TOWARDS MORI'S PROGRAM FOR THE MODULI SPACE OF STABLE MAPS

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ABSTRACT. We introduce and compute the class of a number of effective divisors on the moduli space of stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, which, for small d, provide a good understanding of the extremal rays and the stable base locus decomposition for the effective cone. We also discuss various birational models that arise in Mori's program, including the Hilbert scheme, the Chow variety, the space of k-stable maps, the space of branchcurves and the space of semi-stable sheaves.

CONTENTS

1. INTRODUCTION

This paper studies the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ from the viewpoint of Mori theory. Mori theory for a moduli space consists of the following program:

Given a moduli space $\overline{\mathcal{M}}$, study its effective cone, ample cone and base loci of divisors. For an integral divisor D on $\overline{\mathcal{M}}$, study the map induced by D and the model $M(D) = Proj\left(\bigoplus_{m \geq 0} H^0(\overline{\mathcal{M}}, mD)\right)$. Finally, give a geometric interpretation of the resulting model $M(D)$ and compare it with $\overline{\mathcal{M}}$.

For a (log) canonical divisor on an arbitrary variety, this is the standard Mori's program or the (log) minimal model program. Moreover, when the variety is a moduli space, the induced models can often be interpreted as other moduli spaces, which are closely related to the original moduli space.

Hassett initiated the program to study the log canonical models of the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of stable genus g curves with n-marked points. One can refer to [H1], [H2], [HH1], [HH2], [HL1], [HL2], [Sm], [mSi], [FS] and [AS] for related results.

We can also consider moduli spaces parameterizing curves in an ambient space. The Kontsevich space of stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ is a compactification for the space of degree d rational curves in $\mathbb{P}^r$, cf. [Kon] and [FP]. We work over $\mathbb{C}$ and always assume that $d, r \geq 2$. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ parameterizes isomorphism classes of stable maps $\mu : C \to \mathbb{P}^r$, where μ has degree d and C is a connected at-worst-nodal curve of arithmetic genus zero. The map is stable if and only if every component of C contracted by μ possesses at least three nodes. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ has a nice geometric structure. The intersection theory on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ provides an approach to many enumerative problems in Gromov-Witten theory of rational curves. Here

we focus on the divisor theory of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ is a smooth Deligne-Mumford stack and its coarse moduli scheme is $\mathbb{Q}$-factorial with finite quotient singularities. Hence, a Weil divisor on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ is $\mathbb{Q}$-Cartier.

Let H be the divisor class of maps whose images meet a fixed codimension two linear subspace in $\mathbb{P}^r$. Let $\Delta_{k,d-k}$ be the class of the boundary divisor consisting of maps with reducible domains, where the map has degree k on one component and degree $d - k$ on the other component. For $d, r \geq 2$, $\mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$ is generated by H and $\Delta_{k,d-k}, 1 \leq k \leq [d/2]$, cf. [P1]. Let $\mathrm{Eff}_{d,r}$ denote the cone of effective divisors on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. For $m > 0$, there is a rational map $\pi : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+m}, d) \dashrightarrow \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ by projecting the image of a map to a linear subspace $\mathbb{P}^r$. π is a morphism in codimension one and induces a homomorphism $\pi^* : \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)) \otimes \mathbb{Q} \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+m}, d)) \otimes \mathbb{Q}$. In [CHS1], it was proved that π^* induces an injection from $\mathrm{Eff}_{d,r}$ to $\mathrm{Eff}_{d,r+m}$. Moreover, the injection is a bijection for $r \geq d$. Hence, we use the same notation for an effective divisor on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ and its pull-back on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+m}, d)$. Similarly, we consider $\mathrm{Eff}_{d,r}$ as a subcone of $\mathrm{Eff}_{d,r+m}$ using this injection. When $d = r$, define D_{deg} as the divisor class of the locus parameterizing maps whose images do not span $\mathbb{P}^d$. It was shown in [CHS1] that a divisor class lies in $\mathrm{Eff}_{d,d}$ if and only if it is a non-negative linear combination of D_{deg} and $\Delta_{k,d-k}$ for $1 \leq k \leq [d/2]$. For $d > r$, the structure of $\mathrm{Eff}_{d,r}$ is generally unknown.

Let us introduce more effective divisors. For $r = 2$, let NL be the divisor class consisting of maps whose images have a node on a fixed line. Let TN be the divisor class of the locus corresponding to maps whose images have a tacnode. Let TR be the divisor class parameterizing maps whose images have a triple point. All these divisors can be pulled back to effective divisors on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ when $r > 2$. Similarly for $r = 3$, consider the divisor class NI parameterizing non-isomorphic maps. A general point in NI corresponds to a map whose image is a degree d irreducible rational one-nodal curve in $\mathbb{P}^3$. NI can be pulled back to an effective divisor on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ for $r > 3$. Let T denote the divisor class on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ parameterizing maps whose images are tangent to a fixed hyperplane, cf. [P1], [CHS1] and [CHS2]. Note that we can define a divisor first in the open locus of maps with smooth domains and then take the closure in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ to obtain a $\mathbb{Q}$-Cartier divisor.

One can also define intrinsic divisors as in [DH]. Let $\mathscr{C}$ be the universal curve over the locus $\overline{\mathcal{M}}^0$ of maps that do not have non-trivial automorphisms. The complement of $\overline{\mathcal{M}}^0$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ has codimension ≥ 2 provided $(r, d) \neq (2, 2)$. In particular, $\overline{\mathcal{M}}^0$ and $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ have the same divisor class group, hence the same rational Picard group. Let η be the universal map from $\mathscr{C}$ to $\mathbb{P}^r$, cf. the following diagram:

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\;\eta\;} & \mathbb{P}^r \\ {\scriptstyle \pi}\downarrow & & \\ \overline{\mathcal{M}}^0 & & \end{array}$$

Define a line bundle $\mathscr{L}$ on $\mathscr{C}$ as $\eta^*(\mathscr{O}_{\mathbb{P}^r}(1))$. Let ω_π denote the relative dualizing sheaf of π. Use D and ω to denote the first Chern classes of $\mathscr{L}$ and ω_π, respectively. $D^2, D.\omega$ and ω^2 are codimension two in $\mathscr{C}$. Their images under π form three divisors in $\overline{\mathcal{M}}^0$: $A = \pi_*(D^2)$, $B = \pi_*(D.\omega)$, $C = \pi_*(\omega^2)$. Let $E_m = \mathscr{L}^m$ for $m > 0$. $\pi_*(E_m)$ is locally free of rank $1 + md$ over $\overline{\mathcal{M}}^0$. We can define a divisor $D_m = c_1(\pi_*(E_m))$. Finally, let $K_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)}$ denote the canonical class of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$.

We first express all the above divisor classes as linear combinations of H and $\Delta_{k,d-k}$. In order to simplify the expressions, define $\Delta = \sum_{k=1}^{[d/2]} \Delta_{k,d-k}$ as the total boundary class and define $\Delta_{wt} = \sum_{k=1}^{[d/2]} \frac{k(d-k)}{d} \Delta_{k,d-k}$ as the weighted boundary class.

Theorem 1.1. *On $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, the divisor classes introduced above have the following expressions:*

$$A = H,$$

$$B = T - H,$$

$$C = -\Delta,$$

$$D_m = (\frac{m^2}{12} + m)H - mT,$$

$$K_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,d)} = -\frac{(d+1)(r+1)}{2d}H + \frac{r+1}{2}\Delta_{wt} - 2\Delta,$$

$$D_{deg} = \frac{d+1}{2d}H - \frac{1}{2}\Delta_{wt},$$

$$T = \frac{d-1}{d}H + \Delta_{wt},$$

$$NL = \frac{(d-1)(2d-1)}{2d}H - \frac{1}{2}\Delta_{wt},$$

$$TN = \frac{3(d-1)(d-3)}{d}H + (d-9)\Delta_{wt} + 4\Delta,$$

$$TR = \frac{(d-1)(d-2)(d-3)}{2d}H - \frac{d-6}{2}\Delta_{wt} - \Delta,$$

$$NI = \frac{(d-1)(d-2)}{d}H - \frac{d}{2}\Delta_{wt} + \Delta.$$

The next step in the proposed Mori's program is to study the stable base locus decomposition of the effective cone. Note that the boundary classes give extremal rays of $\mathrm{Eff}_{d,r}$. However, for $d > r$ we do not know the other extremal rays of $\mathrm{Eff}_{d,r}$. The structure of $\mathrm{Eff}_{d,r}$ can be subtle even for small values of r and d. The first unknown case is $d = 4$. Use $\langle D_1, \ldots, D_k \rangle$ to denote the closed cone generated by effective divisors $D_1, \ldots, D_k$. A face M of a convex cone is a subcone such that if $u + v \in M$, then $u, v \in M$. In particular, an extremal ray is a one-dimensional face. Use $\mathbf{B}(D)$ to denote the stable base locus of an effective divisor D. D is called moving if $\mathbf{B}(D)$ does not contain any divisorial components. Our second result reveals all the extremal rays of $\mathrm{Eff}_{4,r}$ and provides a detailed study for the cone decomposition. For $d = r = 4$, since $D_{deg}, \Delta_{1,3}$ and $\Delta_{2,2}$ generate $\mathrm{Eff}_{4,4}$, they are the only possible divisorial components contained in $\mathbf{B}(D)$ for $D \in \mathrm{Eff}_{4,4}$. Introduce two other divisor classes $P = H + \Delta_{1,3} + 4\Delta_{2,2}$ and $Q = 3H + 3\Delta_{1,3} - 2\Delta_{2,2}$ on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 4)$. See section 2 for a chamber decomposition of $\mathrm{Eff}_{4,4}$ by the above divisors.

Theorem 1.2.

(i) $\mathrm{Eff}_{4,3}$ is generated by $NI, \Delta_{1,3}$ and $\Delta_{2,2}$; $\mathrm{Eff}_{4,2}$ is generated by $TR, TN, \Delta_{1,3}$ and $\Delta_{2,2}$.

(ii) Let D be an effective divisor on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 4)$. $\mathbf{B}(D)$ contains $\Delta_{2,2}$ if and only if D lies in the union of the two chambers $\langle P, \Delta_{1,3}, \Delta_{2,2} \rangle \cup \langle D_{deg}, P, \Delta_{2,2} \rangle$; $\mathbf{B}(D)$ contains D_{deg} if and only if D lies in the union of the two chambers $\langle D_{deg}, NI, \Delta_{2,2} \rangle \cup \langle D_{deg}, NI, \Delta_{1,3} \rangle$; $\mathbf{B}(D)$ contains $\Delta_{1,3}$ if D lies in the union of the two chambers $\langle Q, \Delta_{1,3}, \Delta_{2,2} \rangle \cup \langle D_{deg}, Q, \Delta_{1,3} \rangle$. In particular, the cone of moving divisors on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 4)$ is contained in $\langle NI, Q, P, TR \rangle$.

Let us come back to the final step of our program and consider the model

$$M(D) = \mathrm{Proj}\Big(\bigoplus_{m \geq 0} H^0(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d), mD) \Big).$$

For $d = r = 3$, it was proved in [C] that the resulting models are $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$, the component of the Hilbert scheme parameterizing twisted cubics, the normalization of the Chow component, the closure of the locus of twisted cubics in the space of nets of quadrics and the moduli space of 2-stable maps. Before generalizing the result to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ for arbitrary d and r, let us first introduce some special maps in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$.

Definition 1.3. Define a *non-finite map* in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ as a map $[C, \mu]$ such that there is a component C_0 of C mapping to a point under μ. Call a non-finite map $[C, \mu]$ *with moduli* if there exists a connected subcurve C_0 of C contracted by μ with the property that $C \backslash C_0$ has at least four connected components. Define a *degree k tail* of a map $[C, \mu]$ as a component C_1 of C such that C_1 meets $\overline{C \backslash C_1}$ at one point and the degree of μ restricted to C_1 equals k. Define a *multi-image map* in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ as a map $[C, \mu]$ such that μ restricted to a component C_2 of C is a degree ≥ 2 multiple cover of its image $\mu(C_2)$.

By the definition, if $[C, \mu]$ is a non-finite map without moduli, every contracted connected subcurve C_0 has at most three intersection points with $\overline{C \backslash C_0}$. By the stability of the map, C_0 is a smooth rational curve that meets $\overline{C \backslash C_0}$ at three points. Three points on $\mathbb{P}^1$ do not have non-isomorphic moduli structures.

