THE BIRATIONAL GEOMETRY OF THE HILBERT SCHEME OF POINTS ON SURFACES

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To Jim Simons, with gratitude

Abstract. In this paper, we study the birational geometry of the Hilbert scheme of points on a smooth, projective surface, with special emphasis on rational surfaces such as $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$. We discuss constructions of ample divisors and determine the ample cone for Hirzebruch surfaces and del Pezzo surfaces with $K^2 \geq 2$. As a corollary, we show that the Hilbert scheme of points on a Fano surface is a Mori dream space. We then discuss effective divisors on Hilbert schemes of points on surfaces and determine the stable base locus decomposition completely in a number of examples. Finally, we interpret certain birational models as moduli spaces of Bridgeland stable objects. When the surface is $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$, we find a precise correspondence between the Mori walls and the Bridgeland walls, extending the results of [ABCH] to these surfaces.

Contents

1. Introduction

Bridgeland stability brings a new perspective to the study of the birational geometry of moduli spaces of sheaves on surfaces allowing one to construct flips explicitly and to improve classical bounds on nef and effective cones (see, for example, [AB], [ABCH] and [BM2]). In this paper, we study the relation between the stable base locus decomposition of the effective cone and the chamber decomposition of the stability manifold for the Hilbert scheme of points on a smooth, projective surface. We primarily concentrate on rational surfaces such as Hirzebruch and del Pezzo surfaces.

The paper [ABCH] describes a one-to-one correspondence between the Mori walls M_t and Bridgeland walls $W_{x=t-\frac{3}{2}}$ for $\mathbb{P}^{2[n]}$ and proves the correspondence for $n \leq 9$ in complete generality and for all n in the region $t \leq -\frac{n-1}{2}$. Precisely, a scheme Z is in the stable base locus of the

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divisors $D_t = H[n] + \frac{B}{2t}$ for $t < t_0$ if and only if the ideal sheaf $\mathcal{I}_Z$ is destabilized at the Bridgeland wall $W_{x=t_0-\frac{3}{2}}$. Using the correspondence, one can determine the base loci of linear systems on Hilbert schemes that are not apparent from a purely classical point of view. The correspondence is also useful in the other direction allowing classical geometry constructions to determine walls in the stability manifold. In this paper, we extend the correspondence to other surfaces such as $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$.

Let X be a smooth, projective surface over the complex numbers. Let $X^{[n]}$ denote the Hilbert scheme parameterizing zero-dimensional schemes of length n. Let $X^{(n)} = X^n/\mathfrak{S}_n$ denote the n-th symmetric product of X. There is a natural morphism $h : X^{[n]} \to X^{(n)}$, called the *Hilbert-Chow morphism*, that maps a zero dimensional scheme Z of length n to its support weighted by multiplicity.

In [F1], Fogarty proved that if X is a smooth projective surface, then $X^{[n]}$ is a smooth, projective, irreducible variety of dimension $2n$. The locus of n distinct, unordered points is a Zariski-dense open subset of $X^{[n]}$. Furthermore, the Hilbert-Chow morphism $h : X^{[n]} \to X^{(n)}$ is a crepant resolution of singularities.

In this paper, we will study the ample and effective cones of $X^{[n]}$ in the Néron-Severi space of $X^{[n]}$. For simplicity, we always assume that the irregularity of the surface $q(X)$ vanishes. In [F2], Fogarty determined the Picard group of $X^{[n]}$ in terms of the Picard group of X. A line bundle L on X naturally determines a line bundle $L[n]$ on $X^{[n]}$ as follows. The line bundle L on X gives rise to a line bundle $L \boxtimes \cdots \boxtimes L$ on X^n, which is invariant under the action of the symmetric group $\mathfrak{S}_n$. Therefore, $L \boxtimes \cdots \boxtimes L$ descends to a line bundle $L_{X^{(n)}}$ on the symmetric product $X^{(n)}$ under the natural quotient map $\pi : X^n \to X^{(n)}$. The pull-back $L[n] = h^* L_{X^{(n)}}$ under the Hilbert-Chow morphism gives the desired line bundle on $X^{[n]}$.

Let B denote the class of the exceptional divisor of the Hilbert-Chow morphism. Geometrically, the exceptional divisor is the divisor parameterizing non-reduced schemes in $X^{[n]}$. Since $X^{[n]}$ is smooth, we obtain an additional line bundle $\mathcal{O}_{X^{[n]}}(B)$ on $X^{[n]}$. In [F2], Fogarty proves that if the irregularity $q(X) = 0$, then $\mathrm{Pic}(X^{[n]}) \cong \mathrm{Pic}(X) \times \mathbb{Z}$. In particular, the Néron-Severi space of $X^{[n]}$ is spanned by the Néron-Severi space of X and the divisor class B.

In §2, we discuss the ample cone of $X^{[n]}$. If L is an ample line bundle on X, then $L[n]$ is a nef line bundle on $X^{[n]}$. Hence, knowing the ample cone of X allows one to determine the part of the nef cone in the subspace where the coefficient of B is zero. Results of Beltrametti, Sommese, Catanese and Göttsche (see [BSG], [BFS], [CG]) allow one to construct further nef divisors on $X^{[n]}$ from $(n-1)$-very ample line bundles on X. There are good criteria for verifying that a line bundle is $(n-1)$-very ample on a surface. For a large class of surfaces, these criteria allow one to classify the $(n-1)$-very ample line bundles. We will show that for simple surfaces such as $\mathbb{P}^2$, Hirzebruch surfaces and del Pezzo surfaces with $K^2 \geq 2$, these constructions suffice to determine the nef cone of $X^{[n]}$. For example, in Theorem 2.4, we will prove the following:

- The nef cone of $(\mathbb{P}^2)^{[n]}$ is the cone spanned by $H[n]$ and $(n-1)H[n] - \frac{B}{2}$, where H is the hyperplane class in $\mathbb{P}^2$ [LQZ].
- The nef cone of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$ is the cone spanned by $H_1[n]$, $H_2[n]$ and $(n-1)H_1[n] + (n-1)H_2[n] - \frac{B}{2}$, where H_1 and H_2 are the classes of the two fibers in $\mathbb{P}^1 \times \mathbb{P}^1$.
- Let $\mathbb{F}_r$ denote the surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$, $r \geq 1$. Then the nef cone of $\mathbb{F}_r^{[n]}$ is the cone spanned by $E[n] + rF[n]$, $F[n]$ and $(n-1)(E[n] + rF[n]) + (n-1)F[n] - \frac{B}{2}$, where E is the class of the exceptional curve and F is the class of a fiber in $\mathbb{F}_r$.

As a consequence of Theorem 2.4, we prove that if X is a Fano surface, then $X^{[n]}$ is log Fano and, in particular, a Mori dream space.

In §3, we discuss the effective cone of $X^{[n]}$. The effective cone of $X^{[n]}$ is much more subtle and depends on arithmetic properties of n. In many instances, the extremal rays of the cone can be constructed as loci of subschemes in $X^{[n]}$ that fail to impose independent conditions on sections of a given vector bundle on X. Showing that these loci are divisors, in general, is a difficult problem. Recently, Huizenga has made some progress when $X = \mathbb{P}^2$ (see [Hui]). Dually, we will construct moving curves to give upper bounds on the effective cone.

In §4, we will compute the stable base locus decomposition of $X^{[n]}$ for $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $2 \leq n \leq 5$ and $X = \mathbb{F}_1$ and $2 \leq n \leq 4$ in full detail. A quick glance at Figures 4 and 7, will convince the reader that these decompositions become complicated very quickly. These examples were chosen because they have fewer than 30 chambers. As we will discuss below, there is a rich interplay between the Mori chamber decomposition and the Bridgeland chamber decomposition. We hope that these examples will allow readers to explore connections that we will not discuss in this paper. On the other hand, if one restricts one's attention to chambers that are not too far from the nef cone, then it is possible to describe the stable base locus decomposition completely. We will do so for $\mathbb{P}^1 \times \mathbb{P}^1$ in Theorem 4.6 and for $\mathbb{F}_r$ in Theorem 4.15.

In §5, we will recall the definition of Bridgeland stability conditions. We will be interested in very specific stability conditions on X. Let H be an ample line bundle on X, then it is possible to find an abelian subcategory $\mathcal{A}_{s,t}$ of the bounded derived category $\mathcal{D}^b(\mathrm{coh}(X))$ of coherent sheaves on X such that when endowed with the central charge

$$Z_{s,t}(E) = -\int_X e^{-(s+it)H} \cdot \mathrm{ch}(E),$$

the pair $(\mathcal{A}_{s,t}, Z_{s,t})$ is a Bridgeland stability condition ([Br2], [AB]). We will pick a basis of the Néron-Severi space of X and consider the slice of the stability manifold corresponding to these Bridgeland stability conditions.

Fix a numerical class ν. Then the space of stability conditions $\mathrm{Stab}(X)$ has a chamber decomposition into regions where the set of semi-stable objects with class ν in $\mathcal{A}_{s,t}$ remains constant [Br2], [BM2]. In §5, we will derive the basic properties of the chamber structure for the Hilbert scheme of points. When ν is the numerical class of an ideal sheaf of a zero-dimensional scheme of length n, the walls in the second quadrant are nested semi-circles. We will study the walls in detail when X is a Hirzebruch surface or a del Pezzo surface.

In §6, we will describe a precise correspondence between the Bridgeland walls and the Mori walls for $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$. The correspondence is cleanest to state when H is a multiple of the anti-canonical bundle; however, seems to hold much more generally. When $X = \mathbb{P}^1 \times \mathbb{P}^1$, set $H = -\frac{1}{4}K_{\mathbb{P}^1 \times \mathbb{P}^1}$. When $X = \mathbb{F}_1$, set $H = -\frac{1}{6}K_{\mathbb{F}_1}$. Consider the divisor $D_t = H[n] + \frac{B}{2t}$ for $t < 0$ on $X^{[n]}$. Then we find the following conjectural correspondence. The divisor D_t intersects the Mori wall M_{t_0} corresponding to a subscheme Z if and only if $\mathcal{I}_Z$ is destabilized at the wall $W_{t_0 - c(X)}$, where $c(X)$ is a constant depending on the surface ($c(\mathbb{P}^1 \times \mathbb{P}^1) = -2$ and $c(\mathbb{F}_1) = -3$). As in the case of $\mathbb{P}^2$, we can prove the correspondence for small values of n in complete generality and for all n assuming that t is bounded above by a function depending on n and X.

The Hilbert scheme represents the Hilbert functor. When one studies the birational models of a moduli or parameter space, it is natural to ask whether the other birational models also have modular interpretations (see [HH1] and [HH2] for a discussion in the case of moduli spaces of curves and [CC1] and [CC2] for the case of Kontsevich moduli spaces). Classically, it is not clear how to vary the Hilbert functor to get alternative moduli spaces. The key idea is to represent the Hilbert scheme $X^{[n]}$ as a moduli space of Bridgeland stable objects and then to vary the Bridgeland stability. As we will see in §6, in many cases, one thus obtains modular interpretations of log canonical models of $X^{[n]}$.

Abramovich and Polishchuk [AP] have constructed moduli stacks of Bridgeland semi-stable objects parameterizing isomorphism classes of $Z_{s,t}$-semi-stable objects in $\mathcal{A}_{s,t}$. There are many open questions about the geometry of these moduli spaces. For example, it is not in general known whether they are projective. When $X = \mathbb{P}^2$, there are at least two ways of showing that the moduli spaces of Bridgeland stable objects are projective. These moduli spaces are projective because they are isomorphic to moduli spaces of quiver representations and can be constructed by GIT (see [ABCH], [K]). For the surfaces we consider, we do not in general know a GIT construction of the moduli space.

Alternatively, Bayer and Macrì have constructed nef divisors on the moduli space of Bridgeland stable objects [BM2]. Given a stability condition $\sigma = (Z, \mathcal{A})$, a choice of numerical invariants ν and a fine moduli space M parameterizing Bridgeland stable objects in $\mathcal{A}$ with numerical invariants ν, Bayer and Macrì define a nef divisor on M by specifying its intersection number with every curve. Let $\mathcal{E} \in \mathcal{D}^b(X \times M)$ be a universal family. Define $D \cdot C$ by the imaginary part of

$$\Im \left(-\frac{Z(\Phi_{\mathcal{E}}(\mathcal{O}_C))}{Z(\nu)} \right).$$

They prove that the divisor class is nef and if a curve has zero intersection with this divisor class, then for any two closed points on the curve $\mathcal{E}_x$ and $\mathcal{E}_{x'}$ are S-equivalent. In some cases, it can be shown that the divisors they construct are ample, giving a more general and better proof of the projectivity of the moduli space. This allows one to obtain modular interpretations of certain log canonical models of the Hilbert scheme in terms of moduli spaces of Bridgeland stable objects.

The organization of this paper is as follows: In §2, we discuss the nef cones of $X^{[n]}$. In §3, we give constructions of effective divisors and moving curves. In §4, we discuss general features of the stable base locus decomposition of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$ and $\mathbb{F}_r^{[n]}$. We also calculate the complete decomposition for small n. In §5, we recall preliminaries about Bridgeland stability conditions. Finally, in §6, we study the correspondence between Bridgeland walls and Mori walls.

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2. The ample cone of the Hilbert scheme

In this section, we survey the description of the nef cone of $X^{[n]}$ for surfaces such as Hirzebruch and del Pezzo surfaces with $K^2 \geq 2$. In many cases, one can give a complete description of the nef cone of $X^{[n]}$ using the theory of k-very ample line bundles developed by Beltrametti, Sommese, Catanese, Göttsche and others. For more details about k-very ample line bundles, we refer the reader to [BFS], [BSG] and [CG]. We begin by giving two constructions of nef divisors on $X^{[n]}$.

Construction 2.1. Let L be an ample line bundle on X. Then $L^{\boxtimes n} = L \boxtimes \cdots \boxtimes L$ is an ample line bundle on X^n invariant under the action of the symmetric group $\mathfrak{S}_n$. Consequently, $L^{\boxtimes n}$

descends to an ample line bundle $L^{(n)}$ on the symmetric product $X^{(n)}$. Since the Hilbert-Chow morphism is birational, the induced line bundle $L[n]$ is big and nef on $X^{[n]}$. However, since $L[n]$ has degree zero on the fibers of the Hilbert-Chow morphism, $L[n]$ is not ample. Hence, the line bundle $L[n]$ lies on the boundary of the nef cone of $X^{[n]}$.

Construction 2.2. Given a line bundle L on X, the short exact sequence

$$0 \longrightarrow L \otimes I_Z \longrightarrow L \longrightarrow L \otimes \mathcal{O}_Z \longrightarrow 0 \tag{1}$$

induces an inclusion $H^0(X, L \otimes I_Z) \subset H^0(X, L)$. Suppose that $H^0(X, L) = N > n$, then this inclusion induces a rational map

$$\phi_L : X^{[n]} \dashrightarrow G(N-n, N).$$

Let $D_L(n) = \phi_L^*(\mathcal{O}_{G(N-n,N)}(1))$ denote the pull-back of $\mathcal{O}_{G(N-n,N)}(1)$ under the rational map ϕ_L. A straightforward calculation using the Grothendieck-Riemann-Roch Theorem shows that the class of $D_L(n)$ is

$$D_L(n) = L[n] - \frac{B}{2}.$$

Since $\mathcal{O}_{G(N-n,N)}(1)$ is very ample, the base locus of D_L is contained in the indeterminacy locus of ϕ_L. In particular, if ϕ_L is a morphism, then $D_L(n)$ is base-point-free and, consequently, nef.

A line bundle L on X is called *k-very ample* if the restriction map $H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_Z)$ is a surjection for every zero dimensional scheme Z of length at most $k+1$. In [BSG], the authors give a useful characterization of k-very ampleness for adjoint bundles: Let L be a nef and big line bundle on a surface S such that $L^2 \geq 4k+5$. Then either $K_S + L$ is k-very ample or there exists an effective divisor D such that $L - 2D$ is $\mathbb{Q}$-effective, D contains a zero-dimensional subscheme of degree at most $k+1$ for which k-very ampleness fails and

$$L \cdot D - k - 1 \leq D \cdot D < \frac{L \cdot D}{2} < k+1.$$

Let L be an $(n-1)$-very ample line bundle on a surface X and assume that $h^0(X, L) = N$ and $h^1(X, L) = h^2(X, L) = 0$. Then, by the long exact sequence of cohomology associated to the exact sequence (1), we conclude that $H^i(X, L \otimes \mathcal{I}_Z) = 0$ for $i > 0$ and for the ideal sheaf $\mathcal{I}_Z$ associated to every zero-dimensional scheme $Z \in X^{[n]}$. Let

$$\Xi_n \subset X^{[n]} \times X$$

be the universal family and let π_1 and π_2 denote the natural projections. By cohomology and base change, $\pi_{1*}(\pi_2^* L \otimes \mathcal{I}_{\Xi_n})$ is a vector bundle of rank $N - n$ on $X^{[n]}$. Hence, by the universal property of the Grassmannian, $\phi_L : X^{[n]} \to G(N-n, N)$ is a morphism. Therefore, we conclude that $D_L(n) = L[n] - \frac{1}{2}B$ is base-point-free.

