A LITTLEWOOD-RICHARDSON RULE FOR PARTIAL FLAG VARIETIES

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ABSTRACT. This paper studies the geometry of one-parameter specializations in partial flag varieties. The main result is a positive, geometric rule for multiplying Schubert cycles in the cohomology of partial flag varieties. This rule can be interpreted as a generalization of Pieri's rule to arbitrary products and arbitrary partial flag varieties. It has numerous applications to geometry, representation theory and the theory of symmetric functions.

CONTENTS

1. INTRODUCTION

This paper studies the geometry of one-parameter specializations in partial flag varieties. The main result is a positive, geometric rule for computing the structure constants of the cohomology of partial flag varieties in terms of their Schubert basis. The program of giving positive, geometric formulae for expressing classes of products in the cohomology of flag varieties was started in the nineteenth century by Pieri, Schubert and others. Pieri's formula for multiplying a special Schubert cycle with an arbitrary Schubert cycle in the Grassmannian is one of the fundamental results of the theory. The rule presented in this paper is a generalization of Pieri's rule for multiplying arbitrary cycles in arbitrary partial flag varieties. We thus complete the program started more than a century ago for the partial flag varieties.

Partial flag varieties are fundamental objects in algebraic geometry, combinatorics and representation theory. Consequently, their cohomology rings have been studied extensively (see, for example, [BGG], [FPi] or [Ful2]). Although there are many presentations for their cohomology rings, surprisingly, a positive rule for multiplying Schubert cycles in arbitrary partial flag varieties have eluded mathematicians for many

2000 *Mathematics Subject Classification.* Primary 14M15, 14N35, 32M10 .
During the preparation of this article the author was partially supported by a Liftoff Fellowship from the Clay Mathematics Institute and the NSF grant DMS-0737581.

decades. There are many positive rules for multiplying special classes of Schubert cycles. There are also positive rules for multiplying Schubert cycles in special flag varieties. For example, for Grassmannians there are many Littlewood-Richardson rules in terms of Young tableaux, puzzles, checkers and Mondrian tableaux among others (see, for example, [Ful1], [KT], [V1], [C2]). For arbitrary partial flag varieties Monk's rule and its generalizations give a positive rule for multiplying codimension one and more generally special Schubert cycles with an arbitrary Schubert cycle. The first positive, geometric rule for multiplying any two Schubert cycles in two-step flag varieties was given in [C2]. However, before the precursor to the rule presented in this paper was announced in [C1], there was not even a conjectural rule for multiplying arbitrary Schubert cycles in arbitrary partial flag varieties. A. Knutson had conjectured a rule for two-step flag varieties in terms of puzzles. This conjecture was extended by A. Buch to three-step flag varieties.

Recently, A. Knutson and R. Vakil have outlined a program for obtaining a positive rule for multiplying Schubert cycles in arbitrary partial flag varieties. Their program is very similar to the one carried out in this paper. This paper makes their program precise by identifying the varieties that occur in the limit of the degenerations. (The order of degeneration described in this paper is different from the one proposed by Knutson and Vakil. However, the analysis in this paper applies verbatim to their order of degeneration and identifies completely the limits of the specializations in their order. Our order is more efficient and simplifies the geometry whenever possible.)

The structure constants of the cohomology of flag varieties exhibit a very rich structure which is best revealed by positive geometric rules. For instance, in recent years, new Littlewood-Richardson rules for Grassmannians have enabled Klyachko, Knutson, Tao, Woodward and their collaborators to resolve long standing conjectures such as the Saturation Conjecture and the Horn's Conjecture (see [KT] and [KTW]). Vakil using his checker rule resolved the reality of Schubert calculus for Grassmannians [V2]. Similar results follow for partial flag varieties from the rule presented here.

We will phrase our rule in terms of combinatorial objects called Mondrian tableaux. A Mondrian tableau is a very convenient short hand for recording the rank conditions of a vector space with respect to two flags. Recall that the partial flag variety $F(k_1, \dots, k_r; n)$ parameterizes r-tuples $(V_1, \dots, V_r)$ of subspaces of an n-dimensional vector space V, where $V_i \subset V_{i+1}$ and $\dim(V_i) = k_i$ for every i. A Mondrian tableau M for $F(k_1, \dots, k_r; n)$ records k_i vector spaces expressible as a span of basis elements of a fixed ordered basis for each $1 \leq i \leq r$. We can associate an irreducible subvariety Σ_M of $F(k_1, \dots, k_r; n)$ to M. Let W^i be one of the vector spaces recorded by M. Let $\#_i W^i(M)$ denote the number of vector spaces (inclusive) among the k_i that M records that are contained in W^i. We consider r-tuples $(V_1, \dots, V_r)$ where the dimension of intersection of V_i with every vector space W^i recorded by M is at least $\#_i W^i(M)$ and that satisfy certain non-degeneracy and consistency conditions.

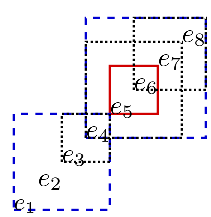

FIGURE 1. An example of a Mondrian tableau.

For example, Figure 1 depicts a typical Mondrian tableau for $F(1, 3, 5; n)$, $n \geq 8$. The ordered basis is placed along the diagonal of a Mondrian tableau from southwest to northeast. (When depicting Mondrian tableaux, we omit the basis from the picture.) We use color and line type to denote constraints imposed on the different vector spaces. In this example, red and solid, blue and dashed, and black and dotted squares depict vector spaces imposing constraints on V_1, V_2 and V_3, respectively. When drawing Mondrian tableau if two squares of different colors overlap, we only draw the square imposing a constraint on the vector space with lowest index. The conditions we impose on Mondrian tableau guarantee that this causes no confusion. With this caveat in mind, the Mondrian tableau in Figure 1 depicts one red and solid vector space $W_1^1 =< e_5, e_6 >$, three blue and dashed vector spaces $W_1^2 =< e_1, e_2, e_3, e_4 >, W_2^2 =< e_4, e_5, e_6, e_7, e_8 >, W_3^2 =< e_5, e_6 >$ and five black and dotted vector spaces $W_1^3 =< e_1, e_2, e_3, e_4 >, W_2^3 =< e_3, e_4 >, W_3^3 =< e_5, e_6 >, W_4^3 =< e_4, e_5, e_6, e_7 >, W_5^3 =< e_6, e_7, e_8 >$. We can associate a subvariety of $F(1, 3, 5; 8)$ to this Mondrian tableau by taking the closure of the locus of triples (V_1, V_2, V_3) in $F(1, 3, 5; 8)$ satisfying the rank constraints imposed by the Mondrian tableau. These conditions are easy to read from the figure. For example, V_1 is required to intersect the red vector space W_1^1. V_2 is required to intersect the blue vector spaces W_1^2 and W_3^2. Since $V_1 \subset V_2$, the intersection of V_2 with W_3^2 coincides with the intersection of V_1 with W_1^1. V_2 is required to have a two-dimensional intersection with W_2^2. Since $V_2 \subset V_3$, the two-dimensional subspace of V_2 contained in W_2^2 must be contained in the subspace of V_3 spanned by the intersections of V_3 with W_3^3, W_4^3 and W_5^3. Similarly, the one dimensional subspace of V_2 contained in W_1^2 must be a subspace of the two dimensional subspace of V_3 contained in W_1^3. V_3 is required to intersect W_2^3, W_3^3 and W_5^3 each in a one dimensional subspace and W_1^3 and W_4^3 each in a two dimensional subspace.

A Schubert variety or the intersection of two Schubert varieties in a partial flag variety can be expressed as the variety associated to a Mondrian tableau. The variety associated to a Mondrian tableau is a Schubert variety exactly when all the squares comprising the Mondrian tableau can be totally ordered by inclusion. Algorithm 3.8 describes how to associate a Mondrian tableau to the intersection of two Schubert varieties. In order to calculate the structure constants of $F(k_1, \dots, k_r; n)$, we begin with the Mondrian tableau M associated to the intersection of two Schubert varieties. The rule consists of changing some of the vector spaces recorded by M and replacing M by one or more Mondrian tableaux. The way we change the vector spaces recorded by M corresponds to a one-parameter specialization arising from a family of the form $(1-t)e_i + te_j$, where e_i and e_j are elements of our special basis. We will describe the order for the degenerations in Rules 3.19 and 3.21. We remark that there is flexibility in the order that the specializations are carried out. One could choose other orders such as the one suggested by Knutson and Vakil. The main advantage of the order described in Rule 3.19 is that it is canonically associated to the variety whose class we would like to compute and does not depend on a choice of basis.

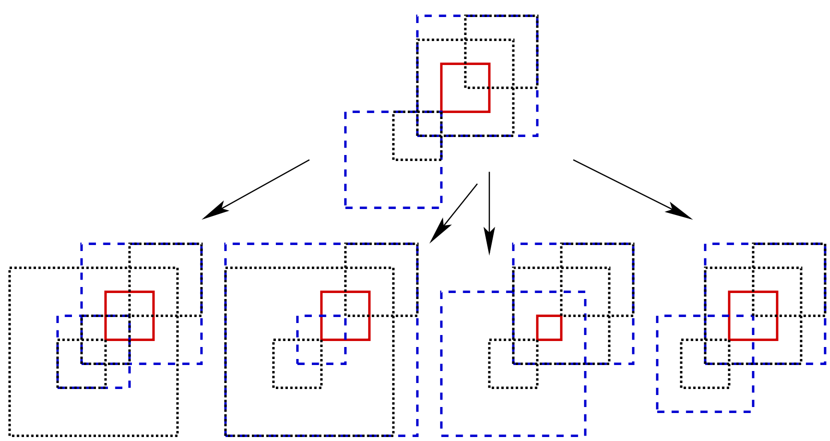

FIGURE 2. A step of the algorithm.

Figure 2 shows a step of the algorithm for the Mondrian tableau depicted in Figure 1. In this example, we change the basis by replacing e_1 with $(1-t)e_5 + te_1$. When $t = 1$, we have the original configuration. As long as $t \neq 0$, the resulting set of vectors is still a basis. We can define a subvariety of the flag variety as before by simply replacing every occurrence of e_1 in the definitions with $(1-t)e_5 + te_1$. We obtain an isomorphic variety. The interesting geometry occurs when $t = 0$. When $t = 0$, we have to determine the flat limit of the family of varieties defined away from $t = 0$. In the limit, the variety breaks into a union of other varieties. In this example, the support of the flat limit of the family is reducible consisting of four irreducible components. Each of these irreducible components is a variety associated to a Mondrian tableau. (These are depicted below the original tableau in Figure 2.) Furthermore, the family is generically reduced along each of the components. (Proving these two claims in general, that the limit is supported along varieties associated to Mondrian tableaux and that the limit is reduced along the generic point of each of these varieties, is the purpose of this paper.) Consequently, the class of the variety associated to the original Mondrian tableau is the sum of the classes of the varieties associated to the resulting Mondrian tableaux.

More generally, the move on a Mondrian tableau M corresponds to a specialization of the variety associated to M. The specialization considered gives rise to a flat family of subvarieties of $F(k_1, \dots, k_r; n)$, where the general member is isomorphic to Σ_M. The rule records the flat limit of such a family. It turns out that the maximal dimensional components of the flat limit are again varieties associated to Mondrian tableaux $M_1, \dots, M_j$ and that the flat limit is generically reduced along each of these components. Hence, the class of the variety Σ_M is equal to the sum of the classes of the varieties associated to Σ_{M_i}. We can then inductively repeat the procedure for each of the M_i until we break all the varieties into a union of Schubert cycles.

The crucial point in this program is to determine the components of the limit $\Sigma_{M_1}, \dots, \Sigma_{M_j}$ of the specialization. The necessary calculations that enable us to achieve this have been carried out in [C2]. The author recommends skimming that paper (at least the rule for Grassmannians) before attempting to read this one. In this paper, we use those calculations and induction to extend the rule from two-step flag varieties to arbitrary partial flag varieties.

The principle that allows us to determine the limits is relatively simple. The specialization increases the dimensions of the intersection of the vector spaces imposing constraints on the r-tuple $(V_1, \dots, V_r)$. Let $W_1^i(0)$ and $W_2^i(0)$ be the limits of two of the vector spaces recorded by M. Suppose $\dim(W_1^i(0) \cap W_2^i(0))$ is larger than $\dim(W_1^i \cap W_2^i)$. Let $(V_1(0), \dots, V_r(0))$ be a general point of an irreducible component of the flat limit. Either $V_j(0) \cap W_1^i(0) \cap W_2^i(0)$ has larger dimension than $V_j \cap W_1^i \cap W_2^i$ or the subspaces of V_j contained in W_1^i and W_2^i specialize to lie in the span of $W_1^i(0)$ and $W_2^i(0)$. This is true for all possible intersections and all the vector spaces $V_1, \dots, V_r$. As one runs through all the possibilities one obtains all the limits. In fact, all the maximal dimensional components of the flat limit correspond to the case when the subspace of at most one of vector spaces in the r-tuple, say V_h, contained in two neighboring vector spaces (see Definition 3.22) become dependent and all other subspaces remain as independent as possible unless forced to become dependent by this dependency and the compatibility conditions. We will make this more precise in §3 and §4. §5 contains some sample calculations for three and four step flag varieties. The reader might want to turn to these examples and the examples in [C2] while reading the paper.

Returning to the example in Figure 2, the degeneration $(1-t)e_5 + te_1$ specializes the blue and dashed vector space W_1^2 and the black and dotted vector space W_1^3. The blue and dashed neighbors of W_1^2 are W_2^2 and W_3^2. The dotted and black neighbors of W_1^3 are W_3^3 and W_4^3. Label the four tableaux in the second row I, II, III, IV from left to right. If in the limit the subspaces of V_2 and V_3 contained in $W_1^2(0)$ and $W_1^3(0)$ remain independent from the subspaces contained in the neighboring vector spaces, we get the limit associated to the Mondrian tableau IV. The subspaces of V_2 contained in $W_1^2(0)$ and W_3^2 may become dependent. In this case, V_2 (and necessarily V_3) intersect $\tilde{W}_3^2 = \tilde{W}_3^3 =< e_5 >$. We denote this by deleting the squares corresponding to W_3^2 and W_3^3 and drawing the squares corresponding to $\tilde{W}_3^2$ and $\tilde{W}_3^3$. In addition, we have to remember that the vector space $\tilde{W}_1^2 = \tilde{W}_1^3 =< e_1, \dots, e_6 >$ contains a two and three dimensional subspace of V_2 and V_3, respectively. We draw the squares corresponding to $\tilde{W}_1^2 = \tilde{W}_1^3$. Finally, V_1 either has to coincide with the intersection of V_2 with $\tilde{W}_3^2$ or V_2 has to intersect W_1^1 in a two-dimensional subspace. The dimension of the latter locus is too small to support a component of the flat limit, so we conclude that V_1 has to intersect $\tilde{W}_1^1$ in a one-dimensional subspace and depict this by replacing the square corresponding to W_1^1 with the square corresponding to $\tilde{W}_1^1$. We obtain the Mondrian tableau III. The subspaces of V_2 contained in $W_1^2(0)$ and W_2^2 may become dependent. This forces the subspaces of V_3 contained in $W_1^3(0)$ and W_4^3 to become dependent. We depict this by deleting the squares corresponding to W_1^2, W_2^2, W_1^3 and W_4^3 and drawing the squares corresponding to $\tilde{W}_2^2 = \tilde{W}_4^3 =< e_4, e_5 >$.

In addition, we draw the squares corresponding to $\tilde{W}_1^3 =< e_1, \dots, e_7 >$ and $\tilde{W}_1^2 =< e_1, \dots, e_8 >$ to denote that $\tilde{W}_1^3$ and $\tilde{W}_1^2$ contain a 5-dimensional subspace of V_3 and a 3-dimensional subspace of V_2, respectively. We obtain the Mondrian tableau II. Finally, the Mondrian tableau I depicts the possibility when the subspace of V_3 contained in $W_1^3(0)$ and W_4^3 become dependent, but the subspaces of V_2 contained in $W_1^2(0)$ and W_2^2 remain independent.

Acknowledgements: I would like to thank S. Billey, A. Buch, J. Harris, A. Knutson and R. Vakil for many fruitful conversations. I am grateful to Craig Desjardins for correcting many mistakes in the earlier versions of the manuscript. A. Buch and R. Vakil have played a crucial role in clarifying, simplifying and correcting earlier formulations of the rule presented here.

2. Preliminaries about partial flag varieties

In this section, we recall the basic facts about the cohomology of partial flag varieties.

