},\mathcal{O}_p) = r$ and $\mathrm{ext}^1(\mathcal{E},\mathcal{O}_p) = 0$ since $\mathcal{E}$ is locally free. Putting this all together,

$$\mathrm{ext}^1(\mathcal{E},\mathcal{E}') = \mathrm{ext}^1(\mathcal{E},\mathcal{E}) + r - 1.$$

Next we apply $\mathrm{Ext}(-,\mathcal{E}')$ to Sequence (1) and get an exact sequence

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{E}') \to \mathrm{Ext}^1(\mathcal{E}',\mathcal{E}') \to \mathrm{Ext}^2(\mathcal{O}_p,\mathcal{E}'),$$

so

$$\mathrm{ext}^1(\mathcal{E}',\mathcal{E}') \le \mathrm{ext}^1(\mathcal{E},\mathcal{E}') + \mathrm{ext}^2(\mathcal{O}_p,\mathcal{E}') = \mathrm{ext}^1(\mathcal{E},\mathcal{E}) + r - 1 + \mathrm{ext}^2(\mathcal{O}_p,\mathcal{E}').$$

Finally $\mathrm{ext}^2(\mathcal{O}_p,\mathcal{E}') = r+1$: by Serre duality, $\mathrm{ext}^2(\mathcal{O}_p,\mathcal{E}') = \hom(\mathcal{E}',\mathcal{O}_p)$. Applying $\mathrm{Ext}(-,\mathcal{O}_p)$ to Sequence (1) we have an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{O}_p,\mathcal{O}_p) \to \mathrm{Hom}(\mathcal{E},\mathcal{O}_p) \to \mathrm{Hom}(\mathcal{E}',\mathcal{O}_p) \to \mathrm{Ext}^1(\mathcal{O}_p,\mathcal{O}_p) \to \mathrm{Ext}^1(\mathcal{E},\mathcal{O}_p) = 0.$$

Here $\hom(\mathcal{O}_p,\mathcal{O}_p) = 1$, $\mathrm{ext}^1(\mathcal{O}_p,\mathcal{O}_p) = 2$, and $\hom(\mathcal{E},\mathcal{O}_p) = r$, so $\mathrm{ext}^2(\mathcal{O}_p,\mathcal{E}') = r+1$, completing the proof. $\square$

3.4. **Comparing components.** We now use our dimension estimates on the components $\mathcal{S}_{\mathcal{R}}(s)$ to show that if $d \gg 0$ then there are moduli spaces of sheaves $M_X(2,1,s)$ with as many components as we like.

Separating two loci. First suppose $\mathcal{R} = \mathcal{R}_{e,g} \subset \mathcal{H}_{e,g}$ and $\mathcal{R}' = \mathcal{R}_{e',g'} \subset \mathcal{H}_{e',g'}$ are two open irreducible subsets, where $e < e'$. Then we have

$$n := n(d,e,g) = e(d-3) - g + 2$$
$$n' := n(d,e',g') = e'(d-3) - g' + 2.$$

Therefore for $d \gg 0$, we have $n < n'$. Then for any $s \ge n'$ we can consider the two components $\mathcal{S}_{\mathcal{R}}(s)$ and $\mathcal{S}_{\mathcal{R}'}(s)$ of $M_X(2,1,s)$.

Theorem 3.13. *With the above notation, if $d \gg 0$ then the components $\mathcal{S}_{\mathcal{R}}(n')$ and $\mathcal{S}_{\mathcal{R}'}(n')$ are distinct.*

Proof. We need only see that $\dim \mathcal{S}_{\mathcal{R}}(n') > \dim \mathcal{S}_{\mathcal{R}'}(n')$. By Proposition 3.11 and our formulas for n and n', we have

$$\dim \mathcal{S}_{\mathcal{R}}(n') \ge 4e + 3(n'-n) = 3(e'-e)d + C_1$$
$$\dim \mathcal{S}_{\mathcal{R}'}(n') \le 5e' + 2g' - 3 = C_2$$

where C_i are constants which depend (at most) on e, g, e', g', but not on d. Since $e' > e$, the required inequality follows for $d \gg 0$. $\square$

If the surface X is fixed and s increases past n', then the components $S_{\mathcal{R}}(s)$ and $S_{\mathcal{R}'}(s')$ must eventually coincide since the moduli space $M_X(2,1,s)$ is irreducible for $s \gg 0$. It is useful to further quantify how large s can be before these components potentially coincide.

Proposition 3.14. *Suppose $d \gg 0$. Then there is a constant C depending on e, g, e', g' such that if*

$$n' \le s \le (4e' - 3e)d + C$$

then the components $\mathcal{S}_{\mathcal{R}}(s)$ and $\mathcal{S}_{\mathcal{R}'}(s)$ are distinct.

Note that $4e' - 3e > e'$ since $e' > e$, while n' grows like $e'd + C$ as d increases. So, the range of numbers s where the components can be separated increases with d.

Proof. Again we use Proposition 3.11 to estimate

$$\dim \mathcal{S}_{\mathcal{R}}(s) \geq 4e + 3(s-n) = -3ed + 3s + C_3$$
$$\dim \mathcal{S}_{\mathcal{R}'}(s) \leq 5e' + 2g' - 3 + 4(s-n') = -4e'd + 4s + C_4$$

where the C_i are constants depending on e, g, e', g'. Then we will have

$$-3ed + 3s + C_3 > -4e'd + 4s + C_4$$

so long as $s < (4e' - 3e)d + C_5$. □

Separating multiple loci. Now suppose we consider a list of k open irreducible sets $\mathcal{R}^i = \mathcal{R}_{e_i,g_i} \subset \mathcal{H}_{e_i,g_i}$, and that the degrees satisfy $e_1 < \cdots < e_k$. Let $n_i = n(d, e_i, g_i)$; then for $d \gg 0$ the largest n_i is n_k. As d increases, the number n_k grows like $e_k d + C$. By Proposition 3.14, if $4e_{i+1} - 3e_i > e_k$ whenever $1 \leq i < k$ then the component $\mathcal{S}_{\mathcal{R}^{i+1}}(n_k)$ will have smaller dimension than $\mathcal{S}_{\mathcal{R}^i}(n_k)$ for large enough d. Thus we have proved the following result.

Proposition 3.15. *Suppose $e_1 < \cdots < e_k$ satisfy $4e_{i+1} - 3e_i > e_k$ for $1 \leq i < k$. Then if $d \gg 0$, the components $\mathcal{S}_{\mathcal{R}^i}(n_k)$ are all distinct for $1 \leq i \leq k$.*

This easily implies the following more qualitative theorem.

Theorem 3.16. *For any integer k, there is a number $d_k \gg 0$ such that if $d \geq d_k$ then a very general surface $X \subset \mathbb{P}^3$ of degree d has some moduli space $M_X(2,1,s)$ with at least k components.*

Proof. By Proposition 3.15 it is enough to see that there are arbitrarily long sequences of positive integers $e_1 < \cdots < e_k$ such that $4e_{i+1} - 3e_i > e_k$. Such sequences are easy to construct. For a crude example, the sequence $4^k - 2^{k-1} < 4^k - 2^{k-2} < \cdots < 4^k - 2^0$ does the trick since

$$4(4^k - 2^i) - 3(4^k - 2^{i+1}) = 4^k + 2^{i+1} > 4^k - 2^0$$

for $i \geq 0$. □

Remark 3.17. When $\Delta(\mathbf{v})$ is small, we would expect the geometry of $M_{X,H}(\mathbf{v})$ to exhibit the same pathologies as the Hilbert scheme of curves in $\mathbb{P}^3$. It would be interesting to make this precise.

4. Brill-Noether Theorems

In this section, we discuss recent progress in Brill-Noether theory of moduli spaces of sheaves on surfaces. This section is based on [CH16b] and [CH17b]. We will first discuss the theory for $\mathbb{P}^2$. We will then discuss the case of Hirzebruch surfaces and del Pezzo surfaces. Finally, we will make some remarks for general surfaces and give a few examples for hypersurfaces in $\mathbb{P}^3$.

Rank 1 sheaves. If $\operatorname{rk} \mathbf{v} = 1$, then any torsion free sheaf with Chern character $\mathbf{v}$ is of the form $L \otimes I_Z$ for a line bundle L and an ideal sheaf of points I_Z. The long exact sequence associated to

$$0 \to L \otimes I_Z \to L \to L \otimes \mathcal{O}_Z \to 0,$$

shows that $H^2(X, L) \cong H^2(X, L \otimes I_Z)$. Furthermore, if Z is a general set of n points, the map $H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_Z)$ has maximal rank. Consequently, if L has no higher cohomology, then $L \otimes I_Z$ has no higher cohomology as long as Z is a general set of points on X with $|Z| \leq h^0(X, L)$. Furthermore, when $|Z| \geq h^0(X, L)$, then $L \otimes I_Z$ has no global sections. We conclude that for a general set of points Z on X, $L \otimes I_Z$ has at most one nonzero cohomology group if and only if one of the following holds

(1) The line bundle L has no higher cohomology, or
(2) We have $h^2(X, L) = 0$ and $|Z| \geq h^0(X, L)$, or
(3) We have $h^0(X, L) = |Z|$ and $h^1(X, L) = 0$.

From now on, we will always assume that $\operatorname{rk} \mathbf{v} \geq 2$.

The projective plane. Let L denote the hyperplane class on $\mathbb{P}^2$. Göttsche and Hirschowitz [GHi94] show that the general sheaf in $M_{\mathbb{P}^2,L}(\mathbf{v})$ has at most one nonzero cohomology group.