After introducing some notation, we will show that the nef divisors defined in Constructions 2.1 and 2.2 suffice to describe the nef cone of $X^{[n]}$ for surfaces like $\mathbb{P}^2$, del Pezzo surfaces with $K^2 \geq 2$ and Hirzebruch surfaces.

Notation 2.3. The Picard group of $\mathbb{P}^2$ is generated by the hyperplane class denoted by H. The Picard group of $\mathbb{P}^1 \times \mathbb{P}^1$ is generated by $H_1 = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and $H_2 = \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$, where π_i are the two projections.

Let $\mathbb{F}_r$, $r \geq 1$, denote the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$. The Picard group of $\mathbb{F}_r$ is the free abelian group generated by E and F, where E is the class of the section of self-intersection of $-r$ and F is the class of a fiber. We have the intersection numbers

$$E^2 = -r, \quad E \cdot F = 1, \quad F^2 = 0.$$

Let D_n denote the del Pezzo surface of degree n. The surface D_n is the blow-up of $\mathbb{P}^2$ at $9-n$ general points. The Picard group is the free abelian group generated by H and $E_1, \dots, E_{9-n}$, where H is the pull-back of the hyperplane class from $\mathbb{P}^2$ and E_i are the exceptional divisors of the blow-up. We have the intersection numbers

$$H^2 = 1, \quad H \cdot E_i = 0, \quad E_i \cdot E_j = -\delta_{i,j},$$

where $\delta_{i,j}$ denotes the Kronecker delta function.

We summarize the nef cone of $X^{[n]}$ for various rational surfaces in the following theorem.

Theorem 2.4. *The nef cone $X^{[n]}$, when X is a minimal rational surface or a del Pezzo surface with $K^2 \geq 2$ is as follows:*

(1) [LQZ, 3.12], [ABCH, 3.1], *The nef cone of $\mathbb{P}^{2[n]}$ is the closed cone bounded by*
$$(n-1)H[n] - \frac{1}{2}B \quad \text{and} \quad H[n].$$

(2) *The nef cone of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$ is the cone $\alpha H_1[n] + \beta H_2[n] + \gamma \frac{B}{2}$ satisfying the inequalities*
$$\gamma \leq 0, \quad \alpha + (n-1)\gamma \geq 0, \quad \text{and} \quad \beta + (n-1)\gamma \geq 0.$$

(3) *The nef cone of the Hilbert scheme $\mathbb{F}_r^{[n]}$ of n-points on the Hirzebruch surface $\mathbb{F}_r$ is the cone $\alpha(E[n] + rF[n]) + \beta F[n] + \gamma \frac{B}{2}$ satisfying the inequalities*
$$\gamma \leq 0, \quad \alpha + (n-1)\gamma \geq 0, \quad \text{and} \quad \beta + (n-1)\gamma \geq 0.$$

(4) *The nef cone of the Hilbert scheme $D_{9-r}^{[n]}$ of n-points on a del Pezzo surface D_{9-r} of degree $9-r = 5, 6$ or 7 is the cone $aH[n] - b_1E_1[n] - \cdots - b_rE_r[n] + c\frac{B}{2}$ satisfying the inequalities*
$$c \leq 0, \quad b_i + (n-1)c \geq 0, \quad \text{and} \quad a + (n-1)c \geq b_i + b_j \quad \text{for} \quad 1 \leq i \neq j \leq r.$$

(5) *The nef cone of the Hilbert scheme $D_{9-r}^{[n]}$ of n-points on the del Pezzo surface of degree $9-r = 4$ or 3 is the cone $aH[n] - b_1E_1[n] - \cdots - b_rE_r[n] + c\frac{B}{2}$ satisfying the inequalities*
$$c \leq 0, \quad b_i + (n-1)c \geq 0, \quad a + (n-1)c \geq b_i + b_j \quad \text{for} \quad 1 \leq i \neq j \leq r,$$
$$\text{and} \quad 2a + (n-1)c \geq \sum_{j=1}^{5} b_{i_j}.$$

(6) *The nef cone of the Hilbert scheme $D_2^{[n]}$ is the cone $aH[n] - b_1E_1[n] - \cdots - b_7E_7[n] + c\frac{B}{2}$ satisfying the inequalities*
$$c \leq 0, \quad b_i + (n-1)c \geq 0, \quad a + (n-1)c \geq b_i + b_j \quad \text{for} \quad 1 \leq i \neq j \leq r,$$
$$2a + (n-1)c \geq \sum_{j=1}^{5} b_{i_j}, \quad \text{and} \quad 3a + (n-1)c \geq 2b_i + \sum_{t=1}^{6} b_{j_t}, \quad j_t \neq i.$$

Proof. By Construction 2.1, if L is an ample line bundle on X, then $L[n]$ is nef on $X^{[n]}$. By Construction 2.2, if L is an $(n-1)$-very ample line bundle on X with vanishing higher cohomology, then $L[n] - \frac{B}{2}$ is nef on $X^{[n]}$. The proof has two parts. We first show that these two constructions generate the cones defined by the inequalities in the theorem. Hence, the nef cone of $X^{[n]}$ contains the cone described in the theorem. We then exhibit curves in $X^{[n]}$ that realize each of the inequalities in the theorem. Since a nef divisor intersects every curve non-negatively, this shows that the nef cone has to be contained in the cone defined by the inequalities in the theorem.

Let R be a general fiber of the Hilbert-Chow morphism over the singular locus of $X^{(n)}$. The curve R has the intersection numbers $R \cdot B = -2$ and $R \cdot L[n] = 0$ for any line bundle L on X. Consequently, the coefficient of B in any nef line bundle on $X^{[n]}$ has to be non-positive. All other curves in $X^{[n]}$ that we exhibit have a very specific form. Let C be a curve in X that admits a g^1_n. The morphism $f : C \to \mathbb{P}^1$ defined by the g^1_n induces a curve $C(n)$ in $X^{[n]}$. For the surfaces described in the theorem, the curve classes dual to the other faces of the nef cone will be of this form. We now carry out the analysis for each of the surfaces.

If $X = \mathbb{P}^2$, then $\mathcal{O}_{\mathbb{P}^2}(n-1)$ is $(n-1)$-ample [BSG], [LQZ]. One may deduce this either using the criterion of Beltrametti and Sommese or using minimal resolutions directly. Let $Z \in \mathbb{P}^{2[n]}$. Then $\mathcal{I}_Z$ admits a resolution

$$0 \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^2}(-a_i) \longrightarrow \bigoplus_{j=1}^{r+1} \mathcal{O}_{\mathbb{P}^2}(-b_j) \longrightarrow \mathcal{I}_Z \longrightarrow 0,$$

where $0 < a_i \leq n+1$ [E, Proposition 3.8]. Let $d \geq n-1$ and tensor the sequence by $\mathcal{O}_{\mathbb{P}^2}(d)$. The associated long exact sequence of cohomology implies that

$$H^1(\mathbb{P}^2, \mathcal{I}_Z(d)) \cong \bigoplus_{i=1}^{r} H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-a_i)) = 0$$

since $d - a_i \geq -2$. We conclude that $\mathcal{O}_{\mathbb{P}^2}(d)$ is $(n-1)$-ample if $d \geq n-1$. Consequently, the nef cone contains the closed cone spanned by $H[n]$ and $(n-1)H[n] - \frac{B}{2}$. On the other hand, let C be a line on $\mathbb{P}^2$, then the induced curve $C(n)$ satisfies the intersection numbers

$$C(n) \cdot H[n] = 1, \quad C(n) \cdot \frac{B}{2} = n-1.$$

Therefore, a nef divisor $aH[n] + b\frac{B}{2}$ satisfies $b \leq 0$ and $a + (n-1)b \geq 0$. We conclude that the nef cone is the cone spanned by $H[n]$ and $(n-1)H[n] - \frac{B}{2}$.

If $X = \mathbb{P}^1 \times \mathbb{P}^1$, using the criterion of Beltrametti and Sommese, it is easy to see that $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)$ is $(n-1)$-ample if $a, b \geq n-1$. We conclude that the nef cone of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$ contains the closed cone spanned by

$$H_1[n], H_2[n] \text{ and } (n-1)H_1[n] + (n-1)H_2[n] - \frac{B}{2}.$$

On the other hand, let F_i be a fiber of the projection π_i. Then the induced curve $F_i(n)$ has intersection numbers

$$F_i(n) \cdot H_j[n] = 1 - \delta_{i,j}, \quad F_i(n) \cdot \frac{B}{2} = n-1.$$

We conclude that a nef divisor $a_1H_1[n] + a_2H_2[n] + b\frac{B}{2}$ satisfies the inequalities $b \leq 0$ and $a_i + (n-1)b \geq 0$. Since any class satisfying these properties is a non-negative linear combination of $H_1[n], H_2[n]$ and $(n-1)H_1[n] + (n-1)H_2[n] - \frac{B}{2}$, we conclude that the nef cone of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$ is the cone spanned by these classes.

By the work of Beltrametti and Sommese ([BS], [ST, Lemma 10]), it is well-known that $M = (n-1)E + (r+1)(n-1)F$ is $(n-1)$-very ample on $\mathbb{F}_r$. Since the nef cone of $\mathbb{F}_r$ is generated by $E + rF$ and F, we conclude that the cone spanned by

$$E[n] + rF[n], \ F[n] \text{ and } M[n] = (n-1)(E[n] + rF[n]) + (n-1)F[n] - \frac{B}{2}$$

is contained in the nef cone of $\mathbb{F}_r^{[n]}$. Consider the curves $E(n)$ and $F(n)$ induced in $\mathbb{F}_r^{[n]}$ by a g^1_n on E and F, respectively. Then we have the intersection numbers

$$E(n) \cdot E[n] = -r, \quad E(n) \cdot F[n] = 1, \quad E(n) \cdot \frac{B}{2} = n-1,$$

$$F(n) \cdot E[n] = 1, \quad F(n) \cdot F[n] = 0, \quad F(n) \cdot \frac{B}{2} = n-1.$$

Consequently, the nef cone is spanned by $E[n] + rF[n]$, $F[n]$ and $M[n]$.

The strategy for the surfaces D_{9-r} is identical. On D_{9-r}, there are finitely many (-1)-curves whose classes can be listed explicitly (see [Ha, V.4] or [dR]). Moreover, the $(n-1)$-very ample line bundles on D_{9-r} have been classified in [dR]. Since D_9 is isomorphic to $\mathbb{P}^2$ and D_8 is isomorphic to $\mathbb{F}_1$, we may assume that $2 \leq r \leq 8$.

When $2 \leq r \leq 4$, then the $(n-1)$-ample line bundles on D_{9-r} are those with class $aH - \sum_{i=1}^r b_iE_i$ such that $b_i \geq n-1$ and $a \geq b_i + b_j + (n-1)$ [dR]. Hence, given a line bundle satisfying these inequalities, $aH[n] - \sum_{i=1}^r b_iE_i[n] - \frac{B}{2}$ is nef on $D_r^{[n]}$. In particular, $-(n-1)K - \frac{B}{2} = 3(n-1)H[n] - (n-1)\sum_{i=1}^r E_i[n] - \frac{B}{2}$ is nef on $D_{9-r}^{[n]}$. Similarly, setting $n = 1$, any line bundle satisfying $b_i \geq 0$ and $a \geq b_i + b_j$ is nef on D_{9-r}. Hence, the divisors $aH[n] - \sum_{i=1}^r b_iE_i[n]$ are nef on $D_{9-r}^{[n]}$ if $b_i \geq 0$ and $a \geq b_i + b_j$. Since every divisor satisfying the inequalities in the theorem is a non-negative linear combination of $-(n-1)K - \frac{B}{2}$ and $L[n]$, where L is nef on D_{9-r}, we conclude that the nef cone of $D_{9-r}^{[n]}$ contains the cone described in the theorem. Conversely, let R be a (-1)-curve. When $2 \leq r \leq 4$, then the only possible classes for R are E_i or $H - E_i - E_j$. If $R = E_i$, we have the intersection condition

$$R(n) \cdot \left(aH[n] - \sum_{i=1}^r b_iE_i[n] + c\frac{B}{2} \right) = b_i + (n-1)c \geq 0.$$

If $R = H - E_i - E_j$, we have the intersection condition

$$R(n) \cdot \left(aH[n] - \sum_{i=1}^r b_iE_i[n] + c\frac{B}{2} \right) = a - b_i - b_j + (n-1)c \geq 0.$$

We conclude that the nef cone is precisely the cone determined by the inequalities in the theorem.

When $5 \leq r \leq 6$, the condition that that a line bundle is $(n-1)$-ample requires the additional inequality $2a \geq b_{i_1} + \cdots + b_{i_5} + n - 1$. Hence, $-(n-1)K - \frac{B}{2}$ is nef on $D_{9-r}^{[n]}$ and every divisor in the cone described in the theorem is a non-negative linear combination of $-(n-1)K - \frac{B}{2}$ and $L[n]$ for a nef line bundle L on D_{9-r}. Conversely, there are new (-1)-curves R with class $2H - E_{i_1} - \cdots - E_{i_5}$. Intersecting a divisor class with $R(n)$, we see that

$$R(n) \cdot \left(aH[n] - \sum_{i=1}^r b_iE_i[n] + c\frac{B}{2} \right) = a - b_{i_1} - \cdots - b_{i_5} + (n-1)c \geq 0.$$

Hence, the nef cone is the cone determined by the inequalities in the theorem.

When $r = 7$, assume first that $n > 2$. Then the same argument as in the previous two paragraphs determines the nef cone. The class $-(n-1)K - \frac{B}{2}$ is nef and the classes satisfying the inequalities in the theorem can be expressed as a non-negative linear combination of $-(n-1)K - \frac{B}{2}$ and $L[n]$ for a nef classes L on D_{9-r}. The new inequalities come from the new types of (-1)-curves. The argument breaks down for $n = 2$ because $-K_{D_2}$ is not very ample, hence one needs to argue that $-K - \frac{B}{2}$ is nef on $D_2^{[2]}$. The linear system $|-K_{D_2}|$ defines a two-to-one map $f : D_2 \to \mathbb{P}^2$. Consequently, schemes of length two fail to impose independent conditions on sections of $-K_{D_2}$ if and only if they are fibers of the map f. Hence, the base locus of $-K - \frac{B}{2}$ is the $\mathbb{P}^2 = Y \in D_2^{[2]}$ parameterizing the fibers of f. The restriction of $-K - \frac{B}{2}$ to Y is trivial since $-K(2) \cdot (-K - \frac{B}{2}) = 0$. Consider the line bundle $\epsilon H[2] - K - \frac{B}{2}$. Since $H[2]$ is base-point-free, the base locus of this line bundle is contained in Y. On the other hand, this line bundle restricts to an ample line bundle on Y, hence is semi-ample by the Theorem of Fujita and Zariski [La, 2.1.32]. Since this is true for every $\epsilon > 0$, we conclude that $-K - \frac{B}{2}$ is nef. Now the rest of the argument follows as before. This concludes the proof of the theorem. □

Remark 2.5. The nef cone of $D_1^{[n]}$ is more complicated and highlights a shortcoming of the method for producing nef divisors described so far. Writing a divisor on $D_1^{[n]}$ as $aH[n] - b_1E_1[n] - \cdots - b_8E_8[n] + c\frac{B}{2}$ we see that the nef cone has to satisfy the inequalities

$$c \leq 0, \quad b_i + (n-1)c \geq 0, \quad a + (n-1)c \geq b_i + b_j \quad \text{for} \quad 1 \leq i \neq j \leq r,$$

$$2a + (n-1)c \geq \sum_{j=1}^{5} b_{i_j}, \quad 3a + (n-1)c \geq 2b_i + \sum_{t=1}^{6} b_{j_t}, \quad 4a + (n-1)c \geq \sum_{t=1}^{3} 2b_{i_t} + \sum_{t=1}^{5} b_{j_t},$$

$$5a + (n-1)c \geq b_{i_1} + b_{i_2} + \sum_{t=1}^{6} 2b_{j_t}, \quad 6a + (n-1)c \geq 3b_{i_1} + \sum_{t=1}^{7} 2b_{j_t} \quad \text{and} \quad 3a + nc \geq \sum_{i=1}^{8} b_i.$$

All the inequalities but the last one arise from curves of type $C(n)$, where C is a (-1)-curve on the surface. The last inequality arises from $-K_{D_1}(n)$. The line bundles $-(n-1)K_{D_1}$ and $-nK_{D_1}$ are no longer $(n-1)$-ample. In fact, the class $-(n-1)K - \frac{B}{2}$ is not nef on $D_1^{[n]}$ since it does not satisfy the last inequality. The class $-nK - \frac{B}{2}$ is nef, even though $-nK_{D_1}$ is not $(n-1)$-ample. The base locus of the linear system $-nK - \frac{B}{2}$ consists of the locus of n points that fail to impose independent conditions on the linear system $|-nK_{D_1}|$. Let p be the base-point of the linear system $|-K_{D_1}|$. A scheme Z of length n fails to impose independent conditions on the linear system $|-nK_{D_1}|$ if and only if Z is contained in a member C of the linear system $|-K_{D_1}|$ and is linearly equivalent to np on C. In other words, the base locus of the linear system $-nK - \frac{B}{2}$ is an n-dimensional scroll Y over $\mathbb{P}^1$. Since $-K_{D_1}(n) \cdot (-nK - \frac{B}{2}) = 0$, the restriction of $-nK - \frac{B}{2}$ to Y is equivalent to a multiple (in fact, $\frac{n^2-n}{2}$) of the class of a fiber. Hence $\epsilon H[n] - nK - \frac{B}{2}$ is semi-ample for every $\epsilon > 0$ by the theorem of Fujita and Zariski [La, 2.1.32]. Hence, $-nK - \frac{B}{2}$ is nef. By [dR], we conclude that any class of the form $L[n]$ or $L[n] - \frac{B}{2}$, where L is a line bundle on D_1, satisfying the inequalities is nef on $D_1^{[n]}$.