Let $0 \leq k_1 < k_2 < \cdots < k_r \leq n$ be a collection of strictly increasing non-negative integers. Given such a sequence, we will set $k_0 = 0$ and $k_{r+1} = n$. The r-step partial flag variety $F(k_1, \dots, k_r; n)$ parameterizes r-tuples $(V_1, \dots, V_r)$ of vector subspaces of an n-dimensional vector space V, where the subspaces are ordered by inclusion $V_1 \subset \cdots \subset V_r$ and V_i has dimension k_i. $F(k_1, \dots, k_r; n)$ is a smooth projective variety of dimension

$$\sum_{i=1}^{r} k_i(k_{i+1} - k_i).$$

Let $F_\bullet : 0 = F_0 \subset F_1 \subset \cdots \subset F_n = V$ be a fixed complete flag in the n-dimensional vector space V. The cohomology of the r-step flag variety $F(k_1, \dots, k_r; n)$ has a $\mathbb{Z}$-basis consisting of the cohomology classes of Schubert varieties. Schubert varieties are parametrized by colored partitions

$$n - k_r \geq \lambda_1[\delta_1] \geq \lambda_2[\delta_2] \geq \cdots \geq \lambda_{k_r}[\delta_{k_r}] \geq 0$$

colored by r colors $1, \dots, r$, where $k_i - k_{i-1}$ of the parts have the color i for $1 \leq i \leq r$. We will denote Schubert cycles by $\sigma^{\delta_1, \dots, \delta_{k_r}}_{\lambda_1, \dots, \lambda_{k_r}}$, where the bottom row denotes the partition and the top row denotes the color. The Schubert variety

$$\Sigma^{\delta_\bullet}_{\lambda_\bullet}(F_\bullet) = \{(V_1, \dots, V_r) \in F(k_1, \dots, k_r; n) | \dim(V_i \cap F_{n-k_r+j-\lambda_j}) \geq \#\{t \leq j | \delta_t \leq i\}\}$$

is defined by requiring the vector space V_i to intersect the flag element $F_{n-k_r+j-\lambda_j}$ in a subspace of dimension at least the number of parts of color less than or equal to i in the first j parts.

Example 2.1. The Schubert cycle $\sigma^{1,2,1,3,2}_{3,2,1,1,0}$ in $F(2,4,5;8)$ is the Poincaré dual of the following Schubert variety. First, we determine the flag elements that have exceptional behavior with respect to V_3. These are determined by the lower sequence $3,2,1,1,0$. Taking $n - k_3 + i - \lambda_i$, we see that they are the flag elements F_1, F_3, F_5, F_6 and F_8. The flag elements that have exceptional behavior with respect to V_2 are those to which the upper sequence $1,2,1,3,2$ assigns an index less than or equal to 2. These flag elements are F_1, F_3, F_5, F_8. Finally, the flag elements that have exceptional behavior with respect to V_1 are those where the upper sequence assigns the index 1. These are F_1 and F_5. In conclusion, the Schubert variety is given by

$$\Sigma^{1,2,1,3,2}_{3,2,1,1,0}(F_\bullet) = \{(V_1, V_2, V_3) \in F(2,4,5;8) | \dim(V_3 \cap F_1) \geq 1, \dim(V_3 \cap F_3) \geq 2, \dim(V_3 \cap F_5) \geq 3,$$
$$\dim(V_3 \cap F_6) \geq 4, \dim(V_3 \cap F_8) \geq 5, \dim(V_2 \cap F_1) \geq 1, \dim(V_2 \cap F_3) \geq 2,$$
$$\dim(V_2 \cap F_5) \geq 3, \dim(V_2 \cap F_8) \geq 4, \dim(V_1 \cap F_1) \geq 1, \dim(V_1 \cap F_5) \geq 2\}.$$

The class of a Schubert variety depends only on the colored partition and not on the flag. Given any two Schubert cycles, their product can be expressed as a linear combination of Schubert cycles. The purpose of this paper is to give a positive, geometric rule for determining the corresponding structure constants.

Some authors represent Schubert cycles in the partial flag variety $F(k_1, \dots, k_r; n)$ as strings of length n consisting of $1, \dots, r, r+1$, where there are $k_i - k_{i-1}$ of digits in the string of value i for every

$i = 1, \ldots, r+1$. The translation between the string notation and our notation is straighforward. Place $r+1$ in every position in the n-string except for the positions $n-k_r+j-\lambda_j$. In the position $n-k_r+j-\lambda_j$ place the digit δ_j. We warn the reader that there are many different conventions for the string notation. Some authors reverse the identification of the vector spaces with the digits, replacing i by $r+1-i$. There are also different conventions about whether the string should be written from right to left or from left to right. Some authors represent Schubert varieties by permutations. The natural translation between permutations and our notation is to assign to the digits $k_{i-1}+1, \ldots, k_i$ the digits $n-k_r+j-\lambda_j$ in increasing order, where λ_j are the parts assigned the color i. We warn the reader that most authors first take the Poincaré dual cycle before applying this construction. In the sequel we will avoid using strings or permutations.

3. Mondrian tableaux.

In this section, we introduce combinatorial objects called Mondrian tableaux for partial flag varieties $F(k_1, \ldots, k_r; n)$. Mondrian tableaux provide a very convenient shorthand for recording the geometry of partial flag varieties.

3.1. **Preliminaries about Mondrian tableaux.**

We first introduce the preliminary definitions about Mondrian tableaux for partial flag varieties.

Notation 3.1. Let V be an n-dimensional vector space. Let $e_1, \ldots, e_n$ be an ordered basis for V. Let $1, \ldots, r$ be r colors ordered $1 < \cdots < r$. A *square* S^i is a pair consisting of a subset of the set of basis elements

$$\{e_1, \ldots, e_n\}$$

and a color $1 \leq i \leq r$. We always denote the color of a square with a superscript. When a statement holds for squares irrespective of their color, we sometimes omit the color from the notation. Let M be a collection of squares of colors $1, \ldots, r$. The side-length of a square S^i (not necessarily contained in M) is the number of basis elements in S^i and is denoted by $\#_{r+1}S^i(M)$. Similarly, the number of squares of M of color j in S^i is denoted by $\#_j S^i(M)$.

Definition 3.2. *We refer to the basis element in S^i with lowest index (respectively, highest index) as the left (respectively, right) corner of S^i and denote it by $l(S^i)$ ($r(S^i)$). We say a square is not chopped if the basis elements contained in it are consecutive. Otherwise, we call the square chopped. The maximal consecutive strings of basis elements in a square S^i are called the* chops *of S^i. We refer to the chop containing the left corner as the left chop and denote it by $lch(S^i)$. Similarly for a chop of a square S^i, the left (respectively, right) corner of the chop refer to the basis element with smallest (respectively, largest) index in that chop. A* gap *of a square S^i is a basis element e_t such that $e_t \notin S^i$ and there exists e_u and e_v in S^i with $u < t < v$. We say a basis element e_u is to the left (respectively, right) of another basis element e_v if $u \leq v$ (respectively, $u \geq v$). We say strictly left or strictly right if the inequalities are strict. We use $\leq, \geq, <, >$ to mean left of, right of, strictly left of and strictly right of, respectively.*

Definition 3.3 (Abuses of notation)**.** *We say a square S^i is the span of the squares $S_1, \ldots, S_m$ if the basis elements contained in S^i is equal to the union of the basis elements contained in $S_1, \ldots, S_m$. We say that two squares S^i and S^j coincide if the set of basis elements in S^i is equal to the set of basis elements in S^j (however, we allow S^i and S^j to have different colors). We often abuse notation and write $S_1^i \subset S_2^j$ to mean that the set of basis elements in S_1^i is a subset of the set of basis elements in S_2^j.*

A square of color i in a Mondrian tableau denotes a vector space that imposes a rank condition on the i-th vector space V_i parameterized by the flag variety.

Definition 3.4. *A Mondrian tableau M for the partial flag variety $F(k_1, \ldots, k_r; n)$ is a collection of squares satisfying the following properties:*

M1 *M contains exactly k_i distinct squares of color i for each $1 \leq i \leq r$.*

M2 *For each $1 \leq i \leq r-1$, every square S^i of M of color i coincides with the span of the squares of M of color $i+1$ contained in S^i. For each $1 \leq i \leq r$, a square S^i of M of color i does not coincide with the span of the squares of M of color i strictly contained in S^i.*

M3 *If S_1^i and S_2^i are two squares of M with $l(S_1^i) \geq l(S_2^i)$, then either $S_1^i \subseteq S_2^i$ or $r(S_1^i) \geq r(S_2^i)$ as well.*

M4 *A square of M of color r may have at most one gap. If $S^r \in M$ has a gap g, then every square T of M with $l(T) \leq l(S^r)$ contains S^r. The collection of squares of M of color r that contain a gap are totally ordered by inclusion, their gaps are equal, and they coincide to the right of the gap. If g is a gap of a square of color r, then M does not contain any square T such that $r(T) \leq g$. If a square S_j^i of M of color $i < r$ has a gap g, then g is either a gap of a square of M of color r contained in S_j^i, or there does not exist a square $S_{j'}^{i+1}$ in M of color $i+1$ such that $l(S_{j'}^{i+1}) \geq l(S_j^i)$ and $g \in S_{j'}^{i+1}$.*

Notation 3.5 (Depicting Mondrian tableaux.). Place the ordered basis $e_1, \dots, e_n$ along the diagonal of an $(n \times n)$-grid ordered from southwest to northeast. Depict a square by drawing a square containing the basis elements that constitute that square. If a square has a gap, we delete the intersection of the square with the the row and column corresponding to the gap. The left (right) corner of a square is the southwest/lower-left (northeast/upper-right) corner of the square in the picture. Every move with Mondrian tableaux will only depend on the basis elements contained in the square. Since there is a linear ordering on them (increasing from left to right) there is no ambiguity in referring to the lower-left (upper-right) corner as simply the left (right) corner. Our examples will involve partial flag varieties with four or fewer steps. We indicate the color of the square with color and line type. In examples, we always order our colors as red (with solid lines), blue (with dashed lines), green (with dashed and dotted lines) and black (dotted lines) in increasing order. In order to not clutter the pictures, if two squares of different colors coincide, we draw only the one with the smallest color. Condition M2 in the definition of a Mondrian tableau guarantees that this does not lead to ambiguities. Figure 3 depicts some examples of Mondrian tableaux for $F(1,2,3,4;6)$.

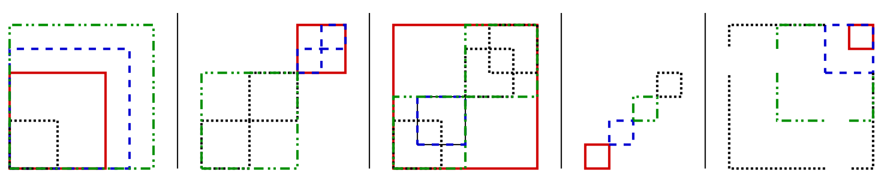

FIGURE 3. Some examples of Mondrian tableaux.

In the next section, we will describe how to associate an irreducible subvariety of $F(k_1, \dots, k_r; n)$ to a certain subset of Mondrian tableaux. Ignoring some subtleties, the subvariety Σ_M associated to a Mondrian tableau M is the closure of the locus of tuples $(V_1, \dots, V_r)$ in $F(k_1, \dots, k_r; n)$ such that

(1) For every $1 \leq i \leq r$ and every square S (not necessarily contained in M), V_i intersects the subspace of V spanned by the basis elements contained in S in dimension at least $\#_i S(M)$; and
(2) The subspace of V_i contained in S^i (where S^i denotes the vector space spanned by the basis elements contained in the square S^i of M) is a subspace of the subspace of V_{i+1} spanned by the intersections of V_{i+1} with the vector spaces corresponding to the squares of color $i+1$ contained in S^i.

We will make the association between M and Σ_M more precise and accurate in the next section. For the purposes of understanding the combinatorial rule, the important information is that (a certain subset of) Mondrian tableaux correspond to irreducible subvarieties of $F(k_1, \dots, k_r; n)$. These subvarieties are determined by imposing rank conditions on the vector spaces $(V_1,\dots,V_r)$ based on the combinatorial data of the Mondrian tableau. Now we will introduce the Mondrian tableaux that occur during the combinatorial rule.

Definition 3.6. *We say that a collection of squares $S_1^{i_1},\dots,S_j^{i_j}$ is totally ordered if for any two of the squares $S_k^{i_k}$ and $S_l^{i_l}$ either $S_k^{i_k}\subseteq S_l^{i_l}$ or $S_l^{i_l}\subseteq S_k^{i_k}$.*

We say that square S^i in a collection of squares M is a nested square if

1. *For every other square $\tilde{S}^j\in M$, either $S^i\subseteq\tilde{S}^j$ or $\tilde{S}^j\subseteq S^i$.*
2. *The collection of squares in M containing S^i is totally ordered.*

A square that is not nested is called an unnested square. Finally, we say a Mondrian tableau M is nested if every square in M is a nested square.

Property M2 implies that if the squares of color r in a Mondrian tableau are totally ordered, then the tableau is nested. It is easy to encode Schubert varieties by Mondrian tableaux.

Definition 3.7. *Let σ_λ^δ be a Schubert cycle for $F(k_1,\dots,k_r;n)$. A Mondrian tableau associated to the Schubert cycle σ_λ^δ is a nested tableau consisting of squares of size $n-k_r+i-\lambda_i$ and colors $\delta_i,\delta_i+1,\dots,r$ for every $1\le i\le k_r$.*

Figure 4 shows a Mondrian tableau associated to the Schubert cycle $\sigma_{2,0,0,0}^{3,1,2,1}$ in $F(2,3,4;6)$.

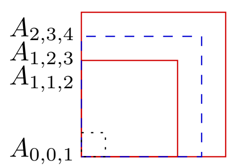

FIGURE 4. The Schubert cycle $\sigma_{2,0,0,0}^{3,1,2,1}$ in $F(2,3,4;6)$.

Note that a Mondrian tableau contains more refined information than the class of the Schubert variety. One can recover the partial flag with respect to which the Schubert variety is defined from the Mondrian tableau.

Next we would like to associate a Mondrian tableau to the intersection of two Schubert varieties in $F(k_1,\dots,k_r;n)$. The diagonal action of $GL(n)$ on the product of two full-flag varieties $Fl(n)\times Fl(n)$ has a dense open orbit. For any point $(F_\bullet,G_\bullet)$ in this dense open orbit, one can choose a basis of the underlying vector space V, so that $F_\bullet$ and $G_\bullet$ are opposite flags with respect to this basis. We will assume that the two Schubert varieties are initially defined with respect to a pair of flags $(F_\bullet,G_\bullet)$ in this dense open subset. Then we can choose an ordered basis of V by setting $e_i=F_i\cap G_{n-i+1}$. Algorithm 3.8 describes how to associate a Mondrian tableau to the intersection of two Schubert varieties defined with respect to $F_\bullet$ and $G_\bullet$.

Algorithm 3.8 (Associating a Mondrian tableau to the intersection of Schubert varieties). We now describe the algorithm that associates a Mondrian tableau $M(\sigma_\lambda^\delta,\sigma_\mu^\kappa)$ to the intersection of two Schubert cycles σ_λ^δ and σ_μ^κ in $F(k_1,\dots,k_r;n)$.

Step 1: Let $M(\lambda,\delta)$ be the Mondrian tableau associated to σ_λ^δ where all the squares are left-justified and none are chopped. Similarly, let $M(\mu,\kappa)$ be the Mondrian tableau associated to σ_μ^κ where all the squares are right-justified and none are chopped. Place $M(\lambda,\delta)$ and $M(\mu,\kappa)$ in the same $(n\times n)$-grid and call the corresponding collection of squares M. Label the squares of color i of $M(\lambda,\delta)$ by $A^i_{m_1,\dots,m_i}$ and the squares of color i of $M(\mu,\kappa)$ by $B^i_{n_1,\dots,n_i}$, where m_j (respectively, n_j) denotes the number of squares of color $j\le i$ (inclusive) contained in $A^i_{m_1,\dots,m_i}$ (respectively, $B^i_{n_1,\dots,n_i}$). Whenever we do not wish to specify an index of the square, we will place a star instead of that index as in $A^i_{*,\dots,*,m_i}$. The A and B squares are totally ordered by inclusion. The smallest A square satisfying property P will mean the A square with the least number of basis elements (equivalently, shortest side-length) that satisfies property P.

Step 2: Let $A^i_{0,\dots,0,1},\dots,A^r_{0,\dots,0,1,\dots,1}$ be the A squares that coincide with the smallest A square of color r. Let $\tilde{B}^r$ be the smallest square among the squares of color r that coincide with $B^i_{*,\dots,*,k_i},\dots,B^r_{*,\dots,*,k_r}$. Let $S^r_1 = A^r_{*,\dots,*,1}\cap\tilde{B}$. If S^r_1 is empty, the intersection of the two Schubert varieties is empty and the algorithm terminates. Otherwise, let M_1 be the collection of squares obtained from M by deleting $A^r_{*,\dots,*,1}$ and $\tilde{B}$. Suppose we have inductively defined S^i_α and obtained a collection of squares M_t by deleting an A and a B square of M of the same color at each step. Suppose i is the largest color of the squares in M_t. Let $A^i_{m_1,\dots,m_i}$ be the smallest A square of color i in M_t and suppose that it coincides with the squares $A^j_{m_1,\dots,m_j}, A^{j+1}_{m_1,\dots,m_j,m_{j+1}},\dots,A^i_{m_1,\dots,m_i}$. For $j\le s\le i$, let $B^{s,i}$ denote the largest B square of color i in M_t that coincides with a B square of color s and contains $B^s_{*,\dots,*,k_s-m_s+1}$. If such a square does not exist, set $B^{s,i}=B^{i,i}$ (note that $B^{i,i}$ always exists). Let $\tilde{B}^i$ be the smallest square among $B^{j,i},\dots,B^{i,i}$. Let $\tilde{S}^i_{m_i} = A^i_{m_1,\dots,m_i}\cap\tilde{B}^i$. If $\tilde{S}^i_{m_i}$ is empty, then the intersection of the Schubert varieties is empty and the algorithm terminates. Otherwise, delete $A^i_{m_1,\dots,m_i}$ and $\tilde{B}^i$ from M_t to obtain M_{t+1}. Let $S^i_{m_i}$ be the square obtained by shrinking $\tilde{S}^i_{m_i}$ to be the span of the squares S^{i+1}_α produced previously in the algorithm and contained in $\tilde{S}^i_{m_i}$. Continue defining the squares S^i_α inductively. The algorithm terminates when there are no A or B squares remaining. Let $M(\sigma^\delta_\lambda,\sigma^\kappa_\mu)$ be the Mondrian tableau consisting of the squares $S^i_{m_i}$, for $1\le i\le r$ and $1\le m_i\le k_i$.