Theorem 4.1 (Göttsche-Hirschowitz [GHi94])**.** *Let* $\mathbf{v}$ *be a stable Chern character with* $\mathrm{rk}(\mathbf{v}) \geq 2$. *Then the general sheaf* $\mathcal{V} \in M_{\mathbb{P}^2,L}(\mathbf{v})$ *has at most one nonzero cohomology group.*

In particular, if $\chi(\mathbf{v}) < 0$, then the general stable sheaf $\mathcal{V}$ has $h^1(\mathcal{V}) = -\chi(\mathbf{v})$. If $\chi(\mathbf{v}) \geq 0$ and $\mu_H(\mathbf{v}) \geq 0$, then $h^0(\mathcal{V}) = \chi(\mathbf{v})$. If $\chi(\mathbf{v}) \geq 0$ and $\mu_H(\mathbf{v}) < 0$, then $h^2(\mathcal{V}) = \chi(\mathbf{v})$. Hence, the Göttsche-Hirschowitz Theorem computes the cohomology of a general stable sheaf on $\mathbb{P}^2$. We will give two simple proofs of the theorem to illustrate the techniques.

Proof Sketch 1. First, by Serre duality, we may assume that $\mu(\mathbf{v}) \geq -\frac{3}{2}$. If the Serre dual sheaf has only one nonzero cohomology group, so does the original sheaf. We can apply Serre duality because the general sheaf of rank at least 2 in $M_{\mathbb{P}^2,L}(\mathbf{v})$ is a vector bundle. This fails when $\mathrm{rk}(\mathbf{v}) = 1$. For example, $\chi(I_p(-3)) = 0$, but $h^1(\mathbb{P}^2, I_p(-3)) = h^2(\mathbb{P}^2, I_p(-3)) = 1$ for any ideal sheaf of a point $p \in \mathbb{P}^2$.

The general stable sheaf $\mathcal{V}$ on $\mathbb{P}^2$ admits a Gaeta resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}(a-2)^k \to \mathcal{O}_{\mathbb{P}^2}(a-1)^l \oplus \mathcal{O}_{\mathbb{P}^2}(a)^m \to \mathcal{V} \to 0, \quad \text{or}$$
$$0 \to \mathcal{O}_{\mathbb{P}^2}(a-2)^k \oplus \mathcal{O}_{\mathbb{P}^2}(a-1)^l \to \mathcal{O}_{\mathbb{P}^2}(a)^m \to \mathcal{V} \to 0,$$

where a is the largest integer such that $\chi(\mathcal{V}(-a)) \geq 0$ but $\chi(\mathcal{V}(-a-1)) < 0$,

$$m = \chi(\mathcal{V}(-a)), \quad k = -\chi(\mathcal{V}(-a-1) \quad \text{and} \quad l = |\,\mathrm{rk}(\mathcal{V}) + k - m|.$$

The sign of $\mathrm{rk}(\mathcal{V}) + k - m$ determines which of the two resolutions $\mathcal{V}$ admits (see [Gae51] for ideal sheaves of general points).

If $a \geq 0$, then $\mathcal{V}$ clearly has no higher cohomology. Since $\mu(\mathcal{V}) \geq -\frac{3}{2}$, $\mu(\mathcal{V}^*(-3)) \leq -\frac{3}{2}$. By Serre duality and stability, $h^2(\mathbb{P}^2, \mathcal{V}) = h^0(\mathbb{P}^2, \mathcal{V}^*(-3)) = 0$. When $a < 0$, then $\mathcal{V}$ clearly has no global sections. Since $h^2(\mathbb{P}^2, \mathcal{V}) = 0$, we conclude that the only nonzero cohomology group can be $H^1(\mathbb{P}^2, \mathcal{V})$. □

Proof Sketch 2. Alternatively, we can prove a slightly more general theorem. By Serre duality, we may assume that $\mu(\mathbf{v}) \geq -\frac{3}{2}$. By the division algorithm, we can write $\mu(\mathbf{v}) = a + \frac{m}{r}$, where a is an integer and $0 \leq m < r$. Then

$$\mathcal{V} = \mathcal{O}_{\mathbb{P}^2}(a)^{r-m} \oplus \mathcal{O}_{\mathbb{P}^2}(a+1)^m,$$

is an L-prioritary sheaf with slope $\mu(\mathbf{v})$. Since $\mu(\mathbf{v}) \geq -\frac{3}{2}$, $a \geq -2$. Consequently, $\mathcal{V}$ has no higher cohomology. A simple computation shows that $\Delta(\mathcal{V}) \leq 0$. By Lemma 2.9, taking general elementary modifications of $\mathcal{V}$, we obtain L-prioritary sheaves with at most one nonzero cohomology group for every integral Chern character $\mathbf{v}$ with $\mathrm{rk}(\mathbf{v}) \geq 2$ and $\Delta(\mathbf{v}) \geq 0$. Since the stack of prioritary sheaves is irreducible and vanishing of cohomology is an open condition, we conclude that the general sheaf in the corresponding stacks also have at most one nonzero cohomology group. In particular, if $\mathbf{v}$ is a stable Chern character, the Gieseker semistable sheaves form an open subset of $\mathcal{P}_L(\mathbf{v})$ and the general semistable sheaf has at most one nonzero cohomology group. □

We obtain the following corollary of the proof.

Corollary 4.2. *Let* $\mathbf{v}$ *be a Chern character such that* $\mathrm{rk}(\mathbf{v}) \geq 2$ *and* $\Delta(\mathbf{v}) \geq 0$. *Then the general prioritary sheaf* $\mathcal{V} \in \mathcal{P}_L(\mathbf{v})$ *has at most one nonzero cohomology group.*

Both of these strategies can be used to obtain Brill-Noether theorems on other surfaces. The weak Brill-Noether theorem has many applications. One application is the classification of globally generated vector bundles. Define a Chern character $\mathbf{v}$ to be a *globally generated Chern character* if the general prioritary sheaf with character $\mathbf{v}$ is globally generated. One needs to exercise some caution with this notion because being globally generated is not an open condition.

Example 4.3. The vector bundle $\mathcal{V}$ defined as the cokernel of the natural map

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \otimes H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \to \mathcal{V} \to 0$$

is semistable and globally generated [LeP97]. However, when $d \geq 3$, the general member of the moduli space is not globally generated. This is easiest to see when $d \geq 4$. In that case, $\chi(\mathbf{v}) < \mathrm{rk}(\mathbf{v})$. The general sheaf has only a $\chi(\mathbf{v})$-dimensional space of sections, so has no chance of being globally generated. When $d = 3$, $\chi(\mathbf{v}) = \mathrm{rk}(\mathbf{v})$ and the moduli space is positive dimensional. The general sheaf has only 9 sections which fail to generate the sheaf along a curve.

However, if the higher cohomology of the sheaves vanishes, then the condition of being globally generated is an open condition

Theorem 4.4 ([BGJ16], [CH17b])**.** *Let $\mathbf{v}$ be an integral Chern character on $\mathbb{P}^2$ such that $\mathrm{rk}(\mathbf{v}) \geq 2$, $\Delta(\mathbf{v}) \geq 0$. Then the Chern character $\mathbf{v}$ is globally generated if and only if $\mu(\mathbf{v}) \geq 0$ and one of the following holds:*

(1) We have $\mu(\mathbf{v}) > 0$ and $\chi(\mathbf{v}(-1)) \geq 0$.
(2) We have $\mu(\mathbf{v}) > 0$, $\chi(\mathbf{v}(-1)) < 0$, and $\chi(\mathbf{v}) \geq \mathrm{rk}(\mathbf{v}) + 2$.
(3) We have $\mu(\mathbf{v}) > 0$, $\chi(\mathbf{v}(-1)) < 0$, and $\chi(\mathbf{v}) \geq \mathrm{rk}(\mathbf{v}) + 1$ and

$$\mathbf{v} = (\mathrm{rk}\,\mathbf{v} + 1)\,\mathrm{ch}(\mathcal{O}_{\mathbb{P}^2}) - \mathrm{ch}(\mathcal{O}_{\mathbb{P}^2}(-2)).$$

(4) We have $\mu(\mathbf{v}) = 0$ and $\mathbf{v} = \mathrm{rk}(\mathbf{v})\,\mathrm{ch}(\mathcal{O}_{\mathbb{P}^2})$.

Proof. If $\mathcal{V}$ is globally generated, then its determinant is also globally generated. We therefore have $\mu(\mathcal{V}) \geq 0$. If $\mu(\mathcal{V}) = 0$, then by Riemann-Roch $\chi(\mathcal{V}) \leq \mathrm{rk}(\mathcal{V})$ with equality if and only if $\Delta(\mathcal{V}) = 0$. Since a globally generated bundle $\mathcal{V}$ needs to have at least $\mathrm{rk}(\mathcal{V})$ independent sections and for the general sheaf there is only one nonzero cohomology group, we conclude that $\mu(\mathcal{V}) = \Delta(\mathcal{V}) = 0$ and $\mathbf{v} = \mathrm{rk}(\mathbf{v})\,\mathrm{ch}(\mathcal{O}_{\mathbb{P}^2})$.