Unfortunately, the cone defined by the inequalities is larger than the cone generated by such classes. There is a further source of nef divisors we have not explored in this paper. A vector bundle E of rank r on a surface X is called k-very ample if the map $H^0(X, E) \to H^0(X, E \otimes \mathcal{O}_Z)$ is surjective for every scheme Z of length $k+1$. As in the case of line bundles, if E is an $(n-1)$-very ample vector bundle of rank r on X with $h^0(X, E) = N$, we get a morphism $\phi_E : X^{[n]} \to G = G(N - rn, N)$. Then $\phi^*\mathcal{O}_G(1)$ is a base-point-free line bundle with class $c_1(E)[n] - \frac{r}{2}B$. To generate the entire cone defined by the inequalities one would have to use this improved construction; however, we will not pursue this here any further. For the next corollary, we simply note that, by [CG], $-mK - \frac{B}{2}$ is very ample on $D_1^{[n]}$ for $m \geq n+3$.

Corollary 2.6. *Let X be surface with ample anti-canonical bundle. Then $X^{[n]}$ is a log Fano variety and a Mori dream space.*

Proof. Recall that the surfaces with ample anti-canonical bundle are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the del Pezzo surfaces D_r. By [BCHM], a log Fano variety is a Mori dream space. Therefore, it suffices to check that $X^{[n]}$ is a log Fano variety when X has ample anti-canonical bundle. By Fogarty's Theorem [F1], the Hilbert-Chow morphism is a crepant resolution. Hence, $-K_{X^{[n]}} = -K_X[n]$. Since $-K_X$ is ample on X, $-K_{X^{[n]}}$ is big and nef. However, $-K_{X^{[n]}}$ is not ample since it has zero intersection with fibers of the Hilbert-Chow morphism. Nevertheless, $-(K_{X^{[n]}} + \epsilon B)$ lies in the ample cone of $X^{[n]}$ by Theorem 2.4 and Remark 2.5 for $1 >> \epsilon > 0$. Let l be the log canonical threshold of B. As long as $l > \epsilon$, the pair $(X, \epsilon B)$ is klt. We conclude that $X^{[n]}$ is log Fano when X has ample anti-canonical bundle. $\square$

Remark 2.7. It would be interesting to compute the Cox ring of $X^{[n]}$ when X is a surface with ample anti-canonical bundle.

3. The effective cone of the Hilbert scheme

In this section, we study the effective cone of $X^{[n]}$. The effective cone of $X^{[n]}$ can be very subtle and often depends on the existence of higher rank vector bundles satisfying interpolation. We give several constructions of effective divisors and a construction of moving curves. In the next section, we show that for small n these constructions determine the entire effective cone.

The Néron-Severi space of $X^{[n]}$ may be identified with the vector space spanned by the Néron-Severi space of X and the divisor class B. Under this identification, we can view the nef and pseudo-effective cones of $X^{[n]}$ for different n as cones in the same abstract vector space. Often the effective cones are easier to determine when n satisfies certain arithmetic properties. For example, the effective cone of $\mathbb{P}^{2[n]}$ is easy to describe when $n = \frac{r(r+1)}{2} + j$ with $-1 \leq j \leq 1$, $j = r-1$ or $j = \frac{r}{2}$ for even r (see [ABCH, Remark 4.6]). For other n, it is harder to construct the extremal effective ray. The following lemma then gives a way of bounding the effective cone in cases when it is not so easy to determine the cone.

Lemma 3.1. *Eff* $(X^{[n+1]}) \subset$ *Eff* $(X^{[n]})$.

Proof. Let p be a point of X. Then there is a rational map $\rho_p : X^{[n]} \dashrightarrow X^{[n+1]}$ defined by mapping a zero-dimensional scheme Z to $Z \cup p$. The map ρ_p is well-defined provided that p is not in the support of Z. The rational map ρ_p induces a map $\rho_p^* : \mathrm{Pic}(X^{[n+1]}) \to \mathrm{Pic}(X^{[n]})$ on the Picard groups. The map ρ_p^* maps $L[n+1]$ to $L[n]$ and B to B, consequently, it induces the identity on the Néron-Severi spaces and is independent of p. Let D be an effective divisor on $X^{[n+1]}$. Let $Z \notin D$ be a union of $n+1$ distinct points $p_1, \dots, p_{n+1}$. Then the pull-back of D by $\rho_{p_{n+1}}$ is an effective divisor on $X^{[n]}$ with the same class as D. We conclude that Eff $(X^{[n+1]}) \subset$ Eff $(X^{[n]})$. $\square$

Express a divisor class on $X^{[n]}$ as $aL[n] + bB$. In the region of the effective cone lying in the half space $b > 0$, the stable base locus contains B as a divisorial component.

Proposition 3.2. *Let $D = aL[n] + bB$ be an effective divisor with $b > 0$. Then the stable base locus of D contains the divisor of non-reduced schemes. If L is ample on X, then the model corresponding to D is the symmetric product $X^{(n)}$ and the induced map is the Hilbert-Chow morphism.*

Proof. Let R be the fiber of the Hilbert-Chow morphism over a general point of the diagonal of $X^{(n)}$. Since the intersection numbers are $R \cdot L[n] = 0, \quad R \cdot B = -2$, we conclude that $R \cdot D < 0$ for any divisor $D = aL[n] + bB$ with $b > 0$. Since curves in the class R cover the divisor B of non-reduced schemes, we conclude that B is in the base locus of D. Hence, the model corresponding to D is the same as the model corresponding to $aL[n]$. If L is ample on X, then $L[n]$ is the pull-back to $X^{[n]}$ of the ample line bundle $L^{(n)}$ on the symmetric product $X^{(n)}$ by the Hilbert-Chow morphism. Consequently, the birational model corresponding to $L[n]$ is $X^{(n)}$ [La, 2.1.B]. □

The geometry of the surface plays a critical role in the half-space $b \leq 0$. We first recall the construction of a large family of effective divisors on $X^{[n]}$ depending on vector bundles on X.

Definition 3.3. Let E be a rank r vector bundle on X such that $h^0(X,E) \geq rn$. Let $W \subset H^0(X,E)$ be an rn-dimensional subspace. We say that W *satisfies interpolation for n points* if $W \cap H^0(X, E \otimes I_Z) = 0$ for a general $Z \in X^{[n]}$. We say that E *satisfies interpolation for n points* if $h^0(X,E) = rn$ and $h^0(X, E \otimes I_Z) = 0$ for a general $Z \in X^{[n]}$.

Construction 3.4. Let W be a subset of $H^0(X,E)$ that satisfies interpolation for n points. Let E have rank r and let $c_1(E) = D$. Let $\Xi_n \subset X^{[n]} \times X$ be the universal family with projections π_1 and π_2. Then the locus where

$$W \otimes \mathcal{O}_{X^{[n]}} \to \pi_{1*}(\pi_2^*(E) \otimes \mathcal{O}_{\Xi_n})$$

fails to be an isomorphism is a determinantal subscheme of codimension one in $X^{[n]}$. Hence, we obtain an effective divisor $D_{E,W}(n)$ on $X^{[n]}$. By the Grothendieck-Riemann-Roch Theorem, the class of $D_{E,W}(n)$ is

$$D[n] - \frac{r}{2}B.$$

In particular, by finding vector bundles on X that satisfy interpolation, we generate a subcone of the effective cone.

More generally, if $W \subset H^0(X,E)$ satisfies interpolation for n points and $n' \geq n$, the locus $D_{E,W}(n')$ of schemes $Z' \in X^{[n']}$ that have a subscheme Z of length n such that $W \cap H^0(X, E \otimes \mathcal{I}_Z) \neq 0$ is a divisor in $X^{[n']}$. If $W = H^0(X,E)$, we omit it from the notation.

Example 3.5. Line bundles satisfy interpolation. Hence, if L is a line bundle on X with $h^0(X,L) = n$, then $L[n] - \frac{B}{2}$ is an effective divisor on $X^{[n]}$. However, classifying vector bundles on X that satisfy interpolation for n-points is a hard problem. Jack Huizenga has made progress in classifying Steiner bundles on $\mathbb{P}^2$ that satisfy interpolation. Let

$$\Phi = \{x \mid x \geq \phi^{-1}\} \cup \{\frac{0}{1}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \cdots\},$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and the fractions are consecutive ratios of Fibonacci numbers. Let $n = \frac{r(r+1)}{2} + s, \quad s \geq 0$. Consider a general vector bundle E given by the resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(r-2)^{\oplus ks} \to \mathcal{O}_{\mathbb{P}^2}(r-1)^{\oplus k(s+r)} \to E \to 0.$$

Huizenga [Hui, Theorem 4.1] proves that for sufficiently large k, E is a vector bundle that satisfies interpolation for n points if and only if $\frac{s}{r} \in \Phi$. Similarly, let F be a general vector bundle given by the resolution

$$0 \to F \to \mathcal{O}_{\mathbb{P}^2}(r)^{\oplus k(2r-s+3)} \to \mathcal{O}_{\mathbb{P}^2}(r+1)^{\oplus k(r-s+1)} \to 0.$$

For sufficiently large k, F has interpolation for n points if and only if $1-\frac{s+1}{r+2}\in\Phi$. We conclude that if $\frac{s}{r}\in\Phi$, then $(r^2-r+s)H[n]-\frac{r}{2}B$ is effective on $\mathbb{P}^{2[n]}$. If $1-\frac{s+1}{r+2}\in\Phi$ and $s\geq 1$, then $(r^2+r+s-1)H[n]-\frac{r+2}{2}B$ is effective on $\mathbb{P}^{2[n]}$.

Construction 3.6. A *moving curve class* C on a variety Y is a curve class whose representatives cover a Zariski dense subset of Y. If D is an effective divisor on Y and C is a moving curve class, then $C\cdot D\geq 0$. Hence, each moving curve class on $X^{[n]}$ gives a bound on the effective cone of $X^{[n]}$.

Let L be a very ample line bundle on X such that $h^0(X,L)>n$. Suppose that a general section of L is a smooth curve of genus $g<n$. Then we obtain a moving curve in $X^{[n]}$ as follows. Since $h^0(X,L)>n$, by Bertini's Theorem, a general scheme Z of length n is contained in a smooth curve $\delta\in|L|$. The scheme Z defines a divisor on the curve δ. By the Riemann-Roch Theorem, $h^0(\delta,Z)\geq 2$, so there exists a map $f:\delta\to\mathbb{P}^1$ such that Z is a fiber of this map. The map f induces a curve $C(n)$ in the Hilbert scheme $X^{[n]}$. Since $C(n)$ passes through a general point $Z\in X^{[n]}$, we conclude that $C(n)$ is a moving curve in $X^{[n]}$. If $D=M[n]+b\frac{B}{2}$ is an effective divisor, then we have the inequality $C(n)\cdot D=\delta\cdot M+b(g-1+n)\geq 0$, where the intersection number $C(n)\cdot\frac{B}{2}$ is computed using the Riemann-Hurwitz formula.

More generally, given a moving curve C in $X^{[n]}$, we obtain a moving curve C' in $X^{[n']}$ for $n'\geq n$ by taking the unions of the schemes parameterized by C with a fixed general scheme Z of length $n'-n$.

Example 3.7. Let $n=\frac{r(r+1)}{2}+s,\quad 0\leq s\leq r$. If $\frac{s}{r}\in\Phi$, let $L=rH$ and if $1-\frac{s+1}{r+2}\in\Phi$, let $L=(r+2)H$ in Construction 3.6. Then combining Constructions 3.4 and 3.6, we obtain the following description of the effective cone of $(\mathbb{P}^2)^{[n]}$.

Theorem 3.8. [ABCH, Theorem 4.5] *Let* $n=\frac{r(r+1)}{2}+s,\quad 0\leq s\leq r$.

(1) *If* $\frac{s}{r}\in\Phi$, *then the effective cone of* $\mathbb{P}^{2[n]}$ *is the closed cone bounded by the rays*
$$H[n]-\frac{r}{2(r^2-r+s)}B \quad \textit{and} \quad B.$$

(2) *If* $1-\frac{s+1}{r+2}\in\Phi$ *and* $s\geq 1$, *then the effective cone of* $\mathbb{P}^{2[n]}$ *is the closed cone bounded by the rays*
$$H[n]-\frac{r+2}{2(r^2+r+s-1)}B \quad \textit{and} \quad B.$$

Construction 3.6 gives bounds on the ratio of the first Chern class and the rank of a vector bundle E on X that can satisfy interpolation for n points. If $L[n]-\alpha\frac{B}{2}$ is not effective, then a vector bundle E with $c_1(E)=aL$ and rank r cannot satisfy interpolation for n points if $\frac{r}{a}\geq\alpha$.

Example 3.9. Let $X=\mathbb{P}^1\times\mathbb{P}^1$ and let $a,b\in\mathbb{Z}$ such that $n=(a+1)(b+1)$. Then, by Construction 3.4, the locus of schemes $Z\in(\mathbb{P}^1\times\mathbb{P}^1)^{[n]}$ that are contained in a curve of type (a,b) is an effective divisor with class $aH_1[n]+bH_2[n]-\frac{B}{2}$. Hence, the effective cone contains the cone generated by these divisors and B.

On the other hand, if $(a+1)(b+1)>n>(a-1)(b-1)$, then every scheme of length n on $\mathbb{P}^1\times\mathbb{P}^1$ is contained in a curve of type (a,b). By Construction 3.6, any effective divisor $\alpha H_1[n]+\beta H_2[n]-\frac{\gamma}{2}B$ has to satisfy $a\beta+b\alpha-\gamma(ab-a-b+n)\geq 0$.

Example 3.10. Let $X=\mathbb{F}_r$. Let a,b be integers such that $b\geq ar\geq 0$. Let $n=(a+1)(b+1-\frac{ra}{2})$. Using induction on i for $0\leq i\leq a-1$ and the exact sequence
$$0\to\mathcal{O}_{\mathbb{F}_r}((a-i-1)E+(b-r(i+1))F)\to\mathcal{O}_{\mathbb{F}_r}((a-i)E+(b-ri)F)\to\mathcal{O}_{\mathbb{P}^1}(b-ri)\to 0,$$

we see that the higher cohomology of $aE+bF$ vanishes. Hence, by the Riemann-Roch formula, $h^0(\mathbb{F}_r, aE+bF) = n$. Then, by Construction 3.4, the locus of schemes $Z \in \mathbb{F}_r^{[n]}$ that are contained in a curve of class $aE+bF$ is an effective divisor with class $aE[n]+bF[n]-\frac{B}{2}$. Hence, the effective cone contains the cone generated by these divisors and B.

4. Stable base locus decomposition of the effective cone of $X^{[n]}$

In this section, we describe the stable base locus decompositions of $X^{[n]}$ for small n when X is $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$. Even when n and the Picard rank of X are small, the stable base locus decomposition of $X^{[n]}$ can be very complicated. Moreover, the number of chambers grows very rapidly with n. We begin by a construction that helps determine the stable base locus.

Construction 4.1. Construction 3.6 can be generalized to study the stable base loci of linear systems on $X^{[n]}$. Let R be a smooth curve on X of genus g and suppose that a general scheme Z_0 of length $m \leq n$ contained in R satisfies $h^0(R, Z_0) \geq 2$. Then we obtain a curve $R(m,n)$ on $X^{[n]}$ through Z_0 by considering $Z_t'' = Z_t \cup Z'$, where Z' is a fixed scheme of length $n-m$ not supported on R and Z_t are the fibers of a map $f: R \to \mathbb{P}^1$ containing Z_0. Then the locus of schemes Z that have a subscheme of length m contained in a curve of type R is in the base locus of any linear system $D = L[n] + b\frac{B}{2}$ such that $L \cdot R + b(g-1+m) < 0$. Curves in the class $R(m,n)$ sweep out this locus. Hence, any effective divisor that has negative intersection with $R(m,n)$ has to contain this locus.