We will later see that the variety associated to the Mondrian tableau produced by Algorithm 3.8 is the intersection of the two Schubert varieties. We give some examples of Algorithm 3.8 in Figure 5.

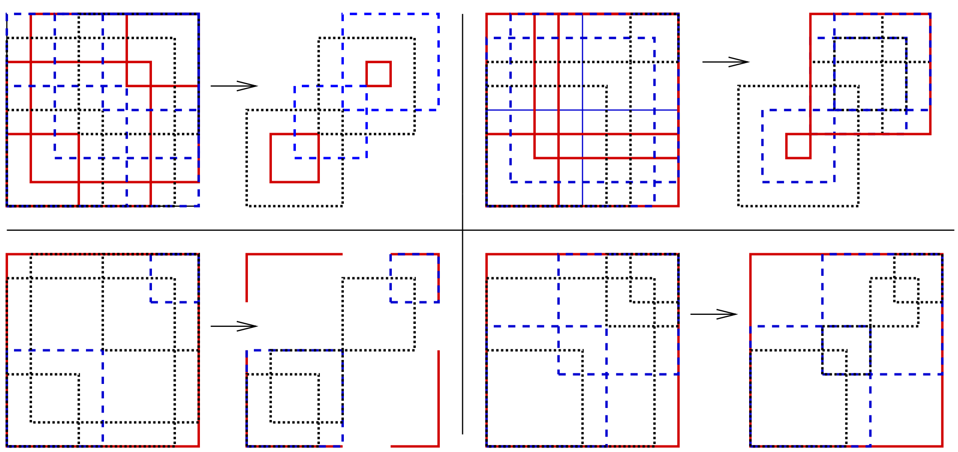

FIGURE 5. Some examples of Algorithm 3.8.

Definition 3.9. *A square F^i of color i of a Mondrian tableau M is called a filler in a square S^j of M if*

(1) *F^i is strictly contained in S^j; and*
(2) *There exists a square T^k of M with $k<i$ such that $l(F^i)=l(T^k)>l(S^j)$ and $T^k\not\subseteq S^j$.*

Definition 3.10 (Admissible Mondrian tableaux)**.** *A Mondrian tableau for $F(k_1,\dots,k_r;n)$ is called admissible if it satisfies the following properties:*

AM1 *If $S^i_1\subsetneq S^i_2$ are two squares of M of color $i>1$, either there exists a square T^{i-1} of M of color $i-1$ such that $S^i_1\subseteq T^{i-1}\subsetneq S^i_2$, or for every square T of M satisfying $l(T)\le l(S^i_2)$ and $T\not\subset S^i_2$,* *we have $S_1^i \subset T$. Furthermore, if S_1^i does not coincide with a square T^{i-1} of M, then every square S of M with $l(S) \leq l(S_2^i)$ and $S \not\subseteq S_2^i$ either contains S_1^i or contains a filler F in S with $l(F) < l(S_1^i)$.*

AM2 *Let $S_2^{i-1} \subsetneq S_1^i$ with $i > 1$ be two squares of M. Let S_3^i be a square of M of color i such that $S_3^i \subset S_2^{i-1}$ and the basis elements contained in S_3^i and S_2^{i-1} to the left of $l(S_3^i)$ are not equal. Then either there exists a square S_4^{i-1} of M of color $i-1$ with $S_3^i \subset S_4^{i-1} \subsetneq S_1^i$ and $l(S_4^{i-1}) \leq l(S_2^{i-1})$, or for every square S of M such that $l(S) \leq l(S_1^i)$ and $S \not\subset S_1^i$ we have that $S_3^i \subset S$.*

Remark 3.11. While calculating the intersection of two Schubert varieties the tableaux that occur satisfy the stronger condition AM1′: If $S_1^1 \subset S_2^1$, then every square T of M with $l(T) < l(S_2^1)$ contains S_1^1. However, this condition is not necessarily preserved under projection to partial flag varieties with fewer steps. Condition AM1 is better suited for induction. The conditions AM1 and AM2 can be used to bound the number of limits that occur during the degenerations. If in addition to AM1 and AM2 we impose the condition AM1′, then the number of irreducible components of the support of the degenerations we will discuss is bounded by a quadratic function of r (the number of steps in the flag variety). For example, for Grassmannnians there are at most 2 components. For two-step flag varieties there are at most 3 components (see [C2]).

Definition 3.12 (Normalized Mondrian tableau)**.** *A Mondrian tableau M is called* normalized *if M is nested or satisfies the following two properties:*

NM1 *If $S_1^i \neq S_2^i$ are any two squares in M of the same color, then $l(S_1^i) \neq l(S_2^i)$.*

NM2 *If $S_1^i \subsetneq S_2^i$ are two squares of M of the same color i such that $r(S_1^i) = r(S_2^i)$, then every square T of M such that $l(T) \leq l(S_2^i)$ contains S_1^i.*

Definition 3.13. *A Mondrian tableau M is called* nice *if for any $1 \leq i \leq r$ and any two squares $S_{h_1}^i$, $S_{h_2}^i$ of M of color i the following inequality holds:*

$$\#_{i+1}(S_{h_1}^i \cup S_{h_2}^i)(M) - \#_{i+1}S_{h_1}^i(M) \geq \#_i(S_{h_1}^i \cup S_{h_2}^i)(M) - \#_i S_{h_1}^i(M).$$

The translation between geometry and combinatorics is especially nice for nice Mondrian tableaux. Normalized Mondrian tableaux are automatically nice. However, there are nice Mondrian tableaux which are not normalized. Unfortunately, it is not possible to run our algorithm keeping all the Mondrian tableaux nice. (The Algorithm in [C2] uses only normalized Mondrian tableaux. The Algorithm we give in this paper will use more general tableaux. The advantage here is that we reduce the amount of chopping of the squares in the tableaux to a minimum.)

Definition 3.14. *The virtual dimension of a Mondrian tableau is given by the following expression*

$$\sum_{i=1}^{r} \sum_{S_h^i \in M} \left(\#_{i+1} S_h^i(M) - \#_i S_h^i(M) \right).$$

The virtual dimension of a nice Mondrian tableau is called its dimension.

Remark 3.15. The virtual dimension of a nice Mondrian tableau will equal the dimension of the variety associated to it. In general, the virtual dimension of a Mondrian tableau M is greater than or equal to (and can be strictly greater than) the dimension of the variety Σ_M associated to it. We will define a more complicated dimension function that calculates the exact dimension of the variety associated to M in the next section, but we will phrase the combinatorial rule in terms of the virtual dimension.

Let σ_λ^δ be a Schubert cycle in $F(k_1, \ldots, k_r; n)$. The sequence δ determines $r-1$ subsequences $s^1, \ldots, s^{r-1}$, where each s^j is the sequence of entries in δ that are less than or equal to $j+1$ listed in the same order as in δ. Let i_h^j denote the index of the h-th digit less than or equal to j in the sequence s^j. If we would like to emphasize that the subsequence is induced by δ, we will write $i_h^j(\delta)$.

For example, if $\delta = 1,2,3,4,4,3,1,2$, then $s^3 = \delta$, $s^2 = 1,2,3,3,1,2$ and $s^1 = 1,2,1,2$. Moreover, $i^1_1 = 1, i^1_2 = 3, i^2_1 = 1, i^2_2 = 2, i^2_3 = 5, i^2_4 = 6$.

Lemma 3.16. *The tableau associated to a Schubert variety is normalized, admissible and of dimension*

$$\sum_{i=1}^{r} \sum_{S^i_j \in M} \#_{i+1} S^i_j(M) - \sum_{i=1}^{r} \frac{k_i(k_i+1)}{2},$$

which is equal to the dimension of the Schubert variety.

Proof. The tableau is nested, hence automatically normalized and admissible. The dimension of the tableau is immediate from the definition. Note that the codimension of the Schubert variety is

$$\sum_{i=1}^{k_r} \lambda_i + \sum_{s=1}^{r-1} \sum_{h=1}^{k_s} (k_{s+1} - k_s + h - i^s_h).$$

□

Lemma 3.17. *The tableau associated to the intersection of two Schubert varieties*

$$\Sigma^{\delta}_{\lambda} \cap \Sigma^{\kappa}_{\mu}$$

in $F(k_1, \dots, k_r; n)$ described by Algorithm 3.8 is normalized, admissible and of dimension equal to the dimension of the intersection of the two Schubert varieties.

Proof. First, note that the intersection of two Schubert varieties in $F(k_1, \dots, k_r; n)$ is empty if and only if one of the intersections of a pair of squares in the construction is empty. If the intersections of any two squares in the construction is empty, then there exists a pair of squares A^i_j and $B^i_{k_i-j+1}$ that do not intersect. The corresponding vector spaces (spanned by the basis elements in A^i_j and $B^i_{k_i-j+1}$) contain a j and $(k_i - j + 1)$-dimensional subspace of V_i, respectively. Since V_i has dimension k_i, these two subspaces must intersect. If there are no common basis elements in these two squares, the two subspaces are disjoint. We conclude that the intersection of the two Schubert varieties is empty. Conversely, if none of the intersections in the construction are empty, we can explicitly build an r-tuple $(V_1, \dots, V_r)$ contained in the intersection of the two Schubert varieties. Let V_i be the vector space spanned by the basis elements that form the left corner of the squares of color i in the Mondrian tableau. By construction, it is clear that $V_i \subset V_{i+1}$ and that the tuple obtained this way is a point in both Schubert varieties. Hence, the intersection is non-empty. From now on we can assume that the tableau is not discarded in Algorithm 3.8 and that the intersection of the two Schubert varieties is non-empty.

Observe the following properties of the construction:

(1) There are k_i squares of color i and the left and right corners of all squares of color i are distinct. Furthermore, by construction squares of color i are the spans of squares of color $i+1$ contained in them. Hence, M1 and M2 hold and the tableau is normalized.

(2) The squares of color r are intersections of squares of color r which are not chopped. Hence the squares of color r are not chopped. Similarly, the squares of color $i < r$ are intersections of squares of color i which are not chopped. They might have gaps if some of the basis elements do not belong to squares of color $i+1$ contained in the square of color i. Hence, M3 and M4 hold.

(3) If a square of color i is contained in another square of color i, then that square coincides with a square of color $i-1$. Hence, the tableau satisfies AM1 and AM2.

It follows from these observations that the initial tableau corresponding to the intersection of two Schubert varieties is a normalized, admissible Mondrian tableau.

Calculating the dimension is straightforward. The codimension of the intersection of the two Schubert varieties is
$$\sum_{i=1}^{k_r} (\lambda_i + \mu_i) + \sum_{s=1}^{r-1} \sum_{h=1}^{k_s} (2k_{s+1} - 2k_s + 2h - i_h^s(\delta) - i_h^s(\kappa)) .$$
By the definition of the dimension of a Mondrian tableau and by the construction in Algorithm 3.8, one can see that the dimension of $M(\sigma_\lambda^\delta, \sigma_\mu^\kappa)$ is given by
$$k_r(n-k_r) - \sum_{i=1}^{k_r}(\lambda_i + \mu_{k_r - i+1}) + \sum_{c=1}^{r-1}\sum_{h=1}^{k_c} \left(i_h^c(\delta) + i_{k_c-h+1}^c(\kappa) - k_{c+1} - 1\right).$$
Recalling that the dimension of the flag variety is $\sum_{i=1}^r k_i(k_{i+1} - k_i)$, the dimension of $M(\sigma_\lambda^\delta, \sigma_\mu^\kappa)$ can be seen to agree with the dimension of the intersection of the two Schubert varieties. □

Even though the initial tableau associated to the intersection of two Schubert varieties is normalized, it is not possible to keep the tableaux normalized in the intermediate stages of the algorithm. We will have to relax the definition of a normalized tableau.

Definition 3.18 (Semi-normalized Mondrian tableau)**.** *A Mondrian tableau M is called semi-normalized if either M is normalized or M satisfies the following properties:*

- SNM1 *At most two squares of the same color share a left corner.*
- SNM2 *Let $S_1^i \subset S_2^i$ be two squares of M of the same color with $e_j = l(S_1^i) = l(S_2^i)$. If $i = r$, then e_{j+1} is the left corner of a filler in S_1^r. Either S_1^r has a gap or contains a filler F with $r(F) = r(S_1^r)$. Every square T^r of M of color r with $l(T^r) \leq l(S_1^r)$ contains S_1^r. If $i < r$, then the first basis element strictly to the right of e_j which is the left corner of a square of color $i+1$ is also the left corner of a filler of S_1^i.*
- SNM3 *If the right corners of two squares $S_1^i \subset S_2^i$ of M of the same color i are equal, then every square T of M with $l(T) \leq l(S_2^i)$ and $T \not\subset S_2^i$ contains S_1^i.*

3.2. **The combinatorial rule for multiplying Schubert cycles.** In this subsection, we state the combinatorial rule for multiplying Schubert cycles. Given an admissible, semi-normalized Mondrian tableau M, we will move some of the squares in M in a given order and replace M with a set of new Mondrian tableaux. In Rule 3.19 we will specify which squares to move. In Rule 3.21 we will specify how to move the squares. We will then describe a set of Mondrian tableaux that will replace M. The algorithm for computing the product of Schubert cycles will consist of starting with the initial tableau described in Algorithm 3.8 and repeating this move on every Mondrian tableau resulting from this initial tableau until every tableau is the tableau associated to a Schubert cycle. The class of the product is then the sum of the Schubert cycles corresponding to the tableaux that result.

As we will see in the next section, geometrically the move on a Mondrian tableau M corresponds to a one-parameter specialization of the flags defining the variety Σ_M. The algorithm records the irreducible components of the flat limit of this degeneration that have dimension equal to the virtual dimension of M. These irreducible components of the flat limit are again varieties associated to Mondrian tableaux. They each occur with multiplicity one in the limit. Hence, the class of the variety associated to M is the sum of the classes of the varieties associated to the Mondrian tableaux occurring in the algorithm. A very simple geometric criterion allows us to decide which varieties associated to a Mondrian tableau occur in the limit. However, in practice this is cumbersome to use. We will also give a much more efficient version of the rule. Now we will make this precise.

Observe that conditions M2, M3 and M4 of the definition of a Mondrian tableau imply that the collection of squares of the same color that have the same left corner is totally ordered by inclusion.

Rule 3.19 (The square to move). Let M be a semi-normalized Mondrian tableau for $F(k_1, \dots, k_r; n)$ which is not nested. Let e_{j_1} be the smallest index basis element that is the left corner of an unnested square of M of color r. Let T_1^r be the smallest square of M of color r with $l(T_1^r) = e_{j_1}$. Define T_i^r

inductively as follows. As long as T^r_{i-1} strictly contains any squares, let e_{j_i} be the least index basis element that is the left corner of a square strictly contained in T^r_{i-1}. Let T^r_i be the smallest square of M of color r strictly contained in T^r_{i-1} with $l(T^r_i) = e_{j_i}$. If none of the squares T^r_i are chopped, set T^r_h to be the square among T^r_i with least index satisfying the following properties:

(1) The collection of squares contained in T^r_h is totally ordered.
(2) There are no fillers in T^r_h.

Set the square to move $S^r = T^r_h$.

If one of the T^r_i is chopped, let T^r_m be the square with largest index among T^r_i which is chopped. Set the square to move $S^r = T^r_m$.

Remark 3.20. Since the tableau is not nested, there exists an unnested square of color r. Hence T^r_1 exists. Since T^r_1 contains finitely many squares of color r, the collection of squares T^r_i is finite. Among them there must be a square T^r_l that does not strictly contain any squares of M (by property M2, once a square does not strictly contain any squares of color r, then it does not strictly contain any squares). Clearly, there are no fillers in T^r_l and the collection of squares contained in T^r_l is totally ordered. It follows that T^r_h exists and S^r is well-defined.

Rule 3.21 (The way to move squares). Let M be a Mondrian tableau which is not nested. Let S^r be the square of M determined by Rule 3.19. Let the left chop of S be $lch(S) = \{e_p, \dots, e_q\}$. Then slide $lch(S^r)$ diagonally left by one unit (i.e., replace it by $\{e_{p+1}, \dots, e_{q+1}\}$). If there are any squares S' (of any color) of M that contain S^r and have a chop whose right corner is e_q, chop S' at e_p and slide the chop of S' that coincides with $lch(S)$ diagonally left by one unit as well. Keep all other squares in M unchanged.

