Geometric positivity in the cohomology of homogeneous varieties

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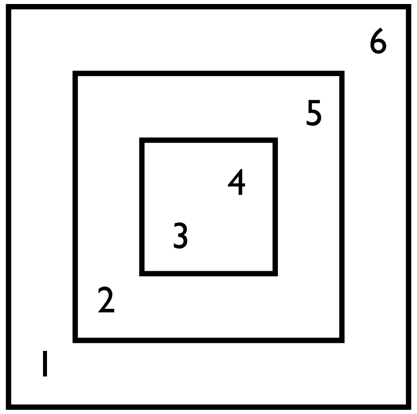
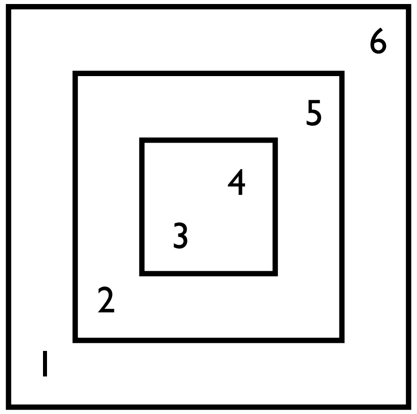
• The cohomology of flag varieties is generated by Schubert classes.



$$\sigma_{2,1} \in G(3,6)$$



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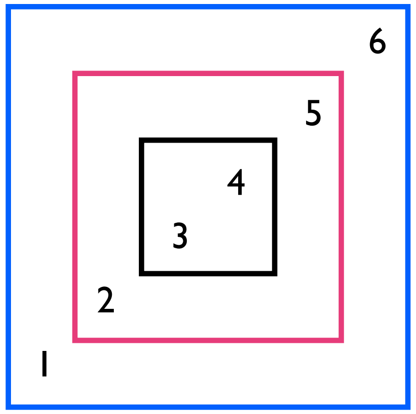

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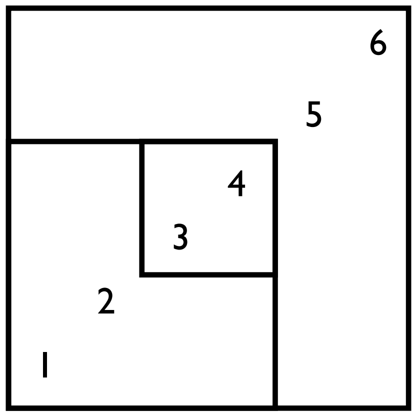

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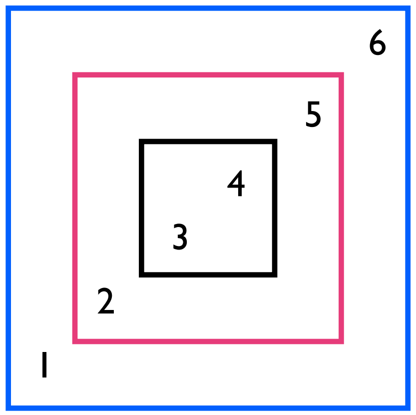

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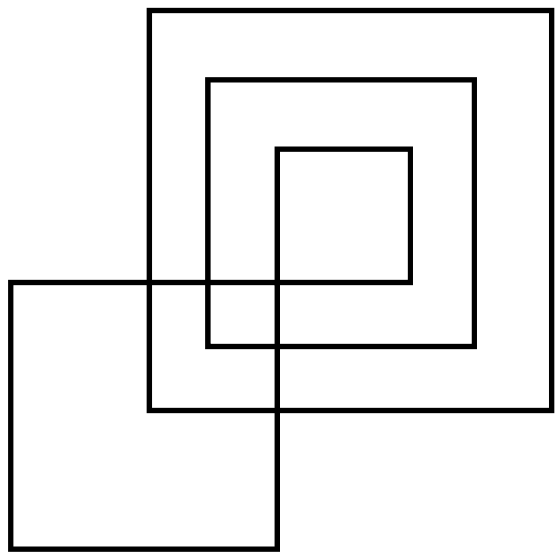
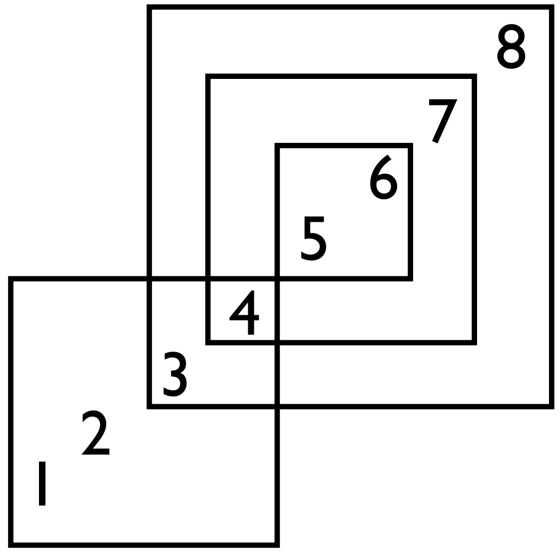
The local Pieri rule explains the Schubert geometry of partial flag varieties.