There are a few other modular compactifications for the space of degree d rational curves in $\mathbb{P}^r$. In the Chow variety parameterizing degree d one-dimensional cycles in $\mathbb{P}^r$, let $\mathrm{Chow}_{d,0,r}$ denote the closure of the locus corresponding to integral rational curves. The relation between $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ and $\mathrm{Chow}_{d,0,r}$ can be interpreted using the divisor class H.

Theorem 1.4. *There is a forgetful morphism $f : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to \mathrm{Chow}_{d,0,r}$ contracting the loci of multi-image maps and non-finite maps with moduli. f factors through the model $M(H)$ and $M(H)$ is the normalization of $\mathrm{Chow}_{d,0,r}$.*

Corollary 1.5. *For $d \geq 3$ and $r \geq 2$, $\mathrm{Chow}_{d,0,r}$ is not normal.*

For $1 \leq k \leq [\frac{d-1}{2}]$, let $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d-k)$ be the moduli space of k-stable maps constructed in [MM] and [Pa]. It differs from $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ in that a degree $e \leq k$ tail of a stable map is replaced by a base point of multiplicity e on the domain curve. There is a morphism $\rho_k : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d-k)$ that contracts boundary components $\Delta_{1,d-1}, \ldots, \Delta_{k,d-k}$. This relation between $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ and $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d-k)$ can be revealed using the divisor T for $k = 1$ and $\Lambda_k(\alpha)$ coming from S_d-equivariant divisors in the log canonical model program of $\overline{\mathcal{M}}_{0,d}$ for $k \geq 2$.

Theorem 1.6. *For $k = 1$, there is a morphism $t : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to M(T)$ contracting $\Delta_{1,d-1}$ and the locus of non-finite maps with moduli. t factors through ρ_1 and the induced morphism $t_1 : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d-1) \to M(T)$ only contracts the locus of non-finite maps with moduli. For $k = 2, \ldots, [\frac{d-1}{2}]$ and any $\alpha \in (\frac{2}{k+2}, \frac{2}{k+1}]$, there is a semi-ample divisor $\Lambda_k(\alpha)$ on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ such that the induced morphism $\lambda_k(\alpha) : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to M(\Lambda_k(\alpha))$ contracts boundary components $\Delta_{1,d-1}, \ldots, \Delta_{k,d-k}$. In particular, $\lambda_k(\alpha)$ factors through $\rho_k : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d-k)$.*

In [AK], the moduli space of branchvarieties parameterizing isomorphism classes of finite morphisms f from reduced (possibly reducible) varieties to $\mathbb{P}^r$ with fixed Hilbert polynomial $\chi(f^* \mathcal{O}_{\mathbb{P}^r}(m))$ for $m \gg 0$ was constructed. Let $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ be the moduli space of branchcurves with Hilbert polynomial $1 + md$, i.e. degree d finite maps from reduced connected arithmetic genus zero curves to $\mathbb{P}^r$. The relation between $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ and $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$, which was observed in [AK, Remark 9.3] and [AL], can be interpreted using a positive linear combination of H and T.

Theorem 1.7. *For a divisor $D = aH + bT$, $a, b > 0$ on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, there is a morphism $\varphi : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to M(D)$ that contracts the locus of non-finite maps with moduli. In particular, there is a morphism $\pi : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to \overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ that factors through $M(D)$ and $M(D)$ is the normalization of $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$.*

Corollary 1.8. *The moduli scheme of branchcurves $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ is projective.*

Finally, let $\mathrm{Simp}^{dm+1}(\mathbb{P}^r)$ be the Simpson moduli space of semi-stable sheaves in $\mathbb{P}^r$ with Hilbert polynomial $dm+1$, cf. [cSi]. For $r \geq 2$, let $[C, \mu]$ denote a stable map that is not a multi-image map. The direct image $\mu_*(\mathcal{O}_C)$ is a stable sheaf parameterized by a point of $\mathrm{Simp}^{dm+1}(\mathbb{P}^r)$, cf. Proposition 3.18.

Denote by $S^{dm+1}(\mathbb{P}^r)$ the closure of the locus corresponding to such sheaves in $\mathrm{Simp}^{dm+1}(\mathbb{P}^r)$. For $d = 3$, we show a wall-crossing phenomenon that flips the space of maps to the space of sheaves.

Theorem 1.9. *Let $D = aH + bNL$, $a, b > 0$ be a divisor class on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. For $d = 3$ and $r \geq 2$, $S^{3m+1}(\mathbb{P}^r)$ is isomorphic to the model $M(D)$, which is the flipping space of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 3)$ over the Chow variety with respect to D. The locus of multi-image maps is flipped to the locus of sheaves with non-reduced supports. The birational transform of D on $S^{3m+1}(\mathbb{P}^r)$ is ample.*

One can run Mori's program for other Kontsevich spaces, for instance, stable maps to the Grassmannians $\mathbb{G}(k, n)$. See [CC] for a complete discussion when the degree of the map is two or three.

This paper is organized as follows. In section 2, we compute various divisor classes and study in detail the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 4)$. In section 3, we discuss models induced by some of the divisors and their geometric meanings. In section 4, we use alternate spaces to compute the volume of a divisor. In the appendix, a Macaulay 2 code written by Charley Crissman is included to verify certain moving curves used in the proof of Theorem 1.2. Throughout the paper, we work over $\mathbb{C}$. We always assume that $d, r \geq 2$. By a divisor we mean a $\mathbb{Q}$-Cartier divisor. Two divisor classes are equal if they are numerically equivalent. We often consider a divisor class up to a positive scalar, since it does not affect the stable base locus and the induced model.

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2. DIVISOR CLASSES

In this section, we prove Theorem 1.1 and discuss the decomposition of the effective cone for $d = 4$.

Proof of Theorem 1.1. The divisor classes T, $K_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)}$ and D_{deg} have been computed in [P1], [P2] and [CHS1], respectively, in a more general setting. The expressions for NL, TN and TR follow from the corresponding formulae in [DH].

The relation $A = H$ is directly from the definition of A. Note that the self-intersection of the first Chern class of the relative dualizing sheaf on a ruled surface drops by one when we blow up a point to get a node on the corresponding fiber. Therefore, the relation $C = -\Delta$ holds on a general one-parameter family of stable maps. Then it holds on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ as well. For the class of B, it suffices to verify that $T = A + B$. Fix a defining hyperplane of T and pull it back by η to the divisor D. The locus T in $\overline{\mathcal{M}}^0$ is the branch locus of the map π restricted to D. By adjunction, $T = \pi_*(\omega + D)|_D = A + B$.

For D_m, apply Grothendieck-Riemann-Roch to a one-parameter family of stable maps with smooth general fibers. We get

$$ch(\pi_! E_m) = \pi_*(ch(E_m) \cdot td(\omega_\pi^\vee)).$$

Notice that

$$td(\omega_\pi^\vee) = 1 - \frac{\omega}{2} + \frac{\omega^2 + \delta}{12} + \dots,$$

where δ is the class of nodes. We have already seen that $\omega^2 + \delta = 0$ for such a family. Moreover, $R^i \pi_* E_m = 0$ for $i > 0$. Therefore, we get

$$D_m = c_1(\pi_* E_m) = \frac{1}{12}\pi_* c_1^2(E_m) - \pi_*(\omega. c_1(E_m)).$$

Since $\pi_* c_1^2(E_m) = m^2 H$ and $\pi_*(\omega. c_1(E_m)) = mB = m(T - H)$, we thus obtain $D_m = (\frac{m^2}{12} + m)H - mT$.

For the divisor NI, let us compute its class on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, d)$. Assume $NI = aH + \sum_{k=1}^{[d/2]} b_k \Delta_{k,d-k}$. Take a test family C_1 as a pencil of lines union a fixed degree $d-1$ rational curve at the base point. We have the following intersection numbers,

$$C_1. H = 1, \ C_1. \Delta_{1,d-1} = -1,$$

$$C_1. \Delta_{i,d-i} = 0 \text{ for } 1 < i \leq [d/2], \ C_1. NI = d - 2,$$

which indicates $a - b_1 = d - 2$.

Take another test family C_2 as a pencil of conics union a fixed degree $d-2$ rational curve at one of the base points. The intersection numbers are as follows,

$$C_2. H = 1, \ C_2. \Delta_{1,d-1} = 3, \ C_2. \Delta_{2,d-2} = -1,$$

$$C_2. \Delta_{i,d-i} = 0 \text{ for } 2 < i \leq [d/2], \ C_2. NI = d - 3,$$

from which we derive $a + 3b_1 - b_2 = d - 3$.

For $2 < k \leq [d/2]$, let C_k be a pencil of degree k rational curves of type $(1, k-1)$ on a fixed quadric surface Q union a fixed degree $d - k$ rational curve at one of its base points. Now we have

$$C_k. H = 2, \ C_k. \Delta_{1,d-1} = 2k - 3, \ C_k. \Delta_{k-1,d-k+1} = 1,$$

$$C_k. \Delta_{k,d-k} = -1, \ C_k. NI = 2(d - k) - 1,$$

hence $2a + (2k - 3)b_1 + b_{k-1} - b_k = 2d - 2k - 1$.

Finally, let C_0 be a pencil of rational curves of type $(1, d-1)$ on Q. Since

$$C_0. H = 2, \ C_0. \Delta_{1,d-1} = 2d - 2,$$

$$C_0. \Delta_{i,d-i} = 0 \text{ for } 1 < i \leq [d/2], \ C_0. NI = 0,$$

we have $2a + (2d - 2)b_1 = 0$.

Solving the above equations for a and b_i, the expression of the divisor class NI follows. $\qquad \square$

When $d = 4$, the divisors in Theorem 1.1 specialize to the following:

$$T = \frac{3}{4}H + \frac{3}{4}\Delta_{1,3} + \Delta_{2,2},$$

$$D_{deg} = \frac{5}{8}H - \frac{3}{8}\Delta_{1,3} - \frac{1}{2}\Delta_{2,2},$$

$$NL = \frac{21}{8}H - \frac{3}{8}\Delta_{1,3} - \frac{1}{2}\Delta_{2,2},$$

$$TN = \frac{9}{4}H + \frac{1}{4}\Delta_{1,3} - \Delta_{2,2},$$

$$TR = \frac{3}{4}H - \frac{1}{4}\Delta_{1,3},$$

$$NI = \frac{3}{2}H - \frac{1}{2}\Delta_{1,3} - \Delta_{2,2}.$$

We need two more divisors:

$$P = H + \Delta_{1,3} + 4\Delta_{2,2},$$

$$Q = 3H + 3\Delta_{1,3} - 2\Delta_{2,2}.$$

P spans an extremal ray for the nef cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 4)$. Up to a scalar, Q is defined as the intersection of the two planes spanned by $\langle NI, \Delta_{1,3} \rangle$ and $\langle T, \Delta_{2,2} \rangle$. Note that $\langle P, H, T \rangle$ generate the nef (semi-ample) cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 4)$, cf. [CHS2, Remark 5.1]. Moreover, the following groups of divisors are coplanar:

$$D_{deg}, \ NL, \ H, \ T, \ \Delta_{wt};$$

$$D_{deg}, \ TR, \ P;$$

$$D_{deg}, \ NI, \ TN;$$

$$TR, \ H, \ \Delta_{1,3};$$
$$TR, \ NL, \ TN;$$
$$NI, \ TR, \ \Delta_{2,2};$$
$$NI, \ Q, \ \Delta_{1,3};$$
$$Q, \ T, \ P, \ \Delta_{2,2}.$$

Up to a scalar, they decompose $\text{Eff}_{4,4}$ as follows:
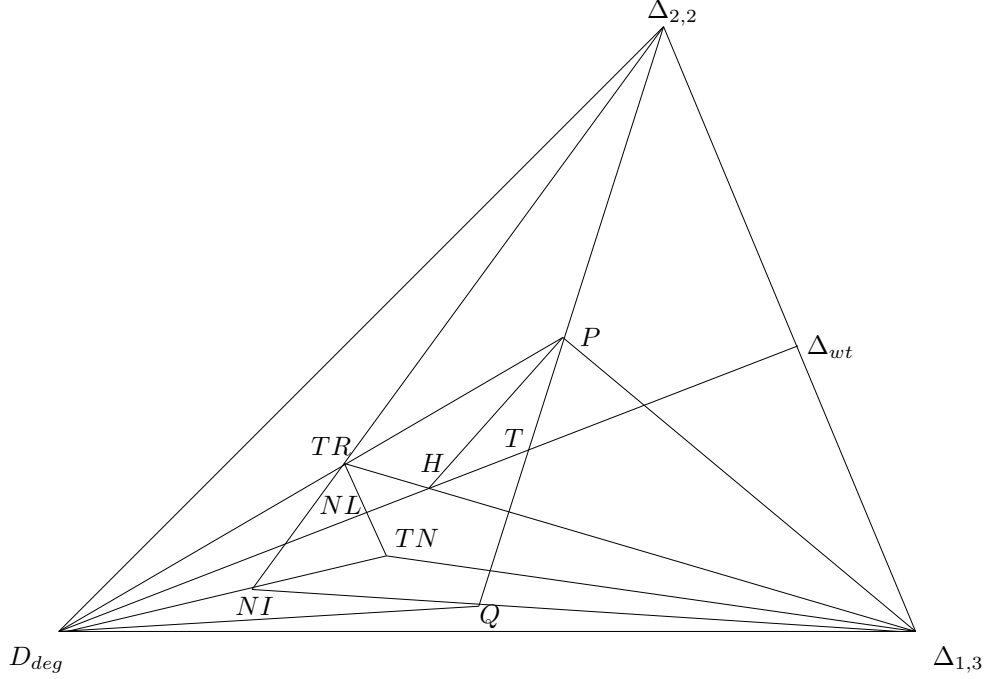
Our strategy to prove Theorem 1.2 is to find moving curves to detect the base locus of a divisor and bound the effective cone. An irreducible curve R is moving on a variety X if the deformation of R covers an open dense subset of X. A moving curve intersects an effective divisor non-negatively.