Definition 3.22. *Let the left chop of a square S^r of a Mondrian tableau M be $lch(S^r) = \{e_p, \dots, e_q\}$. A* neighbor *$N^j$ of color j of a square $\tilde{S}^j$ of M with respect to S^r is a square of M of color j such that*

(1) *$e_{q+1} \in N^j$.*
(2) *N^j does not contain $\tilde{S}^j$.*
(3) *If there exists a square S' of M of color j such that $l(\tilde{S}^j) < l(S') < l(N^j)$, then either $S' \subset \tilde{S}^j$ or $N^j \subset S'$.*

We say two vector spaces spanned by basis elements in $\tilde{S}^j$ and in a neighbor N^j of $\tilde{S}^j$ are neighboring vector spaces.

Algorithm 3.23 (Normalization of right corners). Let M be a Mondrian tableau. Let $S^i_1 \subset S^i_2$ with $r(S^i_1) = r(S^i_2)$ be two squares of M that violate Condition NM2. If $i = r$, replace S^r_2 by the square $S^r_2 - r(S^r_2)$ of color r. If $i < r$, let e_j be the basis element with highest index strictly to the left of $r(S^i_2)$ such that e_j is the right corner of a square of color $i+1$ contained in S^i_2. If there does not exist such a basis element, let $e_j = l(S^i_2)$. Replace S^i_2 with the square $S^i_2 - \{e_{j+1}, e_{j+2}, \dots, r(S^i_2)\}$ of color i. If these operations result in a square without any basis elements, discard M. Call this process the *normalization of the right corner of S^i_2 with respect to S^i_1.*

Let T^j_1, T^j_2 be two squares of M of color j with $r(T^j_1) = r(T^j_2) = e_u$. For the purposes of the next algorithm, let $T^j_1 < T^j_2$ when the number of basis elements in T^j_1 is less than the number of basis elements in T^j_2 or, in case of equality, when $l(T^j_1) > l(T^j_2)$. Note that this is a total order on squares of color j whose right corner is e_u. We can then order pairs of squares $S^j_1 \subset S^j_2$ of squares of M with $r(S^j_1) = r(S^j_2) = e_u$ lexicographically first by S^j_1 then by S^j_2.

Given a Mondrian tableau M that fails to satisfy Condition NM2, *normalize the right corners of M up to color i* as follows.

Step 1. Let $j \geq i$ be the largest color such that M has squares that fail Condition NM2. If M does not have such squares, the process terminates. Otherwise, let e_u be the largest index basis element which is the right corner of two squares of color j that violate Condition NM2 and proceed to the next step.

Step 2. Let $S_1^j \subset S_2^j$ be the smallest two squares of M with $r(S_1^j) = r(S_2^j) = e_u$ that violate Condition NM2. Let M_1 be the tableau obtained by normalizing the right corner of S_2^j with respect to S_1^j. If the tableau is discarded, the algorithm terminates. Otherwise, let $M = M_1$ and return to Step 1.

Algorithm 3.24 (Semi-normalization of left corners). Let M be a Mondrian tableau. Let S_1^i and S_2^i be two squares of M of the same color such that $l(S_1^i) = l(S_2^i)$. First suppose $i = r$. If the left corner of the square $S_2^r - l(S_2^r)$ is not the left corner of a filler of S_1^r, replace S_2^r with the square $S_2^r - l(S_2^r)$ of color r. Otherwise, leave M unchanged. Now suppose $i < r$. Let e_j be the smallest index basis element strictly to the right of $l(S_2^i)$ such that e_j is the left corner of a square of color $i+1$ contained in S_2^i. If there does not exist such a basis element, let $e_j = r(S_2^i)$. If e_j is not the left corner of a filler of S_1^i, replace S_2^i with the square $S_2^i - \{l(S_2^i), \dots, e_{j-1}\}$. Otherwise, leave M unchanged. If these operations result in a square without basis elements, discard M. Call this process the semi-normalization of the left corner of S_2^i with respect to S_1^i.

Given a Mondrian tableau M that violates Condition SNM2, *semi-normalize the left corners of M up to color i* as follows:

Step 1. Let $j \geq i$ be the largest color such that M has two squares of color j violating SNM2. If M does not have such squares, the algorithm terminates. Otherwise, let e_u be the smallest index basis element which is the left corner of two squares of color j that violate SNM2 and proceed to the next step.

Step 2. Let $S_1^j \subset S_2^j$ be the smallest two squares with $l(S_1^j) = l(S_2^j) = e_u$ that violate SNM2. Let M_1 be the tableau obtained by by semi-normalizing the left corner of S_2^j with respect to S_1^j. If the tableau is discarded, the algorithm terminates. Otherwise, let $M = M_1$ and return to Step 1.

We will first state a version of the rule that is hard to use in practice, but highlights the principle behind the rule. We first need some terminology.

Notation 3.25. Let M be a Mondrian tableau. Given a square S^i of M denote by $S^i(1)$ the square S^i in M and by $S^i(0)$ the square after the squares of M have been moved according to Rule 3.21. Similarly, let $S_{j_1}^i, \dots, S_{j_t}^i$ be any subset of the squares of color i of M. Denote by $S_{j_1,\dots,j_t}^i(1)$ the square formed by the union of the basis elements contained in these squares. Note that this square is not in general a square of M. Suppose the lower-left chop of the square determined by Rule 3.19 is $\{e_p, \dots, e_q\}$. If one of the squares $S_{j_h}^i(1)$ contains e_p and another one $S_{j_l}^i(1)$ contains e_{q+1}, let $S_{j_1,\dots,j_t}^i(0)$ denote the union of e_p and the basis elements contained in the squares $S_{j_1}^i(0), \dots, S_{j_t}^i(0)$. Otherwise, let $S_{j_1,\dots,j_t}^i(0)$ denote the union of the basis elements contained in $S_{j_1}^i(0), \dots, S_{j_t}^i(0)$.

Definition 3.26. *Let M be a Mondrian tableau. There is a total ordering on the squares $S_{j_1,\dots,j_t}^i(1)$ of the same color i. A square $S_{j_1,\dots,j_t}^i(1) < S_{l_1,\dots,l_m}^i(1)$ if $\#_i S_{j_1,\dots,j_t}^i(M) < \#_i S_{l_1,\dots,l_m}^i(M)$. If $\#_i S_{j_1,\dots,j_t}^i(M) = \#_i S_{l_1,\dots,l_m}^i(M)$, then the squares are ordered lexicographically according to the basis elements they contain. The squares $S_{j_1,\dots,j_t}^i(0)$ are ordered in the same order that the squares $S_{j_1,\dots,j_t}^i(1)$ are.*

Algorithm 3.27 (An impractical rule.). Let M be a semi-normalized, admissible Mondrian tableau. Let $M(0)$ denote the collection of squares obtained after the squares of M have been moved according to Rule 3.21. Let

$$Q_r = \{T_1^r, \dots, T_{j_r}^r\}$$

be a set (possibly empty) of squares of color r, where $j_r \leq k_r$ and each T_h^r is the intersection of squares $S_{m_1}^r(0) \cap \dots \cap S_{m_h}^r(0)$ of $M(0)$ and there are no containment relations among the squares T_h^r. For each such set Q_r (including the empty set), build a set of squares $R_{Q_r}^r$ of color r of cardinality k_r as follows.

Take all the squares in Q_r and all the squares of $M(0)$ of color r that do not contain any of the squares T_h^r in Q_r. Normalize the upper-right corners of these squares up to color r. For simplicity, even if a square shrinks during the process, keep its label the same. Proceeding in increasing order according to the ordering of the squares, add all the squares $S^r_{l_1,\dots,l_m}(0)$ that have fewer than $\#_r(S^r_{l_1,\dots,l_m}(1))(M)$ squares of color r contained in them. For each of the set of black squares $R^r_{Q_r}$ thus obtained consider all the sets Q_{r-1} consisting of squares of color $r-1$

$$Q_{r-1} = \{T_1^{r-1}, \dots, T_{j_{r-1}}^{r-1}\}$$

where $j_{r-1} \leq k_{r-1}$ and each square T_h^{r-1} contained in Q_{r-1} is

(1) contained in the intersection of squares $S^{r-1}_{m_1}(0) \cap \cdots \cap S^{r-1}_{m_h}(0)$ of color $r-1$ of $M(0)$;
(2) coincides with the span of squares of color r in R_{Q_r};
(3) does not contain or is not contained in any of the other squares $T^{r-1}_{h'}$ in Q_{r-1} for $h \neq h'$.

For every such set Q_{r-1} (including the empty set) of squares of color $r-1$ form a set $R^{r-1}_{Q_{r-1}}$ of squares of color $r-1$ of cardinality k_{r-1} as follows. Take all the squares in Q_{r-1} and all the squares of color $r-1$ in $M(0)$ that do not contain any of the squares T_h^{r-1} in Q_{r-1}. Shrink these squares so that they are the spans of the squares of color r in $R^r_{Q_r}$ contained in them. Normalize the upper-right corners of these squares up to color $r-1$. Proceeding in increasing order according to the ordering of the squares, add all the squares

$$S^{r-1}_{l_1,\dots,l_m}(0)$$

that contain fewer than $\#_{r-1}(S^{r-1}_{l_1,\dots,l_m}(1))(M)$ squares of color $r-1$. Shrink all the squares of color $r-1$, so that they are the spans of the squares of color r in $R^r_{Q_r}$ contained in them. Normalize the upper-right corners of the squares up to color $r-1$. If the side-length of any square shrinks to zero discard that set of squares. Continue building these sets of squares inductively in decreasing order. Suppose we have built sets of squares of colors $r, r-1, \dots, i+1$. For every choice $R^{i+1}_{Q_{i+1}}, \dots, R^r_{Q_r}$ consider any set

$$Q_i = \{T_1^i, \dots, T_{j_i}^i\}$$

of squares of color i where $j_i \leq k_i$ and each square T_h^i contained in Q_i is

(1) contained in the intersection of squares $S^i_{m_1}(0) \cap \cdots \cap S^i_{m_h}(0)$ of color i of $M(0)$;
(2) coincides with the span of squares of color $i+1$ in $R_{Q_{i+1}}$ contained in it;
(3) does not contain or is not contained in any of the other squares $T^i_{h'}$ in Q_i for $h \neq h'$.

Form a set $R^i_{Q_i}$ of squares of color i as follows. Take all the squares in Q_i and all the squares in $M(0)$ of color i that do not contain any of the squares T_h^i in Q_i. Shrink these squares so that they are the spans of squares of color $i+1$ in $R^{i+1}_{Q_{i+1}}$ contained in them. Normalize the upper-right corners of these squares up to color i. Proceeding in increasing order, add the squares $S^i_{l_1,\dots,l_m}(0)$ that have fewer than $\#_i(S^i_{l_1,\dots,l_m}(1))(M)$ squares of color i to $R^i_{Q_i}$. Shrink the resulting squares of color i so that they are the spans of squares of color $i+1$ in $R^{i+1}_{Q_{i+1}}$ contained in them. Normalize the upper-right corners of the squares up to color i. If the side-length of any square shrinks to zero discard that tableau. Continuing we end up with a collection of sets of squares $R^1_{Q_1}, \dots, R^r_{Q_r}$ for every choice of $Q_1, \dots, Q_r$ that are not discarded. Form a tableau consisting of squares of color $1, \dots, r$ produced in this way. Semi-normalize the lower-left corners of the resulting tableau and proceeding from $r-1$ to 1 shrink any square of color i which is not equal to the span of squares of color $i+1$ contained in it so that it is. In this way we obtain a collection of semi-normalized Mondrian tableaux. Replace M with the set of distinct tableaux produced by this procedure and that have the same virtual dimension as the virtual dimension of M. If there are none, simply discard M.

Remark 3.28. It can happen that the virtual dimension of every tableaux produced by the above algorithm is smaller than the virtual dimension of M. It is possible to remedy this situation by introducing a more refined dimension function which computes the actual dimension of the variety associated to M rather than an upper bound on it. We will introduce this function in the next section when we discuss the geometry of Mondrian tableaux. Since this is harmless for calculating the structure constants of the cohomology of flag varieties and the refined dimension function is harder to compute, we postpone introducing it. If M is nice, at least one of the Mondrian tableaux produced by the algorithm will have the same virtual dimension as M.

Let Σ_M be the variety associated to M. Corresponding to the move of Mondrian tableaux, there is a one-parameter family of varieties whose general member is isomorphic to Σ_M. The main claim of this paper is that the locus that supports the components of the flat limit of this degeneration, which have dimension equal to the virtual dimension of M, is equal to the union of the varieties associated to each of the tableaux described in Algorithm 3.27 that have dimension equal to the virtual dimension of M. Furthermore, the flat limit is generically reduced along these components. Hence the class of the variety associated to M is a sum of the classes of the varieties associated to the tableaux that have the same dimension as the variety associated to M. Finally, we claim that those that do have the same virtual dimension are admissible, semi-normalized Mondrian tableaux. Hence, the class of the intersection of two Schubert varieties can be inductively determined. The problem with this formulation is, of course, that this algorithm is very impractical. Very few of the tableaux produced in Algorithm 3.27 will have the same virtual dimension as M. One can narrow down the list of candidates considerably. This allows one to obtain an algorithm that one can easily carry out in practice.

Definition 3.29. *We will say that a square S^i is a minimal square with property P if S^i has property P and none of the squares strictly contained in S^i have property P.*

Now we describe a more streamlined set of tableaux that we will replace M with. Let

$$U_1^{i_1}, \dots, U_{h_r}^{r}$$

be the minimal squares (with respect to inclusion) of colors $i_1, i_1+1, \dots, r$, respectively, that are moved according to Rule 3.21. Let $M(0)$ be the tableau obtained after the squares are moved according to Rule 3.21. Define the following tableaux.

Tableau M_0: Starting with squares of color $r-1$ and proceeding in descending order, if a square in $M(0)$ of color i is not the span of squares of color $i+1$ contained in it, shrink that square so that it is the span of squares of color $i+1$ contained in it. Normalize the upper-right corners of the squares in the resulting tableau. Semi-normalize the lower-left corners of the squares of the resulting tableau and proceeding from $r-1$ to 1 shrink any square of color i which is not equal to the span of squares of color $i+1$ contained in it so that it is. Figure 6 shows some examples.

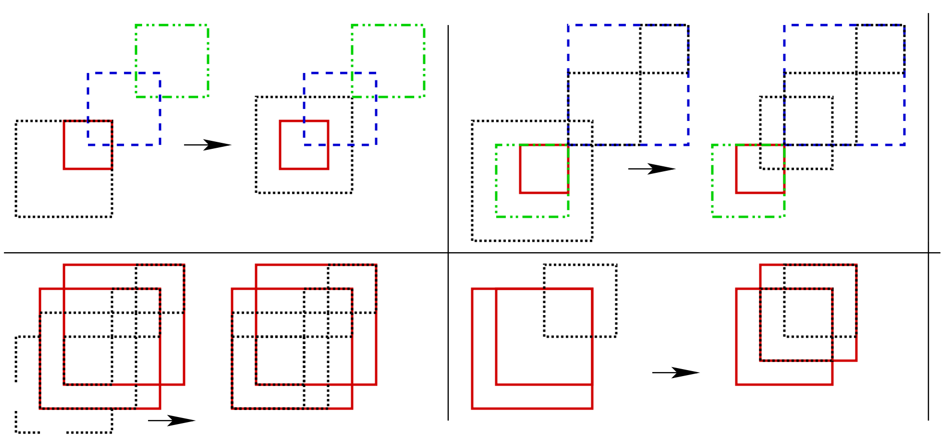

FIGURE 6. Some examples of Tableau M_0.

Tableau $M_1(U_h^j N_s^j)$**:** Let N_s^j be a neighbor of U_h^j of color j with respect to the square determined by Rule 3.19. For $i \geq j$, let $T^i = U_h^j(0) \cap N_s^j(0)$ be the square coinciding with the intersection of $U_h^j(0)$ and $N_s^j(0)$ in color i. Let $M_1(U_h^j N_s^j)$ be the Mondrian tableau whose squares are determined as follows: Starting with r and proceeding in descending order form the following sets $M_1(U_h^j N_s^j)^i$ of squares of color i. If $i \geq j$ take T^i and any square of color i of $M(0)$ that does not contain T^i. Shrink these squares so that they are the spans of the squares of color $i+1$ in $M_1(U_h^j N_s^j)^{i+1}$. Normalize their upper-right corners up to color i. Proceeding in increasing order (according to the ordering of the squares) add all the squares $S^i_{h_1,\dots,h_j}(0)$ that at a given stage have fewer than $\#_i S^i_{h_1,\dots,h_j}(1)$ squares of color i contained in it. Shrink these squares of color i so that they are the spans of the squares of color $i+1$ in $M_1(U_h^j N_s^j)^{i+1}$ contained in them and normalize their upper-right corners. Let the resulting set of squares be $M_1(U_h^j N_s^j)^i$. Now suppose $i < j$. Shrink the squares of color i of $M(0)$ so that they are the spans of the squares in $M_1(U_h^j N_s^j)^{i+1}$ that they contain. Normalize the upper-right corners of the resulting squares up to color i. We obtain in this way a collection of squares of colors $1, \dots, r$. Semi-normalize the lower-left corners of the squares in the resulting tableau and proceeding from $r-1$ to 1 shrink any square of color i which is not equal to the span of squares of color $i+1$ contained in it so that it is. If the side-length of any of the squares shrink to zero during any part of this process discard this tableau. Otherwise, call the resulting tableau $M_1(U_h^j N_s^j)$. Figure 7 depicts some examples.