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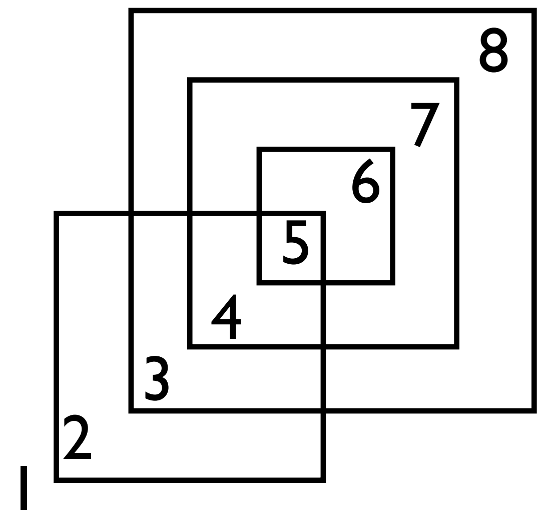
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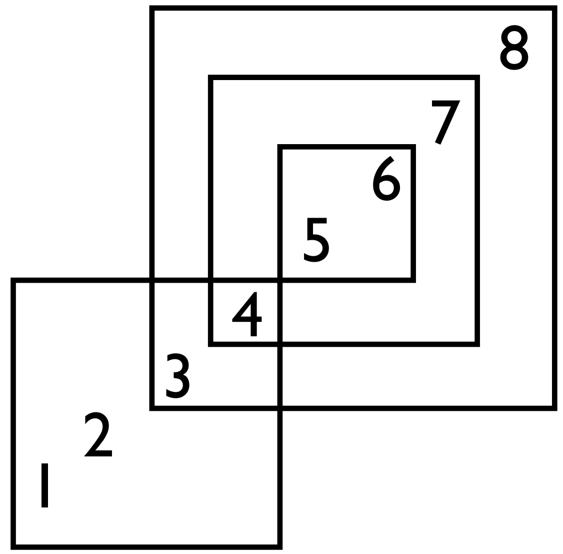
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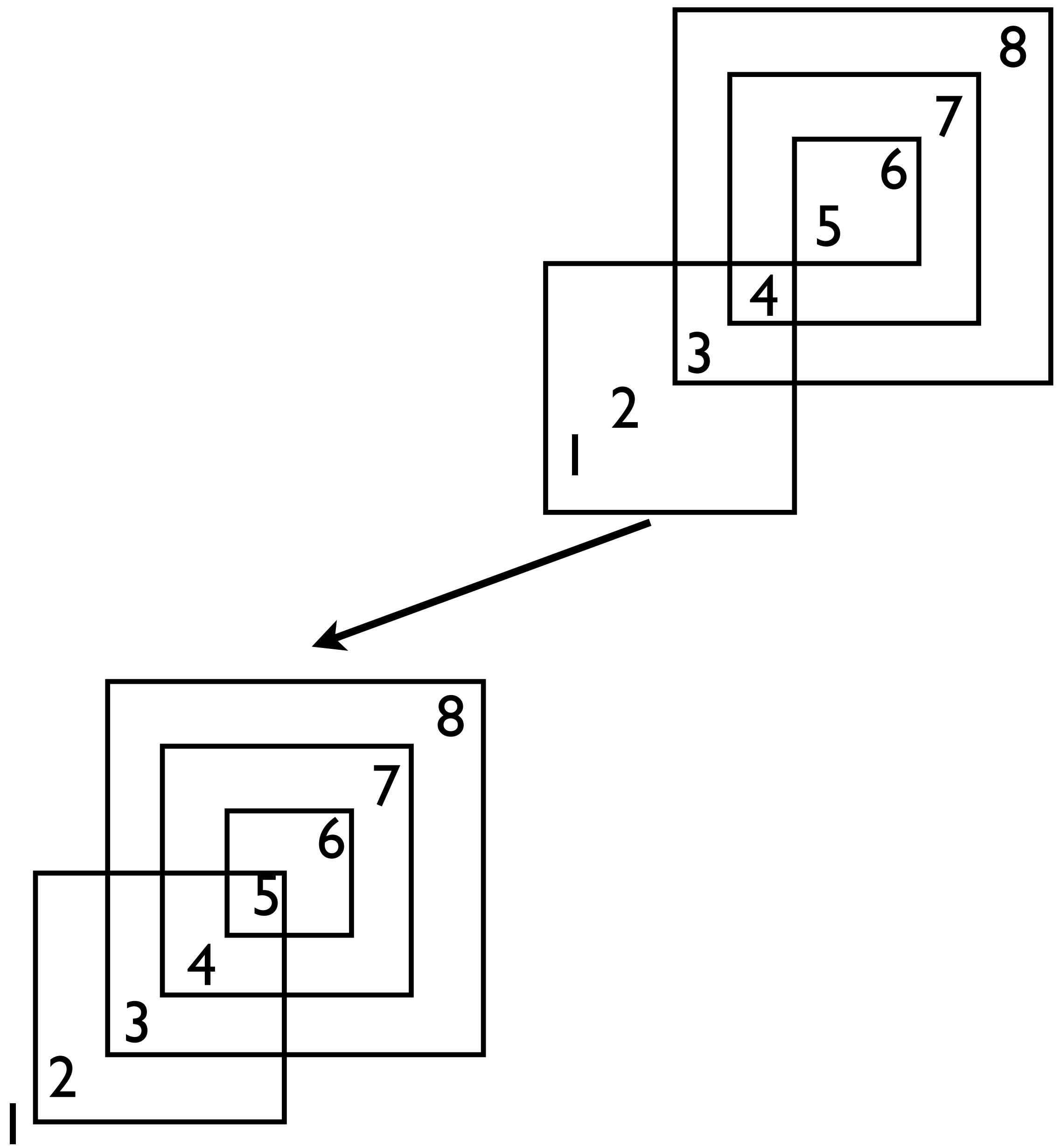