If $\chi(\mathbf{v}(-1)) \geq 0$, then the general sheaf in $\mathcal{P}_{\mathbb{P}^2,L}(\mathbf{v})$ has a Gaeta resolution with $a \geq 1$. Then the general sheaf is clearly a quotient of a globally generated bundle. If $\chi(\mathbf{v}(-1)) < 0$ and $\chi(\mathbf{v}) \geq \mathrm{rk}(\mathbf{v}) + 2$, then the general sheaf in $\mathcal{P}_{\mathbb{P}^2,L}(\mathbf{v})$ has a Gaeta resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^k \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^l \to \mathcal{O}_{\mathbb{P}^2}^m \to \mathcal{V} \to 0, \text{ or}$$
$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^k \to \mathcal{O}_{\mathbb{P}^2}(-1)^l \oplus \mathcal{O}_{\mathbb{P}^2}^m \to \mathcal{V} \to 0.$$

In the first case, $\mathcal{V}$ is the quotient of a globally generated vector bundle, hence globally generated. The most interesting case is the second case. By the assumption that $\chi(\mathcal{V}) \geq \mathrm{rk}(\mathcal{V}) + 2$, we have that $m \geq \mathrm{rk}(\mathcal{V}) + 2$. Therefore, $k \geq l + 2$. To show that $\mathcal{V}$ is globally generated, it suffices to show that $H^1(\mathbb{P}^2, \mathcal{V} \otimes I_p) = 0$ for every point $p \in \mathbb{P}^2$. By the long exact sequence of cohomology, it suffices to show that the map

$$\phi : H^1(\mathbb{P}^2, I_p(-2))^k \to H^1(\mathbb{P}^2, I_p(-1))^l$$

is surjective. Consider the sequence

$$0 \to M \to \mathcal{O}_{\mathbb{P}^2}(-2)^k \to \mathcal{O}_{\mathbb{P}^2}(-1)^l \to 0.$$

Since the map is general, it is surjective and M is a vector bundle. Clearly M does not have any cohomology. Tensoring the standard exact sequence $0 \to I_p \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_p \to 0$ with M, we see that $H^2(\mathbb{P}^2, I_p \otimes M) = 0$. Consequently, the map ϕ is surjective and $\mathcal{V}$ is globally generated. Finally, if $\chi(\mathcal{V}) = \mathrm{rk}(\mathcal{V}) + 1$ and $\mathcal{V}$ is globally generated, then there is a surjective map $\mathcal{O}_{\mathbb{P}^2}^{r+1} \to \mathcal{V}$. The kernel of this map is a line bundle $\mathcal{O}_{\mathbb{P}^2}(-d)$. If $d = 1$, then $\chi(\mathcal{V}(-1)) = 0$. If $d \geq 3$, then $\chi(\mathcal{V}) < r$ and it is not possible for the general prioritary sheaf with Chern character $\mathbf{v}$ to be globally generated. The only remaining possibility is for $d = 2$. In that case, $\chi(\mathcal{V}) = r + 1$ and this is the Gaeta resolution of the general sheaf. This concludes the classification of globally generated Chern characters on $\mathbb{P}^2$. $\square$

The following problem remains open.

Problem 4.5. *Classify the Chern characters* $\mathbf{v}$ *on* $\mathbb{P}^2$ *such that the general prioritary sheaf of character* $\mathbf{v}$ *is ample.*

Note that if $\mathcal{V}$ is a vector bundle such that $\mathcal{V}(-1)$ is globally generated, then $\mathcal{V}$ is ample. Thus the classification of globally generated Chern characters gives a sufficient condition for the general bundle to be ample. In particular, if $\mathrm{rk}(\mathbf{v}) \geq 2$, $\mu(\mathbf{v}) \geq 1$, $\Delta(\mathbf{v}) \geq 0$ and $\chi(\mathbf{v}(-1)) \geq \mathrm{rk}(\mathbf{v})+2$, then the general prioritary sheaf with Chern character $\mathbf{v}$ is ample. However, an ample vector bundle on $\mathbb{P}^2$ does not have to have any sections. For example, Gieseker [Gie71] proves that a general vector bundle with resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d)^2 \to \mathcal{O}_{\mathbb{P}^2}(-1)^4 \to \mathcal{V} \to 0$$

is ample for $d \gg 0$. It is easy to see that we need $d \geq 7$. However, we do not know whether $d = 7$ is sufficient. In general, an ample bundle must satisfy $\mu(\mathbf{v}) \geq 1$ and $\frac{\mu^2}{2} > \frac{\Delta}{r+1}$. It would be interesting to determine conditions under which the converse also holds.

Hirzebruch surfaces. Following [CH16b] and [CH17b], we now explain how to obtain analogues of Corollary 4.2 and Theorem 4.4 for Hirzebruch surfaces.

Let e be a nonnegative integer and let $\mathbb{F}_e$ denote the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$. We refer the reader to [Cos06a] or [Har77] for a detailed description of Hirzebruch surfaces. The surface $\mathbb{F}_e$ admits a natural projection π to $\mathbb{P}^1$. Let F denote the class of a fiber of π. The surface contains a section E with self-intersection $-e$. When $e \geq 1$, this section is unique. The Picard group $\mathrm{Pic}(\mathbb{F}_e) = \mathbb{Z}E \oplus \mathbb{Z}F$ and the intersection product is given by

$$E^2 = -e, \quad E \cdot F = 1, \quad F^2 = 0.$$

Express the total slope of a Chern character $\mathbf{v}$ by

$$\nu(\mathbf{v}) = \frac{k}{r}E + \frac{l}{r}F.$$

Let $\mathcal{V} = \mathcal{O}_{\mathbb{F}_e}(-E-(e+1)F)^a \oplus \mathcal{O}_{\mathbb{F}_e}(-F)^b \oplus \mathcal{O}_{\mathbb{F}_e}^c$. Then a simple calculation shows that $\Delta(\mathcal{V}) \leq 0$ and $\mathcal{V}$ is both F-prioritary and E-prioritary. Furthermore, every slope in the quadrilateral in the $(\frac{k}{r}, \frac{l}{r})$-plane with vertices

$$(-1, -e-1), \quad (0,0), \quad (0,-1), \quad (-1,-e-1)$$

can be expressed as the slope of a bundle $\mathcal{V}$ or $\mathcal{V}^*(-E-(e+1)F)$. Furthermore, these bundles have no higher cohomology. Translates of this quadrilateral by classes of nef line bundles, covers the region defined by the inequalities $\nu(\mathbf{v}) \cdot F \geq -1$ and $\nu(\mathbf{v}) \cdot E \geq -1$. Using elementary modifications and Lemma 2.9, one concludes the following.

Theorem 4.6 ([CH17b], Theorem 3.1)**.** *Let* $\mathbf{v} \in K(F_e)$ *be a Chern character with positive rank* r *and* $\Delta \geq 0$. *Then the stack* $\mathcal{P}_{F_e,F}(\mathbf{v})$ *of* F*-prioritary sheaves is nonempty and irreducible. Let* $\mathcal{V} \in \mathcal{P}_{F_e,F}(\mathbf{v})$ *be a general sheaf.*

(1) *If* $\nu(\mathbf{v}) \cdot F \geq -1$, *then* $h^2(F_e, \mathcal{V}) = 0$. *If* $\nu(\mathbf{v}) \cdot F \leq -1$, *then* $h^0(F_e, \mathcal{V}) = 0$. *In particular, if* $\nu(\mathbf{v}) \cdot F = -1$, *then both* h^0 *and* h^2 *vanish and* $h^1(F_e, \mathcal{V}) = -\chi(\mathbf{v})$.
(2) *If* $\nu(\mathbf{v}) \cdot F > -1$ *and* $\nu(\mathbf{v}) \cdot E \geq -1$, *then* $\mathcal{V}$ *has at most one nonzero cohomology group. Thus if* $\chi(\mathbf{v}) \geq 0$, *then* $h^0(F_e, \mathcal{V}) = \chi(\mathbf{v})$, *and if* $\chi(\mathbf{v}) \leq 0$, *then* $h^1(F_e, \mathcal{V}) = -\chi(\mathbf{v})$.
(3) *If* $\nu(\mathbf{v}) \cdot F > -1$ *and* $\nu(\mathbf{v}) \cdot E < -1$, *then* $H^0(F_e, \mathcal{V}) = H^0(F_e, \mathcal{V}(-E))$, *hence the Betti numbers of* $\mathcal{V}$ *are inductively determined using the previous two parts.*
(4) *If* $\nu(\mathbf{v}) \cdot F < -1$ *and* $\mathrm{rk}(\mathbf{v}) \geq 2$, *then Serre duality determines the Betti numbers of* $\mathcal{V}$.

We call a Chern character $\mathbf{v}$ *nonspecial* if there exists an F-prioritary sheaf $\mathcal{V}$ with Chern character $\mathbf{v}$ such that $\mathcal{V}$ has at most one nonzero cohomology group. In particular, we obtain a classification of nonspecial Chern characters on $\mathbb{F}_e$.

Corollary 4.7 ([CH17b], Corollary 3.9)**.** *Let* $\mathbf{v} \in K(F_e)$ *be a character with positive rank and* $\Delta(\mathbf{v}) \geq 0$*, and suppose* $\nu(\mathbf{v}) \cdot F \geq -1$*. Then* $\mathbf{v}$ *is nonspecial if and only if one of the following holds.*

(1) We have $\nu(\mathbf{v}) \cdot F = -1$*.*
(2) We have $\nu(\mathbf{v}) \cdot F > -1$ *and* $\nu(\mathbf{v}) \cdot E \geq -1$*.*
(3) If $\nu(\mathbf{v}) \cdot F > -1$ *and* $\nu(\mathbf{v}) \cdot E < -1$*, let* m *be the smallest positive integer such that either* $\nu(\mathbf{v}(-mE)) \cdot F \leq -1$ *or* $\nu(\mathbf{v}(-mE)) \cdot E \geq -1$*.*
  (a) If $\nu(\mathbf{v}(-mE)) \cdot F \leq -1$*, then* $\mathbf{v}$ *is nonspecial.*
  (b) If $\nu(\mathbf{v}(-mE)) \cdot F > -1$*, then* $\mathbf{v}$ *is nonspecial if and only if* $\chi(\mathbf{v}(-mE)) \leq 0$*.*

The following corollary when $\chi(\mathbf{v}) \geq 0$ is easier to remember.