Example 4.2. For example, schemes $Z \in \mathbb{P}^{2[n]}$ that have a linear subscheme of length m are contained in the base locus of linear systems $aH[n] - \frac{B}{2}$ if $a < m-1$. Schemes $Z \in \mathbb{P}^{2[n]}$ that have a subscheme of length m in a conic are contained in the base locus of linear systems $aH[n] - \frac{B}{2}$ if $2a < m-1$. More generally, schemes $Z \in \mathbb{P}^{2[n]}$ that have a subscheme of length $\frac{(d-1)(d-2)}{2} + 1 \leq m \leq n$ contained in a curve of degree d are in the base locus of linear systems $aH[n] - \frac{B}{2}$ if $da < \frac{(d-1)(d-2)}{2} - 1 + m$. These observations suffice to describe a large portion of the stable base locus decomposition of $\mathbb{P}^{2[n]}$. These decompositions have been described in detail in [ABCH], so we will turn our attention to other surfaces.

Example 4.3. Let $n \geq m > (a-1)(b-1)$. Then the locus of $Z \in (\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$ that have a subscheme of length m contained on a curve of type (a,b) is in the base locus of a divisor $\alpha H_1[n] + \beta H_2[n] - \gamma\frac{B}{2}$ if $a\beta + b\alpha - \gamma(ab-a-b+m) < 0$.

The stable base locus decomposition of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$. In this subsection, we will compute the stable base locus decomposition of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$ for $2 \leq n \leq 5$ in full detail and discuss some aspects of the decomposition for general n.

By Construction 3.4, the locus of schemes that have a subscheme of length two in a fiber with class H_i is the divisor $D_{H_i}(n)$ on $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$ with class $(n-1)H_i[n] - \frac{B}{2}$. To compute the class of $D_{H_i}(n)$, let F_j, $j \neq i$, be a fiber with class H_j and let R be a curve of type $(1,1)$. Since $F_j \cdot H_i = R \cdot H_i = 1$, the curves $F_j(n,n)$ and $R(n,n)$ defined in Construction 4.1 have intersection number zero with $D_{H_i}(n)$. This determines the class of $D_{H_i}(n)$ up to a constant. The constant can be determined by intersecting with $F_j(1,n)$.

Let $D = a_1H_1[n] + a_2H_2[n] + c\frac{B}{2}$. The moving curve $F_i(1,n)$ has intersection number zero with both B and $(n-1)H_i[n] - \frac{B}{2}$. Hence, the intersection of the effective cone of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$ with the half space $a_1 + a_2 \geq n-1$ is the cone generated by $(n-1)H_1[n] - \frac{B}{2}, (n-1)H_2[n] - \frac{B}{2}$ and B. The stable base locus decomposition is easy to understand in this subcone.

Lemma 4.4. (1) *Let $n \leq a+b+1$ and let $Z \in (\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$. If Z does not have a subscheme of length $b+2$ contained in a fiber with class H_1 or a subscheme of length $a+2$ contained in a fiber with class H_2, then Z imposes independent conditions on sections of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)$.*

(2) *Let $n = a+b+2$ and let $Z \in (\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$. If Z is not contained in a curve of type $(1,1)$ and does not have a subscheme of length $b+2$ contained in a fiber with class H_1 or a subscheme of length $a+2$ contained in a fiber with class H_2, then Z imposes independent conditions on sections of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)$.*

Proof. The lemma follows by induction on a and b and residuation. Consider the exact sequences

$$0 \to \mathcal{I}_{Z_1}(a-1,b) \to \mathcal{I}_Z(a,b) \to \mathcal{I}_{Z\cap F_1 \subset F_1}(a,b) \to 0$$
$$0 \to \mathcal{I}_{Z_2}(a,b-1) \to \mathcal{I}_Z(a,b) \to \mathcal{I}_{Z\cap F_2 \subset F_2}(a,b) \to 0,$$

where Z_i are the residual schemes defined by the ideals $(\mathcal{I}_Z : \mathcal{I}_{F_i})$. By our assumption that none of the fibers with class H_1 (respectively, H_2) contain a subscheme of length $b+2$ $(a+2)$, $H^1(\mathcal{I}_{Z\cap F_i\subset F_i}(a,b)) = 0$. If there exists a fiber F_i with class H_i that contains a subscheme of length $b+1$ when $i=1$ or $a+1$ when $i=2$, then consider the residuation sequence with respect to F_i. Otherwise, consider the residuation sequence with respect to a fiber that contains the maximal length subscheme of Z. For concreteness, let us say that the fiber is F_1. By our choice of F_1, Z_1 does not have a subscheme of length $b+2$ in a fiber with class H_1 and cannot have a subscheme of length $a+1$ in a fiber with class H_2 (otherwise the length of Z would be at least $a+b+2 > n$). Hence, by induction $H^1(\mathcal{I}_{Z'}(a-1,b)) = 0$. The long exact sequence of cohomology implies that $H^1(\mathcal{I}_Z(a,b)) = 0$, proving (1).

The proof of (2) is almost identical. If there are fibers that contain a subscheme of Z of length greater than one, then using the residuation sequence for the fibers, the proof of part (2) reduces to the proof of part (1). Otherwise, the residuation sequence

$$0 \to \mathcal{I}_{Z'}(a-1,b-1) \to \mathcal{I}_Z(a,b) \to \mathcal{I}_{Z\cap R\subset R}(a,b) \to 0$$

applied to a curve R of type $(1,1)$ containing a maximal length subscheme of Z and induction proves (2). □

Notation 4.5. Let $Z(a,b;j)$ denote the locus of schemes in $X^{[n]}$ that have a subscheme of length j supported on a curve of type (a,b). For $i,j \in \mathbb{Z}$, let $X_{i,j}$ denote the divisor class $iH_1[n] + jH_2[n] - \frac{B}{2}$.

Theorem 4.6. (1) *Let $i,j \in \mathbb{Z}$ be such that $i+j > n-1$ and $n-2 \geq i,j > 0$. Then the cone generated by $X_{i,j}, X_{i+1,j}, X_{i+1,j+1}, X_{i,j+1}$ is a chamber of the stable base locus decomposition, where the stable base locus consists of $Z(1,0;j+2) \cup Z(0,1;i+2)$.*

(2) *If $j = n-1$ (respectively, $i = n-1$) and $n-2 \geq i \geq 0$ (respectively, $n-2 \geq j \geq 0$), then the cone generated by $X_{i,n-1}$, $X_{i+1,n-1}$ and $H_2[n]$ (respectively, $X_{n-1,j}, X_{n-1,j+1}$, $H_1[n]$) is a chamber of the stable base locus decomposition, where the stable base locus is $Z(0,1;i+2)$ (respectively, $Z(1,0;j+2)$).*

(3) *If $i+j = n-1$ and $i,j > 0$, then the cone generated by $X_{i,j}, X_{i-1,j+1}$ and $X_{i+1,j+1}$ is a chamber of the stable base locus decomposition, where the stable base locus consists of $Z(1,0;j+2) \cup Z(0,1;i+1)$.*

Proof. By Construction 2.2, if $n > (i+1)(j+1)$, the linear systems $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(i,j)|$ give rise to rational maps to Grassmannians. If $i+j \geq n-1$, then $(i+1)(j+1) = ij+i+j+1 > n$ unless $i+j = n-1$ and one of i or j is zero. Hence, $X_{i,j}$ is an effective divisor with base locus equal to schemes that fail to impose independent conditions on the linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(i,j)|$. The divisors $D_{H_1}(n)$ and $D_{H_2}(n)$ have classes $X_{n-1,0}$ and $X_{0,n-1}$, respectively. Hence, their base locus is contained in $D_{H_1}(n) = Z(1,0;2)$ and $D_{H_2}(n) = Z(0,1;2)$, respectively.

By Lemma 4.4, a scheme Z of length n imposes independent conditions on $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(i,j)|$ as long as Z is not contained in the locus $Z(1,0;j+2) \cup Z(0,1;i+2)$. Since every divisor in the cone generated by $X_{i,j}, X_{i+1,j}, X_{i+1,j+1}, X_{i,j+1}$ is a non-negative linear combination of $X_{i,j}$ and the base-point-free divisors $H_1[n]$ and $H_2[n]$, we conclude that the stable base locus in this cone is contained in the locus $Z(1,0;j+2) \cup Z(0,1;i+2)$. By the same argument, for divisors in the plane spanned by $X_{i+1,j}$ and $X_{i+1,j+1}$ (respectively, $X_{i,j+1}$ and $X_{i+1,j+1}$), the stable base locus is contained in the locus of schemes $Z(1,0;j+2) \cup Z(0,1;i+3)$ (respectively, $Z(1,0;j+3) \cup Z(0,1;i+2)$).

Conversely, using the curves $F_1(j+2,n)$ and $F_2(i+2,n)$ defined in Construction 4.1, where F_1 and F_2 are fibers with classes H_1 and H_2, respectively, we see that the locus $Z(1,0;j+2) \cup Z(0,1;i+2)$ is contained in the base locus of every divisor $a_1H_1[n]+a_2H_2[n]+b\frac{B}{2}$ if $a_1+(i+1)b<0$ and $a_2+(j+1)b<0$. We conclude that the cone generated by $X_{i,j}, X_{i+1,j}, X_{i+1,j+1}, X_{i,j+1}$ is a chamber of the stable base locus decomposition, where the base locus is exactly $Z(1,0;j+2) \cup Z(0,1;i+2)$. This concludes the proof of part (1).

The same argument shows that when $i = n-1$, then in the cone generated by $X_{n-1,j}, X_{n-1,j+1}$ and $H_1[n]$, the stable base locus is equal to $Z(1,0;j+2)$. By the symmetry exchanging the two fibers, this proves part (2).

When $i+j = n-1$, the argument shows that in the cone generated by $X_{i,j}, X_{i-1,j+1}$ and $X_{i+1,j+1}$ the stable base locus is equal to $Z(1,0;j+2) \cup Z(0,1;i+1)$. To conclude that this cone is a chamber of the stable base locus decomposition, we use Construction 4.1 for a curve R of type $(1,1)$. Then $R(n,n)$ has intersection number $a_1 + a_2 + (n-1)b$ with the divisor $a_1H_1[n] + a_2H_2[n] + b\frac{B}{2}$. Hence, if $a_1 + a_2 + (n-1)b < 0$, then the locus $Z(1,1;n)$ is in the base locus of D. We conclude that the cones generated by $X_{i,j}, X_{i-1,j+1}$ and $X_{i+1,j+1}$ with $i+j = n-1$ form chambers of the stable base locus decomposition. This concludes the proof of part (3). □

Remark 4.7. Combining Theorem 4.6, Theorem 2.4 and Proposition 3.2, we obtain the complete stable base locus decomposition of the cone generated by $(n-1)H_1[n] - \frac{B}{2}, (n-1)H_2[n] - \frac{B}{2}$ and B.

We now turn to the explicit decomposition of the effective cone for $2 \le n \le 5$. The automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ exchanging the factors gives rise to a symmetry exchanging $H_1[n]$ and $H_2[n]$. Hence, all the decompositions are symmetric with respect to the vertical axis. We explicitly explain one half of the diagrams and leave it to the reader to exchange H_1 and H_2 to obtain the rest. In each case, we will draw a cross-section of the cone and label the important rays by a meaningful divisor on that ray rather than the point that is contained in the cross-section. In order to avoid cluttering the diagrams, we will write H_i instead of $H_i[n]$.

Example 4.8. Figure 1 shows the stable base locus decomposition of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[2]}$. Recall our convention that $X_{i,j} = iH_1[2] + jH_2[2] - \frac{B}{2}$. The chambers in this decomposition have the following descriptions:

(1) The effective cone is the closed cone spanned by B, $X_{1,0}$ and $X_{0,1}$.
(2) The base-point-free, nef and moving cones coincide and are equal to the closed cone spanned by $H_1[2]$, $H_2[2]$ and $X_{1,1}$.
(3) In the cone spanned by B, $H_1[2]$ and $H_2[2]$, the base locus is equal to B.
(4) In the cone spanned by $H_1[2]$, $X_{1,0}$ and $X_{1,1}$ the base locus is equal to $Z(1,0;2)$.

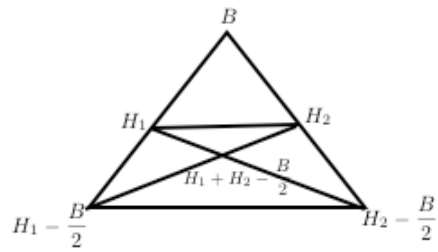

FIGURE 1. The stable base locus decomposition of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[2]}$.

(5) Finally, in the cone spanned by $X_{1,0}$, $X_{0,1}$ and $X_{1,1}$, the base locus is equal to $Z(1,0;2) \cup Z(0,1;2)$.

Proof. Theorem 2.4 (2) describes the base-point-free and nef cones of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$ in general. Proposition 3.2 proves that the base locus in the cone spanned by $H_1[2], H_2[2]$ and B contains the divisor B. Since $H_i[n]$ are base-point-free, part (3) follows. By Theorem 4.6, the cone generated by $X_{1,0}, X_{1,1}$ and $H_1[2]$ is a chamber of the stable base locus decomposition with base locus equal to $D_{H_1}(2) = Z(1,0;2)$. By symmetry, the base locus in the cone spanned by $X_{0,1}, X_{1,1}$ and $H_2[2]$ is the divisor $D_{H_2}(2) = Z(0,1;2)$. By Theorem 4.6, in the cone generated by $X_{0,1}, X_{1,0}$ and $X_{1,1}$ the base locus is the union of the divisors $Z(1,0;2) \cup Z(0,1;2)$. Hence, the moving cone of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[2]}$ is equal to the base-point-free cone in this case.

To complete the proof there remains to show that the effective cone is equal to the cone spanned by $X_{1,0}, X_{0,1}$ and B. We already know that these divisors are effective, so it suffices to give moving curves dual to each face of the cone. The moving curves $F_1(1,2)$ and $F_2(1,2)$ defined in Construction 4.1 are dual to the faces spanned by $X_{1,0}, B$ and $X_{0,1}, B$, respectively. Similarly, let R be a curve of type $(1,1)$. The moving curve $R(2)$ defined in Construction 3.6 is dual to the face spanned by $X_{1,0}$ and $X_{0,1}$. This concludes the discussion of this example. □

Example 4.9. Figure 2 shows the stable base locus decomposition of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[3]}$. The chambers have the following description.

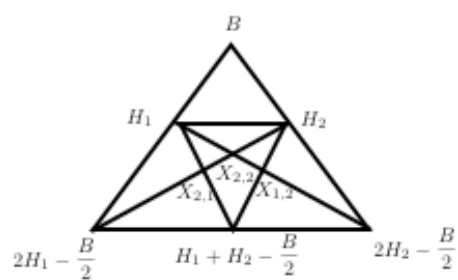

FIGURE 2. The stable base locus decomposition of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[3]}$.

(1) The effective cone is the closed cone spanned by B, $X_{2,0}$ and $X_{0,2}$.
(2) The base-point-free cone is the closed cone generated by $H_1[3]$, $H_2[3]$ and $X_{2,2}$.
(3) The moving cone is the closed cone generated by $H_1[3]$, $H_2[3]$ and $X_{1,1}$.
(4) In the cone generated by B, $H_1[3]$ and $H_2[3]$, the base locus is divisorial equal to B.

(5) For $0 \leq i \leq 1$, in the cone generated by $H_1[3]$, $X_{2,i}$ and $X_{2,i+1}$, the base locus is $Z(1,0;i+2)$.
(6) In the cone generated by $X_{2,0}$, $X_{1,1}$ and $X_{2,1}$ the base locus is $Z(1,0;2) \cup Z(0,1;3)$.
(7) Finally, in the cone generated by $X_{1,1}$, $X_{2,1}$, $X_{2,2}$ and $X_{1,2}$, the base locus is $Z(1,0;3) \cup Z(0,1;3)$.

Proof. Part (2) follows from Theorem 2.4. Part (4) follows from Proposition 3.2. Parts (5), (6) and (7) follow from Theorem 4.6. Since B, $Z(1,0;2)$ and $Z(0,1;2)$ are divisors and $Z(1,0;3)$ and $Z(0,1;3)$ have codimension 2, part (3) follows from parts (4), (5), (6) and (7). Hence, there remains to prove part (1). The divisors $X_{2,0}, X_{0,2}$ and B are effective. To show that the effective cone is equal to the cone generated by them, we exhibit dual moving curves. The moving curves $F_1(1,3)$ and $F_2(1,3)$ are dual to the faces spanned by $X_{2,0}, B$ and $X_{0,2}, B$, respectively. Let R be a curve of type $(1,1)$. Then, the moving curve $R(3)$ is dual to the face spanned by $X_{2,0}$ and $X_{0,2}$. This concludes the discussion of this example. $\square$

Example 4.10. Figure 3 shows the stable base locus decomposition of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[4]}$. The chambers in the decomposition have the following descriptions.