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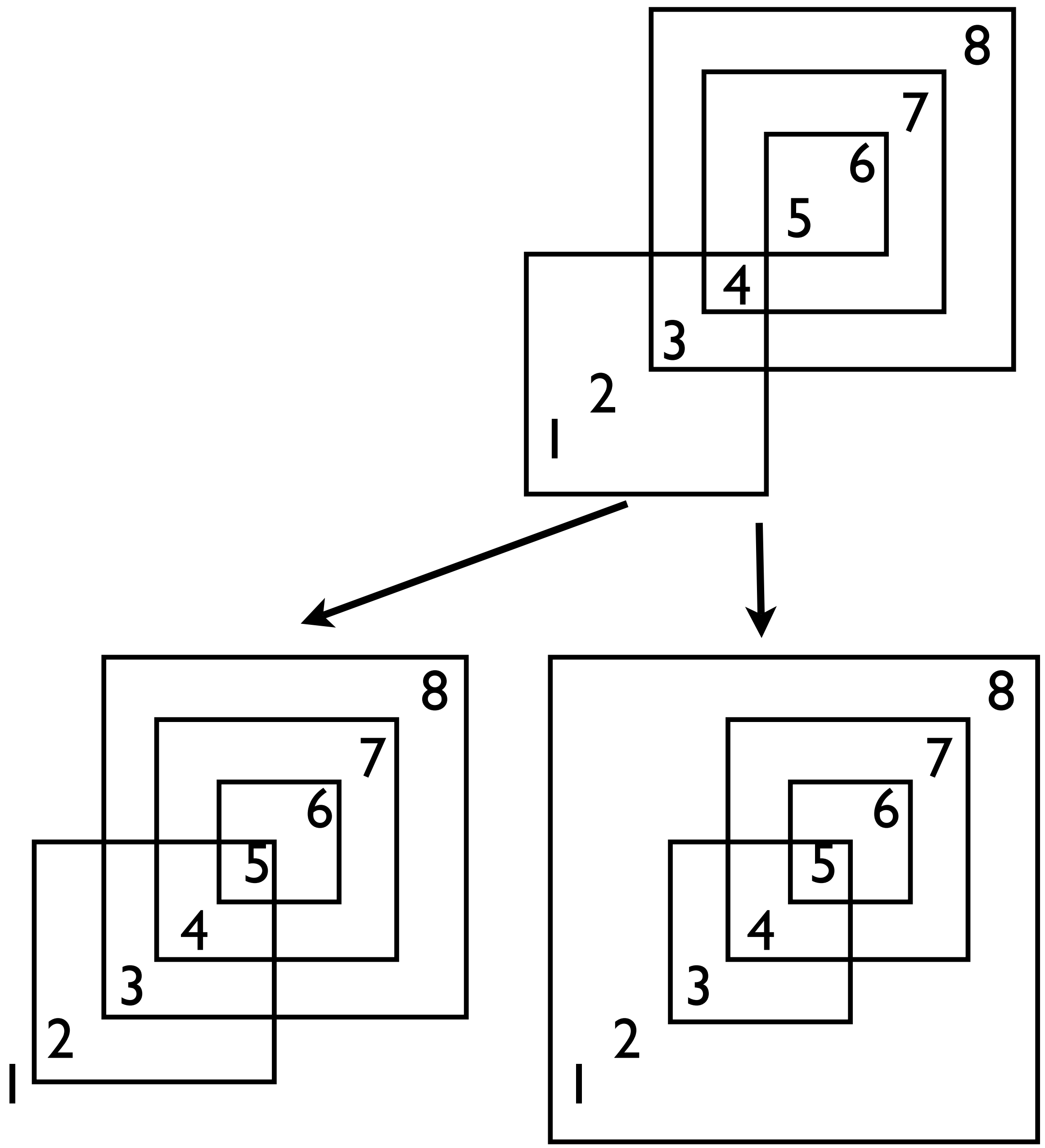
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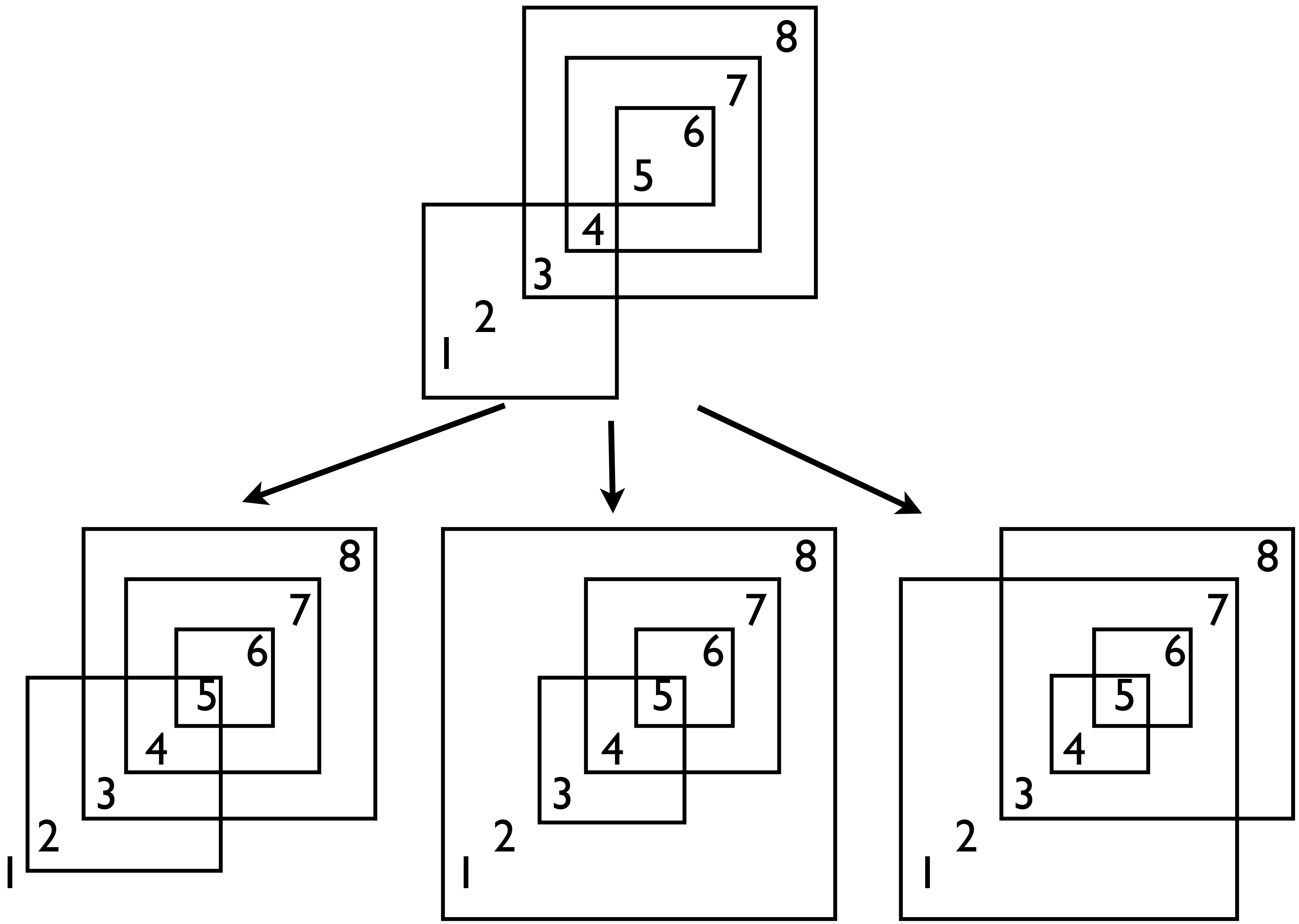
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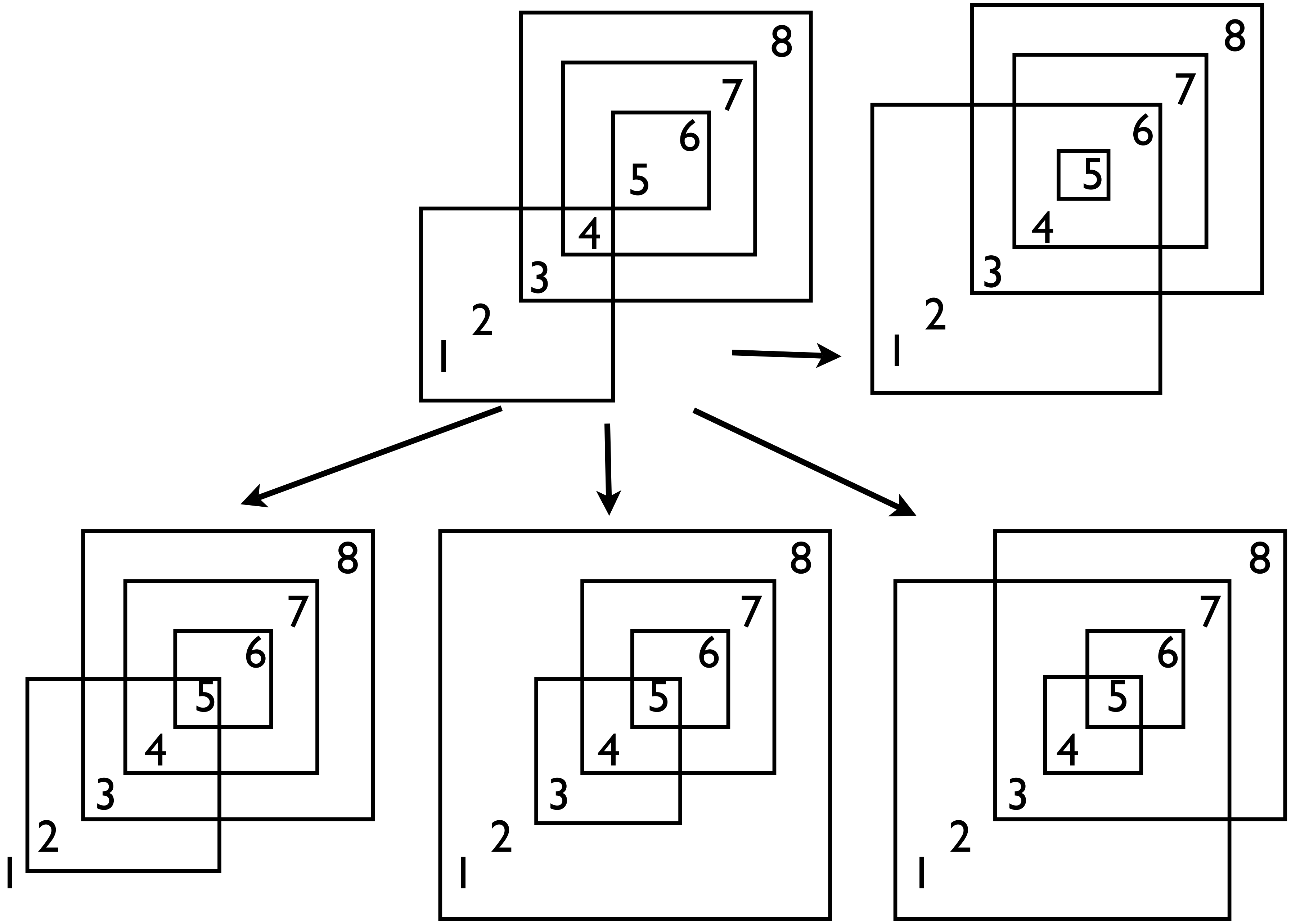