Corollary 4.8 ([CH17b], Corollary 3.10)**.** *Let* $\mathbf{v}$ *be a positive rank Chern character on* $\mathbb{F}_e$ *such that* $\Delta(\mathbf{v}) \geq 0$*,* $\chi(\mathbf{v}) \geq 0$ *and* $F \cdot \nu(\mathbf{v}) \geq -1$*. Then* $\mathbf{v}$ *is nonspecial if and only if* $F \cdot \nu(\mathbf{v}) = -1$ *or* $E \cdot \nu(\mathbf{v}) \geq -1$*.*

As in the case of $\mathbb{P}^2$, we may use the Brill-Noether theorems to characterize the globally generated Chern characters. Let $\mathcal{V}$ a general prioritary sheaf in $\mathcal{P}_{\mathbb{F}_e,F}(\mathbf{v})$ with $\Delta(\mathcal{V}) \geq 0$. If $\mathcal{V}$ is globally generated, then its determinant has to be globally generated and nef. If in addition $\nu(\mathcal{V}) \cdot F = 0$, then the restriction of $\mathcal{V}$ to every fiber must be trivial. Hence, $\mathcal{V}$ must be a pullback from $\mathbb{P}^1$. Since $\mathcal{V}$ is F-prioritary and globally generated, we conclude that $\mathcal{V} = \mathcal{O}_{\mathbb{F}_e}(aF)^m \oplus \mathcal{O}_{\mathbb{F}_e}((a+1)F)^{r-m}$ for some $a \geq 0$ and $m \geq 0$. We may now assume that $\nu(\mathcal{V}) \cdot F > 0$. Since $\mathcal{V}$ is general, the restriction of $\mathcal{V}$ to every fiber will be globally generated. If $\chi(\mathcal{V}(-F)) \geq 0$, then the exact sequence

$$0 \to \mathcal{V}(-F) \to \mathcal{V} \to \mathcal{V}|_F \to 0$$

allows us to lift the section of $\mathcal{V}|_F$ to sections of $\mathcal{V}$ on $\mathbb{F}_e$ since by the cohomology computations $H^1(\mathbb{F}_e, \mathcal{V}(-F)) = 0$. If $\chi(\mathcal{V}(-F)) < 0$, then as in the case of $\mathbb{P}^2$, we need to resort to a Gaeta-type resolution.

Theorem 4.9 ([CH17b], Theorem 4.1)**.** *Let* $\mathbf{v}$ *be an integral Chern character on* $\mathbb{F}_e$ *of positive rank and assume that*

$$\Delta(\mathbf{v}) \geq \frac{1}{4} \text{ if } e = 0, \quad \Delta(\mathbf{v}) \geq \frac{1}{8} \text{ if } e = 1, \quad \Delta(\mathbf{v}) \geq 0 \text{ if } e \geq 2.$$

Then the general sheaf $\mathcal{V} \in \mathcal{P}_{\mathbb{F}_e,F}(\mathbf{v})$ *admits a Gaeta-type resolution*

$$0 \to L(-E-(e+1)F)^a \to L(-E-eF)^b \oplus L(-F)^c \oplus L^d \to \mathcal{V} \to 0, \tag{2}$$

for some line bundle L *and nonnegative integers* a, b, c, d*.*

The exponents in the exact sequence (2) can be formally calculated using the Euler pairing:

$$a = -\chi(\mathcal{V}(-L-E-F)), \quad b = -\chi(\mathcal{V}(-L-E)), \quad c = -\chi(\mathcal{V}(-L-F)), \quad d = \chi(\mathcal{V}(-L)).$$

If we can find a line bundle L such that a, b, c, d are nonnegative, then we can define $\mathcal{V}$ by the sequence (2). An easy check shows that the general sheaf given by such a resolution is F-prioritary and such sheaves provide a complete family of F-prioritary sheaves. It then follows that the general F-prioritary sheaf has such a resolution since the stack of F-prioritary sheaves is irreducible. Finally, the inequalities on Δ guarantee that one can find the desired line bundle L.

If $\chi(\mathcal{V}(-F)) < 0$ and $\mathcal{V}$ is globally generated, we consider

$$0 \to \mathcal{M} \to \mathcal{O}_{\mathbb{F}_e}^{\chi(\mathcal{V})} \to \mathcal{V} \to 0,$$

where $\mathcal{M}$ is a vector bundle with character $\mathbf{v}$. Then $\mathcal{M}$ has no cohomology, $h^1(\mathbb{F}_e, \mathcal{M}(-F)) = 0$ and $\mathcal{M}^*$ is globally generated. Conversely, if we can construct such a vector bundle $\mathcal{M}$, we obtain a globally generated F-prioritary vector bundle $\mathcal{V}$ with Chern character $\mathbf{v}$. As in the case of $\mathbb{P}^2$, one constructs $\mathcal{M}$ and check that $\mathcal{M}^*$ is globally generated directly from the Gaeta-type resolution provided by Theorem 4.9. We obtain the following classification of globally generated Chern characters on $\mathbb{F}_e$.

Theorem 4.10 ([CH17b], Theorem 5.1). *Let $\mathbf{v}$ be a Chern character on $\mathbb{F}_e$, $e \geq 1$ such that $\mathrm{rk}(\mathbf{v}) \geq 2$, $\Delta(\mathbf{v}) \geq 0$ and $\nu(\mathbf{v})$ is nef. Then $\mathbf{v}$ is globally generated if and only if one of the following holds:*

(1) *We have $\nu(\mathbf{v}) \cdot F = 0$ and $\mathbf{v} = \mathrm{ch}(\pi^*(\mathcal{O}_{\mathbb{P}^1}(a)^m \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)^{r-m}))$ for some $a \geq 0$.*
(2) *We have $\nu(\mathbf{v}) \cdot F > 0$ and $\chi(\mathbf{v}(-F)) \geq 0$.*
(3) *We have $\nu(\mathbf{v}) \cdot F > 0$, $\chi(\mathbf{v}(-F)) < 0$ and $\chi(\mathbf{v}) \geq r+2$.*
(4) *We have $e = 1$, $\nu(\mathbf{v}) \cdot F > 0$, $\chi(\mathbf{v}(-F)) < 0$, $\chi(\mathbf{v}) \geq r+1$ and*
$$\mathbf{v} = (\mathrm{rk}(\mathbf{v})+1)\,\mathrm{ch}(\mathcal{O}_{\mathbb{F}_1}) - \mathrm{ch}(\mathcal{O}_{\mathbb{F}_1}(-2E-2F)).$$

Since $\mathbb{P}^1 \times \mathbb{P}^1$ admits two fibration structures, the theorem has to account for both fibrations.

Theorem 4.11 ([CH17b], Theorem 5.2). *Let $\mathbf{v}$ be a Chern character on $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\mathrm{rk}(\mathbf{v}) \geq 2$, $\Delta(\mathbf{v}) \geq 0$ and $\nu(\mathbf{v})$ is nef. Let F_1 and F_2 be the classes of the two rulings. The Chern character $\mathbf{v}$ is globally generated if and only if one of the following holds*

(1) *We have $\nu(\mathbf{v}) \cdot F_i = 0$ for some $i \in \{1,2\}$ and $\mathbf{v} = \mathrm{ch}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(aF_i)^m \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}((a+1)F_i)^{r-m})$ for some $a \geq 0$.*
(2) *We have $\nu(\mathbf{v}) \cdot F_i > 0$ for $i \in \{1,2\}$ and $\chi(\mathbf{v}(-F_i)) \geq 0$ for some $i \in \{1,2\}$.*
(3) *We have $\nu(\mathbf{v}) \cdot F_i > 0$ and $\chi(\mathbf{v}(-F_i)) \geq 0$ for $i \in \{1,2\}$ and $\chi(\mathbf{v}) \geq \mathrm{rk}(\mathbf{v}) + 2$.*

As in the case of $\mathbb{P}^2$, it would be very interesting to classify the Chern characters of ample stable (or F-prioritary) bundles on $\mathbb{F}_e$.

Del Pezzo surfaces and more general rational surfaces. Let X be the blowup of $\mathbb{P}^2$ at r points $p_1, \dots, p_k$. If $k \leq 8$ and the points are in general position, then X is a del Pezzo surface. We refer the reader to [Bea83, Cos06b, Har77] for more detailed information on del Pezzo surfaces. Let L denote the pullback of the hyperplane class from $\mathbb{P}^2$ and let $E_1, \dots, E_r$ denote the exceptional divisors lying over $p_1, \dots, p_k$. Then $\mathrm{Pic}(X) \cong \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_k$ and the intersection form is given by
$$L^2 = 1, \quad L \cdot E_i = 0, \quad E_i \cdot E_j = -\delta_{i,j},$$
where $\delta_{i,j}$ denotes the Krönecker delta function. When X is a del Pezzo surface, the (-1)-curves and $-K_X$ generate the effective cone of curves on X. Furthermore, the cohomology of line bundles is completely known. When X is a more general blowup, even the cohomology of line bundles is not known. Consequently, it is unrealistic to expect a full computation of the cohomology of all higher rank sheaves.