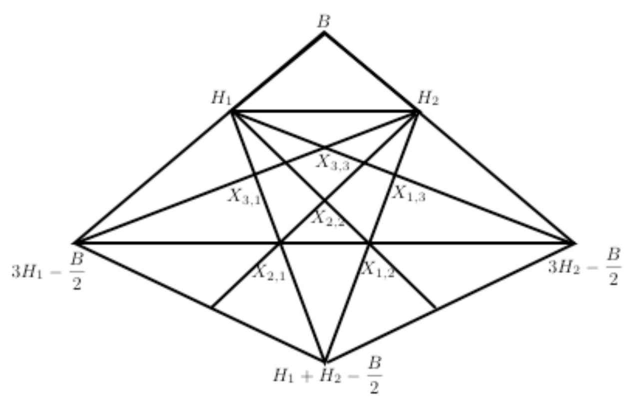

FIGURE 3. The stable base locus decomposition of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[4]}$.

(1) The effective cone is the closed cone generated by B, $X_{3,0}$, $X_{1,1}$ and $X_{0,3}$.
(2) The base-point-free cone is the closed cone spanned by $H_1[4]$, $H_2[4]$ and $X_{3,3}$.
(3) The moving cone is the closed cone spanned by $H_1[4]$, $H_2[4]$, $X_{1,2}$ and $X_{2,1}$.
(4) In the cone spanned by $H_1[4]$, $H_2[4]$ and B the base locus is divisorial equal to B.
(5) The stable base locus contains the divisor $Z(1,0;2)$ (respectively, $Z(0,1;2)$) exactly in the cone spanned by $H_1[4]$, $X_{3,0}$ and $X_{1,1}$ (respectively, $H_2[4]$, $X_{0,3}$ and $X_{1,1}$).
(6) The stable base locus contains the divisor $Z(1,1;4)$ exactly in the cone spanned by $X_{3,0}$, $X_{0,3}$ and $X_{1,1}$. Hence, to give a complete description of the stable base locus decomposition, it suffices to give the stable base locus decomposition of the moving cone.
(7) In the cone spanned by $X_{2,1}$, $X_{1,2}$ and $X_{2,2}$, the stable base locus is $Z(1,0;3) \cup Z(0,1;3)$.
(8) In the cone spanned by $X_{i,j}$, $X_{i+1,j}$, $X_{i+1,j+1}$, $X_{i,j+1}$, for $i+j \geq 3$ and $2 \geq i,j > 0$, the stable base locus is $Z(1,0;j+2) \cup Z(0,1;i+2)$.
(9) In the cone spanned by $H_1[4]$, $X_{3,i}$ and $X_{3,i+1}$, for $1 \leq i \leq 2$, the base locus is $Z(1,0;i+2)$.

Proof. Parts (2) and (4) follow from Theorem 2.4 and Proposition 3.2. Parts (7), (8), (9) follow from Theorem 4.6. Parts (4)-(9) imply (3). Since $H_1[4]$ and $H_2[4]$ are base-point-free, in the cone spanned by $X_{1,1}$, $H_1[4]$ and $H_2[4]$, the base locus is contained in the divisor $Z(1,1;4)$. Similarly, in the cone spanned by $X_{3,0}$, $X_{0,3}$ and $H_1[4]$ and $H_2[4]$, the base locus is contained in the union of divisors $Z(1,0;2) \cup Z(0,1;2)$. The curves $F_1(2,4)$ and $F_2(2,4)$ defined in Construction 4.1 are dual to the faces $H_1[4], X_{1,1}$ and $H_2[4], X_{1,1}$. Therefore, the divisors $Z(1,0;2)$ and $Z(0,1;2)$ are contained in the base locus precisely in the cones generated by $H_1[4], X_{3,0}, X_{1,1}$ and $H_2[4], X_{0,3}, X_{1,1}$, respectively, proving (5). Let $R_{i,j}$ denote a curve of type (i,j). The curve $R_{1,1}(4,4)$ defined in Construction 4.1 is dual to the face spanned by $X_{3,0}$ and $X_{0,3}$. Therefore, the divisor $Z(1,1;4)$ is in the stable base locus exactly in the cone spanned by $X_{1,1}, X_{3,0}$ and $X_{0,3}$, proving (6). Finally, to prove (1), note that the moving curves $F_1(1,4)$, $R_{2,1}(4)$, $R_{1,2}(4)$ and $F_2(1,4)$ defined in Constructions 3.6 and 4.1 are dual to the faces $[B, X_{3,0}], [X_{3,0}, X_{1,1}], [X_{1,1}, X_{0,3}]$ and $[X_{0,3}, B]$, respectively. $\square$

Example 4.11. Figure 4 shows the stable base locus decomposition for $(\mathbb{P}^1 \times \mathbb{P}^1)^{[5]}$. The locus $Z(1,1;4)$ of schemes that have a subscheme of length 4 contained in a curve of type $(1,1)$ is a divisor with class $4H_1[5] + 4H_2[5] - \frac{3}{2}B$. It is easy to calculate this class by intersecting with test curves. Let R be an irreducible curve of type $(1,2)$ or $(2,1)$. Since a curve of type $(1,1)$ has intersection number 3 with R, as long as 5 points vary on R, the scheme they determine does not lie in $Z(1,1;4)$. Hence, the curves $R(5)$ defined in Construction 3.6 are dual to $Z(1,1;4)$ and determine its class up to a multiple, which can easily be determined by pairing with another curve. Since these curves are also dual to the faces $X_{4,0}, Z(1,1;4)$ and $X_{0,4}, Z(1,1;4)$, we conclude that the effective cone of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[5]}$ is the cone spanned by B, $X_{4,0}, Z(1,1;4)$ and $X_{0,4}$. The chambers of the stable base locus decomposition have the following description.

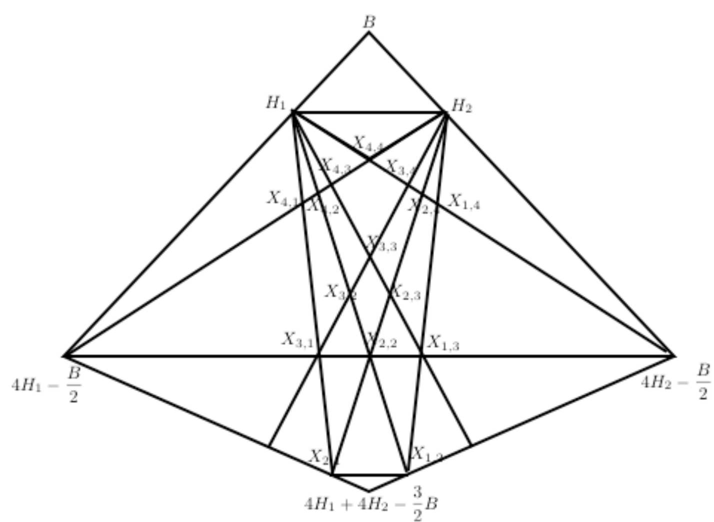

FIGURE 4. The stable base locus decomposition of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[5]}$.

(1) The effective cone is the closed cone spanned by B, $X_{4,0}$, $Z(1,1;4)$ and $X_{0,4}$.
(2) The base-point-free cone is the closed cone spanned by $H_1[5]$, $H_2[5]$ and $X_{4,4}$.
(3) The moving cone is the closed cone spanned by $H_1[5], H_2[5], X_{1,2}$ and $X_{2,1}$.
(4) In the cone generated by B, $H_1[5]$ and $H_2[5]$ the base locus is the divisor B.

(5) The divisor $Z(1,0;2)$ (respectively, $Z(0,1;2)$) is in the stable base locus precisely in the cone generated by $H_1[5], X_{4,0}$ and $X_{2,1}$ (respectively, $H_2[5]$, $X_{0,4}$ and $X_{1,2}$).
(6) The divisor $Z(1,1;4)$ is contained in the base locus precisely in the cone spanned by $X_{1,2}$, $X_{2,1}$ and $Z(1,1;4)$. Hence, it suffices to describe the stable base locus in the moving cone to get a complete description of the stable base locus.
(7) Let $i+j>3$, $3 \geq i,j>0$. In the cone spanned by $X_{i,j}, X_{i+1,j}$, $X_{i+1,j+1}$ and $X_{i,j+1}$, the base locus is $Z(1,0;j+2) \cup Z(0,1;i+2)$.
(8) Let $0 \leq i \leq 3$. In the cone generated by $X_{4,i}, X_{4,i+1}$ and $H_1[5]$, the base locus is $Z(1,0;i+2)$.
(9) In the cone generated by $X_{3,1}, X_{2,2}$ and $X_{3,2}$, the base locus is $Z(1,0;3) \cup Z(0,1;4)$.
(10) In the cone generated by $X_{2,1}, X_{3,1}$ and $X_{2,2}$, the base locus is $Z(1,0;3) \cup Z(0,1;4) \cup Z(1,1;5)$.
(11) Finally, in the cone spanned by $X_{2,1}, X_{1,2}$ and $X_{2,2}$ the base locus is $Z(1,0;3) \cup Z(0,1;3) \cup Z(1,1;5)$.

Proof. We proved (1) before stating the decomposition. Parts (2), (4), (7), (8) and (9) follow from Theorems 2.4, 4.6 and Proposition 3.2. The curves $F_1(2,5)$ and $F_2(2,5)$ are dual to the faces spanned by $X_{2,1}H_1[5]$ and $X_{1,2}H_2[5]$, respectively. Consequently, the divisors $Z(1,0;2)$ and $Z(0,1;2)$ are in the base loci in the cones spanned by $X_{2,1}, X_{4,0}$ and $H_1[5]$ and $X_{1,2}, X_{0,4}$ and $H_2[5]$, respectively. Similarly, $R_{1,1}(4,5)$ is dual to the face spanned by $X_{2,1}$ and $X_{1,2}$, so the divisor $Z(1,1;4)$ is in the base locus in the cone generated by $Z(1,1;4), X_{1,2}$ and $X_{2,1}$. On the other hand, the divisors $X_{1,2}$ and $X_{2,1}$ are pull-backs of $\mathcal{O}(1)$ from the Grassmannian via the rational map induced by the linear systems $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,2)|$ and $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1)|$, respectively. By Lemma 4.4, a scheme Z imposes independent conditions on these linear systems unless Z is contained in $Z(1,1;5) \cup Z(1,0;3) \cup Z(0,1;3)$. Since $H_1[5]$ and $H_2[5]$ are base-point-free, it follows that the moving cone is equal to the cone generated by $X_{2,1}, X_{1,2}, H_2[5]$ and $H_1[5]$. This proves parts (3), (5) and (6). Finally, to conclude the proof, note that the curve $R_{1,1}(5,5)$ is dual to the face generated by $X_{4,0}$ and $X_{0,4}$. Hence, $Z(1,1;5)$ is contained in the base locus of any divisor in the cone generated by $Z(1,1;4), X_{4,0}$ and $X_{0,4}$. Parts (10), (11) follow. This concludes the discussion of this example. □

Remark 4.12. We will refrain from listing the explicit cone decompositions for $n>5$. However, the reader should have no trouble determining these decompositions for the next few cases. The reason for listing these decompositions in such great detail will be apparent when we match these to the Bridgeland walls in the last section.

The stable base locus decomposition of $\mathbb{F}_r^{[n]}$. In this subsection, we discuss some general features of the stable base locus decomposition of $\mathbb{F}_r^{[n]}$. We also compute the complete decomposition for $\mathbb{F}_1^{[n]}$, when $2 \leq n \leq 4$.

Denote divisors on $\mathbb{F}_r^{[n]}$ by $aE[n] + bF[n] + c\frac{B}{2}$. By Construction 3.4, the locus of schemes that have a subscheme of length two contained in a fiber is an effective divisor in $\mathbb{F}_r^{[n]}$ with class $(n-1)F[n] - \frac{B}{2}$. Consequently, the effective cone contains the cone spanned by $E[n]$, $(n-1)F[n] - \frac{B}{2}$ and B. Moreover, the intersection of the effective cone with the half-space $b+(n-1)c \geq 0$ equals this cone. Let R be a curve of class $E+rF$. Then, applying Construction 3.6, we conclude that the face generated by $E[n]$ and B is dual to $R(1)$, which is a moving curve. Similarly, the moving curve $F(1)$ is dual to the face generated by $(n-1)F[n] - \frac{B}{2}$ and B. The stable base locus decomposition in the subcone generated by $E[n]+rF[n]$, $(n-1)F[n] - \frac{B}{2}$ and B is easy to describe.

Lemma 4.13. *Let $|aE + (ar + b)F|$ be a linear system on $\mathbb{F}_r$ such that $0 \leq a, b$.*

(1) *If Z is a scheme of length $n \leq a + b + 1$, then Z imposes independent conditions on the linear system unless Z has a subscheme of length $a + 2$ contained in a fiber or a subscheme of length $b + 2$ contained in the exceptional curve.*
(2) *If Z is a scheme of length $n = a + b + 2$, then Z imposes independent conditions on the linear system unless Z is contained in a curve with class $E + rF$ or has a subscheme of length $a+2$ contained in a fiber or a subscheme of length $b+2$ contained in the exceptional curve.*

Proof. The lemma follows by the residuation and induction on a and b. Consider the three exact sequences,

$$(1) \qquad 0 \to \mathcal{I}_{Z_F}(aE + (ar + b - 1)F) \to \mathcal{I}_Z(aE + (ar + b)F) \to \mathcal{I}_{Z\cap F\subset F}(aE + (ar + b)F) \to 0,$$

$$(2) \quad 0 \to \mathcal{I}_{Z_E}((a-1)E + (ar+b)F) \to \mathcal{I}_Z(aE + (ar+b)F) \to \mathcal{I}_{Z\cap E\subset E}(aE + (ar+b)F) \to 0, \text{ and}$$

$$(3) \qquad 0 \to \mathcal{I}_{Z_R}((a-1)(E + rF)) \to \mathcal{I}_Z(a(E + rF)) \to \mathcal{I}_{Z\cap R\subset R}(a(E + rF)) \to 0.$$

If Z has a subscheme of length $a + 1$ contained in a fiber F or of length $b + 1$ contained in E, we use the exact sequences (1) and (2), respectively, and induction to conclude that $H^1(\mathcal{I}_Z(aE + (ar + b)F)) = 0$. Otherwise, if $b > 0$, we use the exact sequence (1), where F is a fiber containing a maximal length subscheme of Z, and induction to conclude that $H^1(\mathcal{I}_Z(aE + (ar + b)F)) = 0$. If $b = 0$, we use the exact sequence (3), where R is a curve of class $E + rF$ containing a maximal length subscheme of Z, and induction to conclude that $H^1(\mathcal{I}_Z(aE + (ar + b)F)) = 0$. $\square$

Notation 4.14. Let $Z(a, b; m)$ denote the locus of $\mathbb{F}_r^{[n]}$ parameterizing schemes that have a subscheme of length m contained in a curve with class $aE + bF$. Let $X_{i,j}$ denote the divisor $i(E[n] + rF[n]) + jF[n] - \frac{B}{2}$. $X_{i,j}$ is effective if and only if $i \geq 0$ and $j \geq -ir$.

Theorem 4.15. (1) *Let $n - 2 \geq i, j > 0$ and let $i + j > n - 1$. The cone generated by $X_{i,j}, X_{i+1,j}, X_{i+1,j+1}$ and $X_{i,j+1}$ is a chamber of the stable base locus decomposition, where the base locus is $Z(1, 0; j + 2) \cup Z(0, 1; i + 2)$.*

(2) *Let $i = n - 1$ and $n - 2 \geq j \geq 0$ (respectively, $j = n - 1$ and $n - 2 \geq i \geq 0$). The cone generated by $E[n] + rF[n], X_{n-1,j}$ and $X_{n-1,j+1}$ (respectively, $F[n], X_{i,n-1}$ and $X_{i+1,n-1}$) is a chamber of the stable base locus decomposition, where the stable base locus is $Z(1, 0; j + 2)$ (respectively, $Z(0, 1; i + 2)$).*

(3) *The locus $Z(1, r; n)$ is contained in the stable base locus of a divisor $aE[n] + bF[n] + c\frac{B}{2}$ if and only if $b + (n - 1)c < 0$.*

(4) *If $-ir \leq j < 0$, then the stable base locus contains the divisor $E[n] = Z(1, 0; 1)$.*

Proof. The divisor $X_{i,j}$ is the pull-back of $\mathcal{O}(1)$ from the Grassmannian by the rational map induced by the linear system $|i(E + rF) + jF|$. By Lemma 4.13, if $n \leq i + j + 1$, the map is a morphism along the locus of schemes Z that are not contained in $Z(1, 0; j + 2) \cup Z(0, 1; i + 2)$. Hence, the stable base locus of $X_{i,j}$ is contained in $Z(1, 0; j+2) \cup Z(0, 1; i+2)$. Since $E[n]+rF[n]$ and $F[n]$ are base-point-free, we conclude that the stable base locus of the divisors contained in the cone generated by $X_{i,j}, X_{i+1,j}, X_{i+1,j+1}, X_{i,j+1}$ is contained in $Z(1, 0; j + 2) \cup Z(0, 1; i + 2)$.