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In order to obtain an inductive process, one should study a more general problem. There are natural projections between flag varieties

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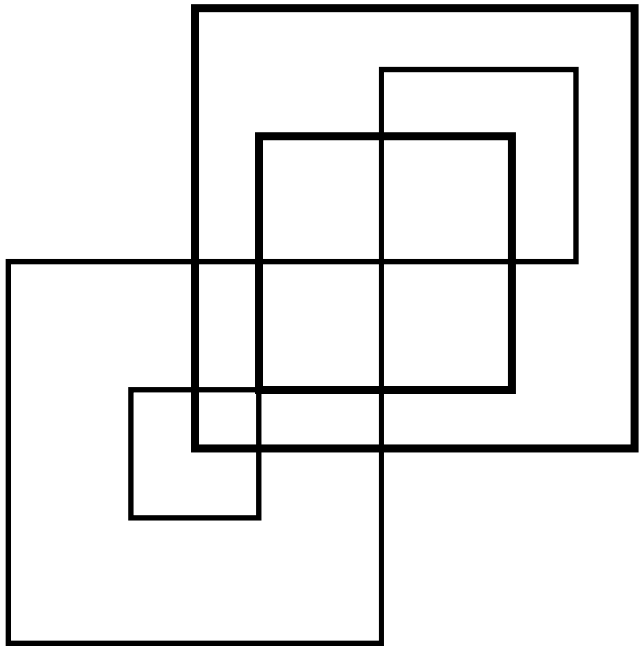
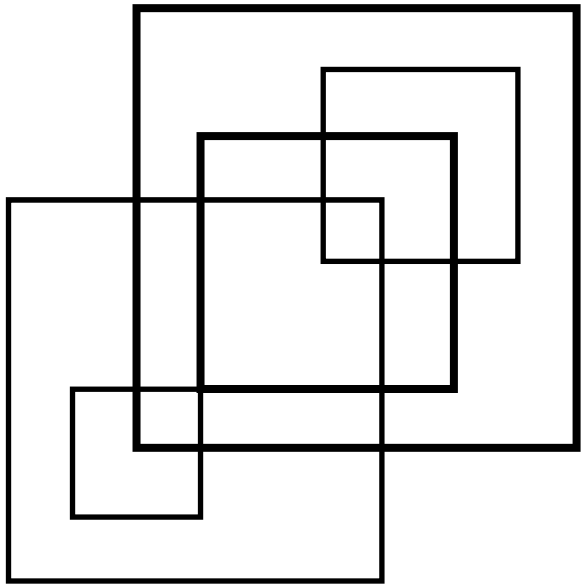
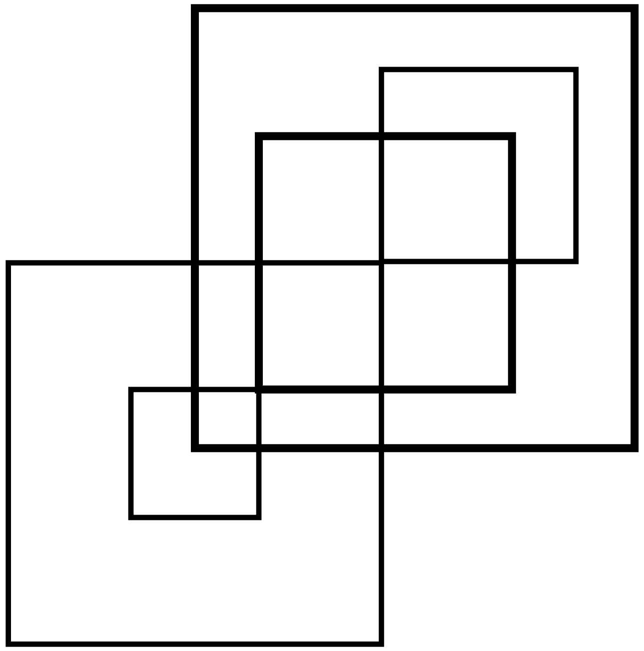
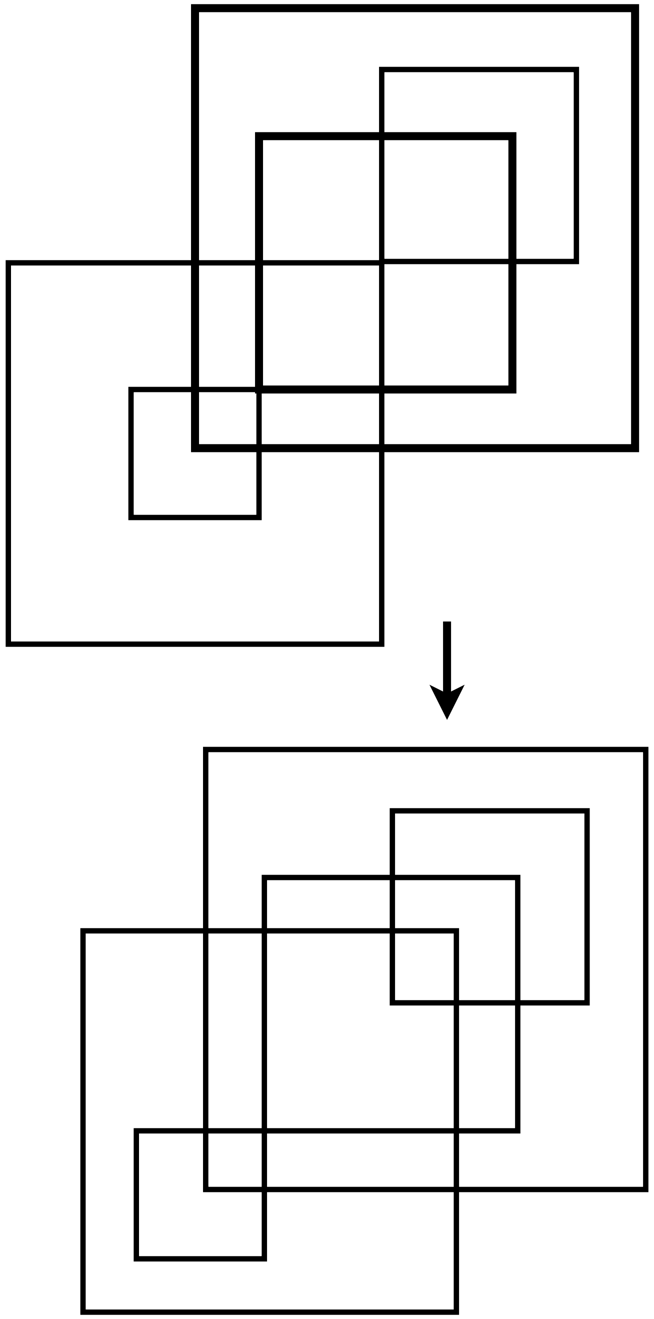
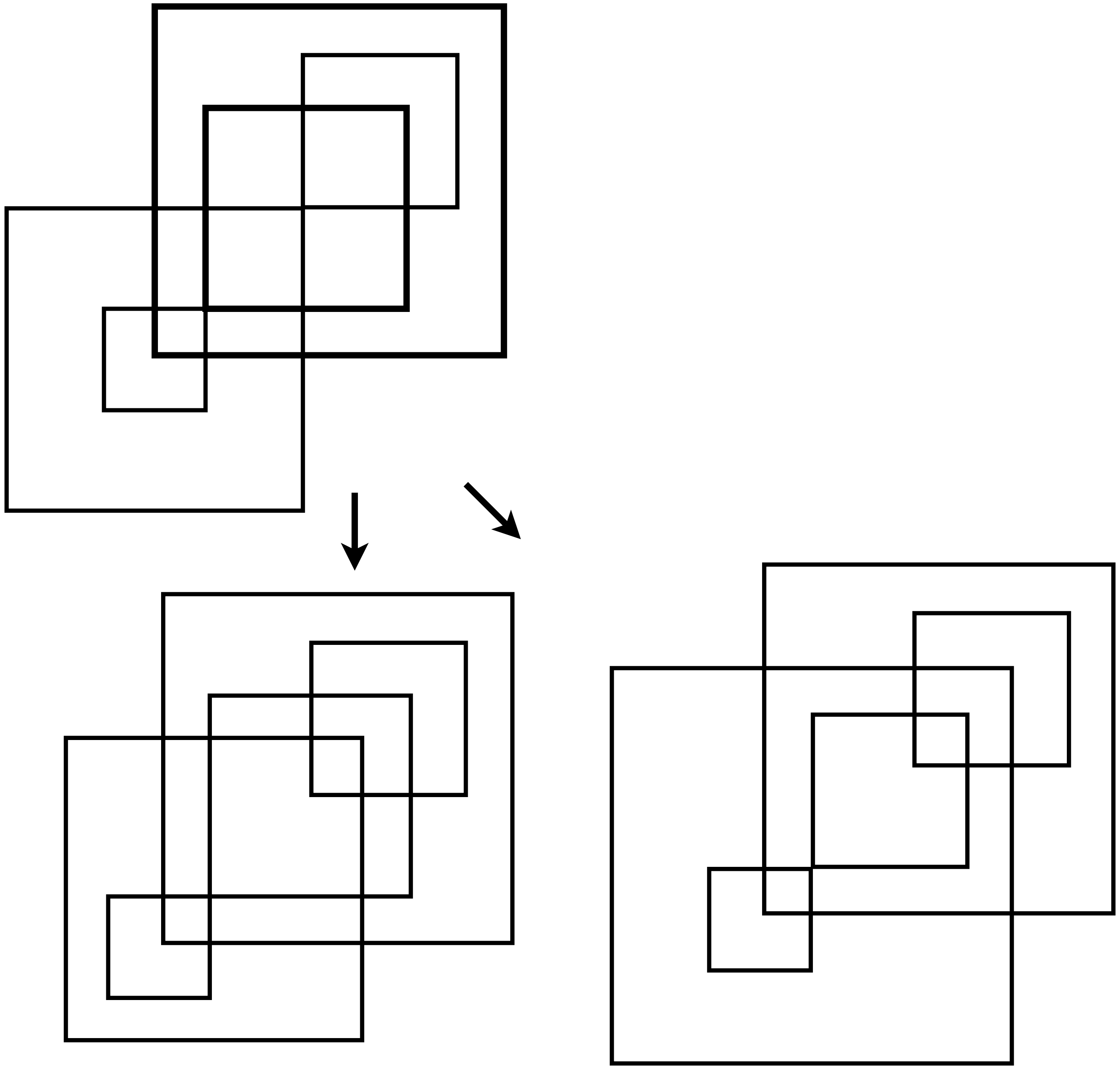
PROBLEM: Given two Schubert varieties Σ_λ and Σ_μ in $F(k_1, \dots, k_h; n)$, compute the class of $\pi(\Sigma_\lambda \cap \Sigma_\mu)$ in $F(k_{i_1}, \dots, k_{i_j}; n)$.