Let $\mathbf{v}$ be a Chern character of rank r and let the total slope be $\nu(\mathbf{v}) = \delta L - \sum_{i=1}^k \alpha_i E_i$. Then we have that
$$\delta = d + \frac{q}{r}, \quad \alpha_i = a_i + \frac{q_i}{r}$$
for some integers d, q, a_i and q_i with $0 \leq q < r$ and $0 \leq q_i < r$. Set
$$\gamma(\mathbf{v}) = \frac{q^2}{2r^2} - \frac{q}{2r} + \sum_{i=1}^k \left(\frac{q_i}{2r} - \frac{q_i^2}{2r^2}\right).$$

Theorem 4.12 ([CH16b], Theorem 4.5). *Let X be the blowup of $\mathbb{P}^2$ at k distinct points. Let $\mathbf{v}$ be a positive rank Chern character on X with total slope*
$$\nu(\mathbf{v}) = \delta L - \alpha_1 E_1 \cdots - \alpha_k E_k$$
with $\delta \geq 0$ and $\alpha_i \geq 0$. Suppose that the line bundle
$$\lfloor \delta \rfloor L - \lceil \alpha_1 \rceil E_1 \cdots - \lceil \alpha_k \rceil E_k$$

does not have higher cohomology. Assume that $\Delta(\mathbf{v}) \geq \gamma(\mathbf{v})$. Then $\mathcal{P}_{X,L-E_1}(\mathbf{v})$ is nonempty and the general sheaf in $\mathcal{P}_{X,L-E_1}(\mathbf{v})$ has at most one nonzero cohomology group.

To prove the theorem it suffices to consider the direct sum of line bundles $\mathcal{V} = L_1 \oplus \cdots \oplus L_k$, where each line bundle L_j has the form

$$n_j L - \sum_{i=1}^{k} m_{j,i} E_i$$

with $n_j \in \{\lfloor \delta \rfloor, \lceil \delta \rceil\}$ and $m_{j,i} \in \{\lfloor \alpha_i \rfloor, \lceil \alpha_i \rceil\}$. By choosing L_j appropriately, we can arrange that $\nu(\mathcal{V}) = \nu(\mathbf{v})$. Then each L_j is a line bundle with no higher cohomology. It is easy to check that $\mathcal{V}$ is $(L - E_1)$-prioritary and has $\Delta(\mathcal{V}) = \gamma(\mathbf{v})$. The theorem follows by taking elementary modifications of $\mathcal{V}$.

It is possible to choose the direct sum of line bundles more carefully to obtain sharper bounds when k is small. We refer the reader to [CH16b, §5] when X is a del Pezzo surface of large degree.

Other surfaces. Determining the cohomology of the general stable sheaf and classifying the Chern characters $\mathbf{v}$ such that the general stable sheaf with Chern character $\mathbf{v}$ is globally generated or stable are important problems on any surface. The theory is most developed for K3 surfaces thanks to the work of Leyenson, Markman, Mukai, O'Grady, Yoshioka and many others. We refer the reader to [O'G97, Ley12, Mrk01] for further details and references. We close this section with a few general remarks on Brill-Noether statements on general surfaces. First, an asymptotic weak Brill-Noether statement holds on any smooth projective surface.

Proposition 4.13. *Let X be a smooth projective surface and let H be an ample divisor. Let $\mathbf{v}$ be a Chern character with $\Delta(\mathbf{v}) \gg 0$. Let $\mathcal{V} \in M_{X,H}(\mathbf{v})$ be a general sheaf. Then the only nonzero cohomology group of $\mathcal{V}$ can be $H^1(X, \mathcal{V})$.*

Proof. Let $\mathbf{v}^D$ denote the Serre dual Chern character of $\mathbf{v}$. Observe that $\Delta(\mathbf{v}) = \Delta(\mathbf{v}^D)$. Assume that $\Delta(\mathbf{v}) \geq \delta$, where δ is the O'Grady bound that guarantees that both $M_{X,H}(\mathbf{v})$ and $M_{X,H}(\mathbf{v}^D)$ are irreducible and the general member is a stable bundle. If the general sheaf $\mathcal{V} \in M_{X,H}(\mathbf{v})$ has only H^1, we are done. If $\mathcal{V}$ has any global sections, replace $\mathcal{V}$ by $h^0(X, \mathcal{V})$ general elementary modifications $\mathcal{V}_1$. Then $\mathcal{V}_1$ is a slope stable sheaf, has no H^0, and has discriminant $\Delta(\mathcal{V}_1) = \Delta(\mathcal{V}) + \frac{h^0(X,\mathcal{V})}{\mathrm{rk}(\mathcal{V})} > \delta$. Hence, the moduli space containing $\mathcal{V}_1$ is irreducible. We can find a locally free slope-stable sheaf $\mathcal{V}_2$ with no H^0 since vanishing of H^0 is an open condition. If $\mathcal{V}_2$ has only H^1, we are done. Otherwise, replace $\mathcal{V}_2$ by its Serre dual $\mathcal{V}_3$. Apply $h^0(X, \mathcal{V}_3) = h^2(X, \mathcal{V}_2)$ general elementary modifications to $\mathcal{V}_3$. The resulting sheaf $\mathcal{V}_4$ has vanishing H^0 and H^2 and is slope stable. A general deformation $\mathcal{V}_5$ of $\mathcal{V}_4$ is locally free and also has vanishing H^0 and H^2. The Serre dual of $\mathcal{V}_5$ gives the desired bundle. For $\mathbf{v}$ with $\Delta(\mathbf{v}) \geq \Delta(\mathcal{V}_5)$, the only nonzero cohomology group of a general sheaf in $M_{X,H}(\mathbf{v})$ can be H^1. □

Since the moduli spaces for small Δ are not necessarily irreducible, even when there is a stable sheaf with at most one nonzero cohomology group, there may still be entire components of the moduli space where more than one cohomology group is nonzero. One can already find such examples on Enriques surfaces. The following example is due to Nuer and Yoshioka.

Example 4.14 ([NY17], §10)**.** Nuer and Yoshioka show that there is a component of the moduli space of rank 2 sheaves on an Enriques surface X whose general element is given by an extension of the form

$$0 \to I_Z \to \mathcal{V} \to \mathcal{O}_X(K_X) \to 0,$$

where Z is a zero-dimensional scheme of length 2. Observe that $h^1(X, \mathcal{V}) = h^2(X, \mathcal{V}) = 1$.

On surfaces of general type, even very ample line bundles can have nonzero higher cohomology. Consequently, we would not expect the higher cohomology of higher rank sheaves to vanish either.

Example 4.15. Let X be a very general hypersurface of degree $d \geq 5$ in $\mathbb{P}^3$. Let Z be $d-1$ collinear points on X. Then a general extensions of the form

$$0 \to \mathcal{O}_X \to \mathcal{V} \to I_Z(1) \to 0$$

is slope stable. By §3, such extensions form a component of the moduli space $M_X((2,1,d-1))$ of rank 2 sheaves with c_1 equal to the hyperplane class and $c_2 = d-1$. The long exact sequence of cohomology associated to the defining sequence of $\mathcal{V}$ and Serre duality show that

$$h^0(X,\mathcal{V}) = 3, \quad h^2(X,\mathcal{V}) = \binom{d-1}{3} + \binom{d-2}{3} - d + 3.$$

The line bundle $\mathcal{O}_X(m)$ has no higher cohomology for $m \geq d-3$. More generally, consider extensions of the form

$$0 \to \mathcal{O}_X(m) \to \mathcal{V}(m) \to I_Z(m+1) \to 0.$$

Since $h^2(X, I_Z(m)) \neq 0$ for $m < d-4$, we conclude that $\mathcal{V}$ has nonvanishing h^0 and h^2 for $0 \leq m < d-4$. On the other hand, the higher cohomology of $\mathcal{V}(m)$ vanishes for $m \geq d-4$. This is immediate for $m \geq d-3$ by the long exact sequence of cohomology. The only case to discuss is $m = d-4$. We have

$$0 \to H^1(X, \mathcal{V}(d-4)) \to H^1(X, I_Z(d-3)) \to H^2(X, \mathcal{O}_X(d-4)) \to H^2(X, V(d-4)) \to 0.$$

Moreover, $H^1(X, I_Z(d-3)) \cong H^2(X, \mathcal{O}_X(d-4)) \cong \mathbb{C}$. The Serre dual $\mathcal{W}$ of $\mathcal{V}(d-4)$ fits in the exact sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{W} \to I_Z \to 0,$$

hence $h^0(X,\mathcal{W}) = h^2(X, \mathcal{V}(d-4)) = 0$. We conclude that the higher cohomology of $\mathcal{V}(m)$ vanishes for $m \geq d-4$. Observe that $\mathcal{V}$ is μ-stable and has $\Delta(\mathcal{V}) = \frac{3}{8}d - \frac{1}{2}$. Let $\mathbf{v}$ be a Chern character of rank 2 on X with $\mu(\mathbf{v}) = \frac{2m+1}{2}$ for $m \geq d-4$. If $\Delta(\mathbf{v}) \geq \frac{3}{8}d - \frac{1}{2}$, there exist stable sheaves with Chern character $\mathbf{v}$ that have at most one nonzero cohomology group.

Remark 4.16. It would be interesting to explore the following questions further.

(1) Let X be a projective surface such that $\mathrm{Pic}(X) = \mathbb{Z}H$ for an ample divisor H. Assume that $H^1(X, mH) = H^2(X, mH) = 0$ for $m \geq m_0$. Assume that $\nu(\mathbf{v}) = \mu H$ for $\mu > m_0$. Does there exist a sheaf $\mathcal{V} \in M_{X,H}(\mathbf{v})$ with at most one nonzero cohomology group? What additional assumptions are necessary for surfaces of higher Picard rank?
(2) Assume that the general sheaf in $M_{X,H}(\mathbf{v})$ has no higher cohomology and $\nu(\mathbf{v})$ is ample. Let $\mathcal{V} \in M_{X,H}(\mathbf{v})$ be a general sheaf. If $h^0(X,\mathcal{V}) \geq r+2$, is $\mathcal{V}$ globally generated?

5. Background on Bridgeland stability

In this subsection, we recall key facts concerning Bridgeland stability conditions. We refer the reader to [AB13, Bri07, Bri08, CH15] for more details. We will review the wall-chamber decomposition of the Bridgeland stability manifold and the Positivity Lemma of Bayer and Macrì.