We first introduce some curves on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 4)$. Let $B_{1,3}$ denote a pencil of lines union a fixed twisted cubic at the base point. $B_{1,3}$ is a moving curve on $\Delta_{1,3}$. Let $B_{2,2}$ denote a pencil of plane conics union a fixed conic at a base point. $B_{2,2}$ is a moving curve on $\Delta_{2,2}$. Let B_1 be a general pencil of plane rational nodal quartics that pass through three fixed nodes. B_1 is moving on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 4)$. Let B_2 be a general pencil in the linear system $|3F_1 + F_2|$ on a quadric surface in $\mathbb{P}^3$, where F_1 and F_2 are the two rulings. B_2 is a moving curve on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 4)$, hence moving on D_{deg}. Their intersection numbers with H, $\Delta_{1,3}$ and $\Delta_{2,2}$ are as follows:

$$B_{1,3}.\, H = 1, \ B_{1,3}.\, \Delta_{1,3} = -1, \ B_{1,3}.\, \Delta_{2,2} = 0;$$
$$B_{2,2}.\, H = 1, \ B_{2,2}.\, \Delta_{1,3} = 3, \ B_{2,2}.\, \Delta_{2,2} = -1;$$
$$B_1.\, H = 1, \ B_1.\, \Delta_{1,3} = 3, \ B_1.\, \Delta_{2,2} = 3;$$
$$B_2.\, H = 2, \ B_2.\, \Delta_{1,3} = 6, \ B_2.\, \Delta_{2,2} = 0.$$

We need two more complicated moving curves as follows. Take the surface $\mathbb{P}^1 \times \mathbb{P}^1$. Let F_1 and F_2 denote the two fiber classes. Consider the linear system $|4F_1 + mF_2|$, which has projective dimension $5(m+1)-1$. Its restriction to F_2 has degree 4. Impose $s = \frac{1}{3}(5(m+1)-1-r)$ double points, where m is chosen so that s is an integer. Consider the line bundle $L = 4F_1 + mF_2 - 2\sum_{i=1}^{s} E_i$ on S, where S is the

blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at s general points and E_i's are the exceptional curves. We still use F_1 and F_2 to denote their birational transforms on S, respectively. If those double points impose independent conditions, the one-parameter family of F_2 under the image of the linear system $|L|$ provides a one-parameter family C_r of rational quartics in $\mathbb{P}^r$. Moreover, we have the intersection numbers:

$$C_r.\, H = L.\, L = 8m - 4s, \ C_r.\, \Delta_{1,3} = 0, \ C_r.\, \Delta_{2,2} = s.$$

When $r = 2$, take $m = 5$, $s = 9$ and we get a curve C_2. When $r = 3$, take $m = 7$, $s = 12$ and we get another curve C_3. C_2 and C_3 satisfy the following intersection numbers:

$$C_2.\, H = 4, \ C_2.\, \Delta_{2,2} = 9, \ C_2.\, TN = 0;$$

$$C_3.\, H = 8, \ C_3.\, \Delta_{2,2} = 12, \ C_3.\, NI = 0.$$

The fact that the double points impose independent conditions in these two cases can be verified using a checker game described in [Y] and [CHS1]. Note that rational quartic curves in $\mathbb{P}^2$ or $\mathbb{P}^3$ have non-constant moduli up to projective equivalence. We use a Macaulay 2 program by Charley Crissman in the appendix to verify that C_2 and C_3 are indeed moving on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 4)$ and $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 4)$, respectively.

Proof of Theorem 1.2 (i). The moving curve B_2 on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 4)$ has zero intersection with NI and $\Delta_{2,2}$. If the sum of two effective divisors $D_1 + D_2$ belongs to the plane spanned by NI and $\Delta_{2,2}$, $B_2.\, (D_1 + D_2) = 0$. Since B_2 is moving, B_2 meets D_1 and D_2 non-negatively. Hence, $B_2.\, D_1 = B_2.\, D_2 = 0$. Since the Picard number of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 4)$ is three, D_1 and D_2 are coplanar with NI and $\Delta_{2,2}$. Hence, NI and $\Delta_{2,2}$ span a face of $\mathrm{Eff}_{4,3}$. Similarly, the moving curve C_3 on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 4)$ has zero intersection with NI and $\Delta_{1,3}$. Hence, NI and $\Delta_{1,3}$ span a face of $\mathrm{Eff}_{4,3}$. Since NI is contained in the intersection of the two faces, NI spans an extremal ray of $\mathrm{Eff}_{4,3}$. Therefore, $\mathrm{Eff}_{4,3}$ is generated by $NI, \Delta_{1,3}$ and $\Delta_{2,2}$.

For $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 4)$, the moving curve B_1 has zero intersection with TR and TN. Hence, TR and TN span a face of $\mathrm{Eff}_{4,2}$. The moving curve C_2 on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 4)$ has zero intersection with TN and $\Delta_{1,3}$. Hence, TN and $\Delta_{1,3}$ span a face of $\mathrm{Eff}_{4,2}$. The projection of B_2 to $\mathbb{P}^2$ is a moving curve on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 4)$, which has zero intersection with $\Delta_{2,2}$ and TR. Hence, TR and $\Delta_{2,2}$ span a face of $\mathrm{Eff}_{4,2}$. Therefore, TR and TN are the only extremal rays of $\mathrm{Eff}_{4,2}$ in addition to the boundary divisors. $\mathrm{Eff}_{4,2}$ is generated by $TR, TN, \Delta_{1,3}$ and $\Delta_{2,2}$. $\qquad\square$

Proof of Theorem 1.2 (ii). Write a divisor class in the form $D = aH + b\Delta_{1,3} + c\Delta_{2,2}$. Since $B_{2,2}$ is moving on $\Delta_{2,2}$, if $B_{2,2}.\, D < 0$, i.e. $a + 3b - c < 0$, $\mathbf{B}(D)$ has to contain $\Delta_{2,2}$. Note that if D is inside the chamber $\langle D_{deg}, P, \Delta_{2,2} \rangle$, the above inequality holds. For D inside the chamber $\langle \Delta_{2,2}, P, \Delta_{1,3} \rangle$, $B_{1,3}.\, D < 0$, so $\mathbf{B}(D)$ always contains $\Delta_{1,3}$. After removing $\Delta_{1,3}$, a positive linear combination of P and $\Delta_{2,2}$ satisfies the inequality $a + 3b - c < 0$. By contrast, if D lies in the chamber $\langle D_{deg}, P, \Delta_{1,3} \rangle$, since P is eventually base-point-free, $\mathbf{B}(D)$ does not contain $\Delta_{2,2}$.

Next, B_2 is a moving curve on D_{deg}. Note that $B_2.\, D < 0$ if and only if $a + 3b < 0$, which implies that $\mathbf{B}(D)$ contains D_{deg} if D is in the chamber $\langle D_{deg}, NI, \Delta_{2,2} \rangle$. Consider the other moving curve C_3 on D_{deg}. For D in the chamber $\langle D_{deg}, NI, \Delta_{1,3} \rangle$, $C_3.\, D_{deg} < 0$, so $\mathbf{B}(D)$ contains D_{deg}. By contrast, if D lies in the chamber $\langle NI, \Delta_{1,3}, \Delta_{2,2} \rangle$, $\mathbf{B}(D)$ does not contain D_{deg}.

Finally, $B_{1,3}$ is a moving curve on $\Delta_{1,3}$. $B_{1,3}.\, D < 0$ if and only if $a - b < 0$. For a divisor D in the chamber $\langle Q, \Delta_{1,3}, \Delta_{2,2} \rangle$, this inequality holds. So $\mathbf{B}(D)$ contains $\Delta_{1,3}$. For D in the chamber $\langle D_{deg}, Q, \Delta_{1,3} \rangle$, we have seen that $\mathbf{B}(D)$ contains D_{deg}. After removing D_{deg}, a positive linear combination of Q and $\Delta_{1,3}$ satisfies the inequality $a - b < 0$. Therefore, $\mathbf{B}(D)$ contains $\Delta_{1,3}$.

Note that the complement of the above chambers in $\mathrm{Eff}_{4,4}$ is $\langle NI, Q, P, TR \rangle$. Hence, the cone of moving divisors on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 4)$ is contained in $\langle NI, Q, P, TR \rangle$. $\qquad\square$

Remark 2.1. If one can interpret Q as a geometrically defined divisor without divisorial stable base loci, then the cone of moving divisors on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 4)$ will be equal to $\langle NI, Q, P, TR \rangle$.

3. Geometric models

In this section, we discuss the model $M(D)$ of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ induced by a divisor class D. We will verify Theorems 1.4, 1.6, 1.7 and 1.9. In order to figure out the relation between $M(D)$ and another known moduli space, the following result will be used frequently.

Lemma 3.1. *Let X be a normal projective variety and L be a semi-ample line bundle on X. The section ring of L is finitely generated. Denote $M(L) = Proj\left(\bigoplus_{m \geq 0} H^0(X, mL)\right)$ and $f : X \to M(L)$. Then $f_*\mathcal{O}_X \cong \mathcal{O}_{M(L)}$. If there is a morphism $g : X \to Y$ such that every curve contracted by f is also contracted by g, then g factors through f. If L' is an ample line bundle on Y, then g factors through $M(g^*L')$ and $M(g^*L')$ is the normalization of Y.*

For a proof of this lemma, one can refer to [L, 2.1.B].

Before getting into the general discussion, recall what is known for small values of d and r. For $r = d = 2$, $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$ is isomorphic to the blow-up of the Hilbert scheme of degree two arithmetic genus zero plane curves along the locus of double lines, i.e. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2) \cong \mathrm{Bl}_{V_2}\mathbb{P}^5$, where V_2 is the image of the degree two Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. The effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$ is generated by D_{deg} and $\Delta = \Delta_{1,1}$. Its nef cone is generated by H and T. These four divisors provide the desired chamber decomposition. It is easy to check that $M(H) \cong \mathbb{P}^5$ the Hilbert scheme of degree two arithmetic genus zero plane curves. For a divisor D in the chamber bounded by D_{deg} and H, $\mathbf{B}(D) = D_{deg}$. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$ is also isomorphic to the space of complete conics. The map sending a conic to its dual transforms H to T and D_{deg} to Δ. So $M(T) \cong \mathbb{P}^5$ and $\mathbf{B}(D) = \Delta$ for a divisor D bounded by T and Δ, cf. [CC].