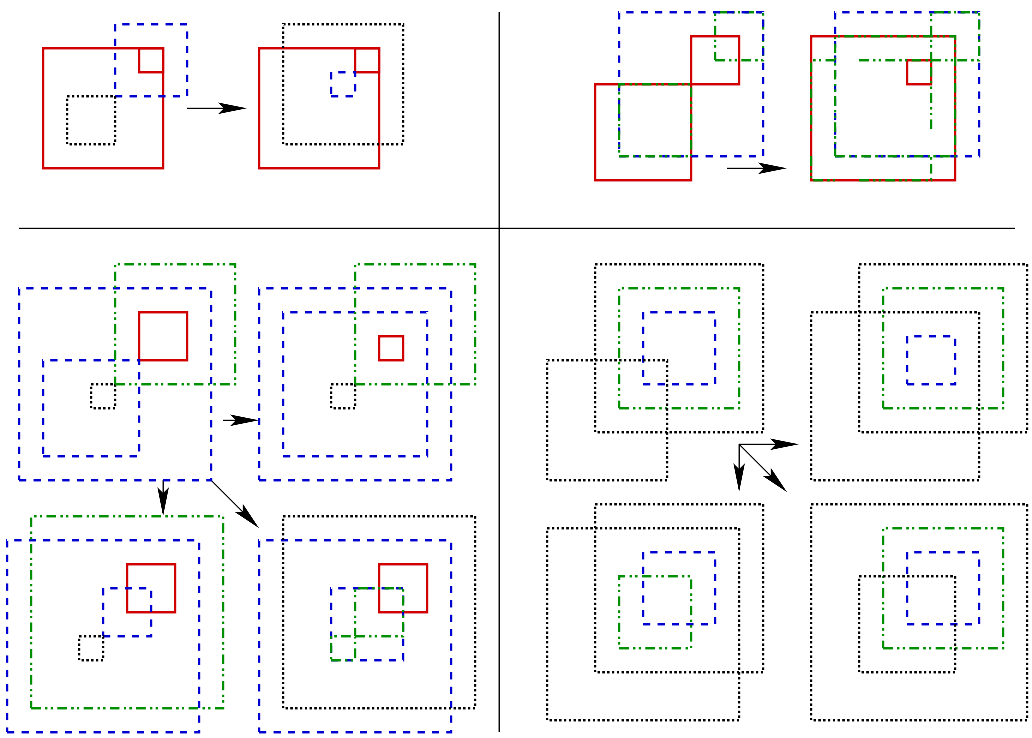

FIGURE 7. Some examples of Tableau M_1.

Remark 3.30. We can say very explicitly which squares of $M(0)$ are modified while drawing Tableau M_1. When drawing the squares of color i for $i \geq j$, form T^i. Normalize the upper-right corners of T^i and the squares of $M(0)$ of color i that do not contain T^i as described above. Label by T^{i*} the square that T^i transforms to under this normalization procedure. Let $B_1^i, B_2^i, \dots, B_s^i$ be the squares of $M(0)$ of color i that minimally contain T^{i*} ordered according to their lower-left corners from southwest to northeast. Replace the squares $B_1^i, \dots, B_s^i$ of $M(0)$ with T^{i*} and $B^i_{1,2}(0), B^i_{2,3}(0), \dots, B^i_{h,h+1}(0), \dots, B^i_{s-1,s}(0)$ in color i. Keep all the other squares of $M(0)$ of color i unchanged. Shrink all the squares of color i so that they are the spans of squares of color $i+1$ contained in them and normalize their upper-right corners. A similar description applies to the tableaux of type M_1' we now describe.

Tableau $M_1'(U_h^j N_s^j)$**:** Let N_s^j be a neighbor of U_h^j of color j with respect to the square determined by Rule 3.19. For $i \geq j$, let $T^i = U_h^j(0) \cap N_s^j(0)$ be the square coinciding with the intersection of $U_h^j(0)$ and $N_s^j(0)$ in color i. Let $M_1'(U_h^j N_s^j)$ be the Mondrian tableau whose squares are determined as follows: Starting with r and proceeding in descending order form the following sets $M_1'(U_h^j N_s^j)^i$ of squares of color i. If $i \geq j$, take T^i and any square of color i of $M(0)$ that does not contain T^i. Shrink these squares so that they are the spans of the squares of color $i+1$ in $M_1'(U_h^j N_s^j)^{i+1}$. Normalize their upper-right corners up to color i. Denote by T^{i*} the square that T^i transforms to under this procedure. Proceeding in increasing order (according to the ordering of the squares) add all the squares $S^i_{h_1,\dots,h_j}(0)$ that at a given stage have fewer than $\#_i S^i_{h_1,\dots,h_j}(1)$ squares of color i contained in it. Shrink the squares of color i so that they are the spans of the squares of color $i+1$ in $M_1'(U_h^j N_s^j)^{i+1}$ contained in them. Let the resulting set of squares be $M_1'(U_h^j N_s^j)^i$. Now suppose $i < j$. If any of the squares of $M(0)$ of color i that minimally contain T^{j*} is not the span of the squares of color $i+1$ of $M_1'(U_h^j N_s^j)^{i+1}$ contained in it, shrink that square so that it is. Take the resulting squares and any square of color i of $M(0)$ that does not contain T^{j*}. Normalize their upper-right corners up to color i. Proceeding in increasing order add all the squares $S^i_{h_1,\dots,h_j}(0)$ that have fewer than $\#_i S^i_{h_1,\dots,h_j}(1)$ squares of color i contained in it. Shrink the squares of color i so that they are the spans of squares of color $i+1$ in $M_1'(U_j N_s^j)^{i+1}$ contained in them and normalize their upper-right corners. Form the tableau consisting of the squares $M_1'(U_h^j N_s^j)^i$ for $i = 1, \dots, r$. Semi-normalize the lower-left corners of the squares in the resulting tableau and proceeding from $r-1$ to 1 shrink any square of color i which is not equal to the span of squares of color $i+1$ contained in it so that it is. If the side-length of any of the squares shrink to zero during any part of this process, discard this tableau. Otherwise, call the resulting tableau $M_1'(U_h^j N_s^j)$. Figure 8 shows some examples.

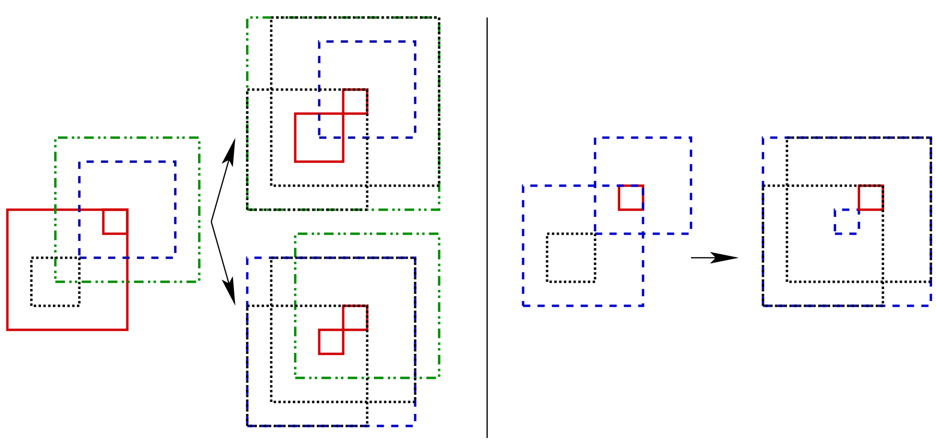

FIGURE 8. Some examples of Tableau M_1'.

Tableau $M_2(U_h^j N_s^{<j})$**:** Suppose that a neighbor $N_s^j = A_1^j$ of color j of U_h^j with southwest most lower-left corner coincides with a square of color $j-1$. Let $A_2^j(0)$ be the minimal square of $M(0)$ satisfying the properties:

1. $A_2^j(0)$ is not contained in A_1^j.
2. $l(A_2^j(0)) \geq l(A_1^j)$.

(3) $A^j_2(0)$ is the minimal square among the squares with southwest most lower-left corner among the squares satisfying the previous two properties.

Suppose $A^j_h(0)$ coincides with a square of color $j-1$. Let $A^j_{h+1}(0)$ be the square of color j satisfying

(1) $A^j_{h+1}(0)$ is not contained in $A^j_h(0)$.
(2) $l(A^j_{h+1}(0)) \geq l(A^j_h(0))$.
(3) $A^j_{h+1}(0)$ is the minimal square among the squares with southwest most lower-left corner among the squares satisfying the previous two properties.

Suppose $A^j_l(0)$ does not coincide with a square of color $j-1$. (If there are no such squares, discard this tableau.) Let $A^j_1(0), \ldots, A^j_l(0)$ be the squares produced by this procedure. Let $B^j_{m,m+1} = A^j_m(0) \cap A^j_{m+1}(0)$. Set $B^j_{0,1} = U^j_h \cap A^j_1(0)$. If any of the intersections $B^j_{m,m+1}$ is empty discard this tableau. Otherwise form the tableau $M_2(U^j_h N^{<j}_s)$ as follows. Proceeding in descending order form the set of squares $M_2(U^j_h N^{<j}_s)^i$ of color i as follows. If $i \geq j$, let $B^i_{m,m+1} = A^j_m(0) \cap A^j_{m+1}(0)$ be the square of color i which coincides with the intersection of $A^j_m(0)$ and $A^j_{m+1}(0)$. Take the squares $B^i_{m,m+1}$ for $m = 0, 1, \ldots, l-1$ and any of the squares that do not contain any of the squares $B^i_{m,m+1}$. Shrink them so that they are the spans of squares of color $i+1$ of $M_2(U^j_h N^{<j}_s)^{i+1}$ contained in them and normalize their upper-right corners. Proceeding in increasing order (according to the ordering of the squares) add all the squares $S^i_{h_1,\ldots,h_j}(0)$ that have fewer than $\#_i S^i_{h_1,\ldots,h_j}(1)$ squares of color i contained in them at the given stage. Shrink the squares of color i so that they are the spans of the squares of color $i+1$ in $M_2(U^j_h N^{<j}_s)^{i+1}$ contained in them. Normalize their upper-right corners up to color i. If $i < j$, shrink the squares of color i so that they are the spans of the squares of color $i+1$ in $M_2(U^j_h N^{<j}_s)^{i+1}$ contained in them. Normalize the upper-right corners of the resulting squares up to color i. Form the tableau consisting of the squares $M_2(U^j_h N^{<j}_s)^i$ for $i = 1, \ldots, r$. Semi-normalize the lower-left corners of the squares in the resulting tableau and proceeding from $r-1$ to 1 shrink any square of color i which is not equal to the span of squares of color $i+1$ contained in it so that it is. If the side-length of any of the squares shrink to zero during any part of this process, discard this tableau. Otherwise, call the resulting tableau $M_2(U^j_h N^{<j}_s)$. Figure 9 depicts some examples.

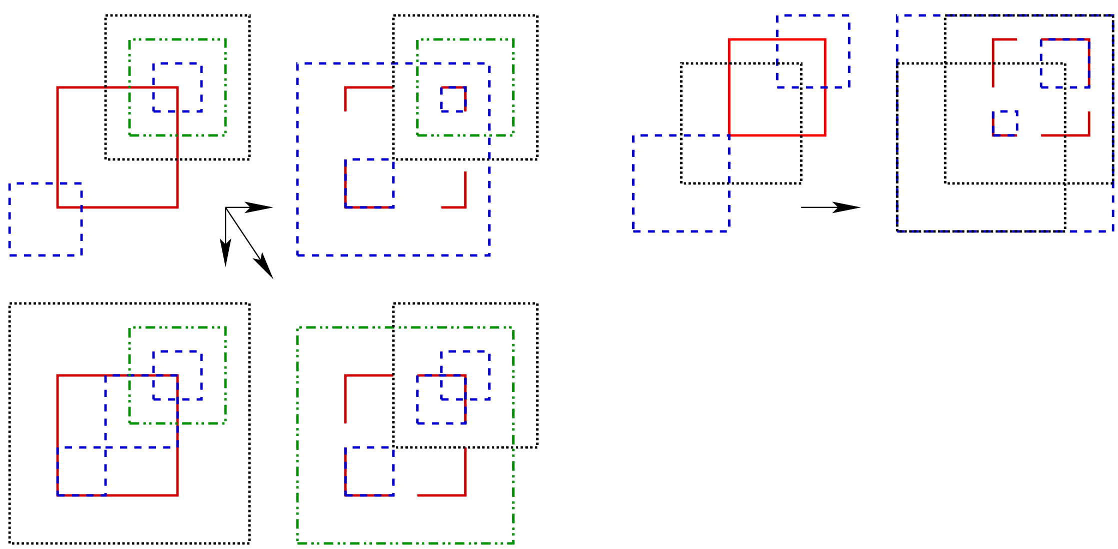

FIGURE 9. Some examples of Tableau M_2.

Remark 3.31. We can be more explicit about the squares that constitute $M_2(U_h^j N_s^{<j})$. Let $A_1^j(0), \dots, A_l^j(0)$ be the squares as above. Let $B_{m,m+1}^j = A_m^j(0) \cap A_{m+1}^j(0)$. If any of the intersections $B_{m,m+1}^j$ is empty discard this tableau. For $i > j$, let the squares $C_1^i(0), \dots, C_{h_i}^i(0)$ be the minimal squares of color i of $M(0)$ that contain at least one $B_{m,m+1}^j$. Suppose $i_1, i_1+1, \dots, i_1+s_1, i_2, \dots, i_2+s_2, \dots, i_q, \dots, i_q+s_q$ are the indices of the squares among the squares $C_t^i(0)$ that coincide with a square $A_p^j(0)$ for some p. Let the square $C_{t,t+1}^i(1)$ be the spans of the squares $C_t^i(1)$ and $C_{t+1}^i(1)$ if neither t nor $t+1$ is one of the indices corresponding to the squares that coincide with the squares A_p^j. Let $C_{t,t+1=i_u}^i(1)$ be the span of the squares $C_t^i(1), C_{i_u}^i(1), C_{i_u+1}^i(1), \dots, C_{i_u+s_u}^i(1)$. Similarly, let $C_{i_u,i_u+s_u+1}^i(1)$ be the span of the squares $C_{i_u}^i(1), C_{i_u+1}^i(1), \dots, C_{i_u+s_u}^i(1)$ and $C_{i_u+s_u+1}^i(1)$. Replace the squares $A_1^j, \dots, A_l^j$ in $M(0)$ with the squares $B_{1,2}^j, \dots, B_{l-1,l}^j$ and the span of $U_j(1), A_1^j, A_2^j, \dots, A_l^j$. For each $i > j$, replace the squares $C_1^i(0), \dots, C_{h_i}^i(0)$ with the squares $B_{1,2}^i, \dots, B_{l-1,l}^i$ in color i and the squares $C_{t,t+1}^i(0)$. If any square of color c is not the span of squares of color $c+1$ contained in it, shrink the square so that it is the span of the squares of color $c+1$ contained in it. Normalize the upper-right hand corners of the resulting tableau. Semi-normalize the lower-left hand corners of the the resulting tableau. If any of the side-length of any of the squares shrink to zero during the process, discard this tableau. Otherwise, label it $M_2(U_h^j N_s^{<j})$.

Remark 3.32. Geometrically the tableaux of type M_0, M_1, M_1' and M_2 correspond to the following cases. In Tableau M_0 the limits of the subspaces of $V_1, \dots, V_r$ contained in the vector spaces corresponding to the squares of the tableaux remain independent. In tableaux of type M_1, M_1' and M_2 the limits of the subspaces of V_j contained in the vector spaces corresponding to U^j and N_s^j become dependent. This forces the subspaces of V_i for $i > j$ contained in these vector spaces to also become dependent in the limit. For $i < j$, the subspaces of V_i contained in the vector spaces corresponding to the minimal squares containing the new intersection may either become dependent or remain independent. Tableaux of type M_1' corresponds to the former case, whereas tableaux of type M_1 and M_2 correspond to the latter. The other intersections in tableaux of type M_2 are forced by the linear algebra fact that in a k-dimensional vector space two subspaces of dimensions l and m intersect in at least a $(l+m-k)$-dimensional subspace. In the limit the vector space corresponding to the square N_s^{j-1} intersects V_j in one more dimension than it previously did. This forces the limiting linear spaces to intersect as in tableaux of type M_2.

The following proposition is straightforward to check.

Proposition 3.33. *The virtual dimension of each of the tableaux $M_0, M_1(U_h^j, N_s^j), M_1'(U_h^j, N_s^j)$ and $M_2(U_h^j, N_s^{<j})$ described above is less than or equal to the virtual dimension of M.*

Remark 3.34. The cases when the tableaux have virtual dimension strictly smaller than that of M can be determined in terms of the combinatorial properties of M. When the flag variety is a two-step flag variety, this analysis has been carried out completely in [C2]. For flag varieties with three or more steps, this is still straightforward. However, remembering the exceptions is more cumbersome than calculating the dimension at each stage. We will, therefore, leave this analysis to the reader.