On the other hand, consider the curves $E(j + 2, n)$ and $F(i + 2, n)$ defined in Construction 4.1. These curves have intersection number zero with divisors along the face generated by $X_{i,j+1}X_{i+1,j+1}$ and $X_{i+1,j}X_{i+1,j+1}$, respectively. Consequently, the base locus in the cone generated by $X_{i,j}, X_{i+1,j}, X_{i+1,j+1}, X_{i,j+1}$ contains $Z(1,0;j+2) \cup Z(0,1;i+2)$. Part (1) of the theorem follows. A similar argument proves (2).

To finish the proof of (3), we observe that the curve $R(n,n)$ defined in Construction 4.1, where R has the class $E+rF$, is dual to the face spanned by $E[n]$ and $(n-1)F[n] - \frac{B}{2}$. We conclude that the locus $Z(1,r;n)$ is contained in the base locus of any divisor with $b+(n-1)c < 0$. If $n \leq r+1$, then every scheme of length n is contained in a curve of class $E+rF$ and we conclude that the effective cone of $\mathbb{F}_r^{[n]}$ is the cone $E[n]$, $(n-1)F[n] - \frac{B}{2}$ and B. We may assume that $n > r+1$. Since the base locus of any divisor with $b+(n-1)c \geq 0$ is contained in the union of the base loci of $E[n], B$ and $(n-1)F - \frac{B}{2}$, which is equal to $Z(1,0;1) \cup B \cup Z(0,1;2)$, we conclude that $Z(1,r;n)$ is not contained in the stable base locus of such a divisor.

Finally, since the curve $E(1,n)$ whose deformations cover the divisor $E[n]$, has negative intersection number with any $X_{i,j}$ such that $-ir \leq j < 0$, $E[n]$ is in the base locus. This concludes the proof of the theorem. $\square$

Example 4.16. Figure 5 shows the stable base locus decomposition of $\mathbb{F}_1^{[2]}$. The chambers have the following description.

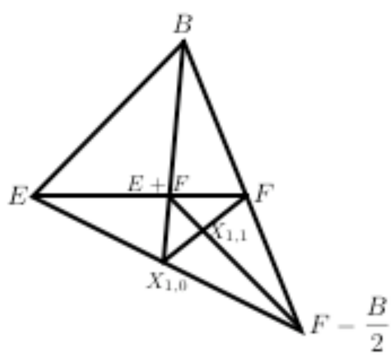

FIGURE 5. The stable base locus decomposition of $\mathbb{F}_r^{[2]}$.

(1) The effective cone is the closed cone spanned by B, $E[2]$ and $X_{0,1}$.
(2) The base-point-free cone is the closed cone spanned by $E[2]+F[2]$, $F[2]$ and $X_{1,1}$.
(3) The moving cone is the closed cone spanned by $X_{1,0}$, $E[2]+F[2]$ and $F[2]$.
(4) In the cone spanned by $X_{1,0}, X_{1,1}$ and $E[2]+F[2]$, the stable base locus is $Z(1,0;2)$.
(5) In the cone spanned by B, $E[2]+F[2]$ and $F[2]$, the base locus is B.
(6) In the cone spanned by $E[2]$, $E[2]+F[2]$ and B, the base locus is $B \cup Z(1,0;1)$.
(7) In the cone spanned by $E[2]$, $E[2]+F[2]$ and $X_{1,0}$, the base locus is $Z(1,0;1)$.
(8) In the cone spanned by $F[2]$, $X_{1,1}$ and $X_{0,1}$ the base locus is $Z(0,1;2)$.
(9) In the cone spanned by $X_{1,0}, X_{1,1}$, and $X_{0,1}$ the base locus is $Z(1,0;2) \cup Z(0,1;2)$.

Proof. Theorem 2.4, Theorem 4.15 and Proposition 3.2 imply (2), (4), (5), (6), (7) and (8). The effective cone contains the cone spanned by $B, E[2]$ and $X_{0,1}$. In view of the discussion preceding Theorem 4.15, to prove (1), it suffices to exhibit a moving curve dual to the face spanned by $E[2], X_{0,1}$. Let R be a curve in the class $E+F$. Then the curve $R(2)$ defined in Construction 3.6 is the required moving curve. Since the base loci described in parts (5)-(9) all contain a fixed divisor and the base locus in (4) is not divisorial, parts (4)-(9) imply (3). The curves $E(1,2), E(2,2)$ and $F(2,2)$ defined in Construction 4.1 are dual to the faces spanned by $[B, E[2]+F[2]]$, $[E[2]+F[2], X_{0,1}]$, and $[F[2], X_{1,1}]$, respectively. Part (9) follows. $\square$

Example 4.17. Figure 6 shows the stable base locus decomposition of $\mathbb{F}_1^{[3]}$. The chambers have the following descriptions.

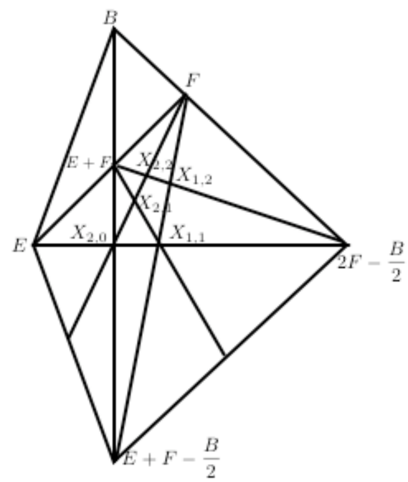

FIGURE 6. The stable base locus decomposition of $\mathbb{F}_1^{[3]}$.

(1) The effective cone is the closed cone generated by $E[3], B, X_{0,2}$ and $X_{1,0}$.
(2) The base-point-free cone is the closed cone generated by $E[3]+F[3]$, $F[3]$ and $X_{2,2}$.
(3) The moving cone is the closed cone generated by $E[3]+F[3]$, $F[3]$, $X_{1,1}$ and $X_{2,0}$.
(4) The divisor B is in the base locus precisely in the cone spanned by $E[3], F[3]$ and B.
(5) The divisor $Z(1,0;1)$ is in the base locus precisely in the cone spanned by $E[3]$, B and $X_{1,0}$.
(6) The divisor $Z(1,1;3)$ is in the base locus precisely in the cone spanned by $E[3], X_{0,2}$, and $X_{1,0}$.
(7) The divisor $Z(0,1;2)$ is the base locus precisely in the cone spanned by $F[3], X_{1,0}$ and $X_{0,2}$. We are thus reduced to describing the stable base locus decomposition of the moving cone.
(8) In the cone spanned by $E[3]+F[3]$, $X_{2,1}$ and $X_{2,2}$, the stable base locus is $Z(1,0;3)$.
(9) In the cone spanned by $F[3]$, $X_{1,2}$ and $X_{2,2}$, the stable base locus is $Z(0,1;3)$.
(10) In the cone spanned by $X_{1,1}, X_{1,2}, X_{2,2}$ and $X_{2,1}$, the stable base locus is $Z(1,0;3) \cup Z(0,1;3)$.
(11) In the cone spanned by $X_{2,0}$, $E[3]+F[3]$ and $X_{2,1}$ the stable base locus is $Z(1,0;2)$.
(12) In the cone spanned by $X_{2,0}, X_{1,1}$ and $X_{2,1}$ the stable base locus is $Z(0,1;3) \cup Z(1,0;2)$.

Since the proof is analogous to the cases of $\mathbb{F}_1^{[2]}$ and $(\mathbb{P}^1 \times \mathbb{P}^1)^{[3]}$, we leave it to the reader.

Example 4.18. We complete our discussion of the stable base locus decomposition of $\mathbb{F}_r^{[n]}$, by describing the stable base locus decomposition of $\mathbb{F}_1^{[4]}$. Figure 7 shows the decomposition. The chambers have the following interpretations.

(1) The effective cone is the closed cone spanned by B, $E[4]$, $3E[4]+3F[4]-B$ and $X_{0,3}$.
(2) The base-point-free cone is the closed cone spanned by $E[4]+F[4], F[4]$ and $X_{3,3}$.
(3) The moving cone is the closed cone spanned by $X_{2,0}, X_{1,1}, F[4]$ and $E[4]+F[4]$.
(4) The divisor B is in the base locus in the cone generated by $B, E[4]$, and $F[4]$. The divisor $Z(1,0;1)$ is in the base locus in the cone generated by $B, E[4]$, and $3E[4]+3F[4]-B$.

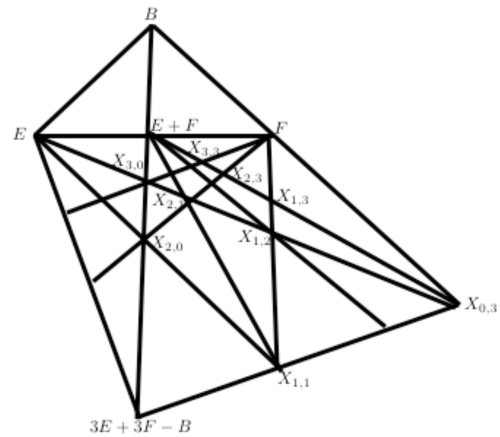

FIGURE 7. The stable base locus decomposition of $\mathbb{F}_1^{[4]}$.

The divisor $Z(1,1;3)$ is in the base locus in the cone generated by $X_{1,1}, E[4]$, and $3E[4]+3F[4]-B$. Finally, the divisor $Z(0,1;2)$ is in the base locus in the cone generated by $X_{1,1}, X_{0,3}$, and $F[4]$. Hence, it suffices to describe the decomposition of the moving cone.

(5) The decomposition in the cone spanned by $X_{3,0}, X_{0,3}, F[4]$ and $E[4]+F[4]$ is as in Theorem 4.15.
(6) In the cone spanned by $X_{2,0}, X_{2,1}, X_{3,0}$, the stable base locus is $Z(1,1;4) \cup Z(1,0;2) \cup Z(0,1;4)$.
(7) In the cone spanned by $X_{2,0}, X_{1,1}, X_{2,1}$, the stable base locus is $Z(1,1;4) \cup Z(1,0;2) \cup Z(0,1;3)$.
(8) Finally, in the cone spanned by $X_{1,1}, X_{1,2}, X_{2,1}$, the stable base locus is $Z(1,1;4) \cup Z(1,0;3) \cup Z(0,1;3)$.

Proof. The divisor $Z(1,1;3)$ of schemes that have a subscheme of length 3 contained in a curve with class $E+F$ has class $3E[4]+3F[4]-B$. The proof of parts (1)-(5) is now analogous to the previous cases. By Lemma 4.13, the base loci in the cones described in parts (6)-(8) are contained in the claimed loci. The curves $E(m,4), F(m,4)$ and $R(4,4)$, where R is a curve of class $E+F$ on $\mathbb{F}_1$, defined in Construction 4.1 show that the claimed loci are contained in the stable base locus. This completes the proof. $\square$

5. Preliminaries on Bridgeland stability and Bridgeland walls

In this section, we recall preliminaries concerning Bridgeland stability. We refer the reader to [AP], [AB], [Br1] for more detailed information. We then determine the general features of Bridgeland walls.

Bridgeland stability conditions. Let X be a smooth projective variety. Let $\mathcal{D}^b(X)$ denote the bounded derived category of coherent sheaves on X. A *Bridgeland stability condition* σ on X consits of a pair $\sigma = (\mathcal{A}, Z)$ such that $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}^b(X)$ and $Z : K(\mathcal{D}^b(X)) \to \mathbb{C}$ is a homomorphism satisfying the following properties:

(1) (Positivity) For every non-zero object E of $\mathcal{A}$, $Z(E)$ lies in the semi-closed upper half-plane:

$$Z(E) = re^{i\pi\theta}, \quad \text{where} \quad r > 0, 0 < \theta \leq 1.$$

Writing $Z = -d(E) + ir(E)$, one may view this condition as two separate positivity conditions requiring $r(E) \geq 0$ and if $r(E) = 0$, then $d(E) > 0$.

(2) (Harder-Narasimhan property) For an object E of $\mathcal{A}$, let the Z-slope of E be defined by setting $\mu(E) = d(E)/r(E)$ with the understanding that $\mu(E) = \infty$ if $r(E) = 0$. An object E is called Z-(semi)-stable, if for every proper subobject F, $\mu(F)(\leq) < \mu(E)$. The pair $(\mathcal{A}, Z)$ is required to satisfy the Harder-Narasimhan property. Namely, every object of $\mathcal{A}$ has a finite filtration

$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_n = E$$

such that $F_i = E_i/E_{i-1}$ is Z-semi-stable and $\mu(F_i) > \mu(F_{i+1})$ for all i.

One also imposes a technical *support property*, which we will not mention here. The set of stability conditions on $\mathcal{D}^b(X)$ satisfying these three properties is called the *stability manifold* of X and is denoted by $\mathrm{Stab}(X)$. In [Br1], Bridgeland proves the following theorem.

Theorem 5.1 (Bridgeland)**.** *The map $(\mathcal{A}, Z) \mapsto \mathbb{C}$ is a local homeomorphism onto an open set in a linear subspace of Hom$(K(\mathcal{D}^b(X)), \mathbb{C})$. In particular, the space Stab(X) of stability conditions on X is a complex manifold.*

When $\dim(X) > 2$, we do not know in general whether the stability manifolds $\mathrm{Stab}(X)$ are non-empty (see [BMT] and [To] for a discussion and references). When X is a surface, Bridgeland [Br2] and Arcara and Bertram [AB] have constructed stability conditions. We will only use the region in the stability manifold corresponding to these special stability conditions.

Example 5.2. If X is a curve, then setting $\mathcal{A}$ to be the category of coherent sheaves on X and $Z(E) = -\deg(E) + i\ \mathrm{rk}(E)$, one obtains a Bridgeland stability condition. If X is a surface and H is an ample line bundle on X, one can still define $\mathcal{A}$ to be the category of coherent sheaves on X and $Z(E) = -\deg_H(E) + i\ \mathrm{rk}(E)$, where the degree is measured with respect to the ample line bundle H. However, this is not a Bridgeland stability condition because Z is zero on sheaves supported on points. The idea of Bridgeland, Arcara and Bertram is to fix this problem by tilting the category.

Bridgeland stability conditions for surfaces. For the remainder of this section, let X be a smooth, projective surface and let H be an ample line bundle. Mumford stability with respect to H gives rise to a Harder-Narasimhan filtration.

Definition 5.3. Given $s \in \mathbb{R}$, define full subcategories $\mathcal{Q}_s$ and $\mathcal{F}_s$ of $\mathrm{coh}(X)$ by the following conditions on their objects:

- $Q \in \mathcal{Q}_s$ if Q is torsion or if each $\mu_i > sH^2$ in the Harder-Narasimhan filtration of Q.
- $F \in \mathcal{F}_s$ if F is torsion-free, and each $\mu_i \leq sH^2$ in the Harder-Narasimhan filtration of F.

By [Br2, Lemma 6.1], each pair $(\mathcal{F}_s, \mathcal{Q}_s)$ of full subcategories satisfies the two properties:

(a) For all $F \in \mathcal{F}_s$ and $Q \in \mathcal{Q}_s$, $\mathrm{Hom}(Q, F) = 0$.
(b) Every coherent sheaf E fits in a short exact sequence $0 \to Q \to E \to F \to 0$, where $Q \in \mathcal{Q}_s$, $F \in \mathcal{F}_s$ and the extension class are uniquely determined up to isomorphism.

A pair of full subcategories $(\mathcal{F}, \mathcal{Q})$ of an abelian category $\mathcal{A}$ satisfying conditions (a) and (b) is called a *torsion pair*. A torsion pair $(\mathcal{F}, \mathcal{Q})$ defines a t-structure on $\mathcal{D}^b(\mathcal{A})$ [HRS] with:

$$\mathcal{D}^{\geq 0} = \{\text{complexes } E \mid \ \mathrm{H}^{-1}(E) \in \mathcal{F} \text{ and } \mathrm{H}^i(E) = 0 \text{ for } i < -1\}$$

$$\mathcal{D}^{\leq 0} = \{\text{complexes } E \mid \ \mathrm{H}^{0}(E) \in \mathcal{Q} \text{ and } \mathrm{H}^i(E) = 0 \text{ for } i > 0\}$$

The heart of the t-structure defined by a torsion pair consists of:

$$\{E \mid \mathrm{H}^{-1}(E) \in \mathcal{F}, \mathrm{H}^0(E) \in \mathcal{Q}, \text{ and } \mathrm{H}^i(E) = 0 \text{ otherwise}\}.$$

The natural exact sequence:

$$0 \to \mathrm{H}^{-1}(E)[1] \to E \to \mathrm{H}^0(E) \to 0$$

for such an object of $\mathcal{D}^b(\mathcal{A})$ implies that the objects of the heart are all given by pairs of objects $F \in \mathcal{F}$ and $Q \in \mathcal{Q}$ together with an extension class in $\mathrm{Ext}^2_{\mathcal{A}}(Q, F)$ [HRS].