Next, let us consider $d = r = 3$. The divisors D_{deg}, NL, H, T and $\Delta = \Delta_{1,2}$ provide the stable base locus decomposition of the effective cone. The nef cone is bounded by H and T. The model $M(H)$ is the normalization of the corresponding Chow component of twisted cubics. The model $M(T)$ is the space of 2-stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3, 2)$. For a divisor D in the chamber bounded by H and NL, $\mathbf{B}(D)$ is the locus of multi-image maps and $M(D)$ is the corresponding Hilbert component containing twisted cubics. Moreover, this Hilbert component is the flip of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ over the Chow variety. $M(NL)$ is the closure of the locus of twisted cubics in the space of nets of quadrics. For a divisor D in the chamber bounded by NL and D_{deg}, $\mathbf{B}(D) = D_{deg}$. If D is contained in the chamber bounded by T and Δ, then $\mathbf{B}(D) = \Delta$, cf. [C].

One more example, which can be analyzed using the methods of [C] and [CC], is $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 3)$. The effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 3)$ is generated by NL and Δ. Again, H and T decompose the cone into three chambers. The nef cone is bounded by H and T. $M(T)$ is the space of 2-stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 3, 2)$. For a divisor D in the chamber bounded by T and Δ, $\mathbf{B}(D) = \Delta$. $M(H)$ is the normalization of the discriminant hypersurface in $\mathbb{P}^9$ parameterizing plane nodal cubics and their degenerations. For a divisor D bounded by H and NL, $M(D) \subset \mathbb{P}^9 \times \mathbb{P}^2$ parameterizes pairs (C, p) where C is a singular plane cubic and p is in the singular locus of C. In particular, $M(D)$ is a $\mathbb{P}^6$-bundle over $\mathbb{P}^2$ so it is smooth. Moreover, NL induces a projection to the base $\mathbb{P}^2$ such that the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ is equivalent to NL. We thus obtain that $M(NL) \cong \mathbb{P}^2$.

After these examples, let us prove Theorem 1.4.

Proof of Theorem 1.4. Take a multi-image map and vary a branch point on its image to get a one-dimensional family R_1 of multi-image maps. All maps parameterized by R_1 have the same image, so $R_1. H = 0$. Similarly, take a general non-finite map with moduli. Vary other domain components along the contracted subcurve to obtain a one-dimensional family R_2 of non-finite maps. All maps parameterized by R_2 have the same image, so $R_2. H = 0$. Conversely, if an irreducible curve R has zero intersection with H, then the images of maps parameterized by R cannot vary. Otherwise their images span a surface in $\mathbb{P}^r$ that meets a general codimension two linear subspace at finitely many points, which implies that $R. H > 0$, a contradiction. Hence, maps parameterized by R are either multi-image maps or non-finite maps. If the maps parameterized by R are non-finite, in order for these maps to have the

same image, there exist contracted subcurves of the maps that have different moduli in this family. Since three points on a rational curve have unique moduli, this subcurve of each map has to meet the rest of the domain curve at four points or more. Hence, R is a family of non-finite maps with moduli.

To see the connection between $M(H)$ and $\mathrm{Chow}_{d,0,r}$, note that there is a natural morphism $f : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to \mathrm{Chow}_{d,0,r}$ that forgets a map and only remembers its image as a cycle class with multiplicity. Since $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ is normal, f is well-defined, cf. [Kol, Chapter I]. The defining ample divisor on the Chow variety pulls back to H on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ up to a scalar, cf. [H, 1.a]. Then the desired statement follows from Lemma 3.1.   □

This implies that $\mathrm{Chow}_{d,0,r}$ is not normal for $d \geq 3$ and $r \geq 2$.

Proof of Corollary 1.5. Consider the union of a line L with a general degree $d - 1$ rational nodal curve C on a plane. As a Chow point, it is unique. However, L meets C at $d - 1 \geq 2$ points. Therefore, its pre-image under f consists of $d - 1$ distinct maps in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, which implies that $\mathrm{Chow}_{d,0,r}$ is not normal by Zariski's Main Theorem, cf. [Ha, Corollary 11.4].   □

Before proving Theorem 1.6, let us consider the following example to get a feel of the space of k-stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d - k)$. Suppose there is a one-parameter family of degree d maps from $\mathbb{P}^1$ to $\mathbb{P}^r$ given by $[F_0(t), \ldots, F_r(t)]$ over a base $B = \mathrm{Spec}\ \mathbb{C}[t]$, where $F_i(t)$'s are degree d homogeneous polynomials in two variables X, Y. Assume that $F_0(0), \ldots, F_r(0)$ have a common zero point p of multiplicity $e \leq k$ on the central fiber. Then the total map $\mathbb{P}^1 \times B \dashrightarrow \mathbb{P}^r$ is only a rational map, since it is not defined at p. If this map can be resolved by blowing up p, then the central fiber would become a union of two components mapping with degree $d - e$ and e, respectively, which can be parameterized by a point in the boundary $\Delta_{e,d-e}$ of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. On the other hand, we can also keep p as a base point of multiplicity e on the central fiber C_0 and not blow it up. By removing the common factors, the resulting map has degree $d - e$ on C_0. Correspondingly, the map restricted to C_0 along with the base point p of multiplicity e is parameterized by a point in the boundary of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d - k)$. There is a natural morphism $\rho_k : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d - k)$ that replaces a tail of degree $e \leq k$ by a base point of multiplicity e, cf. [MM, Proposition 1.3]. For $k = 1$, the morphism ρ_1 can be further analyzed using the divisor T.

Proposition 3.2. *The morphism $t : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to M(T)$ contracts $\Delta_{1,d-1}$ and the locus of non-finite maps with moduli. It factors through ρ_1 and the induced morphism $t_1 : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d - 1) \to M(T)$ only contracts the locus of non-finite maps with moduli.*

Proof. For a curve class R_1 contracted by ρ_1, by the definition of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d-1)$, all maps parameterized by R_1 only differ by degree one tails passing through a common attachment point. Choose a defining hyperplane of T which is away from those finitely many attachment points and also not tangent to the remaining part of the image. Then R_1 does not meet T. Hence, R_1 is contracted by t. By Lemma 3.1, t factors through ρ_1.

Conversely, for a curve R_2 contracted by t, the image of a degree ≥ 2 component of a map parameterized by R_2 cannot vary. Otherwise we can choose a defining hyperplane of T that is only tangent to finitely many images of the maps parameterized by R_2. Then $R_2.T$ is positive and R_2 cannot be contracted under t, contradiction. Hence, only the image of degree one tails can vary. In addition, we can also take a non-finite map with moduli and move other domain components along the contracted subcurve. Such a family of non-finite maps is contracted by t. But it is not contracted by ρ_1 since the domain curves in this family have different moduli.   □

For $k \geq 2$, let us explain how to construct the divisors $\Lambda_k(\alpha)$ in Theorem 1.6. In [CHS2, Theorem 1.1], there is an injection

$$v : Pic(\overline{\mathcal{M}}_{0,d})^{S_d} \otimes \mathbb{Q} \to Pic(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)) \otimes \mathbb{Q},$$

where S_d denotes the symmetric group on d letters. This injection is induced by a rational map $f : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \dashrightarrow \overline{\mathcal{M}}_{0,d}^{S_d}$. Take a hyperplane Λ to intersect the image of a stable map. If the pre-image of the intersection consists of d distinct points, the stable limit of the domain curve marked at these d points corresponds to a point in $\overline{\mathcal{M}}_{0,d}^{S_d}$. In particular, v maps S_d-equivariant nef (semi-ample) divisors on $\overline{\mathcal{M}}_{0,d}$ to nef (semi-ample) divisors on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. Let D_i be the S_d-equivariant boundary divisor of $\overline{\mathcal{M}}_{0,d}$ whose general points parameterize a nodal union of two rational curves with i and $d-i$ marked points, respectively, $2 \leq i \leq [d/2]$. v acts on these boundary classes as follows:

$$v(D_i) = \Delta_{i,d-i} \text{ for } i > 2 \text{ and } v(D_2) = \frac{1}{2}T + \Delta_{2,d-2},$$

cf. [CHS2, Figure 4].

Let $\overline{\mathcal{M}}_{0,\mathcal{A}}$ be the moduli space of weighted stable d-pointed curves with symmetric weight $\mathcal{A} = \{a, a, \ldots, a\}$, where $\frac{1}{k+1} < a \leq \frac{1}{k}$. In [H1], it was shown that there is a birational morphism $\overline{\mathcal{M}}_{0,d} \to \overline{\mathcal{M}}_{0,\mathcal{A}}$ that contracts boundary components D_i with $i \neq 2$ and $ia < 1$. Let $D = \sum D_i$ be the total boundary class of $\overline{\mathcal{M}}_{0,d}$. Define the model

$$\overline{\mathcal{M}}_{0,d}(\alpha) = \text{Proj}\Big(\bigoplus_{m \geq 0} H^0(\overline{\mathcal{M}}_{0,d}, m(K_{\overline{\mathcal{M}}_{0,d}} + \alpha D)))\Big).$$

In [mSi], [FS] and [AS], the following result was established: if $\frac{2}{k+2} < \alpha \leq \frac{2}{k+1}$ for some $k = 1, \ldots, [\frac{d-1}{2}]$, then $\overline{\mathcal{M}}_{0,d}(\alpha) \cong \overline{\mathcal{M}}_{0,\mathcal{A}}$ with $\mathcal{A} = \{1/k, \ldots, 1/k\}$; if $\frac{2}{d-1} < \alpha \leq \frac{2}{[d/2]+1}$, then $\overline{\mathcal{M}}_{0,d}(\alpha) \cong (\mathbb{P}^1)^n // SL_2$. Consequently, the divisor $K_{\overline{\mathcal{M}}_{0,\mathcal{A}}} + \alpha E$ is ample on $\overline{\mathcal{M}}_{0,\mathcal{A}}$ in the above range, where E is the total boundary of $\overline{\mathcal{M}}_{0,\mathcal{A}}$. Then the pull-back of $K_{\overline{\mathcal{M}}_{0,\mathcal{A}}} + \alpha E$ is a semi-ample S_d-equivariant divisor A_α on $\overline{\mathcal{M}}_{0,d}$. A_α has the following expression:

$$A_\alpha = \sum_{i=2}^{k} \Big(\frac{i(i-1)}{2}\alpha - \frac{i(i-1)}{d-1}\Big)D_i + \sum_{i>k}^{[d/2]} \Big(\frac{i(d-i)}{d-1} - 2 + \alpha\Big)D_i.$$

Using the map v above, we thus get a semi-ample divisor $\Lambda_k(\alpha) = v(A_\alpha)$ on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ with the following expression:

$$(\frac{\alpha}{2} - \frac{1}{d-1})T + \sum_{i=2}^{k} \Big(\frac{i(i-1)}{2}\alpha - \frac{i(i-1)}{d-1}\Big)\Delta_{i,d-i} + \sum_{i>k}^{[d/2]} \Big(\frac{i(d-i)}{d-1} - 2 + \alpha\Big)\Delta_{i,d-i}.$$

Proposition 3.3. *For $k = 1, \ldots, [\frac{d-1}{2}]$ and $\frac{2}{k+2} < \alpha \leq \frac{2}{k+1}$, the morphism $\lambda_k(\alpha) : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to M(\Lambda_k(\alpha))$ induced by the divisor $\Lambda_k(\alpha)$ contracts the boundary components $\Delta_{1,d-1}, \ldots, \Delta_{k,d-k}$. In particular, $\lambda_k(\alpha)$ factors through $\rho_k : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d-k)$.*

Proof. There is another way to obtain $\Lambda_k(\alpha)$ by the following diagram:

$$
\begin{array}{ccc}
\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) & \xrightarrow{\rho_k} & \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d-k) \\
\Big\downarrow{f} & & \Big\downarrow{g} \\
\overline{\mathcal{M}}_{0,d}^{S^d} & \xrightarrow{\pi} & \overline{\mathcal{M}}_{0,\mathcal{A}}^{S_d}
\end{array}
$$

The horizontal morphisms ρ_k and π contract $\Delta_{1,d-1}, \ldots, \Delta_{k,d-k}$ and $D_3, \ldots, D_k$, respectively. The vertical maps are only rational. f was constructed in [CHS2] by taking a general hyperplane to slice the image of a map and marking the pre-image of the intersection to get a d-marked rational curve. g can be constructed in the same vein. We only need to assign weight $\frac{1}{k}$ to the pre-image of each intersection point.