Algorithm 3.35. Let M be a semi-normalized, admissible Mondrian tableau. If M is nested, the algorithm terminates. Otherwise, let S^r be the square of M determined by Rule 3.19. Move S^r according to Rule 3.21. Let

$$U_1^i, \dots, U_{w_r}^r = S^r$$

be the minimal squares (with respect to inclusion) of each color that have a part that moves according to Rule 3.21. For all the squares $U_{h_c}^c$ with $c \geq i$ and for every neighbor of them relative to S^r form the tableaux $M_0, M_1(U_{h_c}^c N_s^c), M_1'(U_{h_c}^c N_s^c), M_2(U_{h_c}^c N_s^{<c})$ whenever appropriate. Replace M with the set of distinct tableaux among these that have the same virtual dimension as M.

Figure 10 depicts an example of Algorithm 3.35.

The following proposition is straightforward to verify.

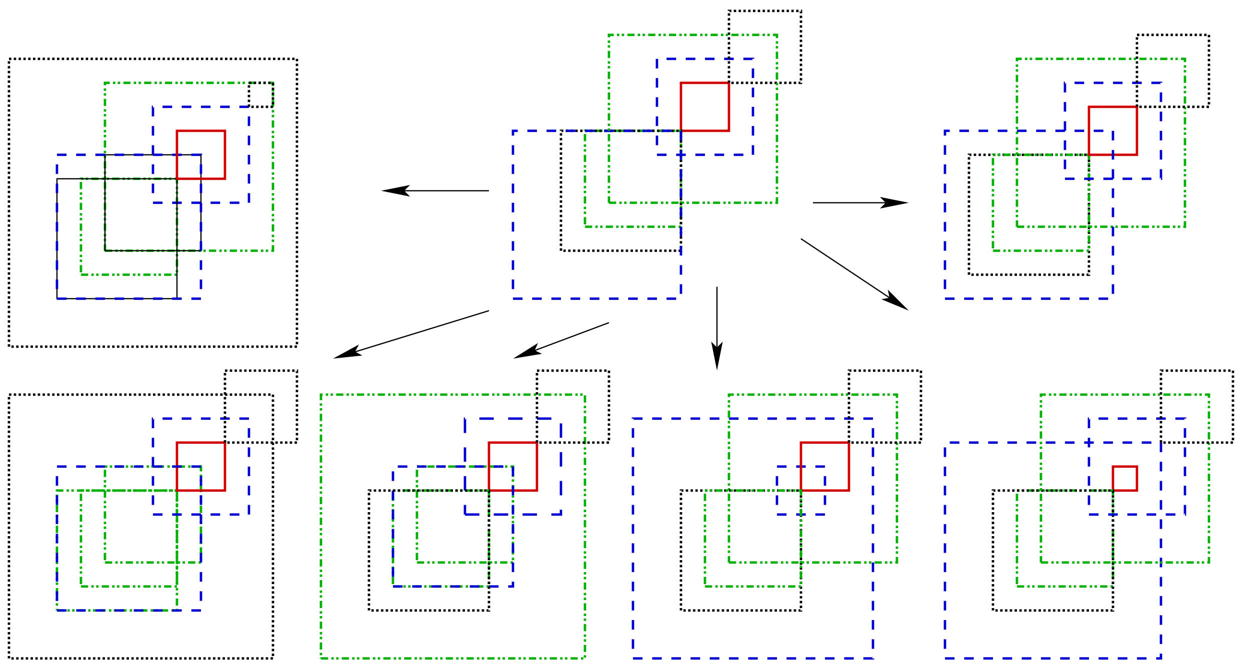

FIGURE 10. An example of Algorithm 3.35.

Proposition 3.36. *Let M be a semi-normalized, admissible Mondrian tableau for $F(k_1, \ldots, k_r; n)$. Any tableau produced when applying Algorithm 3.35 to M is an admissible, semi-normalized Mondrian tableaux for $F(k_1, \ldots, k_r; n)$.*

We can now inductively determine the class of the product of two Schubert varieties.

Algorithm 3.37 (Algorithm for determining the structure constants of the cohomology of partial flag varieties). Let σ_λ^δ and σ_μ^κ be two Schubert cycles in $F(k_1, \ldots, k_r; n)$.

Step 1: Form the tableau $M(\sigma_\lambda^\delta, \sigma_\mu^\kappa)$ according to Algorithm 3.8. If the product is zero, stop. Otherwise, set $M = M(\sigma_\lambda^\delta, \sigma_\mu^\kappa)$ and proceed to Step 2.

Step 2: If M is nested, stop. Otherwise, apply Algorithm 3.35 to M. If none of the tableaux produced by Algorithm 3.35 have the same virtual dimension as M, stop. Otherwise proceed to Step 3.

Step 3: For every tableau produced in Step 2, return to Step 2 and run the Algorithm again.

Definition 3.38. *A degeneration path for an admissible, semi-normalized Mondrian tableau M is a sequence of Mondrian tableaux*

$$M^1 \to M^2 \to \cdots \to M^p$$

where $M^1 = M$, M^p is a Mondrian tableau associated to a Schubert variety and for every $1 < i \leq p$ the tableau M^i is a tableau in the set of tableaux that replaces M^{i-1} in Algorithm 3.35.

It is easy to see that Algorithm 3.37 terminates with the tableaux associated to Schubert varieties. The main theorem of this paper is the following.

Theorem 3.39. *Let σ_λ^δ and σ_μ^κ be two Schubert cycles for $F(k_1, \ldots, k_r; n)$. Let $\sigma_\lambda^\delta \cdot \sigma_\mu^\kappa = \sum c_{\lambda,\delta,\mu,\kappa}^{\nu,\alpha} \sigma_\nu^\alpha$ be their product in the cohomology ring of $F(k_1, \ldots, k_r; n)$. Then the coefficient $c_{\lambda,\delta,\mu,\kappa}^{\nu,\alpha}$ is equal to the number of degeneration paths starting with $M(\sigma_\lambda^\delta, \sigma_\mu^\kappa)$ in an $(n \times n)$-grid and ending in a Mondrian tableau associated to σ_ν^α produced by the Algorithm 3.37.*

The proof of this theorem will be by interpreting the moves on the Mondrian tableaux as degenerations. We will determine the flat limit of the degeneration by induction on the number of steps r.

Let $k_{j_1} < \cdots < k_{j_s}$ be a subsequence of the sequence $k_1 < \cdots < k_r$. Then there is a natural projection

$$\pi_{j_1,\ldots,j_s} : F(k_1, \ldots, k_r; n) \to F(k_{j_1}, \ldots, k_{j_s})$$

sending the r-tuple $(V_1, \ldots, V_r)$ to the s-tuple $(V_{j_1}, \ldots, V_{j_s})$. We would like to associate a Mondrian tableau for $F(k_{j_1}, \ldots, k_{j_s})$ to the image of the projection of a variety associated to a Mondrian tableau. This is easy to do.

Algorithm 3.40. Let $\pi_{j_1,\ldots,j_s} : F(k_1, \ldots, k_r; n) \to F(k_{j_1}, \ldots, k_{j_s})$ be a natural projection between two partial flag varieties. Let $M(k_1, \ldots, k_r)$ be a Mondrian tableau for $F(k_1, \ldots, k_r; n)$. Then the Mondrian tableau associated to the image of the projection is the tableau consisting of the squares of colors $j_1, \ldots, j_s$.

Note that the semi-normalization of the projected tableau is an admissible, semi-normalized tableau for $F(k_{j_1}, \ldots, k_{j_s})$. This is the mechanism that allows us to carry out the dimension counts inductively.

Remark 3.41. In fact, the algorithm computes not only the Littlewood-Richardson coefficients, but also the class of the projection of the intersection of two Schubert varieties in the full-flag variety to any partial flag variety $F(k_1, \ldots, k_r; n)$.

3.3. **Proofs of combinatorial statements.**

Proof of Proposition 3.33. This proposition is straightforward. Its proof is analogous (but easier) to the proofs of the Lemmas leading to the proof of Proposition 4.41 in [C2]. Here we briefly recall the main steps.

We begin the analysis with the tableau M_0. If $S^r(0)$, the square determined by Rule 3.19 does not contain the smallest neighbor of S^r or is not contained in the largest neighbor of S^r, then clearly $M(0)$ has the same virtual dimension as M. The side-lengths of the squares and containment relations among them do not change. Starting with squares of color $r-1$ and proceeding in decreasing order, shrinking the side-length of squares of color i so that they are the spans of squares of color $i+1$ contained in them clearly preserves the dimension. Finally, we assert that the normalization of the upper-right corners and semi-normalization of the lower-left corners of the tableaux cannot increase the virtual dimension of a Mondrian tableau. This is easy to see. Each time we shrink a square of color i, we decrease the number of squares of color $i+1$ that it contains by at least 1 and decrease the number of squares of color i it contains by at most 1. Also note that since we do not allow the normalization if the square becomes lower-left justified with a filler, if a square of color i is contained in a square of color $i-1$ as a result of shrinking it, it must also be contained in a square of color i. From these observations it is clear that normalization of upper-right corners and the semi-normalization of lower-left corners preserves or decreases the virtual dimension. The analysis in the cases when $S^r(0)$ contains the smallest neighbor of S^r or it is contained in the largest neighbor of S^r is similar. We leave it to the reader.

Let U^j be a minimal square of color j containing S^r that moves according to rule 3.21 and let N^j be a neighbor of U^j. Let $T^j = U^j(0) \cap N^j$ in $M_1(U^j, N^j)$. Suppose that either T^j does not coincide with a square of color less than j or if it does T^j is not contained in any squares of $M(0)$ of color less than j that did not contain N^j. Then for $i > j$, the number of squares of color $i+1$ in squares of color i increases by at most the number of squares of color $i+1$ not contained in either N^j or U^j, but contained in the union of N^j and U^j. The number of containment relations among squares of color $i+1$ increases by at least the same amount. The total side-lengths of squares of color r increases by one. The number of containment relations among squares of color j also increases by at least one. Since normalization of the upper-right corners and semi-normalization of lower-left corners does not increase the virtual dimension, we conclude that the virtual dimension of $M_1(U^j, N^j)$ is at most equal to the virtual dimension of M. Now suppose that the smallest color square that T^j coincides with has color i_0. Suppose that i_1 is the least color of a square which coincides with T^j and is contained in a minimal square $S_h^{i_1}$ of color i_1 containing T^j and S^k, but not N^j. Then for $i > i_1$, the number of squares of color $i+1$ in squares of color i increases by at most the number of squares of color $i+1$ not contained in either N^j or $S_h^{i_1}$, but contained in the union of N^j and $S_h^{i_1}$. The number of containment relations among squares of color $i+1$ increases by at least the same amount. The total side-lengths of squares of color r increases by one. The number of containment relations among squares of color i_1 also increases by at least one. It follows that the virtual dimension of $M_1(U^jN^j)$ and $M_1'(U^jN^j)$ is less than or equal to the virtual dimension of M. The analysis for Tableaux of type M_2 is almost identical, hence left to the reader. □

Proof of Proposition 3.36. By construction each of the tableaux of type M_0, M_1, M_1' and M_2 have k_i squares of color i. Hence they all satisfy M1. Again by construction every square of color $1 \leq i \leq r-1$ is the span of squares of color $i+1$ contained in them. Hence the first half of M2 holds by construction. If the tableau is normalized, the second half of M2 also holds automatically. If a square $S_1^i \subset S_2^i$ have the same lower-left corner in one of the tableaux, then there cannot be a square of color i that has the same upper-right corner as S_2^i and contained in S_2^i. Otherwise, SNM2 would be violated for M. Hence, M2 holds for the tableaux resulting in Algorithm 3.35. M3 is clear by construction. By Rule 3.19 it follows that if a square of color r has a gap, then we move the square which has a gap. By Rule 3.21 it follows that M4 is preserved for squares of color r. Since M satisfies M4 for squares of color i, M4 follows by construction for the tableaux of type M_0, M_1, M_1' and M_2. It is straightforward to check that if M is admissible and semi-normalized that the tableaux of type M_0, M_1, M_1' and M_2 are also admissible and semi-normalized. □

4. The geometry of Mondrian tableaux

4.1. **Geometric preliminaries.** In this subsection we will explain how to associate an irreducible subvariety of $F(k_1, \dots, k_r; n)$ to a Mondrian tableau.

Notation 4.1. The squares of a Mondrian tableau represent vector spaces spanned by the basis elements they contain. In the rest of the paper we will distinguish between the squares of Mondrian tableaux and the vector spaces they represent by using the ordinary math font for the squares (such as S^i) and the Roman font for the corresponding vector space (such as S^i).

We begin by defining the variety associated to a nice, semi-normalized, admissible Mondrian tableau. If the Mondrian tableau is not nice, the definition needs to be slightly modified.

Definition 4.2 (The variety associated to a nice Mondrian tableau)**.** *Let M be a nice Mondrian tableau. The variety Σ_M associated to M is the Zariski closure in $F(k_1, \dots, k_r; n)$ of the locus of r-tuples $(V_1, \dots, V_r)$ that satisfy the following conditions:*

1. *For every $1 \leq i \leq r$ and every square S_h^i of M, the dimension of intersection of V_i with S_h^i is $\#_i S_h^i(M)$.*
2. *For every $1 \leq i \leq r$ and any two squares $S_{h_1}^i, S_{h_2}^i$ of M, the dimension of intersection of V_i with $\mathrm{S}_{h_1}^i \cap \mathrm{S}_{h_2}^i$ is $\#_i(S_{h_1}^i \cap S_{h_2}^i)(M)$.*
3. *For every $1 \leq i \leq r-1$ and for every square S_h^i of M, the subspace of V_i contained in S_h^i is a subspace of the subspace of V_{i+1} spanned by the subspaces of V_{i+1} contained in S_l^{i+1}, where $S_l^{i+1} \subset S_h^i$.*

If the Mondrian tableau is not nice, then the previous definition has to be modified. If there are two squares $S_{h_1}^i$ and $S_{h_2}^i$ that fail the inequality

$$(1) \qquad \#_{i+1}(S_{h_1}^i \cup S_{h_2}^i)(M) - \#_{i+1}S_{h_1}^i(M) \geq \#_i(S_{h_1}^i \cup S_{h_2}^i)(M) - \#_i S_{h_1}^i(M),$$

then V_i has to intersect $\mathrm{S}_{h_1}^i$ in dimension greater than $\#_i S_{h_1}^i(M)$. We modify the definition as follows:

Definition 4.3 (The variety associated to a Mondrian tableau in general)**.** *Let M be an admissible, semi-normalized Mondrian tableau. The variety Σ_M is the closure of the locus of r-tuples $(V_1, \dots, V_r)$ that satisfy the following properties:*

(1) *For every* $1 \leq i \leq r$ *and every square* S *(not necessarily a square of* M*), the vector space* V_i *intersects the vector space* S *in a vector space of dimension at least* $\#_i S(M)$.

(2) *For every* $1 \leq i \leq r-1$ *and for every square* S_h^i *of* M*, the subspace of* V_i *contained in* S_h^i *is a subspace of the subspace of* V_{i+1} *spanned by the subspaces of* V_{i+1} *contained in* S_l^{i+1}*, where* $S_l^{i+1} \subset S_h^i$.

(3) *For every* $1 \leq i \leq r-1$ *and every pair of squares* $S_{h_1}^i$ *and* $S_{h_2}^i$ *and any square* S_m^{i+1} *contained in either of* $S_{h_1}^i$ *or* $S_{h_2}^i$*, the span of the subspaces of* V_i *contained in* $\mathrm{S}_{h_1}^i$ *and* $\mathrm{S}_{h_2}^i$ *intersects the subspace of* V_{i+1} *spanned by the vector spaces corresponding to the squares of color* $i+1$ *of* M *except for* S_m^{i+1} *contained in* $S_{h_1}^i$ *or* $S_{h_2}^i$ *in dimension at least* $\#_i S_{h_1}^i(M) + \#_i S_{h_2}^i(M) - \#_i(S_{h_1}^i \cap S_{h_2}^i)(M) - 1$.

Remark 4.4. The variety associated to a Mondrian tableau M has a stratification by varieties associated to Mondrian tableaux. We will not need this in what follows, so we leave the description to the reader. However, when one would like to extend the rule presented here to K-theory or equivariant cohomology this stratification is useful.

We can make this definition a little bit more explicit. We first associate a new collection M' of squares to M.

Algorithm 4.5. Let M be an admissible, semi-normalized Mondrian tableau. Associate to M a set of squares M' as follows. If the tableau is nice, associate to M, M itself. If not, let S_1^r be the larger of the two minimal squares of color r that share a lower-left corner and have the northeast most lower-left corner among such squares. If S_1^r and the first square S_2^r with its lower-left corner northeast of S_1^r and which is not contained in S_1^r fail inequality (1) for $i = r$, shrink the lower-left corner of S_1^r until the lower-left corner of the shrunk square coincides with $l(S_2^r)$. Otherwise proceed to the next pair of squares of color r that share a lower-left corner. Repeat this procedure until the set of squares of color r are nice. If during the process the side-length of any square of color r shrinks to zero, discard the tableau and associate the empty set as Σ_M to M. Proceeding in decreasing order from r to 1 apply the same procedure to squares of color i. That is let S_1^i be the larger of the two minimal squares of color i that share a lower-left corner and have the northeast most lower-left corner among such squares. If S_1^i and the first square S_2^i with its lower-left corner northeast of S_1^r and which is not contained in S_1^r fail the inequality (1), then shrink the lower-left corner of S_1^i until it coincides with $l(S_2^i)$. Repeat until the inequality is satisfied for every square. If the side-length of a square shrinks to zero during the process, discard the tableau and assign to M the empty set. Let M' be the collection of squares obtained by this procedure.