Definition 5.4. Let $\mathcal{A}_s$ be the heart of the t-structure on $\mathcal{D}^b(\mathrm{coh}(X))$ obtained from the torsion-pair $(\mathcal{F}_s, \mathcal{Q}_s)$ in Definition 5.3. Define the central charge by setting

$$Z_{s,t}(E) = -\int_X e^{-(s+it)H} \mathrm{ch}(E).$$

With this definition, $(\mathcal{A}_s, Z_{s,t})$ is a Bridgeland-stability condition.

Theorem 5.5 (Bridgeland [Br2], Arcara-Bertram [AB], Bayer-Macrì [BM1])**.** *For each $s \in \mathbb{R}$ and $t > 0$, the pair $(\mathcal{A}_s, Z_{s,t})$ define Bridgeland stability conditions on $\mathcal{D}^b(\mathrm{coh}(X))$.*

Bridgeland walls. Fix a class ν in the numerical Grothendieck group. Then there exists a locally finite set of walls in $\mathrm{Stab}(X)$, depending only on ν, such that as the stability condition σ varies in a chamber, the set of σ-(semi)-stable objects of class ν does not change ([Br2], [BM1], [BM2]). We will call these walls *Bridgeland walls*.

We are interested in calculating the Bridgeland walls in the case of an ideal sheaf $\mathcal{I}_Z$ of n points on X. We will record the numerical invariant by $(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$. If $Z \in X^{[n]}$, then the corresponding invariant is $(1, 0, -n)$. It has been worked out in several contexts ([ABCH], [Ma]) that the potential walls are lines or non-intersecting nested semi-circles. Let (s, t) be a point of a Bridgeland wall. Then there exists an object E destabilizing an object of the category $\mathcal{A}_s$ with invariant $(1, 0, -n)$. Hence, the $Z_{s,t}$-slope of E has to equal the $Z_{s,t}$-slope of an object with invariant $(1, 0, -n)$.

$$\mu_{s,t}(E) = -\frac{\Re(Z_{s,t}(E))}{\Im(Z_{s,t}(E))} = \frac{\mathrm{ch}_2(E) - s\ \mathrm{ch}_1(E) \cdot H + \frac{s^2-t^2}{2}\ \mathrm{ch}_0(E) H^2}{t\ \mathrm{ch}_1(E) \cdot H - st\ \mathrm{ch}_0(E) H^2}.$$

In particular,

$$\mu_{s,t}(\mathcal{I}_Z) = \frac{n - \frac{s^2-t^2}{2} H^2}{st\ H^2}.$$

Equating the two slopes and assuming that $t > 0$ and $s < 0$, we get the equation of a semi-circle

$$(s - x)^2 + t^2 = r^2,$$

where the center is $(x, 0)$ with

$$x = \frac{n\ \mathrm{ch}_0(E) + \mathrm{ch}_2(E)}{\mathrm{ch}_1(E) \cdot H}, \quad \text{and the radius} \quad r = \sqrt{x^2 - \frac{2n}{H^2}}.$$

Observe that two distinct semi-circles do not intersect. We will index the Bridgeland walls by their centers and denote them by W_x.

Example 5.6. When $X = \mathbb{P}^2$ and H is the hyperplane class, then the potential Bridgeland walls have center $(x, 0)$ with

$$x = \frac{n\ \mathrm{ch}_0(E) + \mathrm{ch}_2(E)}{\mathrm{ch}_1(E)} \quad \text{and radius} \quad r = \sqrt{x^2 - 2n}.$$

Example 5.7. When $X = \mathbb{P}^1 \times \mathbb{P}^1$ and H is the ample class $aH_1 + bH_2$, then the potential Bridgeland wall corresponding to E with $c_1(E) = \alpha H_1 + \beta H_2$ has center $(x, 0)$ with

$$x = \frac{n\ \mathrm{ch}_0(E) + \mathrm{ch}_2(E)}{a\beta + b\alpha} \quad \text{and radius} \quad r = \sqrt{x^2 - \frac{n}{ab}}.$$

Example 5.8. When $X = \mathbb{F}_r$ is a Hirzebruch surface and H is the ample class $aE + bF$ with $b > ra$, then the potential Bridgeland wall corresponding to E with $c_1(E) = \alpha E + \beta F$ has center $(x, 0)$ with

$$x = \frac{n\ \mathrm{ch}_0(E) + \mathrm{ch}_2(E)}{-a\alpha r + a\beta + b\alpha} \quad \text{and radius} \quad r = \sqrt{x^2 - \frac{2n}{-a^2 r + 2ab}}.$$

Example 5.9. When $X = D_{9-r}$ is a del Pezzo surface and H is an ample class $aH - \sum_{i=1}^r b_i E_i$, then the potential Bridgeland wall corresponding to E with $c_1(E) = \alpha H - \sum_{i=1}^r \beta_i E_i$ has center $(x, 0)$ with

$$x = \frac{n\ \mathrm{ch}_0(E) + \mathrm{ch}_2(E)}{a\alpha - \sum_{i=1}^r b_i \beta_i} \quad \text{and radius} \quad r = \sqrt{x^2 - \frac{2n}{a^2 - \sum_{i=1}^r b_i^2}}.$$

Rank one walls. The key problem is to determine which of these walls are actual Bridgeland walls in $\mathrm{Stab}(X)$. By [ABCH, Proposition 6.2(d)] line bundles L are stable objects of $\mathcal{A}_s$ for $L \cdot H > s$ and all $t > 0$. More generally, any destabilizing subsheaf of $\mathcal{I}_Z$ of rank one has the form

$$\mathcal{I}_{Z'} \otimes L \subset \mathcal{I}_Z$$

for some ideal sheaf $\mathcal{I}_{Z'}$ and some line bundle L on X. Any such subsheaf is a subobject in the category $\mathcal{A}_s$ as long as $s < L \cdot H$. Hence, these sheaves give rise to rank one walls W_x with

$$x = \frac{n + \frac{L^2}{2} - l(Z')}{L \cdot H},$$

where $l(Z')$ denotes the length of Z'.

Example 5.10. Taking $X = \mathbb{P}^2$, H the hyperplane class and $L = \mathcal{O}_{\mathbb{P}^2}(-k)$, we get the rank one walls with center

$$x = -\frac{n}{k} - \frac{k}{2} + \frac{l(Z')}{k}.$$

Example 5.11. Taking $X = \mathbb{P}^1 \times \mathbb{P}^1$, $H = aH_1 + bH_2$ and $L = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-\alpha, -\beta)$, we get the rank one walls with center

$$x = \frac{-n - \alpha\beta + l(Z')}{a\beta + b\alpha}.$$

Example 5.12. Taking $X = \mathbb{F}_r$, $H = aE + bF$ and $L = \mathcal{O}_{\mathbb{F}_r}(-\alpha E - \beta F)$, we get the rank one walls with center

$$x = \frac{-n + \frac{\alpha^2 r}{2} - \alpha\beta + l(Z')}{-a\alpha r + b\alpha + a\beta}.$$

Higher rank walls. The geometry of the moduli spaces of Bridgeland semi-stable objects becomes harder to understand once we cross a higher-rank wall. Hence, it is important to bound the centers of higher rank walls that can occur. We use an observation from [ABCH] to get the desired bound.

Suppose that $\mathcal{F} \to \mathcal{I}_Z$ is a destabilizing subsheaf of rank at least two giving rise to a Bridgeland wall W_x. Let $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{I}_Z$ be the kernel of the morphism. Then, by [ABCH, Corollary 6.4], we have that both $\mathcal{F}$ and $\mathcal{K}[1]$ have to be contained in all the categories $\mathcal{A}_s$ along the wall

W_x. Even though Proposition 6.2, Lemma 6.3 and Corollary 6.4 of [ABCH] are stated for $\mathbb{P}^2$, the proofs do not use the fact that the surface is $\mathbb{P}^2$, but only use categorical properties and the fact that walls are nested semi-circles.

We conclude that

$$x - r \geq \frac{c_1(\mathcal{K}) \cdot H}{\mathrm{rk}(\mathcal{K})}, \quad x + r \leq \frac{c_1(\mathcal{F}) \cdot H}{\mathrm{rk}(\mathcal{F})}.$$

Combining this with $\mathrm{rk}(\mathcal{K}) = \mathrm{rk}(\mathcal{F}) - 1$ and $c_1(\mathcal{K}) \cdot H \geq c_1(\mathcal{F}) \cdot H$, we obtain that

$$x + r \leq \frac{c_1(\mathcal{F}) \cdot H}{\mathrm{rk}(\mathcal{F})} \leq \frac{c_1(\mathcal{K}) \cdot H}{\mathrm{rk}(\mathcal{F})} \frac{\mathrm{rk}(\mathcal{K})}{\mathrm{rk}(\mathcal{K})} \leq (x - r)\frac{\mathrm{rk}(\mathcal{K})}{\mathrm{rk}(\mathcal{F})}.$$

Hence, we obtain the following inequality on the centers of potential walls of higher rank

$$\mathrm{rk}(\mathcal{F})(x + r) \leq c_1(\mathcal{F}) \cdot H \leq (\mathrm{rk}(\mathcal{F}) - 1)(x - r).$$

In particular, one obtains the bound

$$x^2 \leq \frac{n}{2H^2} \frac{(2\mathrm{rk}(\mathcal{F}) - 1)^2}{\mathrm{rk}(\mathcal{F})(\mathrm{rk}(\mathcal{F}) - 1)}.$$

Example 5.13. When $X = \mathbb{P}^2$ and H is the hyperplane class, the inequality translates to

$$x^2 \leq \frac{n(2\mathrm{rk}(\mathcal{F}) - 1)^2}{2\mathrm{rk}(\mathcal{F})(\mathrm{rk}(\mathcal{F}) - 1)}$$

.

Example 5.14. When $X = \mathbb{P}^1 \times \mathbb{P}^1$ and H is $aH_1 + bH_2$, the inequality translates to

$$x^2 \leq \frac{n \ (2\mathrm{rk}(\mathcal{F}) - 1)^2}{4ab \ \mathrm{rk}(\mathcal{F})(\mathrm{rk}(\mathcal{F}) - 1)}.$$

Example 5.15. When $X = \mathbb{F}_r$ and $H = aE + bF$, the inequality translates to

$$x^2 \leq \frac{n \ (2\mathrm{rk}(\mathcal{F}) - 1)^2}{2(-a^2 r + 2ab) \ \mathrm{rk}(\mathcal{F})(\mathrm{rk}(\mathcal{F}) - 1)}.$$

We call the Bridgeland wall where all ideal sheaves are destabilized the *collapsing wall.* We remark that the inequalities become strictly sharper as the rank of $\mathcal{F}$ increases. In particular, if the inequalities force the centers of potential walls of rank r to be larger than that of a collapsing wall, then for every $r' > r$ the centers of the potential walls are larger than that of the collapsing wall. This observation will help us eliminate potential higher rank walls.

6. The correspondence between Bridgeland walls and Mori walls

In this section, we calculate the Bridgeland walls for the examples we discussed in §4 and find that there is a precise correspondence between the Bridgeland walls and the Mori walls. The correspondence is cleanest when H is a multiple of the anti-canonical bundle. The most interesting aspect of this correspondence is that it does not depend on the number of points, making it a powerful tool for studying base loci decompositions. However, the correspondence appears to be much more general: Traversing the Bridgeland walls for a specific H corresponds to running a log minimal model program along a face $[-K, D]$ in the Néron-Severi space of $X^{[n]}$, where there is a precise relation between H and D . When H is a multiple of $-K$, then the corresponding face is $[-K, B]$.

The Bridgeland walls for $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$**.** We let $H = \frac{1}{2}H_1 + \frac{1}{2}H_2$. Since the Hilbert-Chow morphism is a crepant resolution, we see that $H[n] = -\frac{1}{4}K_{(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}}$. The coefficient of $\frac{1}{4}$ is a normalization chosen so that the transformation is a nice integer. We will denote a Bridgeland wall with center x by W_x. The radius of W_x is $\sqrt{x^2 - 4n}$. We now calculate the Bridgeland walls for $n = 2, 3$ and all the walls with sufficiently small center for all n.

Example 6.1. When $n = 2$, the Bridgeland walls are as follows:

(1) The wall W_{-4} given by the destabilizing objects $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1)$ and $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, 0)$.
(2) The collapsing wall W_{-3} given by the destabilizing object $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$.

Proof. Since every sheaf of length two on $\mathbb{P}^1 \times \mathbb{P}^1$ is contained in a curve of type $(1, 1)$, there exists a map $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \to \mathcal{I}_Z$ destabilizing every ideal sheaf. Hence, W_{-3} is a collapsing wall. We have already described the rank one walls in Example 5.11. It suffices to show that there are no higher rank walls. By Example 5.14, the center of any higher rank wall has to satisfy the inequality $x^2 \leq 9 = (-3)^2$. Hence, any potential higher rank wall either coincides or is contained in W_{-3}. $\square$

Consider the divisors $D_t = H[2] + \frac{B}{2t}$, $t < 0$, in Figure 1. Then the divisor crosses Mori walls at $t = -2$ and -1. At a Mori wall M_t, the divisor D_t picks up as base locus the sheaves that are destabilized at the Bridgeland wall $W_{x=t-2}$. We will see that this picture persists for all $n \geq 2$.

Example 6.2. When $n = 3$, the Bridgeland walls are as follows:

(1) The wall W_{-6} given by the destabilizing objects $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1)$ and $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, 0)$.
(2) The wall W_{-4} given by the destabilizing objects $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$, $\mathcal{I}_p(0, -1)$ and $\mathcal{I}_p(-1, 0)$.

Proof. Since every scheme of length 3 is contained in a curve of type $(1, 1)$, W_{-4} is a collapsing wall. Using Example 5.14, we see that any higher rank wall satisfies $x^2 \leq \frac{27}{2} < 16$. Hence, all the walls are rank one walls. The description follows from Example 5.11. $\square$

As in the previous example, setting $D_t = H[2] + \frac{B}{2t}$, $t < 0$, in Figure 2, we see that the Mori walls occur at $t = -4, -2$. At the Mori wall M_t, the divisor D_t picks up the locus of sheaves destabilized at $W_{x=t-2}$ in its base locus.

Unfortunately, we do not know how to compute all the Bridgeland walls as n increases. Two difficulties arise. We do not know how to control higher rank walls in general. We also do not know how to control walls where none of the objects destabilized are ideal sheaves. In particular, we do not know whether walls of the latter kind exist. However, if we bound x from above (depending on n), then we can compute all the Bridgeland walls and show that the correspondence persists for all n. We have the following proposition.

Proposition 6.3. *Let $D_t = \frac{1}{2}H_1[n] + \frac{1}{2}H_2[n] + \frac{B}{2t}$, for $t < 0$, be a divisor on $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$. Assume that $1 - n \geq t$. Then there is a one-to-one correspondence between the Mori walls M_{t_i} and the Bridgeland walls $W_{x_i = t_i - 2}$ when $1 - n \geq t$ and $-1 - n \geq x$. An ideal sheaf $\mathcal{I}_Z$ is in the base locus of D_t if and only if $\mathcal{I}_Z$ is destabilized at $W_{x_i = t_i - 2}$ for $t > t_i$.*

Proof. Since $t_i \leq 1 - n$, $x_i = t_i - 2 \leq -1 - n$. By Example 5.14, we have that the centers of higher rank walls satisfy $x^2 \leq \frac{9n}{2}$. Since $(n + 1)^2 \geq \frac{9n}{2}$ for all $n \geq 2$, we conclude that there cannot be any higher rank walls in this range. Consequently, all the Bridgeland walls are rank one walls, which have been determined in Example 5.11. We see that the Bridgeland walls occur at $W_{-2n}, W_{-2n+2}, \cdots, W_{-n-1}$ and correspond to destabilizing objects $\mathcal{I}_{Z'}(0, -1)$ or $\mathcal{I}_{Z'}(-1, 0)$, where the length of Z' giving the wall W_{-2n+2j} has length j. Finally, W_{-n-1} also corresponds to the destabilizing object $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$.