If there is a base point of multiplicity $i \leq k$ on the domain curve, we assign weight $\frac{i}{k}$ to it. For any stable map $[C, \mu]$, there exists a hyperplane such that the map f is a local morphism around $[C, \mu]$, cf. [CHS2]. From this fact, it follows that f pulls back S_d-equivariant semi-ample divisors on $\overline{\mathcal{M}}_{0,d}$ to semi-ample divisors on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. The same conclusion holds for g. Moreover, $\overline{\mathcal{M}}_{0,\mathcal{A}}$ is smooth, so $\overline{\mathcal{M}}_{0,\mathcal{A}}^{S_d}$ has finite quotient singularities. Therefore, we can alternatively obtain $\Lambda_k(\alpha)$ by pulling back the ample divisor $K_{\overline{\mathcal{M}}_{0,\mathcal{A}}} + \alpha E$ via $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d - k)$. Note that $g^*(K_{\overline{\mathcal{M}}_{0,\mathcal{A}}} + \alpha E)$ is semi-ample on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d - k)$. Hence, by Lemma 3.1, the morphism $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to M(\Lambda_k(\alpha))$ factors through $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d - k)$. $\qquad\square$

Remark 3.4. For $k = 1$, $\Lambda_1(\alpha)$ is proportional to T, which recovers the result in Proposition 3.2.

Proposition 3.2 and 3.3 together complete the proof of Theorem 1.6.

Corollary 3.5.

(i) *The rational Picard group of* $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d - k)$ *is generated by* $\rho_{k*}H$ *and* $\rho_{k*}\Delta_{j,d-j}, k < j \leq [d/2]$.

(ii) *The following push-down pull-back formulae hold:*

$$\rho_k^* \rho_{k*} \Delta_{j,d-j} = \Delta_{j,d-j} \ \text{for } j > k \ \text{and} \ \rho_k^* \rho_{k*} H = H + \sum_{i=1}^{k} i^2 \Delta_{i,k-i}.$$

(iii) *The canonical class of* $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d - k)$ *has the expression:*

$$K_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d-k)} = -\frac{(r+1)(d+1)}{2} \rho_{k*} H + \sum_{j=k+1}^{[d/2]} \left(\frac{(r+1)j(d-j)}{2d} - 2 \right) \rho_{k*} \Delta_{j,d-j}.$$

Proof. (i) can be seen from the fact that ρ_k contracts exactly boundary components $\Delta_{i,d-i}$ for $i \leq k$. Note that the images of H and $\Delta_{j,d-j}$ for $j > k$ under ρ_k are still divisors on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d - k)$. But the image of $\Delta_{i,d-i}$ for $i \leq k$ has higher codimension in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d - k)$. An alternative verification can be done by a direct analysis as in [P1].

For (ii), take a pencil of lines on a plane and embed this plane by a degree i Veronese map to $\mathbb{P}^N$. Attach a degree $d - i$ rational curve at the image of the base point of this pencil. Project the whole family to $\mathbb{P}^r$ and we obtain a one-dimensional family C_i of maps contained in $\Delta_{i,d-i}$. For $i \leq k$, the curve class C_i is contracted by ρ_k. Hence, $C_i . \rho_k^* \rho_{k*}(D) = 0$ for any divisor D on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. Note that $C_i . H = i^2, C_i . \Delta_{j,d-j} = 0$ for $j \neq i$ and $C_i . \Delta_{i,d-i} = -1$. Then the formulae in (ii) follow immediately.

Finally, for the canonical class of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d - k)$, one can follow Pandharipande's calculation for the canonical class of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, cf. [P2, 2.3]. Take a rational map $\mu : S = \mathbb{P}^1 \times C \dashrightarrow \mathbb{P}^r$ with simple base points of degree e such that $k < e \leq [d/2]$, cf. [P2, 2.1]. In this setting, everything works identically except that boundary divisors $\Delta_{i,d-i}$ for $i \leq k$ do not appear in the expression of $K_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d, d-k)}$. $\qquad\square$

Next we will prove Theorem 1.7. Let $D = aH + bT$, $a, b > 0$ be an effective divisor on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. D is semi-ample since H and T are semi-ample.

Lemma 3.6. *The morphism* $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to M(D)$ *only contracts the locus of non-finite maps with moduli.*

Proof. If an irreducible curve R on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ has zero intersection with D, it must satisfy $R . H = 0$ and $R . T = 0$, which implies that the maps parameterized by R have the same image and the branch points of the maps cannot vary. The only possible way to obtain a positive dimensional family of such maps is to have a domain component C_0 mapping to a point, so that other components can move along C_0 to vary the moduli of the domain curve. Hence, those maps in this family are non-finite maps with moduli. $\qquad\square$

Recall that the moduli space of branchvarieties constructed in [AK] parameterizes isomorphism classes of finite maps f from a connected (reduced but possibly reducible) variety to $\mathbb{P}^r$ such that the Hilbert polynomial of $f^*\mathcal{O}_{\mathbb{P}^r}(m)$ is fixed for $m \gg 0$. We are interested in the finite maps with the fixed Hilbert polynomial $1 + md$. By Riemann-Roch, the domain curve has arithmetic genus zero. This forces the curve to possess a special class of singularities.

Definition 3.7. A curve singularity p is a *rational k-fold point* if it is locally cut out by k smooth branches whose tangent line directions are linearly independent at p.

For instance, the k coordinate axes of $\mathbb{A}^k$ meeting at the origin form a rational k-fold point. Locally, a rational k-fold point is isomorphic to this form.

Lemma 3.8. *Let C be a connected reduced curve of arithmetic genus zero. Then every irreducible component of C is a smooth rational curve. Moreover, C can only have rational k-fold points as its singularities.*

Proof. Let $\pi : \widetilde{C} \to C$ be the normalization of C. We have the following short exact sequence:

$$0 \to \mathcal{O}_C \to \pi_*\mathcal{O}_{\widetilde{C}} \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is a sheaf supported on the singularities of C. Note that $h^1(\mathcal{O}_C) = p_a(C) = 0$ and $h^1(\mathcal{F}) = 0$. Thus $h^1(\mathcal{O}_{\widetilde{C}}) = 0$ and $p_a(\widetilde{C}) = 0$. It implies that every component of C is rational. An irreducible component of C still has arithmetic genus zero. Since it is rational, it must be isomorphic to $\mathbb{P}^1$.

To see how those components meet at a singularity, let us take a singular point p of C. Suppose k branches of smooth rational curves $C_1, \ldots, C_k$ pass through p. Denote their union by C'. The arithmetic genus $p_a(C')$ is zero. Let L_i be a line bundle on C' such that $L_i|_{C_j}$ is trivial for $j \neq i$ and $L_i|_{C_i} \cong \mathcal{O}_{\mathbb{P}^1}(1)$. Let $L = L_1 \otimes \cdots \otimes L_k$. By Riemann-Roch, $\chi(L) = 1 - p_a(C') + k = 1 + k$, so $h^0(L) \geq k + 1$. On the other hand, $h^0(L) \leq k + 1$ since $h^0(L_i) = 2$ and those sections over each C_i can only be glued to form a global section of L if they take the same value on the fiber over p, which imposes at least $k-1$ conditions. Therefore, $h^0(L) = k + 1$ and $h^1(L) = 0$. C' maps into $\mathbb{P}^k$ by the linear system $|L|$ and each component C_i maps isomorphically to a line. These k lines span $\mathbb{P}^k$. Hence, $C_1, \ldots, C_k$ have linearly independent tangent directions at p. $\qquad\qquad\square$

By Lemma 3.8, we make the following definition.

Definition 3.9. A connected reduced curve is a *multi-nodal rational curve* if it has arithmetic genus zero.

Use $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ to denote the moduli space of isomorphism classes of finite maps from multi-nodal rational curves to $\mathbb{P}^r$ with $1 + md$ as the Hilbert polynomial. $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ is a Deligne-Mumford moduli stack over a characteristic zero ground field, cf. [AK, Remark 3.3]. The following result shows that on a set-theoretic level, $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ admits a map to $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ that contracts the locus of non-finite maps with moduli.

Lemma 3.10. *Let $[C, \mu] \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ be a stable map and $L = \mu^*(\mathcal{O}_{\mathbb{P}_r}(1))$ be a semi-ample line bundle on C. Denote $C' = \mathrm{Proj}\big(\bigoplus_{m \geq 0} H^0(C, mL)\big)$ and $f : C \to C'$. If μ contracts a component C_0 of C, C_0 is also contracted by f and the resulting singularity in C' is a rational k-fold point, where $k = |C_0 \cap \overline{C \backslash C_0}|$. In particular, C' is a multi-nodal rational curve. C' admits a degree d finite map g to $\mathbb{P}^r$ such that $\mu = g \circ f$.*

Proof. Let C_0 denote a maximal connected subcurve of C contracted by μ. Suppose $C_1, \ldots, C_k$ are k irreducible components of C attached to C_0 at k distinct points. The map μ has positive degree d_i restricted to C_i for $0 < i < k$. Therefore, L restricted to C_i is a degree d_i line bundle for $i > 0$ and L is trivial on C_0. Let D be the union of $C_0, C_1, \ldots, C_k$ and let $d_0 = \sum d_i$. By the same argument in

Lemma 3.8, we know that $h^0(D, mL) = 1 + md_0$. The sections of mL induce a map from D to $\mathbb{P}^{md_0}$ such that C_i maps to a degree md_i rational normal curve spanning a subspace $\mathbb{P}^{md_i}$ for $i > 0$ and C_0 maps to a point p contained in all those $\mathbb{P}^{md_i}$. Note that $md_1 + \cdots + md_k = md_0$. So those linear subspaces $\mathbb{P}^{md_i}$ cannot contain a common line. It implies that the tangent directions of the k rational normal curves at p are linearly independent. Therefore, f restricted to D contracts C_0 to a rational k-fold point.

Since any component of C contracted by μ is also contracted by f, μ factors through f and g is a finite map of degree d. $\qquad\square$

The picture below illustrates an example of the above situation. μ contracts C_0 and maps C to three concurrent lines in a plane. But the map factors through three concurrent lines that span $\mathbb{P}^3$.
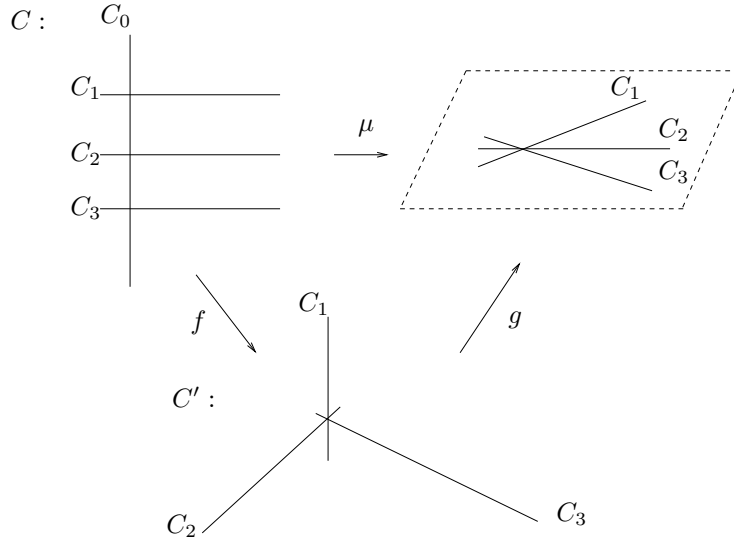
Now let us state a global version of Lemma 3.10.

Proposition 3.11. *There is a natural morphism $\pi : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to \overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ contracting the locus of non-finite maps with moduli. In particular, let $[C', g]$ be a branchcurve such that the domain C' has m rational singularities of type k_1-fold, $\ldots$, k_m-fold. Then $\pi^{-1}([C', g])$ is isomorphic to $\overline{M}_{0,k_1} \times \cdots \times \overline{M}_{0,k_m}$.*

Proof. Suppose there is a family of degree d stable maps from rational curves to $\mathbb{P}^r$ over a base B as follows:

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\phi} & \mathbb{P}^r_B \\ {\scriptstyle h}\downarrow & & \\ B & & \end{array}$$

Let $\mathcal{L} = \phi^*\mathcal{O}(1)$ be the pull-back of the twisted sheaf from $\mathbb{P}^r_B$, cf. [Ha, II 5]. Define $f : \mathscr{C} \to \mathscr{C}' = \mathrm{Proj}\big(\bigoplus_{m \geq 0} H^0(\mathscr{C}, m\mathcal{L})\big)$ relative to the base B. Every curve contracted by f is contained in a fiber of h and is also contracted by ϕ. So $\mathscr{C}'$ admits a morphism g to $\mathbb{P}^r_B$ such that $\phi = g \circ f$. $\mathscr{C}'$ can be viewed as a family of finite maps over B. Below we will show that the arithmetic genus of a fiber of $\mathscr{C}'$ over B is always zero. Then $(\mathscr{C}', g)$ is a family of branchcurves with Hilbert polynomial $1 + md$.