Definition 4.6. *We define the actual dimension (or simply the dimension) of a Mondrian tableau* M *as follows. If in the above procedure, the side-length of any square shrinks to zero, we set the dimension equal to* -1*. Otherwise, let* a_i *be the total number of times a square of color* i *starts being contained in a square of color* i*, but not of smaller color while runnning Algorithm 4.5. Let* $a = \sum_{i=1}^r a_i$*. Let* b_i *be the total number of fillers of color* $i+1$*, which have lower-left corner strictly northeast of* $l(S^i)$ *and are contained in a square* S^i *of color* i*, that cease to be contained in* S^i *after the lower-left corner of* S^i *is shrunk in Algorithm 4.5. Let* $b = \sum_{i=1}^r b_i$*. Define the discrepancy* d *to be* $d = \max(0, a-b)$*. The actual dimension of a semi-normalized, admissible Mondrian tableau is equal to the virtual dimension minus the discrepancy. In particular, since the discrepancy is at least* 0*, the actual dimension is always less than or equal to the virtual dimension of* M*.*

Proposition 4.7. *The variety* Σ_M *associated to a semi-normalized, admissible Mondrian tableau* M*, if non-empty, is irreducible of dimension equal to the (actual) dimension of the Mondrian tableau.*

Proof. We prove this proposition by induction on r. When $r = 1$, then the Mondrian tableau is a generalized Mondrian tableau in the sense of [C2]. The irreducibility and the dimension for Grassmannians is proved in Lemma 3.8 of [C2]. Briefly one may construct a non-empty Zariski open subset of the variety associated to a generalized Mondrian tableau as an open set in a tower of projective bundles. If the tableau consists of one square, then the associated variety is projective space. Hence irreducible and of the claimed dimension. Take the largest square with southwest most lower-left corner. If we delete this square, we obtain a tableau for $G(k-1,n)$. By induction we can assume that this variety is irreducible of the claimed dimension. To obtain the variety in $G(k,n)$ for every point in the variety in $G(k-1,n)$ we have to choose a vector in the vector space represented by the square which does not lie in the $(k-1)$-plane represented by the point in $G(k-1,n)$. Over the open set described in the definition this is an open set in a projective bundle over the variety in $G(k-1,n)$ of the claimed dimension. The proposition follows in the case $r=1$.

The proposition now follows by induction on r. First suppose that the tableau is nice. There are projection maps

$$\pi_s : F(k_1,\dots,k_r;n) \to F(k_s,\dots,k_r;n)$$

for every $1 \leq s < r$. Given a variety associated to a Mondrian tableau, the image of the variety under the projection π_s is again a variety associated to a Mondrian tableau in $F(k_s,\dots,k_r;n)$. In fact, this variety is easy to describe. We simply take the variety associated to the Mondrian tableau where we only take the squares of color $s, s+1,\dots,r$. Restricted to the open set described in the definition of Σ_M,the map

$$\pi_2 : F(k_1,\dots,k_r;n) \to F(k_2,\dots,k_r;n)$$

realizes Σ_M as an open set in a tower of projective bundles over an open subset of the variety $\Sigma_{\pi_2(M)}$. The fiber dimension of the projection is

$$\sum_{S_h^1 \in M} (\#_2 S_h^1(M) - \#_1 S_h^1(M)).$$

The irreducibility and the dimension calculation follow by induction. If the tableau is not nice the argument is almost identical. We can still construct an open subset of the variety inductively as an open set in a tower of projective space and Grassmannian bundles, so the variety is still irreducible. We now calculate the dimension. Associate to M the collection of squares M' as in Algorithm 4.5. When we shrink a square S^i of color i, the virtual dimension of the tableau increases if the shrinking of S^i is contained in a square T^{i-1} of color $i-1$, but not of color i. In that case the shrinking of S^i starts sharing a lower-left corner with a filler (whose lower-left corner is the same as $l(T^{i-1})$) contained in S^i. The vector space V_{i-1} contained in T^{i-1} still intersects the subspace of V_i contained in T^{i-1} along the subspace spanned by the subspaces of V_i corresponding to the squares of color i of M contained in T^{i-1}. Also V_{i-1} has to intersect the span of the subspaces contained in squares of color i omitting those in M' that are contained in S_2^i as in item (3) of the definition. Hence, the actual dimension of the variety remains constant. Now suppose we shrink a square S^i of color i so that it is contained in a square U^i of color i, but not in a square of color $i-1$. In this case the virtual dimension of the tableau decreases. The actual dimension of the variety strictly decreases as well except when S^i is contained in T^{i-1} and it is the shrinking of a square that was not previously contained in T^{i-1}. In the latter case the dimension of the variety does not change. The actual dimension of the variety associated to M follows from this calculation. □

Remark 4.8. One can improve the efficiency of Algorithm 3.35 and Algorithm 3.37 by replacing M with tableaux whose actual dimension (as opposed to virtual dimension) is equal to the dimension of M. Another advantage of using the actual dimension is that every branch of the degeneration tree ends in a Schubert variety. No tableau is ever discarded midway through the algorithm. The disadvantage is that calculating the actual dimension of a tableau which is not nice is more cumbersome.

4.2. **Degenerations.** In this section, we interpret the move described by Rule 3.21 as a degeneration of the vector spaces defining the variety associated to a Mondrian tableau. Using this geometric description we prove Theorem 3.39.

Let S^r denote the square determined by Rule 3.19. Suppose that the lower-left chop of S^r is given by $lch(S^r) = \{e_p, \dots, e_q\}$. Then there is a one-parameter family of bases parameterized by an open set in $\mathbb{P}^1$ given by

$$\{e_1, \dots, e_{p-1}, te_p + (1-t)e_{q+1}, e_{p+1}, \dots, e_q, e_{q+1}, \dots, e_n\}.$$

These set of vectors form a basis of V as long as t is not 0. Corresponding to this family of bases and for every vector space T^i for a square T^i of V, there exists a flat family of vector spaces $\mathrm{T}_i^h(t)$. Suppose $T^i = \{e_{i_1}, \dots, e_{i_m}\}$. Set $e_i(t) = e_i$ if $i \neq p$. Set $e_p(t) = te_p + (1-t)e_{q+1}$. Then $\mathrm{T}^i(t)$ for $t \neq 0$ is the vector space spanned by the basis elements

$$\{e_{i_1}(t), \dots, e_{i_m}(t)\}.$$

We will denote by $\mathrm{T}^i(0)$ the flat limit of these vector spaces. Explicitly, if at most one of e_p and e_{q+1} are contained in T^i, then $\mathrm{T}^i(0)$ is the vector space spanned by the basis elements

$$\{e_{i_1}(0), \dots, e_{i_m}(0)\}.$$

If both e_p and e_{q+1} are in T^i, then $\mathrm{T}^i(0)$ is equal to T^i. Note that this agrees with Rule 3.21 for moving squares.

Let M be a Mondrian tableau. The degeneration just described leads to flat families of vector spaces corresponding to the squares of M. For $t \neq 0$, let $\Sigma_M(t)$ denote the variety defined exactly as in Definition 4.3, but replacing every occurrance of e_p with $te_p + (1-t)e_{q+1}$. Note that for $t \neq 0$, the variety associated to M is projectively equivalent to $\Sigma_M(t)$ under a simple change of basis. Hence $\Sigma_M(t)$ forms a flat family over a Zariski open subset of $\mathbb{P}^1$. By the properness of the Hilbert scheme, there exists a flat limit $\Sigma_M(0)$ of this family. We will refer to this specialization as *the standard specialization of the variety associated to the Mondrian tableau M*. Our task in this subsection is to describe the flat limit $\Sigma_M(0)$.

Notation 4.9. Let M be a Mondrian tableau. For any collection $S^i_{h_1}, \dots, S^i_{h_m}$ of squares of M of color i, we will denote by $S^i_{h_1,\dots,h_m}$ the square of color i given by the union of the basis elements in $S^i_{h_1}, \dots, S^i_{h_m}$. We will denote by $\mathrm{S}^i_{h_1,\dots,h_m}$ or sometimes by $\mathrm{S}^i_{h_1,\dots,h_m}(1)$ the corresponding vector space in V spanned by those basis elements. We will use $\mathrm{S}^i_{h_1,\dots,h_m}(t)$ to denote the vector space over the fiber at $t \neq 0$ and $\mathrm{S}^i_{h_1,\dots,h_m}(0)$ to denote the flat limit of the corresponding vector space under the degeneration described above.

Theorem 4.10 (The Geometric Littlewood-Richardson Rule)**.** *Let M be an admissible, semi-normalized Mondrian tableau whose virtual dimension equals its actual dimension. Let $\Sigma_M(t)$ be the standard specialization of the variety associated to M. Then the support of the flat limit $\Sigma_M(0)$ is equal to the union of the varieties associated to the Mondrian tableaux described in Algorithm 3.37 that have the same dimension as the variety associated to M. Moreover, $\Sigma_M(0)$ is generically reduced along each of these varieties.*

The combinatorial rule for computing the structure constants of the cohomology of partial flag varieties immediately follows from Theorem 4.10.

Theorem 4.10 implies Theorem 3.39. The class $\sigma^\delta_\lambda \cdot \sigma^\kappa_\mu$ can be computed as the Poincaré dual of the class of the intersection of two Schubert varieties Σ^δ_λ and Σ^κ_μ defined with respect to opposite flags. The variety associated to the initial Mondrian tableau $M(\sigma^\delta_\lambda, \sigma^\kappa_\mu)$ is precisely such an intersection. By Lemma 3.17 and Proposition 4.7, the variety associated to $M(\sigma^\delta_\lambda, \sigma^\kappa_\mu)$ is irreducible of dimension equal to the dimension of $\Sigma^\delta_\lambda \cap \Sigma^\kappa_\mu$. Since a Zariski open subset of the variety associated to $M(\sigma^\delta_\lambda, \sigma^\kappa_\mu)$ is clearly contained in the intersection of the two Schubert varieties, we conclude that the two coincide. By Theorem 4.10, the class of the variety is equal to the sum of the classes of the varieties associated to the Mondrian tableaux described by the Algorithm 3.37. In turn, the classes of each of these are the sum of the classes of the varieties associated to the Mondrian tableaux described by Algorithm 3.37. The algorithm clearly terminates with a collection of tableaux corresponding to Schubert varieties. (Note for instance that if we partially order the tableaux lexicographically by the positions of the lower-left corners of the squares of color $1, 2, \dots, r$ and the total side-length of the squares of color r in that order, then each stage of the algorithm the order of the resulting tableaux is strictly larger than the initial tableau. Since all the tableaux reside in an $(n \times n)$-grid this process cannot go on indefinitely.) Hence, Algorithm 3.37 expresses the class of the variety associated to $M(\sigma_\lambda^\delta, \sigma_\mu^\kappa)$ as a sum of Schubert classes. This concludes the proof of Theorem 3.39. □

Proof of Theorem 4.10. The proof of Theorem 4.10 has two main steps. The first step is to determine the support of the flat limit of the standard specialization of the variety associated to M. This is the hardest step. Luckily the main work has been carried out in [C2]. We simply build on it by induction. The second step is to show that flat limit is generically reduced along each maximal dimensional irreducible component of the flat limit. Given that the support of the flat limit has a very simple description given by Algorithm 3.37, the second step is a consequence of an easy tangent space calculation. In fact, that the multiplicities are one can be reduced to the fact that in Pieri's Rule or Monk's Rule every summand occurs with multiplicity one.

We begin with step 1. On the r-tuples $(V_1, \dots, V_r)$ there are no visible conditions other than the rank conditions that we impose on them. However, a priori in the limit the r-tuples that lie in the flat limit $\Sigma_M(0)$ could satisfy some non-apparent conditions. The main content of the theorem is that the obvious conditions are the only conditions that the r-tuples of vector spaces satisfy. We will prove the theorem by writing down the necessary conditions that the linear spaces need to satisfy. We will then observe that these conditions already force the dimension of the locus that satisfy them to be at most of dimension equal to the dimension of the variety associated to M. We thus bound the support of $\Sigma_M(0)$. We then inductively see that the components that have the same dimension as Σ_M are the varieties that correspond to tableaux of type M_0, M_1, M_1' and M_2.

Observation 4.11 (The main geometric observation). Let M be an admissible, semi-normalized Mondrian tableau. Let $(V_1(t), \cdots, V_r(t))$ be an r-tuple of vector spaces lying in $\Sigma_M(t)$ for $t \neq 0$. Then for every $1 \leq i \leq r$ and any collection of indices $h_1, \dots, h_m$, the vector space $V_i(t)$ intersects the vector space $\mathrm{S}^i_{h_1,\dots,h_m}(t)$ in dimension at least $\#_i S^i_{h_1,\dots,h_m}(M)$. Since intersecting a given vector space in at least a given dimension is a closed condition, any r-tuple $(V_1(0), \dots, V_r(0))$ contained in the flat limit $\Sigma_M(0)$ has to satisfy the following condition. For every $1 \leq i \leq r$ and any collection of indices $h_1, \dots, h_m$, the vector space $V_i(0)$ intersects $\mathrm{S}^i_{h_1,\dots,h_m}(0)$ in dimension at least $\#_i S^i_{h_1,\dots,h_m}(M)$.

Furthermore, the subspace of $V_i(t)$ contained in $\mathrm{S}^i_{h_1,\dots,h_m}(t)$ is a subspace of the subspace of $V_{i+1}(t)$ spanned by the subspaces of $V_{i+1}(t)$ contained in $\mathrm{S}^{i+1}_{l_r}(t)$, where $S^{i+1}_{l_r}$ is contained in $S^i_{h_j}$ for at least one h_j. In the limit the subspace of $V_i(0)$ contained in $\mathrm{S}^i_{h_1,\dots,h_m}(0)$ has to be contained in the limit of this vector space.

Note that Observation 4.11 is consistent with the way we draw the tableaux of types M_0, M_1, M_1' and M_2. In fact, the way tableaux are drawn in Rule 3.27 and the way the tableaux M_0, M_1, M_1' and M_2 are drawn is simply a combinatorial encoding of this observation.

We can now determine the support of the flat limit using Observation 4.11. We will do this by induction on r. When $r = 1$, then the Theorem is proved in [C2]. For the convenience of the reader we very briefly recapitulate the main points in the argument.

Let Y be an irreducible component of the flat limit $\Sigma_M(0)$. If the k_1-dimensional vector space V_1 parameterized by a general point of Y intersects the vector spaces $\mathrm{S}^1_h(0)$ in dimension $\#_1 S^1_h(0)$ and the vector spaces $\mathrm{S}^1_h(0) \cap \mathrm{S}^1_j(0)$ in dimension $\#_1(S^1_h \cap S^1_j(0))(M(0))$, then Y has to be contained in Σ_{M_0}. If the k_1-dimensional vector space V_1 parameterized by a general point of Y intersects a vector space $\mathrm{S}^1_\mathrm{h}(0) \cap \mathrm{S}^1_\mathrm{j}(0)$ in dimension greater than $\#_1(S^1_h \cap S^1_j(0))(M(0))$, then the dimension of Y can be bound from above as follows. Let T be the smallest dimensional vector space spanned by consecutive basis elements such that $\dim(V_1 \cap \mathrm{T}) > \#_1 T(M(0))$ (if there are two such vector spaces, let T be the one with southwest most lower-left corner). Suppose that the number of vector spaces S^1_h that minimally (with respect to inclusion) contain T is $i \geq 2$. Order them according to their lower-left corners $S^1_1(0), \dots, S^1_i(0)$. Denote the virtual dimension of M by vdim. Then the dimension of Y is at most $\mathrm{vdim}(M) - i + 1$ unless one of the vector spaces is $\mathrm{S}^1(0)$, the limit of the vector space associated to the square determined by

Rule 3.19. In the latter case, the dimension of the locus is at most $\mathrm{vdim}(M) - i + 2$. To see this we argue as follows. We add T to $M(0)$. But then the conditions imposed by the vector spaces $\mathrm{S}^1_\mathrm{h}(0)$, where $S^1_h(0)$ are the squares of $M(0)$ that minimally contain T, are automatically satisfied. We can then delete those squares. By Observation 4.11, the conditions imposed by the spans of $\mathrm{S}^1_h(0)$ and $\mathrm{S}^1_{h+1}(0)$ are not recorded. We therefore add the squares $S^1_{h,h+1}(0)$ as h varies from 1 to $i-1$. It is easy to calculate the locus of k_1-planes satisfying these constraints. It follows from this dimension calculation that the only loci of dimension equal to $\mathrm{vdim}(M)$ that can support Y has to be among Σ_{M_0} and $\Sigma_{M(S^1 N^1_s)}$.

The case $r > 1$ is argued similarly. First we make the following simple observation which follows by induction building on the proof of Theorem 3.32 in [C2].