Let $\mathcal{D}^b(X)$ denote the bounded derived category of coherent sheaves on X. A *Bridgeland stability condition* on $\mathcal{D}^b(X)$ is a pair $\sigma = (Z, \mathcal{A})$, where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}^b(X)$ and $Z : K_0(X) \to \mathbb{C}$ is a group homomorphism called the *central charge* satisfying the following properties:

(1) (Positivity) The homomorphism Z maps any nonzero object $E \in \mathcal{A}$ to the extended upper-half plane $\{re^{i\pi\theta} | r > 0, \theta \in (0, 1]\}$. The homomorphism Z allows us to define the *Bridgeland slope* of objects $E \in \mathcal{A}$.
$$\mu_\sigma(E) := -\frac{\Re Z(E)}{\Im Z(E)}.$$
An object E is called μ_σ-(semi)stable if for every nonzero proper subobjects we have
$$\mu_\sigma(F) \underset{(-)}{<} \mu_\sigma(E).$$
(2) (Harder-Narasimhan Property) Every non-zero object $E \in \mathcal{A}$ admits a finite filtration
$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$$
with $E_i \in \mathcal{A}$ such that the quotient $F_i = E_i/E_{i-1}$ are σ-semistable with
$$\mu_\sigma(F_1) > \mu_\sigma(F_2) > \cdots > \mu_\sigma(F_m).$$
(3) (Support Property) Fix a norm $||\cdot||$ on $K_{num} \otimes \mathbb{R}$. Then there exists a constant $C > 0$ such that $||\operatorname{ch}(E)|| \le C|Z(E)|$ for all σ-semistable objects $E \in \mathcal{A}$.

We will only need very special Bridgeland stability conditions for our purposes. These were constructed by Bridgeland [Bri08], Arcara, Bertam [AB13], and Toda [Tod13]. Let s be a real number and define two subcategories of $\operatorname{Coh}(X)$ depending on s as follows
$$\mathcal{T}_s := \{\mathcal{V} \in \operatorname{Coh}(X) \mid \overline{\mu}_{H,D}(\mathcal{W}) > s \text{ for every quotient } \mathcal{V} \twoheadrightarrow \mathcal{W}\}$$
$$\mathcal{F}_s := \{\mathcal{V} \in \operatorname{Coh}(X) \mid \overline{\mu}_{H,D}(\mathcal{W}) \le s \text{ for every subsheaf } \mathcal{W} \subset \mathcal{W}\}$$
The pair $(\mathcal{T}_s, \mathcal{F}_s)$ is a torsion pair for $\operatorname{Coh}(X)$ and allows us to define a *tilt* of $\operatorname{Coh}(X)$ as follows
$$\mathcal{A}_s := \{E^\bullet \in \mathcal{D}^b(X) \mid \mathrm{H}^{-1}(E^\bullet) \in \mathcal{F}_s, \mathrm{H}^0(E^\bullet) \in \mathcal{T}_s, \mathrm{H}^i(E^\bullet) = 0, i \neq -1, 0\}.$$
Then the category $\mathcal{A}_s$ is an abelian subcategory of $\mathcal{D}^b(X)$ which is the heart of a bounded t-structure. For a positive real number t define the central charge
$$Z_{s,t} = -\overline{\operatorname{ch}}_2^{D+tH} + \frac{t^2H^2}{2}\overline{\operatorname{ch}}_0^{D+tH} + iH\overline{\operatorname{ch}}_1^{D+tH}.$$

Theorem 5.1 (Bridgeland, Arcara-Bertram, [Bri08], [AB13])**.** *The pair $\sigma_{s,t} = (\mathcal{A}_s, Z_{s,t})$ is a Bridgeland stability condition for $s, t \in \mathbb{R}$, $t > 0$.*

The $\sigma_{s,t}$-slope of an object $\mathcal{V}$ is given by the formula
$$\nu_{\sigma_{s,t}}(\mathcal{V}) = -\frac{\Re(Z_{s,t}(\mathcal{V}))}{\Im(Z_{s,t}(\mathcal{V}))} = \frac{(\overline{\mu}_{H,D}(\mathcal{V}) - s)^2 - t^2 - 2\overline{\Delta}_{H,D}(\mathcal{V})}{\overline{\mu}_{H,D}(\mathcal{V}) - s}.$$

These stability conditions define a half-plane in the stability manifold $\operatorname{Stab}(X)$ called the (H, D)-*slice.*

Fix a numerical invariant $\mathbf{v} \in K_{num}(X)$. Then there is a locally finite wall and chamber decomposition of $\operatorname{Stab}(X)$ where for stability conditions σ in a chamber the σ-(semi)stable objects remain constant. We are interested in describing this wall and chamber decomposition in the (H, D)-slice.

Problem 5.2. *Given a surface X and a Chern character $\mathbf{v}$, describe explicitly the wall and chamber decomposition in the (H, D)-slice of* $\operatorname{Stab}(X)$*.*

Given a numerical invariant $\mathbf{w} \in K_{num}(X)$ that does not have the same $\sigma_{s,t}$-slope everywhere in the (H, D)-slice, we define the *numerical wall* $W(\mathbf{v}, \mathbf{w})$ as the locus of (s, t) in the (H, D)-slice where $\mathbf{v}$ and $\mathbf{w}$ have the same $\sigma_{s,t}$-slope. A numerical wall is an *actual wall* if there exists a point $(s, t) \in W(\mathbf{v}, \mathbf{w})$ and an exact sequence
$$0 \to F \to E \to G \to 0$$

in $\mathcal{A}_s$ such that E, F, G are $\sigma_{s,t}$-semistable of the same slope.

Assume $\mathbf{v}$ and $\mathbf{w}$ have positive rank. Equating the $\sigma_{s,t}$-slope of $\mathbf{v}$ and $\mathbf{w}$, one can compute the equation of the wall $W(\mathbf{v},\mathbf{w})$ and see the following basic facts:

(1) The numerical walls $W(\mathbf{v},\mathbf{w})$ in the (H,D)-slice are disjoint. If $\overline{\mu}_{H,D}(\mathbf{v}) = \overline{\mu}_{H,D}(\mathbf{w})$, then $W(\mathbf{v},\mathbf{w})$ is the vertical line $s = \overline{\mu}_{H,D}(\mathbf{v})$. If $\overline{\mu}_{H,D}(\mathbf{v}) \neq \overline{\mu}_{H,D}(\mathbf{w})$, then $W(\mathbf{v},\mathbf{w})$ is a semicircle with center $(c,0)$ on the s-axis and radius ρ given by

$$c = \frac{1}{2}\left(\overline{\mu}_{H,D}(\mathbf{v}) + \overline{\mu}_{H,D}(\mathbf{w})\right) - \frac{\overline{\Delta}_{H,D}(\mathbf{v}) - \overline{\Delta}_{H,D}(\mathbf{w})}{\overline{\mu}_{H,D}(\mathbf{v}) - \overline{\mu}_{H,D}(\mathbf{w})},$$

$$\rho^2 = (c - \overline{\mu}_{H,D}(\mathbf{v}))^2 - 2\overline{\Delta}_{H,D}(\mathbf{v}).$$

It may happen that $\rho^2 < 0$, in which case $W(\mathbf{v},\mathbf{w})$ is empty. Also note that, in the formula for ρ^2 one may replace $\mathbf{v}$ with $\mathbf{w}$ and see that the expression does not change using the formula for c.

(2) The semicircular walls to the left of the vertical half-line $s = \overline{\mu}_{H,D}(\mathbf{v})$ are nested. Let W_1, W_2 be numerical walls with centers $(c_1, 0)$ and $(c_2, 0)$, respectively. Then W_1 is nested in W_2 if and only if $c_1 > c_2$.

Using the fact that the numerical walls are disjoint, one may prove the following important fact. If $0 \to F \to E \to G \to 0$ is a destabilizing sequence at some point $(s_0, t_0) \in W(\mathbf{v},\mathbf{w})$, then the sequence is a destabilizing sequence for every point $(s,t) \in W(\mathbf{v},\mathbf{w})$ [ABCH13, Mac14]. We will refer to this fact as *coherence along Bridgeland walls*. Several walls are especially important. The largest semicircular actual wall to the left of $s = \overline{\mu}_{H,D}(\mathbf{v})$ where a Gieseker semistable sheaf is destabilized is called the *Gieseker wall*. If it exists, the largest semicircular actual wall to the left of $s = \overline{\mu}_{H,D}(\mathbf{v})$ below which all Gieseker semistable sheaves are destabilized is called the *collapsing wall*.

Example 5.3 (Hilbert schemes of points on $\mathbb{P}^2$)**.** In [ABCH13], it is shown that $\mathcal{O}_{\mathbb{P}^2}(-1)$ defines the Gieseker wall for Hilbert schemes of points on $\mathbb{P}^2$.

Example 5.4 (The collapsing wall for moduli spaces of sheaves on $\mathbb{P}^2$)**.** The collapsing wall for all moduli spaces of sheaves on $\mathbb{P}^2$ have been computed in [CHW17, Theorem 5.7]. The collapsing wall is always defined by an exceptional bundle, which can be explicitly computed in terms of the Drézet-Le Potier classification of stable bundles on $\mathbb{P}^2$.

Example 5.5. It is also possible to compute the Bridgeland wall where a particular sheaf is destabilized. For example, [CH14] computes the Bridgeland wall where the ideal sheaf of a monomial zero-dimensional scheme is destabilized in terms of the combinatorics of the monomial scheme.