Let C_b be a fiber of $\mathscr{C}$ over $b \in B$. $h^0(m\mathcal{L}|_{C_b}) = 1 + md$ does not depend on b. By cohomology and base change, cf. [Ha, Corollary 12.9], we get $H^0(\mathscr{C}, m\mathcal{L})|_{C_b} \cong H^0(C_b, m\mathcal{L}|_{C_b})$. Therefore, the corresponding

fiber C_b' in $\mathscr{C}'$ is isomorphic to $\mathrm{Proj}\left(\bigoplus_{m\geq 0} H^0(C_b, m\mathcal{L}|_{C_b})\right)$. By Lemma 3.10, C_b is a multi-nodal rational curve and $g|_{C_b'}$ is a finite map of degree d. Hence, $g : \mathscr{C}' \to \mathbb{P}^r_B$ is a family of degree d finite maps from multi-nodal rational curves to $\mathbb{P}^r$. The construction of this family is compatible with base change. In particular, it yields the desired morphism from $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ to $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$.

Let $[C', g]$ be a branchcurve such that the domain C' has m rational singularities $p_1, \ldots, p_m$ of type k_1-fold, $\ldots$, k_m-fold, respectively. Let $[C, f]$ be a stable map contained in $\pi^{-1}([C', g])$. Then over p_i, there is a rational subcurve glued to k_i components of C and contracted by f. We can move the k_i components along this subcurve without affecting their image $[C', g]$ in $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$. Therefore, we get $\pi^{-1}([C', g]) \cong \overline{M}_{0,k_1} \times \cdots \times \overline{M}_{0,k_m}$. It has positive dimension if and only if some of the k_i is bigger than three. So the locus of non-finite maps with moduli gets contracted by π. $\qquad\square$

Proposition 3.12. *For a divisor $D = aH + bT$, $a, b > 0$ on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, let $\varphi : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to M(D)$ be the corresponding contraction. Then the morphism $\pi : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to \overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ factors through φ. In particular, $M(D)$ is normal and bijective to $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$.*

Proof. By Lemma 3.6, a curve is contracted by φ if and only if it is contained in the locus of non-finite maps with moduli. It must also be contracted under π by the construction of π. Moreover, $\varphi_*\mathcal{O}_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,d)} \cong \mathcal{O}_{M(D)}$ so π factors through φ. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ is $\mathbb{Q}$-factorial, which implies that $M(D)$ is normal. The map $M(D) \to \overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ is bijective since π has connected fibers by Proposition 3.11. Therefore, $M(D)$ is the normalization of $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$. $\qquad\square$

We thus completed the proof of Theorem 1.7.

Remark 3.13. We do not know whether $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ is normal. The two spaces $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ and $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ are different if and only if $d \geq 4$. If $d = 2$, there is no stable map that fails to be finite. If $d = 3$, the only non-finite stable maps are those whose domain curves consist of a rational subcurve C_0 attached with three rational tails, where C_0 gets contracted under the map. Since three points on $\mathbb{P}^1$ have unique moduli, $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 3)$ is isomorphic to $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, 3)$.

If $\pi : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to \overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ is a small contraction, $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ is singular. For instance, it may not be $\mathbb{Q}$-factorial. It is unclear which Weil divisors on $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ are $\mathbb{Q}$-Cartier. Nevertheless, it is still possible to get line bundles on $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ by pushing down divisors from $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. We will use the same notation to denote a divisor on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ and its image in $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$.

Proposition 3.14. *H and T are $\mathbb{Q}$-Cartier divisors on $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$. In particular, a positive linear combination of H and T is ample on $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$.*

Proof. For a family B of degree d finite maps from multi-nodal rational curves to $\mathbb{P}^r$, the vector bundle E_m in Theorem 1.1 can be defined over B. It remains locally free due to the fact $h^1(C, f^*\mathcal{O}_{\mathbb{P}^r}(m)) = 0$ for a branchcurve $[C, f]$. Therefore, $L_m = \mathrm{Det}\,(E_m)$ is a natural line bundle on the moduli space $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$. The divisor class $c_1(L_m)$ on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ is $D_m = (\frac{m^2}{12} + m)H - mT$. Moreover, $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ and $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ are isomorphic in codimension one. Hence, by varying m we get both H and T as $\mathbb{Q}$-Cartier divisors on $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$. Take $a, b > 0$ such that $D = aH + bT$ is Cartier on $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$. By Proposition 3.12, the contraction $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to \overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ factors through $M(D)$. Moreover, D is ample on $M(D)$ and $M(D) \to \overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ is a bijection. It follows that $D = aH + bT$ as a divisor on $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ is ample, cf. [L, 1.2.28, 1.2.29]. $\qquad\square$

Proposition 3.14 implies Corollary 1.8. Finally, let us consider Theorem 1.9.

Proposition 3.15. *If a divisor D on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ is a positive linear combination of H and NL, then $\mathbf{B}(D)$ consists of the locus of multi-image maps.*

Proof. Take a multi-image map and vary a branch point on its image. All maps parameterized by such a one-dimensional family R have the same image, so $R.H = 0$. On the other hand, if the moving branch point meets the defining hyperplane of T, then $R.T > 0$. This implies that $R.NL < 0$ since H is a positive linear combination of T and NL, cf. Theorem 1.1. Hence, $R.D < 0$ and $\mathbf{B}(D)$ contains the locus of multi-image maps.

Conversely, if a map is not a multi-image map, after a general projection to $\mathbb{P}^2$, it is still not a multi-image map. We can pick a defining line L for NL such that no singular point of the image is contained in L. Therefore, this map is not contained in the base locus of NL. We already know that H is semi-ample. So this map is not contained in $\mathbf{B}(D)$. $\qquad\square$

It would be interesting to find the flipping space with respect to D. Unlike the case $d = r = 3$, in general the Hilbert scheme is not the flipping space. For instance, when $d = r = 4$, a rational one-nodal quartic may have different embedded scheme structures at the node. However, the corresponding configuration in the Kontsevich space or the Chow variety is unique and does not depend on the embedded point. A more plausible candidate is provided by the Simpson moduli space of semi-stable sheaves with 1-dimensional support.

Definition 3.16. A coherent sheaf E is *pure* if any non-zero subsheaf of E has the same dimensional support as E. A pure sheaf E is *stable (semi-stable)* if $\frac{\chi(E(m))}{r(E)} < (\leq)\frac{\chi(F(m))}{r(F)}$, $m \gg 0$ for any non-trivial pure quotient sheaf F of the same dimension, where $\chi(E(m))$ is the Hilbert polynomial of E and r is its leading coefficient.

In [cSi], it was proved that there exists a moduli space $\mathrm{Simp}^P(\mathbb{P}^r)$ parameterizing semi-stable sheaves on $\mathbb{P}^r$ with Hilbert polynomial P. Here we are interested in the case when the sheaves have 1-dimensional support and P equals $dm + 1$. Let us first justify the stability of certain sheaves.

Lemma 3.17. *The structure sheaf of a Cohen-Macaulay curve of arithmetic genus zero in $\mathbb{P}^r$ is stable.*

Proof. Let C be a Cohen-Macaulay curve of arithmetic genus zero in $\mathbb{P}^r$. Let F be a pure quotient sheaf of $\mathcal{O}_C$. F can also be regarded as the structure sheaf of a closed subcurve C' of C. Suppose C and C' have degree d and d', respectively, $d > d'$. By the exact sequence

$$0 \to I_{C'/C} \to \mathcal{O}_C \to \mathcal{O}_{C'} \to 0,$$

we have $0 = h^1(\mathcal{O}_C) \geq h^1(\mathcal{O}_{C'}) = p_a(C')$. Therefore,

$$\frac{\chi(\mathcal{O}_C(m))}{d} = m + \frac{1}{d} < m + \frac{1 - p_a(C')}{d'} = \frac{\chi(\mathcal{O}_{C'}(m))}{d'}$$

for $m \gg 0$, which implies that $\mathcal{O}_C$ is stable. $\qquad\square$

For a smooth connected degree d rational curve C in $\mathbb{P}^r$, $r \geq 3$, the correspondence $[C] \to [\mathcal{O}_C]$ in Lemma 3.17 induces an injection from the locus of maps whose images are smooth connected degree d rational curves to $\mathrm{Simp}^{dm+1}(\mathbb{P}^r)$. Let $S^{dm+1}(\mathbb{P}^r)$ denote the closure of its image in $\mathrm{Simp}^{dm+1}(\mathbb{P}^r)$.

Proposition 3.18. *For $r \geq 3$, let $[C, \mu] \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ be a stable map which is not a multi-image map. Then $\mu_*\mathcal{O}_C$ is a stable sheaf represented by a point of $S^{dm+1}(\mathbb{P}^r)$. This yields a birational morphism from the complement of the locus of multi-image maps in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ to $S^{dm+1}(\mathbb{P}^r)$. In particular, this morphism factors through the complement of the locus of multi-image maps in $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ and the resulting morphism to $S^{dm+1}(\mathbb{P}^r)$ is injective.*

Proof. If μ contracts a component of C, we can use the factorization $g : C' \to \mathbb{P}^r$ of μ obtained in Lemma 3.10 instead, due to the fact $\mu_*\mathcal{O}_C \cong g_*\mathcal{O}_{C'}$. So from now on, assume that μ is a finite map which is generically one-to-one from a multi-nodal rational curve C to its image D in $\mathbb{P}^r$.

Since μ restricted to any component of C is not a multiple covering map, D is of degree d and $\mu_* \mathcal{O}_C$ is a purely 1-dimensional sheaf supported on D. Since μ is finite, $R^i \mu_* E = 0$ for all $i > 0$ and $h^j(D, \mu_* E) = h^j(C, E)$ for all $j \geq 0$, where E is an arbitrary coherent sheaf on C. In particular, $h^0(\mu_* \mathcal{O}_C(m)) = dm + 1$ and $h^1(\mu_* \mathcal{O}_C(m)) = 0$ for all $m \geq 0$. Hence, the Hilbert polynomial of $\mu_* \mathcal{O}_C$ equals $dm + 1$.

Suppose F is a non-trivial pure quotient sheaf of $\mu_* \mathcal{O}_C$ supported on a subcurve D' of D with degree $d' < d$. We have two exact sequences as follows:

$$0 \to I \to \mu_* \mathcal{O}_C(m) \to F(m) \to 0,$$

$$0 \to \mathcal{O}_{D'}(m) \to F(m) \to Z \to 0,$$

where Z is a sheaf with support in D'. From the first sequence we have $h^1(F(m)) = 0$. Combining with the second we get $h^0(Z) \geq h^1(\mathcal{O}_{D'}(m))$. Therefore, $h^0(F(m)) = 1 + md' - h^1(\mathcal{O}_{D'}) + h^0(Z) \geq 1 + md'$. Then we have

$$\frac{\chi(F(m))}{d'} \geq m + \frac{1}{d'} > m + \frac{1}{d} = \frac{\chi(\mu_* \mathcal{O}_C(m))}{d},$$

which implies the stability of $\mu_* \mathcal{O}_C(m)$.

The fact that $R^i \mu_* \mathcal{O}_C = 0$ for all $i > 0$ guarantees that the above correspondence can be done for a family of finite generically one-to-one maps. Therefore, we get an injection from the complement of the locus of multi-image maps in $\overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$ to $S^{dm+1}(\mathbb{P}^r)$. Using the morphism $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to \overline{\mathcal{B}}_{0,0}(\mathbb{P}^r, d)$, we also get a morphism from the complement of the locus of multi-image maps in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ to $S^{dm+1}(\mathbb{P}^r)$ that contracts the locus of non-finite maps with moduli. $\qquad\square$

Remark 3.19. For a multi-image map $[C, \mu] \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, $\mu_* \mathcal{O}_C$ may fail to be semi-stable. For instance, let $\mu : C \to L$ be a degree $d > 1$ branched cover from a smooth rational curve C to a line L in $\mathbb{P}^r$. Since μ is finite, $\mu_* \mathcal{O}_C$ is locally free of rank d on L and $h^0(\mu_* \mathcal{O}_C(m)) = dm + 1$ for all $m \geq 0$. $\mu_* \mathcal{O}_C$ splits into $\mathcal{O}_L \oplus E$ by the trace map, where E is a rank $d - 1$ subbundle. Then the subbundle $\mathcal{O}_L$ destabilizes $\mu_* \mathcal{O}_C$.