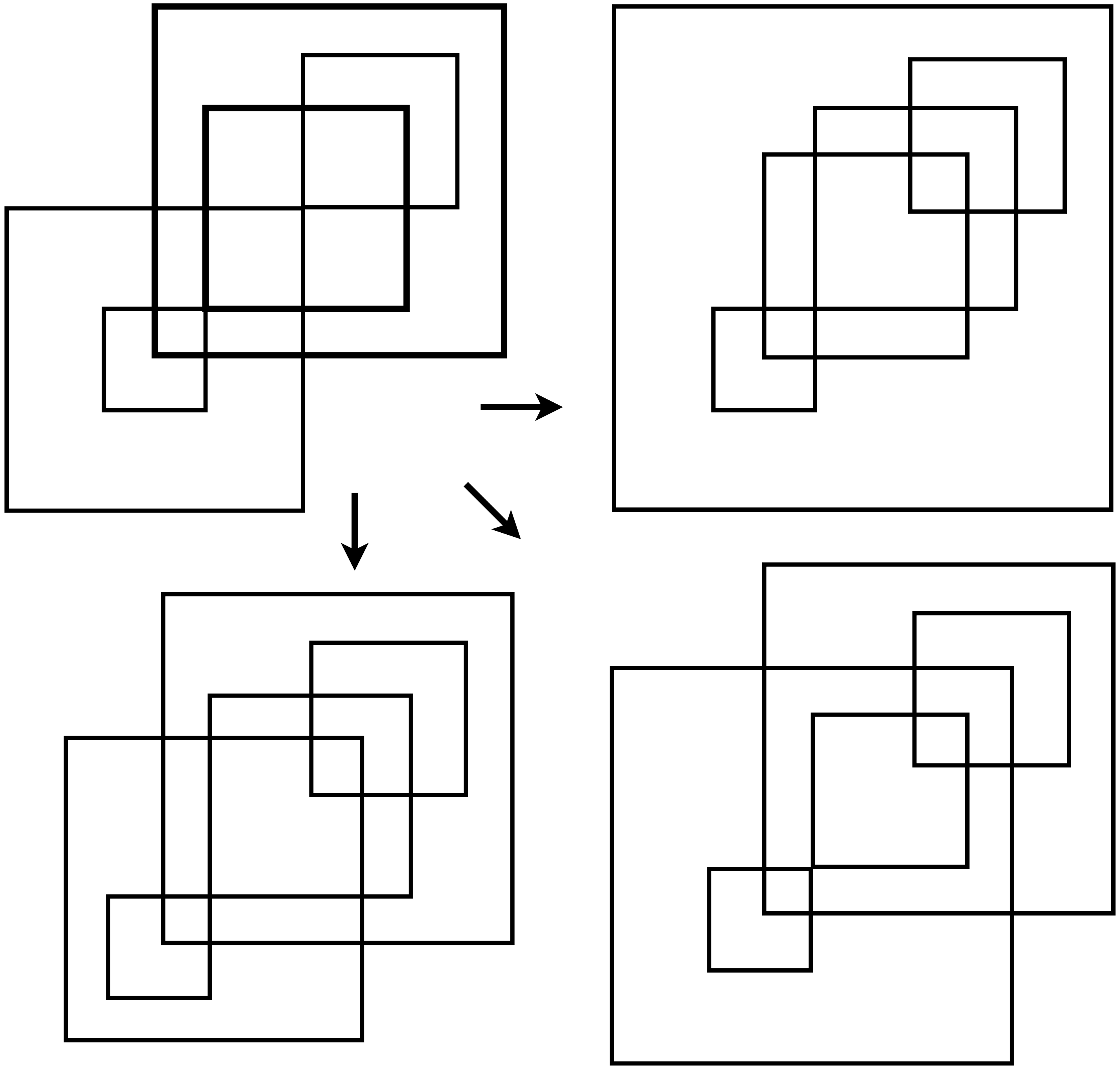


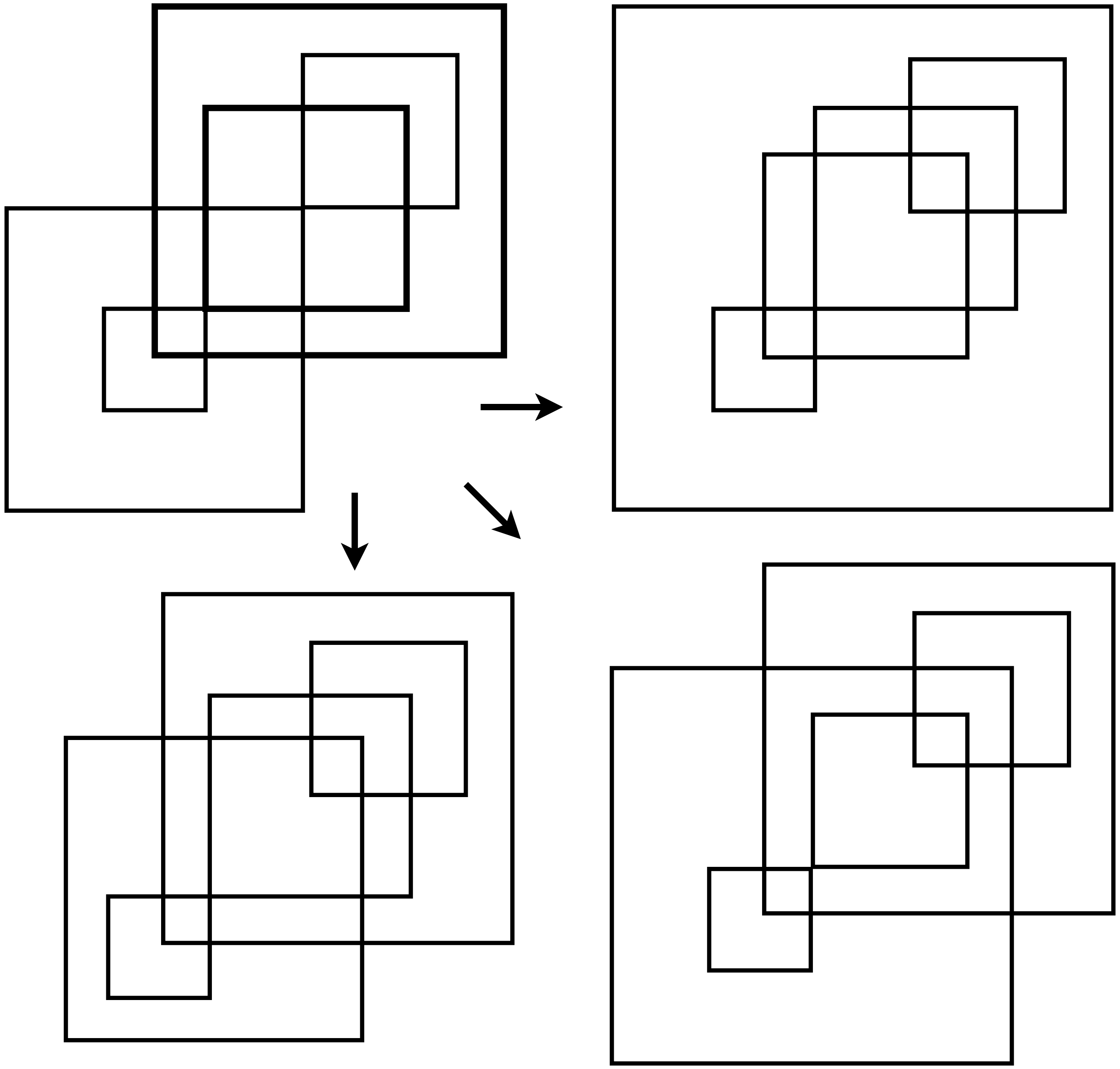






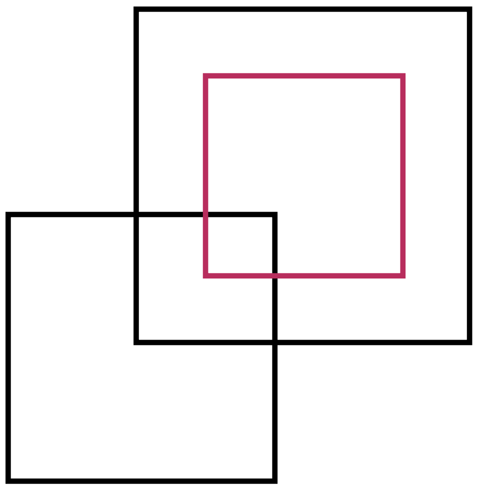


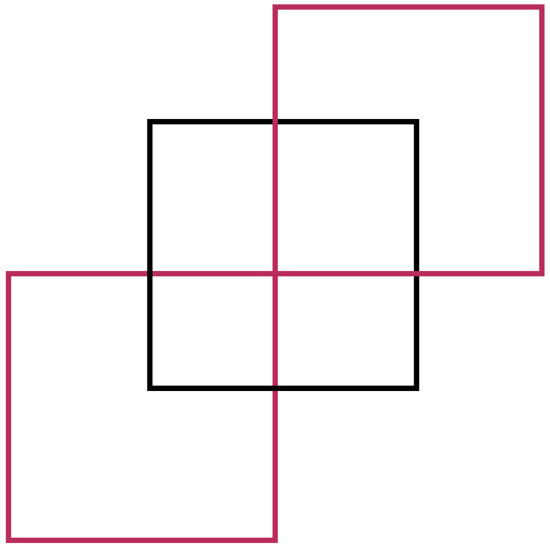


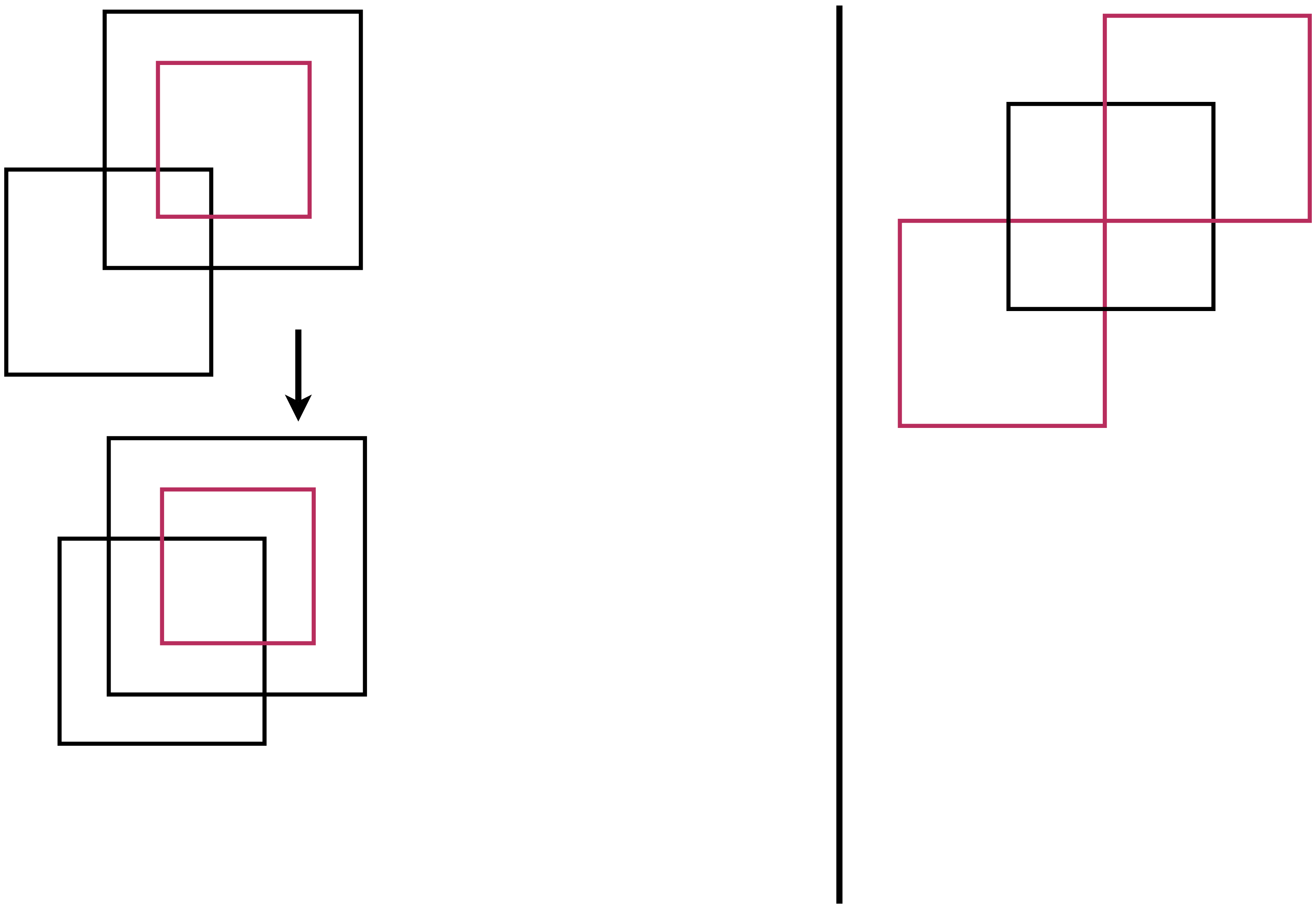