On the other hand, by Theorem 4.6, the Mori walls occur at $t = -2i$ for $\frac{n-1}{2} \leq i \leq n-1$. When $-2i \leq t < -2i - 2$, the stable base locus consists of schemes that have subschemes of length at least $i+2$ contained in a fiber. When D_t crosses the value $t = -n+1$, then D_t contains the locus of schemes contained in a curve of type $(1,1)$. This concludes the proof of the proposition. □

Remark 6.4. The correspondence seems to work in greater generality and can be proved for a larger part of the cone than covered in Proposition 6.3. For example, when $n = 4$ the rank one Bridgeland walls are $W_{-8}, W_{-6}, W_{-5}, W_{-4}$ corresponding to destabilizing objects $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1)$, $\mathcal{I}_p(0,-1)$, $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$ and $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-2)$ or $\mathcal{I}_{Z'}(0,-1)$ with $l(Z') = 2$, respectively. The sheaves obtained by switching the two fiber classes also give rise to the same walls. These correspond precisely to the Mori walls in Figure 3, which occur at $t = -6, -4, -3, -2$. Similarly, when $n = 5$ the rank one Bridgeland walls occur at $W_{-10}, W_{-8}, W_{-6}, W_{-5}, W_{-\frac{14}{3}}$. These correspond precisely to the Mori walls in Figure 4, which occur at $t = -8, -6, -4, -3$ and $-\frac{8}{3}$.

One may conjecture that there is always a one-to-one correspondence between Bridgeland walls and Mori walls given by the relation $x = t - 2$. Even when one does not a priori know this correspondence, it is still very useful for guessing base loci of linear systems on $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$.

Remark 6.5. Recently, Bayer and Macrì have constructed nef divisors on the moduli spaces of Bridgeland semi-stable objects. For the moduli spaces discussed here, their arguments show that their nef divisor is ample. Hence, these moduli spaces are projective. Therefore, one obtains a modular interpretation of the log anti-canonical models of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$ with respect to the boundary divisor B, at least in the ranges covered by Proposition 6.3.

The Bridgeland walls for $\mathbb{F}_1^{[n]}$. We describe the correspondence between Bridgeland walls and Mori walls for the Hilbert schemes $\mathbb{F}_1^{[n]}$. Let $H = \frac{1}{3}E + \frac{1}{2}F$. Observe that $H = -\frac{1}{6}K_{\mathbb{F}_1}$. In this subsection, we will see that if we set $D_t = -\frac{1}{6}K_{\mathbb{F}_1^{[n]}} + \frac{B}{2t}$ for $t < 0$, then there is a correspondence between the Mori wall M_t and the Bridgeland wall $W_{x=t-3}$–at least in certain ranges. The coefficient $\frac{1}{6}$ is chosen to make the correspondence be given by an integer. We begin by listing the Bridgeland walls for $n = 2$ and $n = 4$.

Example 6.6. The Bridgeland walls when $n = 2$ are as follows:

(1) The wall W_{-9} corresponding to the destabilizing object $\mathcal{O}_{\mathbb{F}_1}(-E)$.
(2) The wall W_{-6} corresponding to the destabilizing object $\mathcal{O}_{\mathbb{F}_1}(-F)$.
(3) The wall W_{-5} corresponding to the destabilizing object $\mathcal{O}_{\mathbb{F}_1}(-E-F)$.

Proof. Every scheme of length two is contained in a curve of class $E + F$. Hence, W_{-5} is a collapsing wall. By Example 5.15 any higher rank wall W_x satisfies $x^2 \leq \frac{81}{4} < 25$. Hence, the only walls are rank one walls and have been described in Example 5.12. □

The Mori walls corresponding to the divisor $D_t = -\frac{1}{6}K_{\mathbb{F}_1^{[2]}} + \frac{B}{2t}$ occur at $t = -6, -3, -2$ as can be seen from Figure 5. Using the given descriptions of the base loci, the reader can easily check that the divisor D_t picks up a subscheme in its base locus at t if and only if the corresponding ideal sheaf is destabilized at the Bridgeland wall W_{t-3}.

Example 6.7. The Bridgeland walls when $n = 4$ are as follows:

(1) The wall W_{-21} corresponding to the destabilizing object $\mathcal{O}_{\mathbb{F}_1}(-E)$.
(2) The wall W_{-15} corresponding to the destabilizing object $\mathcal{I}_p(-E)$.
(3) The wall W_{-12} corresponding to the destabilizing object $\mathcal{O}_{\mathbb{F}_1}(-F)$.

(4) The wall W_{-9} corresponding to the destabilizing objects $\mathcal{I}_{Z'}(-E)$, where $l(Z')=2$ and $\mathcal{I}_p(-F)$.
(5) The wall W_{-7} corresponding to the destabilizing object $\mathcal{O}_{\mathbb{F}_1}(-E-F)$.
(6) The wall $W_{-\frac{33}{5}}$ corresponding to the destabilizing object $\mathcal{O}_{\mathbb{F}_1}(-E-2F)$.

Proof. Every scheme of length 4 is contained in a curve with class $E+2F$. Hence, $W_{-\frac{33}{5}}$ is a collapsing wall. By Example 5.15 any higher rank wall W_x satisfies $x^2 \leq \frac{81}{2} < \left(\frac{33}{5}\right)^2$. Hence, all the Bridgeland walls are rank one walls and are described in Example 5.12. $\square$

The reader can compare this to Figure 7. The divisor $D_t = -\frac{1}{6}K_{\mathbb{F}_1^{[2]}} + \frac{B}{2t}$ crosses walls precisely when $t = -18, -12, -9, -6, -4, -\frac{18}{5}$. From the description of the base loci, we see that the divisor D_t picks up a subscheme in its base locus at t if and only if the corresponding ideal sheaf is destabilized at the Bridgeland wall W_{t-3}.

As in the case of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, while we do not know how to prove this correspondence for every wall, we can prove it for walls in certain ranges.

Proposition 6.8. *Let $D_t = \frac{1}{3}E[n] + \frac{1}{2}F[n] + \frac{B}{2t}$, for $t<0$, be a divisor on $\mathbb{F}_1^{[n]}$. Assume that $2-2n \geq t$. Then there is a one-to-one correspondence between the Mori walls M_{t_i} and the Bridgeland walls $W_{x_i=t_i-3}$ when $2-2n \geq t$ and $-1-2n \geq x$. An ideal sheaf $\mathcal{I}_Z$ is in the base locus of D_t if and only if $\mathcal{I}_Z$ is destabilized at $W_{x_i=t_i-3}$ for $t > t_i$.*

Proof. The proof of this proposition is analogous to the proof of Proposition 6.3. Since $(-2n-1)^2 > \frac{81n}{8}$ for $n \geq 2$, in this range there are only rank one Bridgeland walls, which have been described in Example 5.12. We see that they occur at $x = -6n+3+6l(Z')$, when $\alpha=1, \beta=0$, or at $x=-3n+3l(Z')$, when $\alpha=0,\beta=1$, or at $x=-2n-2$ when $\alpha=\beta=1$. On the other hand, in this range the stable base locus decomposition is described by Theorem 4.15. The divisor D_t intersects the Mori wall spanned by $X_{i,n-1-i}$ and $E[n]+F[n]$ when $t=-6n+6+6i$. Similarly, D_t intersects the Mori wall spanned by $X_{n-i-1,i}$ and $F[n]$ at $t=-3n+3+3i$. Finally, D_t intersects the wall spanned by $X_{n-1,0}$ and $X_{0,n-1}$ at $t=-2n+1$. One obtains the proposition by matching the two descriptions. $\square$

Remark 6.9. One can speculate that the relation $x=t-3$ gives a one-to-one correspondence between Bridgeland and Mori walls in general. As in the case of $\mathbb{P}^1\times\mathbb{P}^1$, at least in the cases covered by Proposition 6.8, one obtains a modular interpretation of the log canonical models $\mathrm{Proj}(R(-K_{\mathbb{F}_1^{[n]}} - cB))$, where $R(-K_{\mathbb{F}_1^{[n]}} - cB)$ is the log canonical ring associated to the divisor $-K_{\mathbb{F}_1^{[n]}} - cB$.

The correspondence for other slices. The correspondence between the two sets of walls extends beyond the slice we have studied so far. We can decompose the ample cone of X into chambers such that for ample classes in a chamber the Bridgeland walls with respect to the central charge $-\int_X e^{-(s+it)H}\mathrm{ch}(E)$ have the same order with respect to the same destabilizing objects.

Example 6.10. For $(\mathbb{P}^1\times\mathbb{P}^1)^{[2]}$ the rank one Bridgeland walls with respect to aH_1+bH_2 are:

- $W_{-\frac{2}{a}}, W_{-\frac{1}{a}}$ corresponding to $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1)$ and $\mathcal{I}_p(0,-1)$.
- $W_{-\frac{2}{b}}, W_{-\frac{1}{b}}$ corresponding to $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0)$ and $\mathcal{I}_p(-1,0)$.
- $W_{-\frac{3}{a+b}}$ corresponding to $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$.

Correspondingly, the ample cone of $\mathbb{P}^1 \times \mathbb{P}^1$ decomposes into regions separated by hyperplanes $a = \frac{b}{2}$, $a = b$ and $a = 2b$. When $2b > a > b$, then the rank one walls are $W_{-\frac{2}{b}}, W_{-\frac{2}{a}}, W_{-\frac{3}{a+b}}$ ordered by increasing centers and $W_{-\frac{3}{a+b}}$ is a collapsing wall. Whereas, when $a > 2b$, the relevant walls are $W_{-\frac{2}{b}}, W_{-\frac{1}{b}}$, where $W_{-\frac{1}{b}}$ is a collapsing wall. The threshold value $a = 2b$ is the value where the three walls $W_{-\frac{2}{a}}$, $W_{-\frac{3}{a+b}}$ and $W_{-\frac{1}{b}}$ become equal. Similarly, at the threshold value $a = b$, the walls $W_{-\frac{2}{a}}$ and $W_{-\frac{2}{b}}$ become equal.

We can similarly divide the ample cone of $X^{[n]}$ into regions, where in a fixed region the face spanned by $-K$ and D intersect the same Mori walls in the same order.

Example 6.11. For $(\mathbb{P}^1 \times \mathbb{P}^1)^2$ these regions are separated by the planes spanned by $[-K, X_{1,2}]$, $[-K, X_{2,2}]$ and $[-K, X_{2,1}]$. Note that if D is in the region bounded by $[-K, X_{2,2}]$ and $[-K, X_{1,2}]$, then it intersects the walls $[X_{1,0}, H_2[2]], [X_{0,1}, H_1[2]]$ and $[X_{1,0}, X_{0,1}]$ in order (see Figure 1). These precisely correspond to the (rank one) Bridgeland walls in the region $2b > a > b$, where a scheme Z defines a Mori wall if and only if the ideal sheaf $\mathcal{I}_Z$ is destabilized at the corresponding Bridgeland wall. If D is in the region bounded by $[-K, X_{1,2}]$ and $[-K, H_2[2]]$, then $[-K, D]$ intersects the walls $[X_{1,0}, H_2[2]]$ and $[X_{0,1}, H_2[2]]$. These precisely correspond to the (rank one) Bridgeland walls in the region $a > 2b$.

Example 6.12. As a final example, we work out the two decompositions for $(\mathbb{P}^1 \times \mathbb{P}^1)^{[3]}$. The rank one Bridgeland walls with respect to $aH_1 + bH_2$ are the walls $W_{-\frac{3}{a}}, W_{-\frac{2}{a}}, W_{-\frac{1}{a}}$ corresponding to $\mathcal{I}_{Z'}(0, -1)$ with length of $Z' = 0, 1$ or 2, the walls $W_{-\frac{3}{b}}, W_{-\frac{2}{b}}, W_{-\frac{1}{b}}$ obtained by symmetry and the wall $W_{-\frac{4}{a+b}}$ corresponding to $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$. The ample cone of $\mathbb{P}^1 \times \mathbb{P}^1$ decomposes into chambers where the boundaries are given by the hyperplanes $a = b$, $2a = 3b$, $a = 3b$, $3a = 2b$ and $3a = b$. If $\frac{3}{2}b > a > b$, then the rank one Bridgeland walls are $W_{-\frac{3}{b}}, W_{-\frac{3}{a}}, W_{-\frac{2}{b}}$ and $W_{-\frac{4}{a+b}}$ listed in the order of increasing centers. When $2a = 3b$, the walls $W_{-\frac{3}{a}}$ and $W_{-\frac{2}{b}}$ coincide. When $3b > a > \frac{3}{2}b$, then the rank one Bridgeland walls are $W_{-\frac{3}{b}}, W_{-\frac{2}{b}}, W_{-\frac{3}{a}}$ and $W_{-\frac{4}{a+b}}$ listed in the order of increasing centers. When $a = 3b$, the three walls $W_{-\frac{3}{a}}$, $W_{-\frac{1}{b}}$ and $W_{-\frac{4}{a+b}}$ coincide. If $a > 3b$, then the walls are $W_{-\frac{3}{b}}, W_{-\frac{2}{b}}, W_{-\frac{1}{b}}$.

Correspondingly, the ample cone of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[3]}$ decomposes into regions bounded by the faces $[-K, X_{4,2}], [-K, X_{3,2}], [-K, X_{2,2}], [-K, X_{2,3}], [-K, X_{2,4}]$. By symmetry, let us assume that $a > b$. If D is in the region bounded by $[-K, X_{2,2}]$ and $[-K, X_{2,3}]$, then $[-K, D]$ intersects the walls $[X_{2,0}, H_2[3]]$, $[X_{0,2}, H_1[3]]$ and $[X_{2,0}, X_{0,2}]$ in order (see Figure 2). These correspond precisely to the (rank one) Bridgeland walls in the region $\frac{3}{2}b > a > b$. If D is in the region bounded by $[-K, X_{2,3}]$ and $[-K, X_{2,4}]$, then $[-K, D]$ intersect the Mori walls that correspond to the Bridgeland walls in the region $3b > a > \frac{3}{2}b$. Similarly, if D is in the region bounded by $[-K, X_{2,4}]$ and $[-K, H_2[3]]$, then the Mori walls correspond to the walls in the region $a > 3b$.

Remark 6.13. One can conjecture that there is always a one-to-one correspondence between the walls decomposing the ample cone of $\mathbb{P}^1 \times \mathbb{P}^1$ into chambers and the walls decomposing the ample cone of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$. Since the decompositions are symmetric about $a = b$, we may assume that $a \geq b$. Based on the examples, one can predict that the critical ratio

$$\frac{-n - i_1 j_1 + m_1}{j_1 a + i_1 b} = \frac{-n - i_2 j_2 + m_2}{j_2 a + i_2 b},$$

where two rank one walls become equal corresponds to the wall $[-K, D]$, where $D = \alpha H_1[n] + \beta H_2[n] - \gamma \frac{B}{2}$ with (α, β, γ) satisfying the two equations

$$j_s\alpha + i_s\beta - (i_sj_s + n - m_s - i_s - j_s)\gamma = 0,$$

for $s = 1, 2$. For example, the walls $(n-k)a = nb$ correspond to the walls $[-K, D]$, where $D = (n-1)H_1[n] + (n-1+k)H_2[n] - \frac{B}{2}$.

Using Figure 3 and Example 5.11, the reader can check that for $(\mathbb{P}^1 \times \mathbb{P}^1)^{[4]}$ the two sets of walls occur at $a = b, 3a = 4b, 2a = 3b, a = 2b$ and $a = 4b$ and at $[-K, 3H_1[4]+(3+k)H_2[4]-\frac{B}{2}]$ with $k = 0, 1, \frac{3}{2}, 2, 3$. Similarly, by Figure 4 and Example 5.11, for $(\mathbb{P}^1 \times \mathbb{P}^1)^{[5]}$ the two sets of walls occur at $a = b, 4a = 5b, 3a = 4b, 3a = 5b, 2a = 5b$ and $a = 5b$ and at $[-K, 4H_1[4] + (4+k)H_2[4] - \frac{B}{2}]$ with $k = 0, 1, 2, \frac{5}{2}, 3, 4$.

Suppose that an ample divisor D is contained in a chamber determined by two walls $[-K, D_1]$ and $[-K, D_2]$ in the ample cone of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$. Suppose that $a = \frac{n}{k_1}b$ and $a = \frac{n}{k_2}b$ are the corresponding chambers in the ample cone of $\mathbb{P}^1 \times \mathbb{P}^1$. Then, in the examples, there is a one-to-one correspondence between the Mori walls that intersect $[-K, D]$ and the (rank one) Bridgeland walls that occur for an ample divisor $aH_1 + bH_2$ satisfying $\frac{n}{k_1}b < a < \frac{n}{k_2}b$. One can speculate that there is a one-to-one correspondence between Mori walls intersecting $[-K, D]$ and Bridgeland walls in full generality. Furthermore, one can expect that running the log minimal model program in the face $[-K, D]$ corresponds to the birational transformations that take place as one crosses the Bridgeland walls, giving modular interpretations to all the models.

We leave it to the reader to check that a similar story holds for $\mathbb{F}_1^{[n]}$ for $2 \leq n \leq 4$ using §4. It would be interesting to explore the connection between the Bridgeland walls and Mori walls for other smooth, projective surfaces, especially those with ample canonical bundle.

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