Li and Zhao have computed the Bridgeland walls for $\mathbb{P}^2$ [LZ16]. Bayer and Macrì have computed the Bridgeland walls for moduli spaces of K3 surfaces [BM14a, BM14b]. Minamide, Yanagida, Yoshioka have analyzed the case of abelian surfaces [MYY14, MYY15]. Nuer has analyzed the case of Enriques surfaces [Nue16a]. Finally, there are partial results for $\mathbb{P}^1 \times \mathbb{P}^1$ and other del Pezzo and Hirzebruch surfaces (see [Abe17, BC13, Rya16]). We will discuss the problem of computing the Gieseker wall when $\Delta \gg 0$ in the next section.

The positivity lemma. The link between the ample cone of the moduli space $M_{X,H,D}(\mathbf{v})$ and the Gieseker wall in the (H,D)-slice is provided by Bayer and Macrì's Positivity Lemma. Let $\sigma = (Z, \mathcal{A})$ be a Bridgeland stability condition on X and let $\mathcal{E}/S$ be a flat family of σ-semistable objects parameterized by an algebraic space S. Let p, q denote the two projections of $S \times X$ to S and X, respectively. The Fourier-Mukai transform $\Phi_{\mathcal{E}} : \mathcal{D}^b(S) \to \mathcal{D}^b(X)$ with kernel $\mathcal{E}$ is defined by

$$\Phi_{\mathcal{E}}(A) : q_*(p^*(A) \otimes \mathcal{E}).$$

Define a numerical divisor class $D_{\sigma,\mathcal{E}} \in N^1(S)$ on S by specifying the intersection number of $D_{\sigma,\mathcal{E}}$ on every curve $C \subset S$ as follows:

$$D_{\sigma,\mathcal{E}} \cdot C = \Im\left(-\frac{Z(\Phi_{\mathcal{E}}(\mathcal{O}_C))}{Z(\mathbf{v})}\right).$$

Theorem 5.6 (Positivity Lemma, [BM14a])**.** *The class $D_{\sigma,\mathcal{E}}$ is a nef divisor class on S. A complete integral curve $C \subset S$ satisfies $D_{\sigma,\mathcal{E}} \cdot C = 0$ if and only if the objects parameterized by general points of C are S-equivalent with respect to the stability condition σ.*

The divisor $D_{\sigma,\mathcal{E}}$ can be explicitly expressed in terms of the Donaldson morphism. Given a family of sheaves $\mathcal{E}/S$ on X parameterized by S, the *Donaldson homomorphism*

$$\lambda_{\mathcal{E}} : K(X) \to \mathrm{Pic}(S)$$

associates to a character in $K(X)$ a line bundle on S. Let $\pi_1 : X \times S \to X$ and $\pi_2 : X \times S \to S$ denote the two natural projections. Then $\lambda_{\mathcal{E}}$ is defined by the composition of the maps

$$\lambda_{\mathcal{E}} : K(X) \xrightarrow{\pi_1^*} K^0(X \times S) \xrightarrow{\cdot[\mathcal{E}]} K^0(X \times S) \xrightarrow{\pi_{2!}} K(S) \xrightarrow{\det} \mathrm{Pic}(S),$$

where $\pi_{2!} = \sum_i (-1)^i R^i \pi_{2*}$. Since the Euler pairing $(-,-)$ is nondegenerate on $K_{num} \otimes \mathbb{R}$, for any linear functional ϕ on $K_{num} \otimes \mathbb{R}$ vanishing on $\mathbf{v}$, there exists a vector $\mathbf{w}_\phi \in \mathbf{v}^\perp$ such that

$$\phi(\mathbf{u}) = (\mathbf{w}_\phi, \mathbf{u}).$$

Therefore, there is a unique vector $\mathbf{w}_Z \in \mathbf{v}^\perp \subset K(X)$ such that

$$\Im\left(-\frac{Z(\mathbf{w})}{Z(\mathbf{v})}\right) = (\mathbf{w}_Z, \mathbf{w}) \quad \text{for all } \mathbf{w} \in K(X).$$

Bayer and Macrì prove that their divisor $D_{\sigma,\mathcal{E}}$ is expressible in terms of the Donaldson homomorphism by

$$D_{\sigma,\mathcal{E}} = \lambda_{\mathcal{E}}(\mathbf{w}_Z).$$

It is useful to have explicit formulae for the character $\mathbf{w}_Z$ in the (H,D)-slice of the stability manifold.

Proposition 5.7 ([BHL$^+$15], Proposition 3.8)**.** *Let X be a smooth, complex projective surface, H an ample divisor and D a $\mathbb{Q}$-divisor. If $\sigma \in \mathrm{Stab}(X)$ is a Bridgeland stability condition in the (H,D)-slice on the potential wall for $\mathbf{v}$ with center $(c,0)$, then the character $\mathbf{w}_Z$ is a multiple of*

$$(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2) = (-1, -K_X + cH + D, m),$$

where m is determined by the property that $\mathbf{w}_Z \in \mathbf{v}^\perp$.

6. The ample cone

In this section, using Bridgeland stability we show that the problem of computing the ample cone of moduli spaces of sheaves $M_{X,H}(\mathbf{v})$ is intimately tied to the classification of stable Chern characters on X. We compute the ample cones for several families of moduli spaces as an illustration of the techniques. This section follows [CH17a] closely.

Fix an (H,D)-slice of the stability manifold of X. Let e be the largest positive integer dividing H in $\mathrm{Pic}(X)$. Define the reduced slope

$$\tilde{\mu}_H(\mathbf{v}) = \frac{H^2}{e}\mu_H(\mathbf{v}) = \frac{\mathrm{ch}_1(\mathbf{v}) \cdot H}{e\,\mathrm{ch}_0(\mathbf{v})}.$$

Definition 6.1. An *extremal Chern character* $\mathbf{w}$ for $\mathbf{v}$ is any Chern character satisfying the following properties:

(E1) The rank $r(\mathbf{w})$ satisfies $0 < r(\mathbf{w}) \leq r(\mathbf{v})$, and if $r(\mathbf{w}) = r(\mathbf{v})$, then $c_1(\mathbf{v}) - c_1(\mathbf{w})$ is effective.

(E2) The reduced slope $\tilde{\mu}_H(\mathbf{w})$ satisfies $\tilde{\mu}_H(\mathbf{w}) < \tilde{\mu}_H(\mathbf{v})$, and $\tilde{\mu}_H(\mathbf{w})$ is as close to $\tilde{\mu}_H(\mathbf{v})$ as possible subject to (E1).
(E3) The moduli space $M_{H,D}(\mathbf{w})$ is nonempty.
(E4) The discriminant $\overline{\Delta}_{H,D}(\mathbf{w})$ is as small as possible, subject to (E1)-(E3).
(E5) The rank $r(\mathbf{w})$ is as large as possible, subject to (E1)-(E4).

Since the denominators of the reduced slopes are bounded by $\mathrm{rk}(\mathbf{v})$, it makes sense in (E2) to require $\tilde{\mu}_H(\mathbf{w})$ to be as close as possible to $\tilde{\mu}_H(\mathbf{v})$. Then the conditions (E1) and (E2) uniquely determine $\tilde{\mu}_H(\mathbf{w})$. By the Bogomolov inequality, the discriminant is at least zero. Furthermore, by O'Grady's Theorem, if the discriminant is sufficiently large, the moduli space is nonempty. Hence, condition (E4) uniquely determines $\overline{\Delta}_{H,D}(\mathbf{w})$. Finally, property (E5) uniquely determines $r(\mathbf{w})$. Most importantly, the discriminant $\overline{\Delta}_{H,D}(\mathbf{v})$ plays no role in the determination of $\mathbf{w}$. Thus the triple

$$(r(\mathbf{w}), \tilde{\mu}_H(\mathbf{w}), \overline{\Delta}_{H,D}(\mathbf{w}))$$

is uniquely determined by $r(\mathbf{v})$ and $c_1(\mathbf{v})$; on the other hand, there may be several possible choices for $c_1(\mathbf{w})$. The requirement that $\overline{\Delta}_{H,D}(\mathbf{w})$ is as small as possible may restrict which first Chern classes $c_1(\mathbf{w})$ are permissible.

Recall §5 that the center $(c,0)$ of a Bridgeland wall $W(\mathbf{v},\mathbf{u})$ is given by

$$c = \frac{1}{2}\left(\overline{\mu}_{H,D}(\mathbf{v}) + \overline{\mu}_{H,D}(\mathbf{u})\right) - \frac{\overline{\Delta}_{H,D}(\mathbf{v}) - \overline{\Delta}_{H,D}(\mathbf{u})}{\overline{\mu}_{H,D}(\mathbf{v}) - \overline{\mu}_{H,D}(\mathbf{u})}.$$

The extremal Chern character $\mathbf{w}$ is defined to minimize this quantity assuming that $\mathrm{rk}(\mathbf{u}) \leq \mathrm{rk}(\mathbf{v})$ and $\overline{\Delta}_{H,D}(\mathbf{v}) \gg 0$. The main result of [CH17a] shows that the Gieseker wall in the (H,D)-slice is determined by the extremal character $\mathbf{w}$ assuming that $\overline{\Delta}_{H,D}(\mathbf{v}) \gg 0$.

Theorem 6.2 ([CH17a], Theorem 3.2)**.** *If $\overline{\Delta}_{H,D}(\mathbf{v}) \gg 0$, then the Gieseker wall for $M_{H,D}(\mathbf{v})$ in the (H,D)-slice is $W(\mathbf{v},\mathbf{w})$. There are curves in $M_{H,D}(\mathbf{v})$ parameterizing sheaves which become S-equivalent along this wall.*

If the complete classification of stable Chern characters $\mathbf{u}$ with $0 < \mathrm{rk}(\mathbf{u}) \leq \mathrm{rk}(\mathbf{v})$ is known, then it is routine to compute the extremal Chern character. Bayer and Macrì first used this strategy to compute ample cones when X is a K3 surface [BM14a, BM14b]. Similarly, when $X = \mathbb{P}^2$, the Gieseker walls can be computed using the Drézet-Le Potier classification and one recovers the main theorem of [CH16a]. Similar computations have been carried out for Enriques and abelian surfaces (see [Nue16a], [MYY14], [MYY15]).