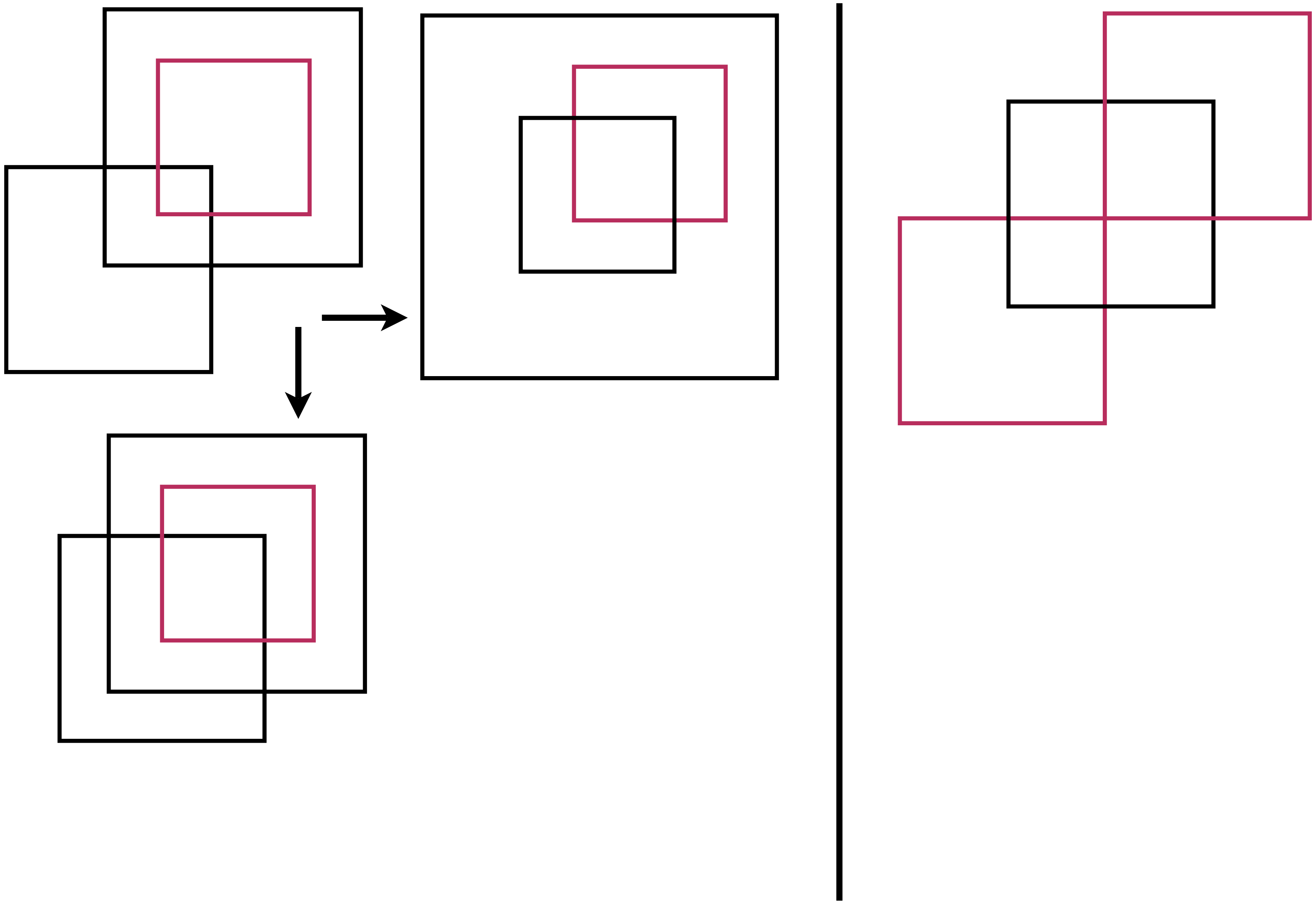
Theorem

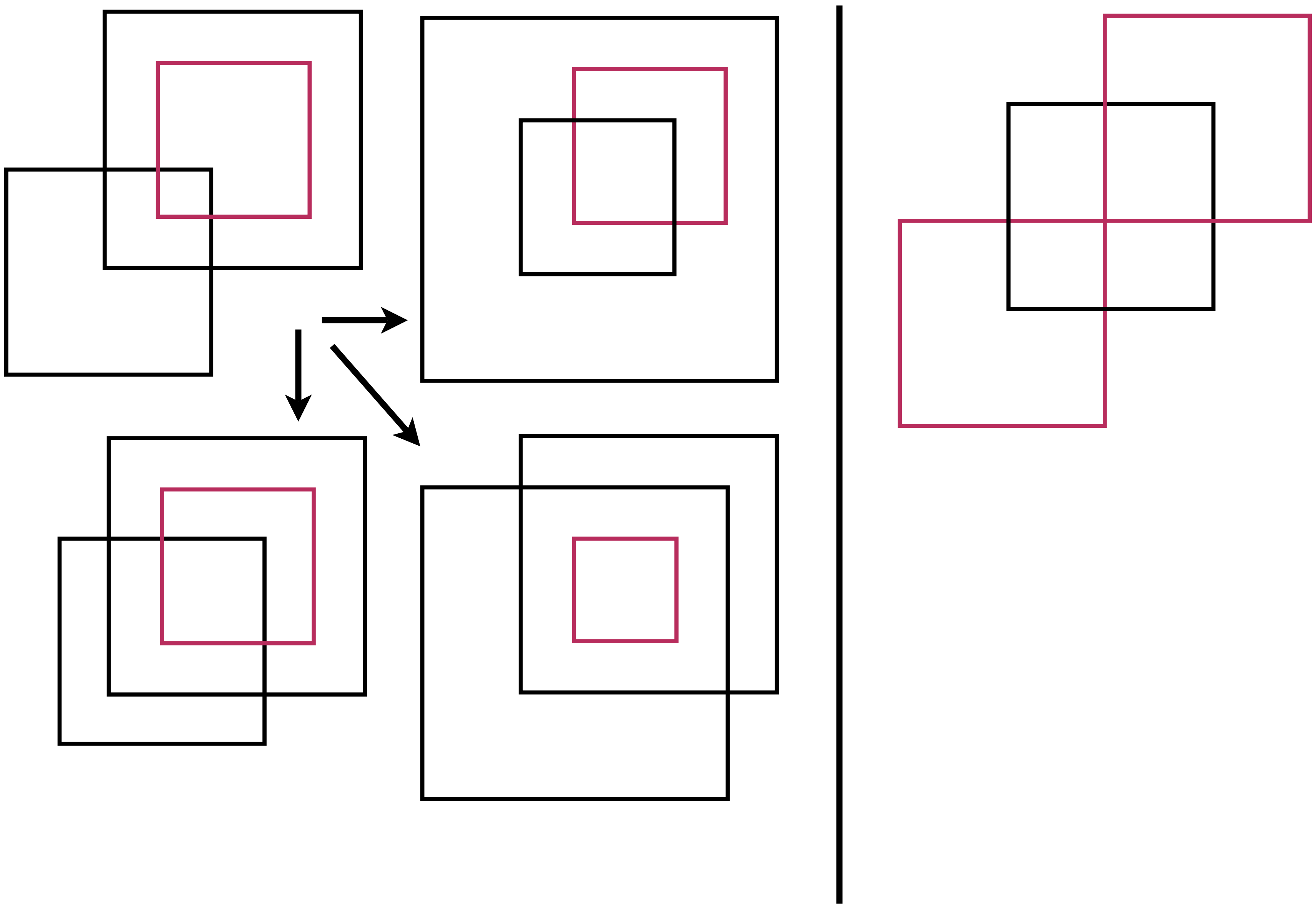
The flat limit of the degeneration is supported along the varieties just described. Each one occurs with multiplicity one in the limit.