The case $d = 3$ is particularly interesting.

Proposition 3.20. *For $r \geq 3$, $S^{3m+1}(\mathbb{P}^r)$ is the flip of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 3)$ over the Chow variety with respect to the divisor $D = aH + bNL$, $a, b > 0$. In particular, $S^{3m+1}(\mathbb{P}^r)$ is smooth and has Picard number two.*

Proof. The prototype is $r = 3$. In that case, it was shown in [FT, Theorem 1.1 (3)] that $S^{3m+1}(\mathbb{P}^3)$ is one of the two components of $\text{Simp}^{3m+1}(\mathbb{P}^3)$. Moreover, $S^{3m+1}(\mathbb{P}^3)$ is smooth and isomorphic to the closure $H^{3m+1}(\mathbb{P}^3)$ of the locus of twisted cubics in the Hilbert scheme $\text{Hilb}^{3m+1}(\mathbb{P}^3)$. In [C], it was shown that the Hilbert component $H^{3m+1}(\mathbb{P}^3)$ is the flip of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ over the Chow variety with respect to the divisor class $aH + bNL$, $a, b > 0$.

For $r \geq 4$, let $H^{3m+1}(\mathbb{P}^r)$ be the closure of the locus of smooth rational cubic curves in the Hilbert scheme $\text{Hilb}^{3m+1}(\mathbb{P}^r)$. In [CK, Proposition 1.3], it was proved that the morphism $H^{3m+1}(\mathbb{P}^r) \to S^{3m+1}(\mathbb{P}^r)$ is the blow-up along the locus of stable sheaves with planar support. Note that $H^{3m+1}(\mathbb{P}^r)$ is a fiber bundle over the Grassmannian $\mathbb{G}(3, r)$ with fiber isomorphic to $H^{3m+1}(\mathbb{P}^3)$. Since $H^{3m+1}(\mathbb{P}^3)$ has Picard number two, the Picard numbers of $H^{3m+1}(\mathbb{P}^r)$ and $S^{3m+1}(\mathbb{P}^r)$ are three and two, respectively. By Proposition 3.18, the birational map $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 3) \dashrightarrow S^{3m+1}(\mathbb{P}^r)$ is an isomorphism along the complement of the locus of multi-image maps. We still denote by D the birational transform of a divisor D from $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 3)$ to $S^{3m+1}(\mathbb{P}^r)$ or $H^{3m+1}(\mathbb{P}^r)$. The divisor class H is semi-ample on all three spaces. Moreover, for a sheaf $[F]$ in $S^{3m+1}(\mathbb{P}^r)$, it can be regarded as a sheaf in a subspace $\mathbb{P}^3$. The projection from $\mathbb{P}^r$ to this $\mathbb{P}^3$ induces a rational map $\pi : S^{3m+1}(\mathbb{P}^r) \dashrightarrow S^{3m+1}(\mathbb{P}^3)$. π is a morphism in codimension one and well-defined in a local neighborhood of $[F]$. Since NL is base-point-free on $S^{3m+1}(\mathbb{P}^3)$, its pull-back under π does not contain $[F]$ in the base locus. Hence, as a divisor on $S^{3m+1}(\mathbb{P}^r)$, NL is also base-point-free. Since $S^{3m+1}(\mathbb{P}^r)$ has Picard number two and HL, H are both semi-ample, its ample

cone is bounded by NL and H. Therefore, $S^{3m+1}(\mathbb{P}^r)$ is the flip of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 3)$ with respect to a divisor $aH + bNL$, $a, b > 0$. $\hfill\square$

Proposition 3.20 completes the proof of Theorem 1.9 for $r \geq 3$. For $r = 2$, it was shown in [LP] that $\mathrm{Simp}^{3m+1}(\mathbb{P}^2)$ is isomorphic to the universal cubic over $\mathbb{P}^2$. A rational plane cubic C has a node p. We associate C with a coherent sheaf E such that there exists a non-split extension

$$0 \to \mathcal{O}_C \to E \to \mathbb{C}_p \to 0,$$

cf. [FT, Lemma 3.2]. Note that E is isomorphic to the direct image sheaf $\mu_*(\mathcal{O}_{\widetilde{C}})$, where $\mu : \widetilde{C} \to C$ is the normalization of C. Via this, $S^{3m+1}(\mathbb{P}^2)$ as a subvariety of $\mathrm{Simp}^{3m+1}(\mathbb{P}^2)$ is isomorphic to the incidence correspondence $\{(p, C) | p \in C_{sing}\}$, where C is a singular plane cubic. We discussed at the beginning of this section that the incidence correspondence $S^{3m+1}(\mathbb{P}^2)$ is the model $M(D)$ for a divisor $D = aH + bNL$, $a, b > 0$. $M(D)$ is the flipping space of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 3)$ over the Chow variety with respect to D. This proved Theorem 1.9 for $r = 2$. Overall, what we proved can be summarized as a wall-crossing phenomenon that flips the ample cone (H, T) of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 3)$ to the ample cone (H, NL) of $S^{3m+1}(\mathbb{P}^r)$.

A variety on which Mori's program can be carried out for every effective divisor is called a Mori dream space, cf. [HK]. Equivalently, a Mori dream space is a space whose Cox ring is finitely generated. By [BCHM, Corollary 1.3.1], a log Fano variety is a Mori dream space. For $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$, $-K_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)}$ is big since it is proportional to $D_{deg} + \frac{2}{d+1}\Delta$, cf. Theorem 1.1. But it is not ample except for very small r when d is two or three. In general, we can ask the following fundamental question.

Question 3.21. *Is $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ a Mori dream space?*

Remark 3.22. The assumption that $-K_X$ is big does not guarantee that a variety X is a Mori dream space. For instance, let X be the blow-up of $\mathbb{P}^3$ at 12 general points on an plane cubic C. Denote H as the pullback of $\mathcal{O}(1)$ and E as the sum of the exceptional divisors. In this case $-K_X = 4H - 2E = 2H + 2(H - E)$ is big. Moreover, the divisor $D = 4H - E$ is big and nef, but it has the proper transform of C contained in its stable base locus. The section ring $R(D)$ is not finitely generated, since for a big and nef divisor, its section ring is finitely generated if and only if it is semi-ample.

4. VOLUME OF A DIVISOR

Let D be an effective $\mathbb{Q}$-Cartier divisor on a variety X of dimension n. The volume of D is an important numerical invariant defined by the expression

$$\mathrm{vol}(D) = \lim_{m \to \infty} \sup \frac{h^0(X, mD)}{m^n/n!}.$$

If D is nef, $\mathrm{vol}(D) = D^n$, cf. [L, 2.2.C]. The study of the effective cone decomposition and birational models of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ yields a useful tool for calculating $\mathrm{vol}(D)$.

To illustrate the idea, consider the Kontsevich space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 3)$ discussed in section 2. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 3)$ is of dimension 8. Its effective cone is bounded by Δ and NL. H and T further decompose the cone into three Mori Chambers. If a divisor D lies in the nef cone $\langle H, T \rangle$, $\mathrm{vol}(D) = D^8$ can be calculated using Pandharipande's algorithm, cf. [P1, 4.2]. If D is in the chamber $\langle T, \Delta \rangle$, then $D = aT + b\Delta$, $a, b > 0$ and $\mathrm{vol}(D) = a^8 \mathrm{vol}(T)$. If D is inside the chamber $\langle NL, H \rangle$, D is not nef and $\mathbf{B}(D)$ contains the locus of multi-image maps. However, we can flip $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 3)$ to the birational model $M(D) = \{(p, C)\} \subset \mathbb{P}^2 \times \mathbb{P}^9$, where p belongs to the singular locus of a plane nodal cubic or its degeneration. Denote by $\widetilde{A}$ the birational transform of a divisor A from $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 3)$ to $M(D)$. On $M(D)$, $\widetilde{D}$ is ample. Since $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 3)$ and $M(D)$ are isomorphic in codimension 1, $\mathrm{vol}(D) = \mathrm{vol}(\widetilde{D}) = \widetilde{D}^8$. Write $\widetilde{D}$ as a linear combination of $\widetilde{H}$ and $\widetilde{NL}$ on $M(D)$. In order to compute $\widetilde{D}^8$, it suffices to know all the top intersections $\widetilde{H}^k \widetilde{NL}^{8-k}$. $M(D)$ has the structure of a $\mathbb{P}^6$-bundle over $\mathbb{P}^2$ as follows. Let E_k be a vector bundle over $\mathbb{P}^2$ whose fiber over a point p corresponds to the space of plane cubics with vanishing order $\geq k$ at p. Then $M(D)$

is isomorphic to $\mathbb{P}E_2$. Let π_1 and π_2 be the projections from $\mathbb{P}E_2$ to $\mathbb{P}^2$ and to $\mathbb{P}^9$ the space of plane cubics, respectively. Let S be the tautological line bundle with Chern class $1 - \eta$ on $\mathbb{P}E_2$ and let Q be the quotient bundle. Let l denote the pullback of a line class via π_1. We have the following exact sequences:

$$0 \to S \to \pi_1^* E_2 \to Q \to 0,$$
$$0 \to E_2 \to E_1 \to T_{\mathbb{P}^2}^* \otimes \mathcal{O}_{\mathbb{P}^2}(3) \to 0,$$
$$0 \to E_1 \to E_0 \to \mathcal{O}_{\mathbb{P}^2}(3) \to 0.$$

A Chern class calculation shows that

$$c(\pi_1^* E_2) = 1 - 6l + 24l^2$$

and

$$c(Q) = \frac{1 - 6l + 24l^2}{1 - \eta}.$$

Since Q has rank six, the relation $c_7(Q) = 0$ yields

$$\eta^7 - 6\eta^6 l + 24\eta^5 l^2 = 0.$$

Since $\eta^6 l^2 = 1$ and $l^3 = 0$, we get $\eta^7 l = 6$ and $\eta^8 = 12$. The class $\widetilde{NL}$ is just l. Moreover, S can be identified as $\pi_2^* \mathcal{O}_{\mathbb{P}^9}(-1)$, which implies $\widetilde{H} = \eta$. Now the top intersection numbers of $\widetilde{H}$ and $\widetilde{NL}$ are:

$$\widetilde{H}^8 = 12, \ \widetilde{H}^7 \widetilde{NL} = 6, \ \widetilde{H}^6 \widetilde{NL}^2 = 1, \ \widetilde{H}^a \widetilde{NL}^{8-a} = 0 \text{ for } 0 \le a \le 5.$$

The situation for $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ is similar. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ is of dimension 12. Its effective cone is bounded by D_{deg} and Δ. H, T and NL decompose the cone into Mori chambers. If a divisor D lies in the nef cone $\langle H, T \rangle$, $\text{vol}(D) = D^{12}$ can be calculated using Pandharipande's algorithm. If D is in the chamber $\langle T, \Delta \rangle$, $\mathbf{B}(D)$ contains Δ. Then $D = aT + b\Delta$, $a, b > 0$ and $\text{vol}(D) = a^{12}\text{vol}(T)$. If D is in the chamber $\langle D_{deg}, NL \rangle$, $\mathbf{B}(D)$ contains D_{deg}. Then $D = aNL + bD_{deg}$, $a, b > 0$ and $\text{vol}(D) = a^{12}\text{vol}(NL)$. The remaining chamber is $\langle NL, H \rangle$. A divisor D in this chamber is not nef and $\mathbf{B}(D)$ contains the locus of multi-image maps. But we have seen that the corresponding Hilbert component $H^{3m+1}(\mathbb{P}^3)$ is the flip of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ over the Chow variety. Denote by $\widetilde{A}$ as the birational transform of a divisor A from $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ to $H^{3m+1}(\mathbb{P}^3)$. The chamber $\langle \widetilde{NL}, \widetilde{H} \rangle$ is the nef cone of $H^{3m+1}(\mathbb{P}^3)$. So $\text{vol}(D) = \text{vol}(\widetilde{D}) = \widetilde{D}^{12}$ for a divisor in this chamber. Write D as a linear combination of H and NL. Since H is nef on both $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ and $H^{3m+1}(\mathbb{P}^3)$, $\widetilde{H}^{12} = H^{12} = 80160$, cf. [P1, 4.3]. Moreover, NL is numerically equivalent to $H + D_{deg}$. So the top intersection of $\widetilde{H}$ and $\widetilde{NL}$ can be worked out once we know the top intersection of $\widetilde{H}$ and $\widetilde{NL}$ restricted to $\widetilde{D_{deg}}$. $\widetilde{D_{deg}}$ has a simple structure. It is isomorphic to a $\mathbb{P}^6$-bundle over the point-plane flag variety $F = \{p \in \mathbb{P}^2 \subset \mathbb{P}^3\}$. The fiber over a point $(p, \mathbb{P}^2)$ in F parameterizes the space of nodal cubics on that plane and their degenerations that contain p as a singular point. There is a unique embedded point supported at p on those cubics. The flag variety F admits natural projections to $\mathbb{P}^3$ and $\mathbb{P}^{3*}$. Use λ and κ to denote the pull-back of $\mathcal{O}(1)$ from $\mathbb{P}^3$ and $\mathbb{P}^{3*}$ to $\widetilde{D_{deg}}$, respectively. Denote by $-\eta$ the first Chern class of the tautological line bundle of $\widetilde{D_{deg}}$. Note that λ, κ and η generate the Chow ring of $\widetilde{D_{deg}}$. By using similar exact sequences as in the last paragraph, these classes satisfy the following relations:

$$\lambda^4 = \kappa^4 = 0, \ \kappa^3 - \kappa^2\lambda + \kappa\lambda^2 - \lambda^3 = 0,$$
$$\eta^7 + (9\kappa - 6\lambda)\eta^6 + (45\kappa^2 - 52\kappa\lambda + 24\lambda^2)\eta^5$$
$$+ (85\lambda^3 + 35\lambda^2\kappa - 85\lambda\kappa^2)\eta^4 + (40\kappa^2\lambda^2 + 240\kappa\lambda^3)\eta^3 + 280\kappa^2\lambda^3\eta^2 = 0.$$