Observation 4.12. Let M be a Mondrian tableau for $F(k_1, \dots, k_r; n)$. Let S^r be the square of M moved according to Rule 3.21. Let $X \subset \Sigma_M(0)$ be a locus where the subspaces V_i remain as independent as dictated by $M(0)$ for $i < i_0$. Suppose for the points parameterized by X the subspaces of V_{i_0} minimally (with respect to inclusion) contained in the vector spaces $\mathrm{S}^{i_0}_1, \dots, \mathrm{S}^{i_0}_m$ become dependent along a subspace T. Suppose there exists N vector spaces corresponding to the squares of $M(0)$ of color i_0 minimally containing T and contained in the span of $\mathrm{S}^{i_0}_1, \dots, \mathrm{S}^{i_0}_m$. Suppose for the points parameterized by X the subspaces V_i for $i \geq i_0$ contained in vector spaces corresponding to squares of $M(0)$ remain independent except if the dependence is implied by Observation 4.11. Then the dimension of X is at most $\dim(\Sigma_M) - N + 1$ unless T is the vector space corresponding to the new intersection of a square of M moved by Rule 3.21 with another square of M. In the latter case, the dimension of X is at most $\dim(\Sigma_M) - N + 2$.

Now we can determine the support of $\Sigma_M(0)$ by induction on the number of steps in $F(k_1, \dots, k_r; n)$. The induction step is almost identical to the proof of the Geometric Littlewood-Richardson Rule for two-step flag varieties in [C2]. Consider the projection

$$\pi : F(k_1, \dots, k_r; n) \to F(k_2, \dots, k_r; n).$$

As discussed, the image of a variety associated to a Mondrian tableau M is the variety associated to the Mondrian tableau obtained by taking all the squares of colors $2, 3, \dots, r$. Let Y be an irreducible component of the support of $\Sigma_M(0)$. If the general fiber dimension of the projection of Y is equal to the general fiber dimension of the projection of Σ_M, then the dimension of the image has to be equal to the dimension of the image of Σ_M. The possible images of the projection are determined by induction. Otherwise, we have to bound the dimension of the image of the projection over a component where the general fiber dimension increases by a given amount. Recall that the fiber dimension of the projection of Σ_M is calculated by

$$\sum_{S^1_h \in M} \left(\#_2 S^1_h(M) - \#_1 S^1_h(M) \right).$$

Suppose in an irreducible component Y of the support of the flat limit, the general fiber dimension of the projection of Y is equal to the general fiber dimension of the projection of Σ_M. The projection of Σ_M is a variety $\Sigma_{\pi(M)}$ associated to an admissible Mondrian tableau $\pi(M)$. By induction, the projection of Y has to be contained in one of the varieties associated to $\pi(M(0))_0, \pi(M(0))_1, \pi(M(0))'_1$ or $\pi(M(0))_2$. By Observation 4.11, V_1 has to intersect each $\mathrm{S}^1_h(0)$ in dimension at least $\#_1 S^1_h(0)(M(0))$. If at a general point of Y the subspaces of V_1 contained in $\mathrm{S}^1_h(0)$ are as independent as dictated by $M(0)$, then Y has to be contained in the variety associated to the Mondrian tableau M', where M' is the Mondrian tableau corresponding to the projection of Y (prior to semi-normalization of the lower-left corners) together with the squares of color 1 of $M(0)$ shrunk so that they are the spans of the squares of color 2 contained in them. This Mondrian tableau is one of types M_0, M_1, M'_1 or M_2 depending on M'. Since both Y and the variety associated to this Mondrian tableau are irreducible of the same dimension, we conclude that they are equal. In case the subspaces of V_1 contained in $\mathrm{S}^1_h(0)$ are not as independent as dictated by $M(0)$, add to M' a minimal square T^1_h for which the vector space V_1 parameterized by Y intersects T^1_h in dimension greater than $\#_1 T^1_h(M(0))$. By Observation 4.11, the conditions imposed by squares of $M(0)$

that minimally contain T_h^1 are automatically satisfied, so they can be deleted from $M(0)$. However, the conditions imposed by the consecutive spans still need to be satisfied. As in the case of the Grassmannian, by Observation 4.12, it is easy to see that except for tableaux of type M_1 or M_1' the variety associated to the resulting tableau has dimension strictly smaller than $\dim(\Sigma_M)$.

Now we can assume that the general fiber dimension of the projection of Y to $F(k_1, \dots, k_r; n)$ is larger than the general fiber dimension of the projection of Σ_M. Suppose first that at a general point of Y, the subspaces of V_1 contained in $\mathrm{S}_{h_1}^1(0)$ and $\mathrm{S}_{h_2}^1(0)$ remain independent for any two squares of color 1. There are a few cases. Since the fiber dimension increases, the total number of squares of color 2 contained in squares of color 1 has to increase.

Let $S_0^1(0)$ be the square with lower-left most corner such that $\mathrm{S}_0^1(0)$ has a larger dimensional intersection with V_2 than $\#_2 S_0^1(1)(M)$. Let W be the vector space spanned by the basis elements northeast of (and including) $l(S_0^1(0))$. This case splits to a few cases.

Case A There exists a square $S^2(0) \notin S_0^1(0)$ of color 2 with lower-left corner southwest of $S_0^1(0)$ such that the dimension of $V_2 \cap \mathrm{S}^2(0) \cap \mathrm{S}_0^1(0)$ is greater than $\#_2(S^2(1) \cap S_0^1(1))(M)$. Suppose B^2 is the minimal square (with respect to inclusion) among such squares. This case splits to two cases.

- i Either the dimension of $V_2 \cap \mathrm{W} = \#_2 W(M)$; or
- ii The dimension of $V_2 \cap \mathrm{W} > \#_2 W(M)$.

Case B There does not exist a square $S^2(0) \notin S_0^1(0)$ with lower-left corner southwest of $S_0^1(0)$ such that the dimension of $V_2 \cap \mathrm{S}^2(0) \cap \mathrm{S}_0^1(0)$ is greater than $\#_2(S^2(0) \cap S_0^1(0))(M)$. In that case let $B^2(0)$ be the smallest square (with respect to inclusion) such that $V_2 \cap \mathrm{B}^2(0) \cap \mathrm{S}_0^1(0)$ is greater than $\#_2(B^2(1) \cap S_0^2(1))(M)$.

The possibilities Case A ii and Case B have strictly smaller dimension unless a square of color 2 starts being contained in a square of color 1 as a result of the move or as a result of shrinking the square of color 2 because of normalization of upper-right corners or to make it the span of squares of color 3 contained in it. These cases lead to the tableaux of types M_0 or M_1 in the respective cases. All other such loci have strictly smaller dimension. The argument is identical to the argument given for the correponding cases in the proof of Theorem 4.45 in [C2]. Case A i corresponds to the variety associated to a tableau of type M_2. The argument that all other loci have lower dimension is indentical to the argument given for the corresponding case in the proof of Theorem 4.45 in [C2].

Finally we can assume that at least two of the subspaces of V_1 contained in two vector spaces corresponding to squares of color 1 of $M(0)$ become dependent in the limit. Using Observation 4.12 it is easy to see that the dimension strictly decreases unless the two squares of color 1 are neighbors. The corresponding tableaux are either of type M_1 or M_1'. It follows from this discussion that the support of the limit is contained in the union of the varieties corresponding to the tableaux in Algorithm 3.35.

Note that it is easy to see that the support of $\Sigma_M(0)$ contains each of the varieties described by Algorithm 3.35. One can explicitly write down a sequence of vector spaces that specialize to a general point of each of the varieties associated to the Mondrian tableaux. In each of the limits either all the vector spaces contained in the vector spaces corresponding to the squares of the tableaux remain independent or there exists a smallest index i such that two of the subspaces contained in neighboring vector spaces become dependent. All other subspaces remain as independent as possible given M and the fact that these two subspaces became dependent. This is clear for the tableaux M_0, M_1 and M_1', but is also true for the tableaux of type M_2. In tableaux of type M_2 except for the intersection of U^i with the neighbor N_s^{i-1}, the other intersections are a consequence of the linear algebra fact that in a vector of dimension k two subspaces of dimension l and m have to intersect in a subspace of dimension at least $l+m-k$. The other intersections are an application of this fact to the subspace of V_i contained in the vector spanned by the basis elements northeast of $l(N_s^{i-1})$. (See also Remark 3.32.)

Finally there remains to check that $\Sigma_M(0)$ is generically reduced along each of the maximal dimensional varieties. There are many ways of checking this. To see that the variety associated to the Tableau $M_1(U^j N_s^j)$ occurs with multiplicity one, we can assume that the Tableau M consists of U^j and the neighbors of N_s^j of color j. There exists a smooth morphism from a Zariski open subset of the total space of the family $\Sigma_M(t)$ to an open subset of a family $\Sigma_{M'}(t)$, where M' is the tableau consisting of U^j and the neighbors of U^j of color j. The restriction of $\Sigma_{M_1(U^j N_s^j)}$ is the pull-back of the corresponding divisor on $\Sigma_{M'}(t)$. To check the multiplicity it suffices to carry out the calculation on the family $\Sigma_{M'}(t)$. But now consider the projection to the variety $G(V_j, n)$. $\Sigma_{M_1(U^j N_s^j)}$ is the pull-back of a corresponding divisor in the family for $G(V_j, n)$. The multiplicity of the divisor in the latter family is one. This is immediate from Pieri's rule for the Grassmannian of lines and induction on the number of neighbors of U^j (see the proof of Theorem in [C2]). It follows that the miltiplicity of $\Sigma_{M_1(U^j N_s^j)}$. The multiplicity calculations for the other tableaux are similar and left to the reader.

It is also possible to give a proof that does not depend on Pieri's rule or Monk's rule. There are at least two ways of doing this. For each of the tableaux it is easy to write curves in the total space of the family that intersects the fiber with multiplicity one and intersects only one of the tableaux that occur. Alternatively one can calculate the dimension of the Zariski tangent space to the fiber at a general point of the central fiber directly. These are both easy and left to the reader. □

Remark 4.13. It is possible to prove many variations of Theorem 4.10. It is possible to alter the order of degeneration. For instance, a similar analysis applies to the order proposed by Knutson and Vakil. One can also desingularize the total space of the family using a Bott-Samelson type resolution and obtain restrictions on the singularities of the central fiber. This has been carried out in similar situations.

5. Examples.

In this section we give a few examples to illustrate the algorithm by calculating some simple products in three and four step flag varieties.

5.1. **Example 1.** The first example calculates

$$\sigma^{2,1,3,2}_{1,0,0,0} \cdot \sigma^{2,1,3,2}_{0,0,0,0} = \sigma^{1,2,3,2}_{2,1,0,0} + \sigma^{1,2,2,3}_{2,0,0,0} + \sigma^{1,2,2,3}_{1,1,0,0} + \sigma^{2,1,3,2}_{2,2,0,0}$$

in $F(1,3,4;6)$. Figure 11 shows how the algorithm calculates this product. As usual we have used red, blue and black for the colors $1, 2$ and 3, respectively. We also remind our short hand that if two squares of different colors coincide we only depict the one with the smaller color.

5.2. **Example 2.** The second example calculates

$$\sigma^{2,3,1,4}_{0,0,0,0} \cdot \sigma^{1,3,4,2}_{0,0,0,0} = \sigma^{1,3,2,4}_{1,1,1,0} + \sigma^{1,2,3,4}_{1,1,0,0} + \sigma^{1,3,2,4}_{2,1,0,0}$$

in $F(1,2,3,4;6)$. Figure 12 shows how the algorithm calculates this product. We have used red, blue, green and black to depict the colors $1, 2, 3$ and 4, respectively.

5.3. **Example 3.** Our third example is a more involved calculation in $F(1,3,4;7)$. We calculate

$$\sigma^{3,2,1,2}_{2,1,0,0} \cdot \sigma^{1,2,3,2}_{1,1,1,0} = \sigma^{1,3,2,2}_{2,2,2,2} + \sigma^{1,2,3,2}_{2,2,2,1} + 2\sigma^{1,3,2,2}_{3,2,2,1} + 2\sigma^{1,2,3,2}_{3,2,1,1} + \sigma^{3,1,2,2}_{3,3,3,0} + \sigma^{2,1,3,2}_{3,3,1,0}$$
$$+\sigma^{1,2,3,2}_{3,2,2,0} + \sigma^{2,1,3,2}_{3,3,1,1} + \sigma^{3,1,2,2}_{3,3,2,1} + \sigma^{1,3,2,2}_{3,3,2,0} + \sigma^{1,2,3,2}_{3,3,1,0}$$

using the algorithm. Figure 13 shows the calculation.

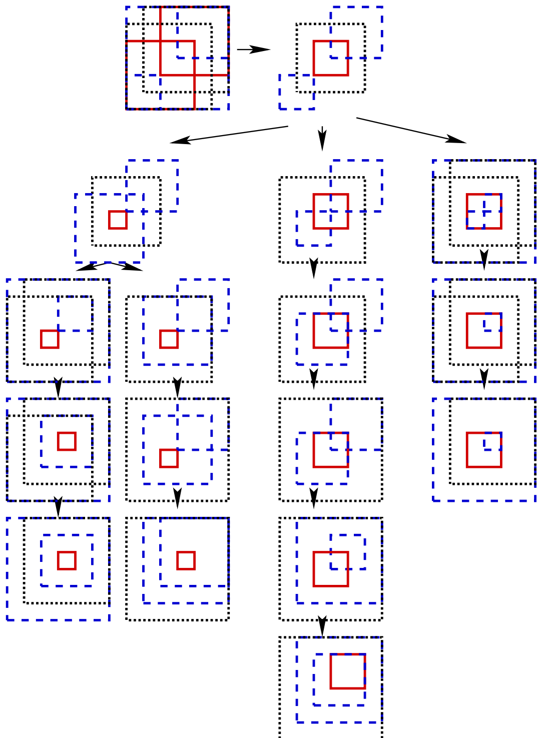

FIGURE 11. A calculation in $F(1,3,4;6)$.

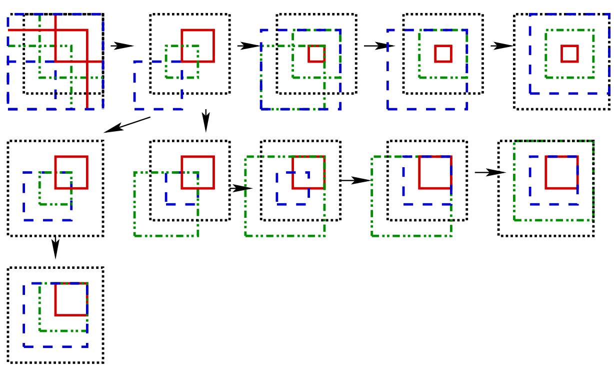

FIGURE 12. A calculation in $F(1,2,3,4;6)$.

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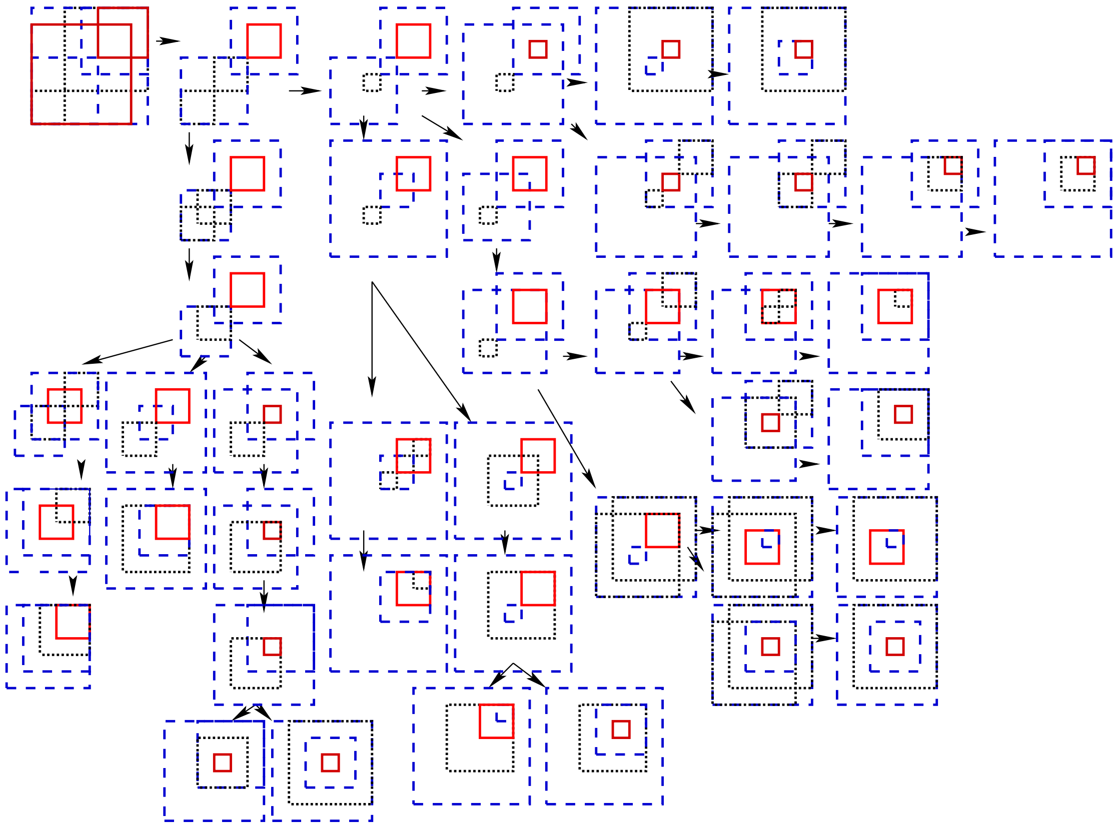

FIGURE 13. A calculation in $F(1,3,4;7)$.

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