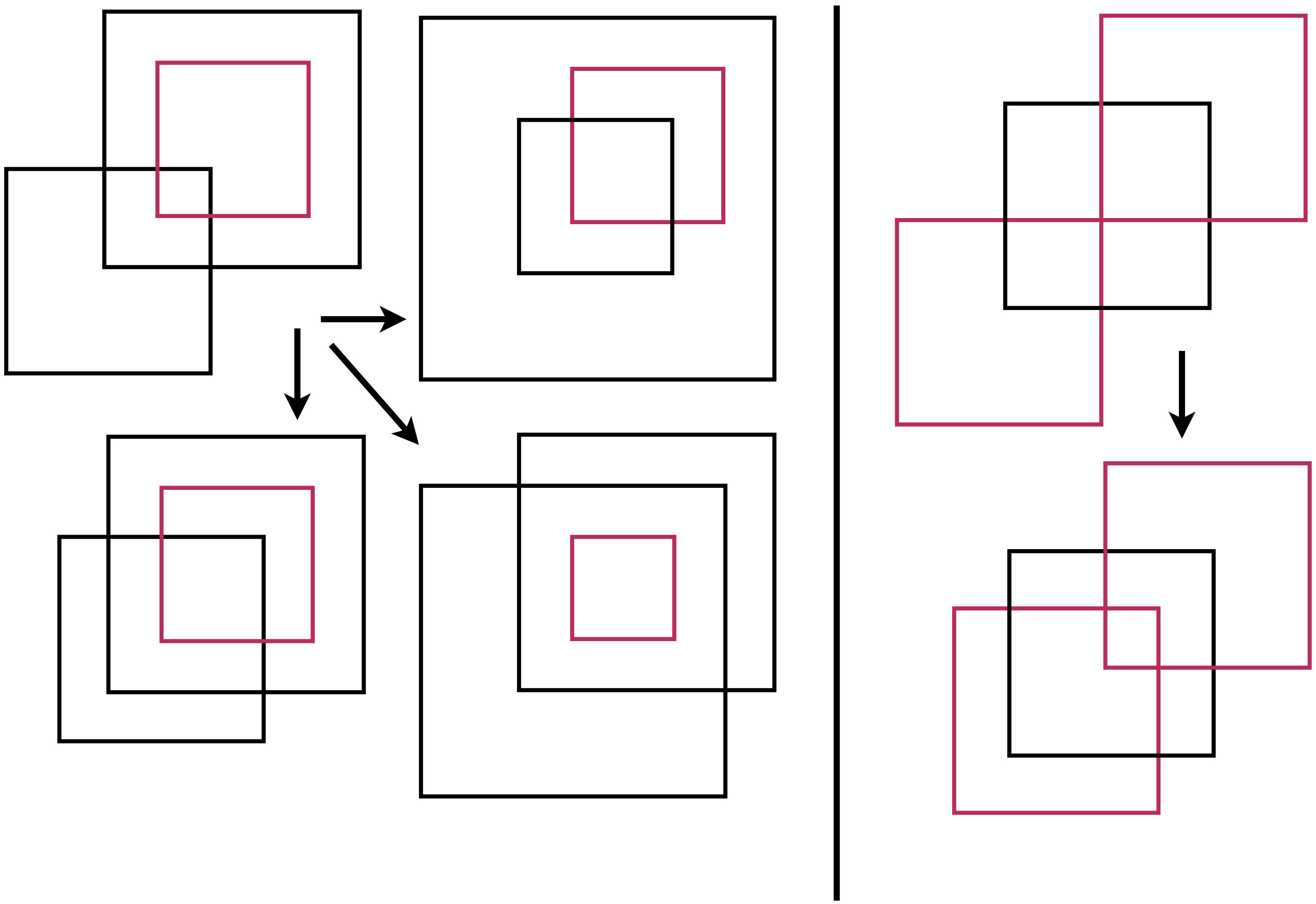


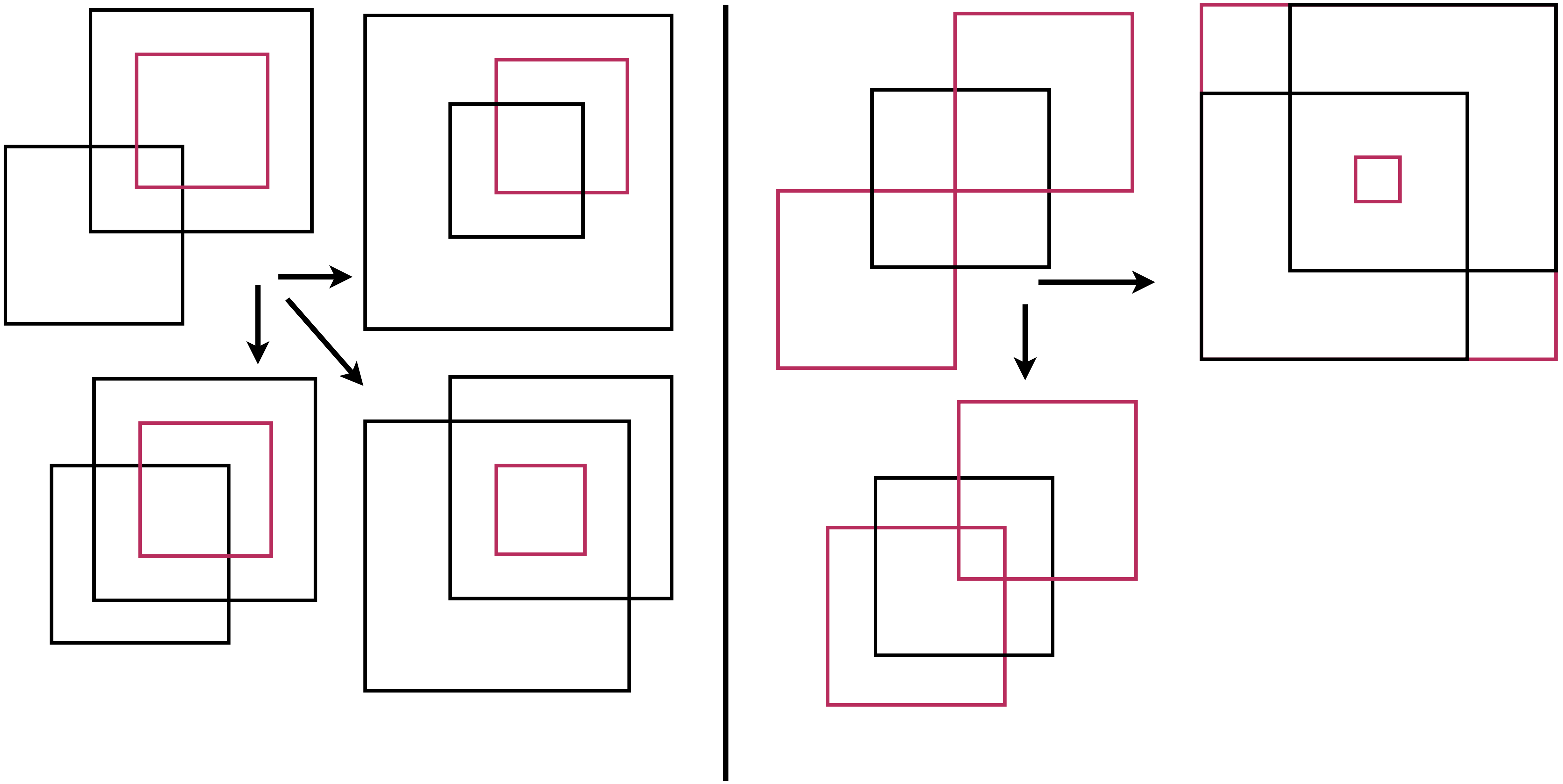


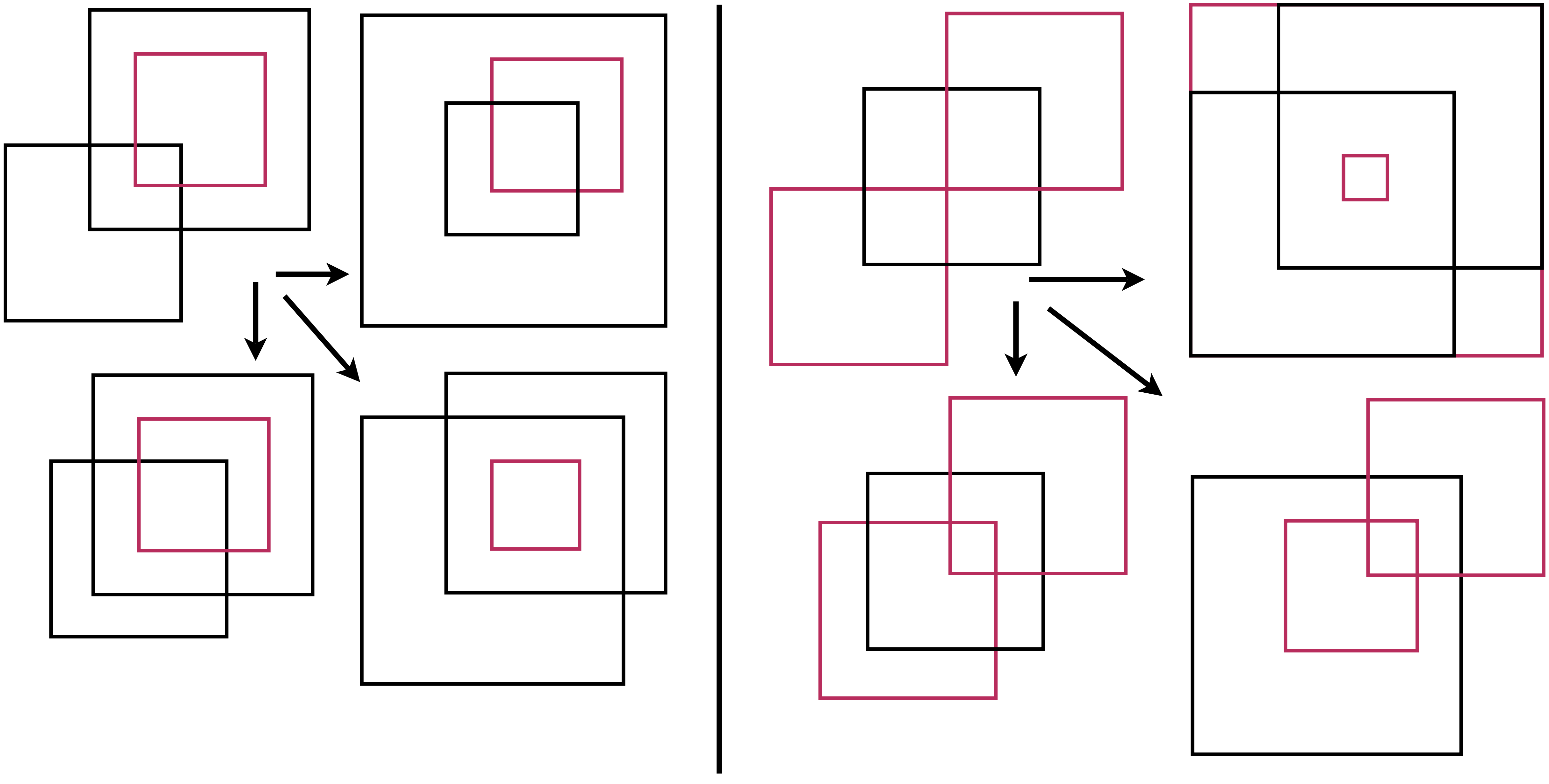


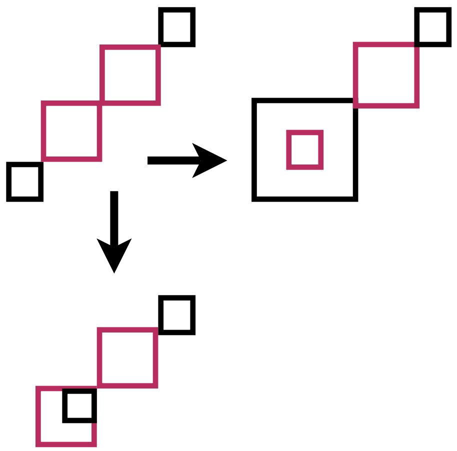


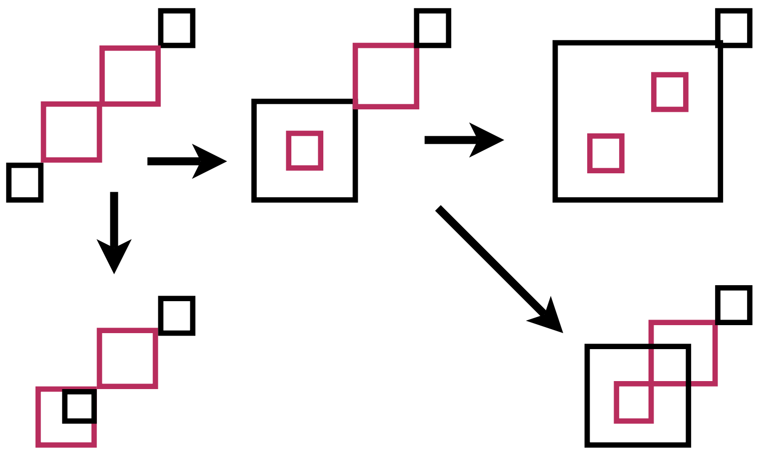