Their top intersection numbers on $\widetilde{D_{deg}}$ can be worked out from the above relations. Moreover, using test curves in $\widetilde{D_{deg}}$, one can check that $\widetilde{H}$ and $\widetilde{NL}$ restricted to $\widetilde{D_{deg}}$ are numerically equivalent to $\eta + 3\kappa$ and $\lambda + 3\kappa$, respectively. Hence, we obtain the following intersection numbers:

$$\widetilde{H}^{12} = 80160, \ \widetilde{H}^{11}\widetilde{NL} = 93120, \ \widetilde{H}^{10}\widetilde{NL}^2 = 104280,$$

$$\widetilde{H}^9\widetilde{NL}^3 = 112360, \ \widetilde{H}^8\widetilde{NL}^4 = 116896, \ \widetilde{H}^7\widetilde{NL}^5 = 118660,$$

$$\widetilde{H}^a\widetilde{NL}^{12-a} = 119020 \text{ for } 0 \leq a \leq 6.$$

5. Appendix (Charley Crissman)

A Macaulay 2 program is included to verify that C_3 and C_2 used in the proof of Theorem 1.2 are moving curves on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 4)$ and $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 4)$, respectively.

Reinterpret the construction of C_3 as follows. Consider the space of polynomials of bidegree up to 7 and 4 in two variables x and y. It is a 40-dimensional vector space. Take 12 random pairs $p_i = (x_i, y_i)$. Consider the subspace U of polynomials that have a zero of order at least two at each p_i. Since a double point imposes 3 conditions, the expected dimension of U is $40 - 12 \times 3 = 4$ for a general choice of the points p_i. Restrict the polynomials in U to $x = 0$. Then we get a 4-dimensional space V of degree 4 polynomials in the single variable y. Since the space of polynomials of degree 4 in y is 5-dimensional, V can be regarded as a point $[V]$ in the Grassmannian $G(4, 5)$. We thus get a map from the union of the points p_i to $G(4, 5)$. To check that C_3 is moving on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 4)$, it suffices to verify that if we vary the points p_i, the corresponding $[V]$ covers an open dense subset of $G(4, 5)$, namely, the associated map of differentials is onto. The Macaulay 2 code to verify this is the following:

```
kk = ZZ/32003

R = kk[u,x,y,z]

P = random(R^2,R^12)

M = matrix({flatten(entries(matrix(apply(5, j->{y^j*z^(4-j)}))
*matrix({apply(8, j-> x^j*z^(7-j))}))))})

for i from 0 to 11 do A_i = (ideal(x - z*P_(0,i), y - z*P_(1,i)))^2

for i from 0 to 11 do (
D = intersect(if i==0 then ideal(1_R) else intersect(apply(i, j-> A_j)),
if i==11 then ideal(1_R) else intersect(apply(11-i, j->A_(i+j+1))),
(ideal(x-z*u - z*P_(0,i),y-z*P_(1,i)))^2);
S = R^1/module(D); T = R^40; F = map(S, T, M); K = mingens(kernel(F));
V_i = apply(5, j->apply(4, k->(K)_(8*j, 95-k)))
)

Y = apply(12, i-> flatten(entries(exteriorPower(4,matrix(V_i)))))

5 == rank(sub(diff(u, matrix(Y)), u=>0))
```

For C_2, consider the space of polynomials of bidegree up to 5 and 4 in two variables x and y. It is a 30-dimensional vector space. Take 9 random pairs $p_i = (x_i, y_i)$. Consider the subspace U of polynomials that have a zero of order at least two at each p_i. The expected dimension of U is $30 - 9 \times 3 = 3$ for a general choice of the points p_i. Restrict the polynomials in U to $x = 0$. Then we get a 3-dimensional space V of degree 4 polynomials in the single variable y. Since the space of polynomials of degree 4 in y is 5-dimensional, V is represented by a point $[V]$ in the Grassmannian $G(3, 5)$. We thus get a map from the union of the points p_i to $G(3, 5)$. To check that C_2 is moving on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 4)$, it suffices to verify that the associated map of differentials is onto. The Macaulay 2 code to verify this is the following:

```
kk = ZZ/32003
```

```
R = kk[u,x,y,z]

P = random(R^2,R^9)

M = matrix({flatten(entries(matrix(apply(5, j->{y^j*z^(4-j)}))
*matrix({apply(6, j-> x^j*z^(5-j))})))})

for i from 0 to 8 do A_i = (ideal(x - z*P_(0,i), y - z*P_(1,i)))^2

for i from 0 to 8 do (
D = intersect(if i==0 then ideal(1_R) else intersect(apply(i, j-> A_j)),
if i==8 then ideal(1_R) else intersect(apply(8-i, j->A_(i+j+1))),
(ideal(x-z*u - z*P_(0,i),y-z*P_(1,i)))^2);
S = R^1/module(D); T = R^30; F = map(S, T, M); K = mingens(kernel(F));
V_i = apply(5, j->apply(3, k->(K)_(6*j, 69-k)))
)

Y = apply(9, i-> flatten(entries(exteriorPower(3,matrix(V_i)))))

7 == rank(sub(diff(u, matrix(Y)), u=>0))
```

For polynomials of bidegree (a, b) with k_i points of given vanishing order m_i, in case they yield a 1-dimensional family of maps in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, where $d = a$ and $r = (a+1)(b+1) - \sum \binom{m_i+1}{2} - 1$, a general program has been written to check whether the family is moving on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, cf. [Cr]. This provides a useful tool for understanding the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ for larger d.

REFERENCES

[AK]   V. Alexeev and A. Knutson, *Complete moduli spaces of branchvarieties*, arXiv:math/0602626, to appear in J. Reine Angew. Math.
[AL]   V. Alexeev and J. Lin, *The Betti numbers of moduli spaces of pointed branchvarieties*, preprint
[AS]   V. Alexeev and D. Swinarski, *Nef divisors on $\overline{\mathcal{M}}_{0,n}$ from GIT*, arXiv:0812.0778
[ACGH]   E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves*, Grundlehren der Mathematischen Wissenschaften, 267, Springer-Verlag, New York, 1985
[BCHM]   C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, arXiv:math/0610203
[C]   D. Chen, *Mori's Program for the Kontsevich Moduli Space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$*, Internat. Math. Res. Notices, Volume 2008, article ID rnn067
[CC]   D. Chen and I. Coskun, *Stable base locus decompositions of Kontsevich moduli spaces*, arXiv:0902.1785
[CK]   K. Chung and Y.-H. Kiem, *Hilbert scheme of rational cubic curves via stable maps*, arXiv:0812.1492
[CHS1]   I. Coskun, J. Harris and J. Starr, *The effective cone of the Kontsevich moduli space*, Canad. Math. Bull., vol. 51 no. 4 (2008), 519–534
[CHS2]   I. Coskun, J. Harris and J. Starr, *The ample cone of the Kontsevich moduli space*, Canad. J. Math., vol. 61 no. 1 (2009), 109–123
[Cr]   C. Crissman, http://math.berkeley.edu/~charleyc/code/movingcurves.m2
[DH]   S. Diaz and J. Harris, *Geometry of the Severi variety*, Trans. Amer. Math. Soc. 309 (1988), no. 1, 1–34
[FS]   M. Fedorchuk and D. Smyth, *Ample divisors on moduli spaces of weighted pointed rational curves, with applications to log MMP for $\bar{M}_{0,n}$*, arXiv:0810.1677
[FT]   H. Freiermuth and G. Trautmann, *On the moduli scheme of stable sheaves supported on cubic space curves*, Amer. J. Math. 126 (2004), no. 2, 363–393
[FP]   W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry–Santa Cruz 1995, 45–96, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997
[H]   J. Harris, *Curves in projective spaces*, Montréal, Les Presses de L'Université de Montreal, 1982
[Ha]   R. Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977

[H1]   B. Hassett, *Moduli spaces of weighted pointed stable curves*, Adv. Math. 173 (2003), no. 2, 316–352

[H2]   B. Hassett, *Classical and minimal models of the moduli space of curves of genus two*, Geometric methods in algebra and number theory, 169–192, Progr. Math., 235, Birkhuser Boston, Boston, MA, 2005

[HH1]   B. Hassett and D. Hyeon, *Log canonical models for the moduli space of curves: First divisorial contraction*, Trans. Amer. Math. Soc. 361 (2009), 4471–4489

[HH2]   B. Hassett and D. Hyeon, *Log minimal model program for the moduli space of stable curves: The first flip*, arXiv:0806.3444

[HK]   Y. Hu and S. Keel, *Mori dream spaces and GIT*, Michigan Math. J. 48 (2000), 331–348

[HL1]   D. Hyeon and Y. Lee, *Stability of tri-canonical curves of genus two*, Math. Ann. 337 (2007), no. 2, 479–488

[HL2]   D. Hyeon and Y. Lee, *Log minimal model program for the moduli space of stable curves of genus three*, arXiv:math/0703093

[Kol]   J. Kollár, *Rational curves on algebraic varieties*, Springer, 1996

[Kon]   M. Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves (Texel Island, 1994), 335–368, Progr. Math. 129, Boston, MA, 1995

[L]   R. Lazarsfeld, *Positivity in algebraic geometry I, classical setting: line bundles and linear series*, Springer-Verlag, 2004

[LP]   J. Le Potier, *Faisceaux semi-stables de dimension* 1 *sur le plan projectif*, Rev. Roumaine Math. Pures Appl. 38 (1993), no. 7-8, 635–678

[MM]   A. Mustaţă and M. A. Mustaţă, *Intermediate moduli spaces of stable maps*, Invent. Math. 167 (2007), no. 1, 47–90

[P1]   R. Pandharipande, *Intersections of $\mathbb{Q}$-divisors on Kontsevich moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ and enumerative geometry*, Trans. Amer. Math. Soc. 351 (1999), no. 4, 1481–1505

[P2]   R. Pandharipande, *The canonical class of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ and enumerative geometry*, Internat. Math. Res. Notices (1997), no. 4, 173–186

[Pa]   A. E. Parker, *An elementary GIT construction of the moduli space of stable maps*, Illinois J. Math. 51 (2007), no. 3, 1003–1025

[cSi]   C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety*, Inst. Hautes Études Sci. Publ. Math., no. 79 (1994), 47–129

[mSi]   M. Simpson, *On log canonical models of the moduli space of stable pointed curves*, arXiv:0709.4037

[Sm]   D. Smyth, *Compact moduli of singular curves: a case study in genus one*, arXiv:0808.0177

[Y]   S. Yang, *Linear systems in $\mathbb{P}^2$ with base points of bounded multiplicity*, J. Algebraic Geom. 16 (2007), no. 1, 19–38

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