Conversely, suppose $\mathrm{Pic}(X) = \mathbb{Z}H$ with H effective. In that case, knowing the Gieseker walls for all Chern characters provides sharp Bogomolov inequalities. Fix a rank r and a slope μ such that $r\mu \in \mathbb{Z}$. Let $\delta(r,\mu)$ be the minimal discriminant of a stable bundle $\mathcal{V}$ on X such that $\mu_H(\mathcal{V}) = \mu$ and $\mathrm{rk}(\mathcal{V}) \leq r$. Then the inequality $\Delta \geq \delta(r,\mu)$ is a sharp Bogomolov inequality for the discriminant of any stable bundle with slope μ and rank at most r on X. Recall that the *Farey sequence* of order n consists of all the reduced fractions with denominator at most n. Express $\mu = \frac{p}{q}$ in reduced terms and let $\frac{p}{q} < \frac{p'}{q'}$ be two consecutive terms in the Farey sequence of order r. Then the median of these $\frac{p+p'}{q+q'}$ is reduced and

$$\frac{p}{q} < \frac{p+p'}{q+q'} < \frac{p'}{q'}$$

are consecutive terms in the Farey sequence of order $q+q'$. It is easy to check that the extremal character $\mathbf{w}$ for the character $\mathbf{v}$ with rank $q+q'$, slope $\frac{p+p'}{q+q'}$, and $\Delta \gg 0$ has rank r and slope $\frac{p}{q}$. Consequently, $\Delta(\mathbf{w}) = \delta(r,\mu)$. We conclude the following corollary.

Corollary 6.3 ([CH17a], Corollary 3.5)**.** *Suppose* $\mathrm{Pic}(X) = \mathbb{Z}H$ *with* H *effective. Computing the Gieseker wall for all* $\mathbf{v}$ *with sufficiently large discriminant is equivalent to computing the function* $\delta(r,\mu)$ *for all* $r > 0$ *and* $\mu \in \mathbb{Q}$ *with* $r\mu \in \mathbb{Z}$*.*

The proof of Theorem 6.2 has two steps. First, we need to bound the Gieseker wall from above. Suppose the destabilizing sequence

$$0 \to \mathcal{W}' \to \mathcal{V} \to \mathcal{F}' \to 0$$

arises from an injection in a category $\mathcal{A}_s$ but is not an injection of sheaves. Then there is an estimate on the size of the corresponding Bridgeland wall proved in [ABCH13] for $\mathbb{P}^2$ and generalized in [BHL+15]. Let $\mathbf{w}' = \mathrm{ch}(\mathcal{W}')$. If $\rho_{\mathbf{w}'}$ is the radius of the wall $W(\mathbf{v},\mathbf{w}')$, then

$$\rho_{\mathbf{w}'}^2 \le \frac{(\min\{\mathrm{rk}(\mathbf{w}')-1, \mathrm{rk}(\mathbf{v})\})^2}{2\,\mathrm{rk}(\mathbf{w}')}\overline{\Delta}_{H,D}(\mathbf{v}).$$

Using this bound, it is easy to check that $W(\mathbf{w},\mathbf{v})$ is larger than any wall defined by a non-injective sheaf morphism if $\overline{\Delta}_{H,D}(\mathbf{v}) \gg 0$. Since $\mathbf{w}$ was chosen to make the wall $W(\mathbf{v},\mathbf{w})$ as small as possible whenever the destabilizing subobject is a subsheaf, it follows that the Gieseker wall is contained in the potential Bridgeland wall $W(\mathbf{v},\mathbf{w})$.

Second, we need to produce a curve C of semistable sheaves that are not S-equivalent in the sense of Gieseker that become S-equivalent for the Bridgeland stability conditions on $W(\mathbf{v},\mathbf{w})$. This shows that the potential wall $W(\mathbf{v},\mathbf{w})$ is an actual wall, hence is the Gieseker wall. In fact, we produce a curve C of non-isomorphic (H,D)-Gieseker stable sheaves that become S-equivalent on $W(\mathbf{v},\mathbf{w})$ by induction on the rank $\mathbf{v}$. Let $\mathbf{u}$ be the quotient Chern character. If $\overline{\Delta}_{H,D}(\mathbf{v}) \gg 0$, then $\overline{\Delta}_{H,D}(\mathbf{u})$ is sufficiently large. Hence, by O'Grady's theorem, there are $\mu_{H,D}$-stable sheaves in $M_{X,H,D}(\mathbf{u})$. Furthermore, the extension group $\mathrm{Ext}^1(\mathcal{U},\mathcal{W})$ is nonzero if $\mathcal{U} \in M_{X,H,D}(\mathbf{u})$ and $\mathcal{W} \in M_{X,H,D}(\mathbf{w})$. The curve C is obtained by varying the extension class between two general objects in these moduli spaces. We need to check that the general extension is (H,D)-Gieseker stable and these extensions are not all isomorphic. The latter is straightforward. To check stability of an extension $\mathcal{V}$ we induct on the rank of $\mathbf{v}$. A calculation shows that the Gieseker wall for $\mathbf{u}$ is nested inside $W(\mathbf{v},\mathbf{w})$. Using this, one checks that $\mathcal{V}$ is Bridgeland stable for stability conditions just above $W(\mathbf{v},\mathbf{w})$. Combined with an easy verification that $\mathcal{V}$ is $\mu_{H,D}$-semistable, we conclude that $\mathcal{V}$ is an (H,D)-Gieseker stable sheaf. We refer the reader to [CH17a, Theorem 6.4] for the details.

The Bayer-Macrì Positivity Lemma allows us to use Theorem 6.2 to obtain nef divisors on the boundary of the nef cone of the moduli spaces of sheaves.

Corollary 6.4. *If the moduli space of* (H,D)*-twisted Gieseker stable sheaves* $M_{X,H,D}(\mathbf{v})$ *does not contain any strictly semistable sheaves and* $\overline{\Delta}_{H,D}(\mathbf{v}) \gg 0$*, then the Bayer-Macrì divisor associated to the Bridgeland wall* $W(\mathbf{v},\mathbf{w})$ *defines a divisor lying on the boundary of the nef cone of* $M_{X,H,D}(\mathbf{v})$*.*

Moreover, Theorem 6.2 and Corollary 6.4 can be applied with only partial knowledge of strong Bogomolov inequalities: one needs to know the classification of stable Chern characters only up to rank r and only for nearby slopes. Consequently, one can compute the ample cones of Hilbert schemes of points or low rank moduli spaces.

Example 6.5 ([BHL+15], Proposition 4.5)**.** Let X be a very general hypersurface of degree $d \ge 4$ in $\mathbb{P}^3$. If $n \ge d-1$, then the nef cone of the Hilbert scheme of n points $\mathrm{Nef}(X^{[n]})$ is spanned by

$$\left(\frac{d-3}{2} + \frac{n}{d}\right) H^{[n]} - \frac{B}{2} \quad \text{and} \quad H^{[n]},$$

where $H^{[n]}$ is the class of the divisor of points whose support intersects a fixed hyperplane section and B is the class of the divisor of nonreduced schemes.

Example 6.6 ([BHL+15], Proposition 4.8)**.** Let X be a very general degree d cyclic cover of $\mathbb{P}^2$ branched along a curve of degree $e \geq \frac{3d}{d-1}$. If $n \geq d$, then $\mathrm{Nef}(X^{[n]})$ is spanned by

$$\left(\frac{ed-2d-e}{2d}+\frac{n}{d}\right)H^{[n]}-\frac{B}{2} \quad \text{and} \quad H^{[n]}.$$

Example 6.7 ([CH17a], Example 7.1)**.** Let X be a very general hypersurface in $\mathbb{P}^3$ of degree $d \geq 4$ and let H denote the hyperplane class. Then the nef cone of the moduli space $M_{X,H}(\mathbf{v})$ with $\mathrm{rk}(\mathbf{v}) = 2$, $\mathrm{ch}_1(\mathbf{v}) = H$ and $\Delta(\mathbf{v}) \gg 0$ is given by the images of

$$(0, H, n), \left(-1, \left(-\frac{d-4}{2}+\frac{\mathrm{ch}_2}{d}\right)H, m\right) \in \mathbf{v}^{\perp},$$

where n and m are determined by the requirement that the vectors lie in $\mathbf{v}^{\perp}$. The first ray defines the Donaldson-Uhlenbeck-Yau morphism and the second ray is the ray computed by Theorem 6.2.

Example 6.8 ([CH17a], Example 7.2)**.** Let X be a very general double cover of $\mathbb{P}^2$ branched along a curve of degree d and let H denote the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$. Then the nef cone of the moduli space $M_{X,H}(\mathbf{v})$ with $\mathrm{rk}(\mathbf{v}) = 2$, $\mathrm{ch}_1(\mathbf{v}) = H$ and $\Delta(\mathbf{v}) \gg 0$ is given by the images of

$$(0, H, n), \left(-1, \left(-\frac{d-3}{2}+\frac{\mathrm{ch}_2}{2}\right)H, m\right) \in \mathbf{v}^{\perp},$$

where n and m are determined by the requirement that the vectors lie in $\mathbf{v}^{\perp}$. The first ray defines the Donaldson-Uhlenbeck-Yau morphism and the second ray is the ray computed by Theorem 6.2.

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