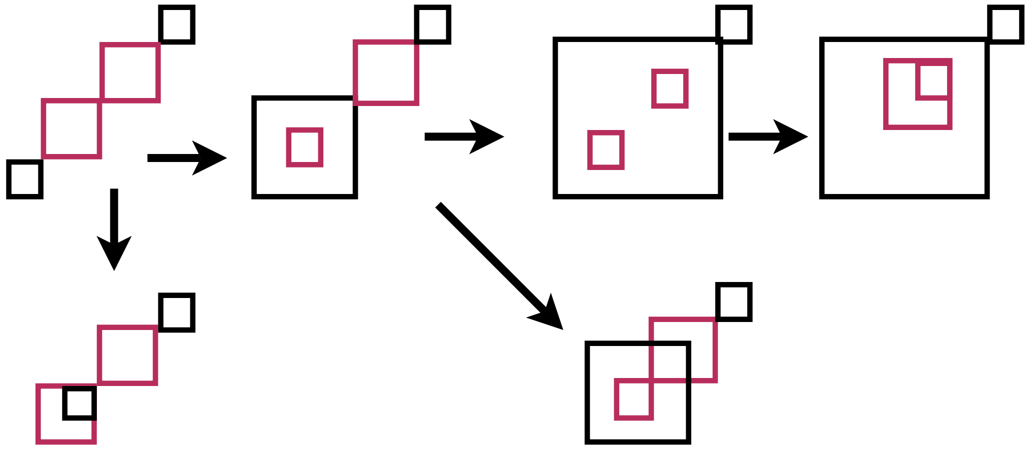


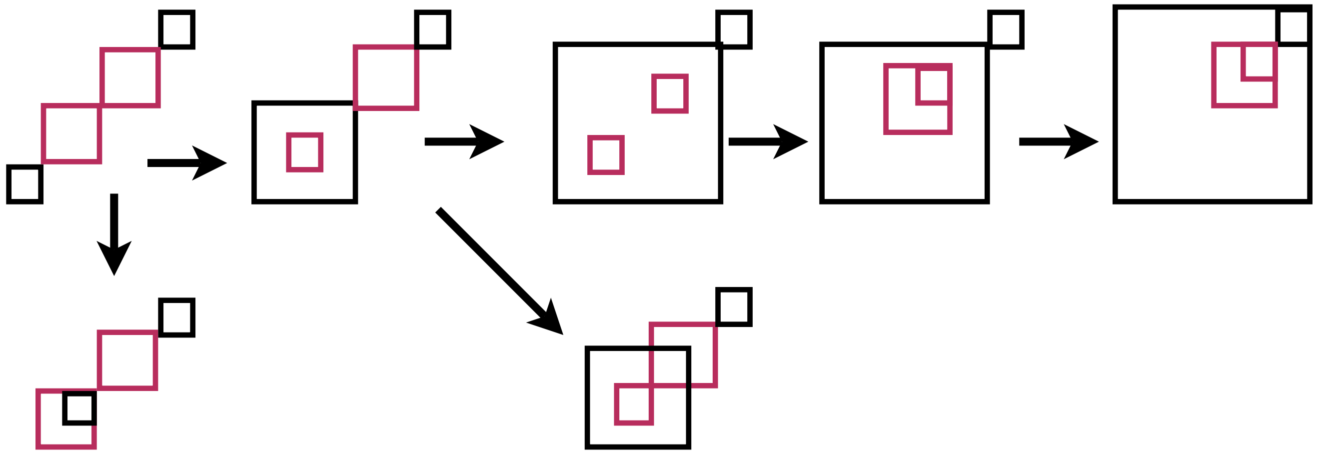


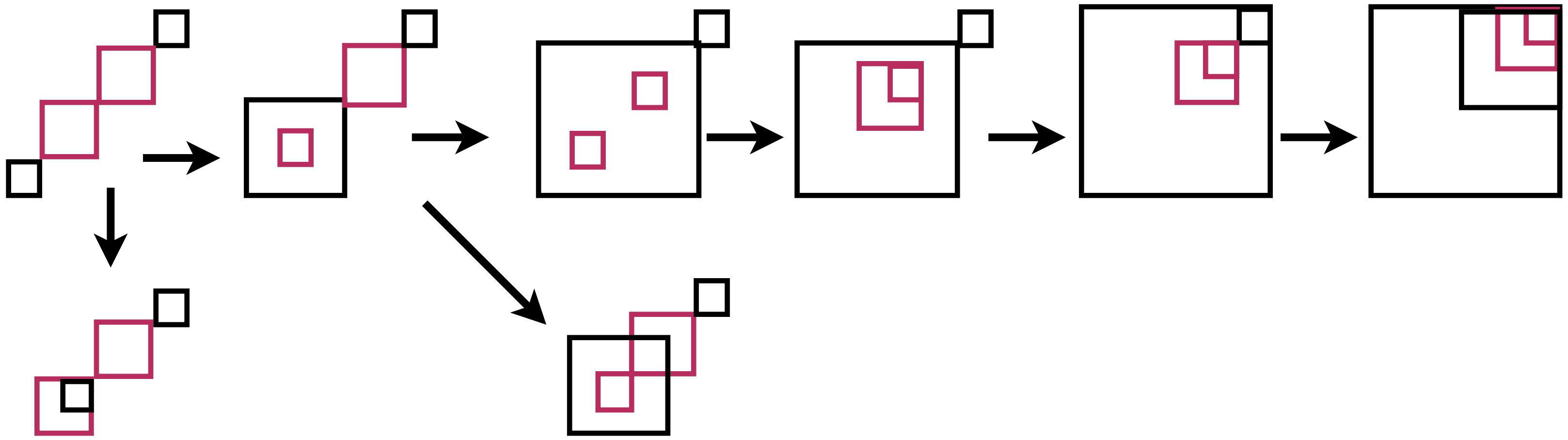


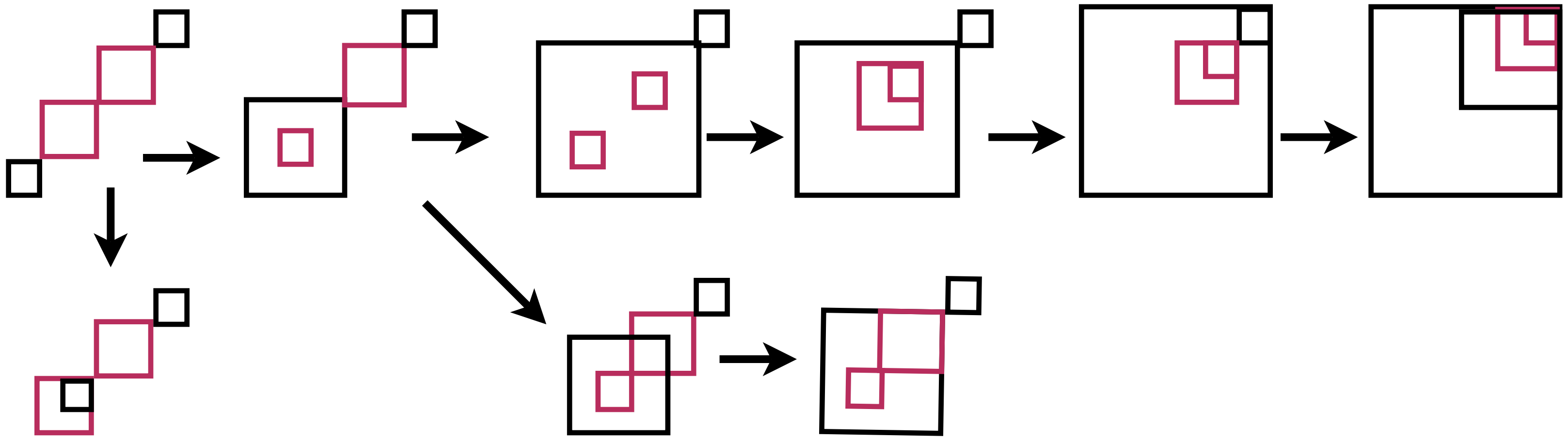


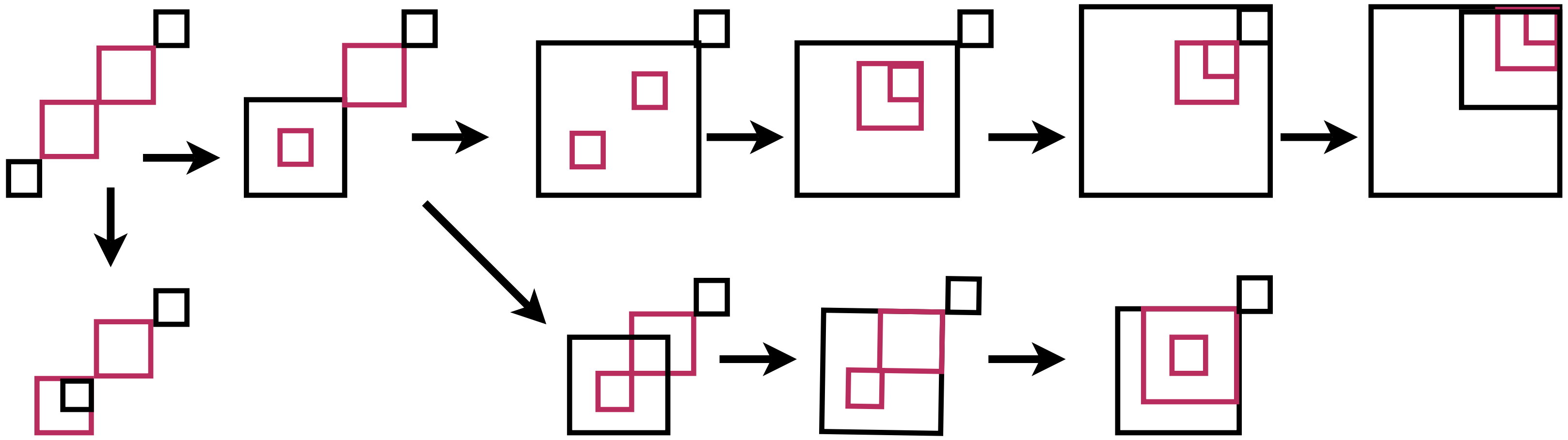


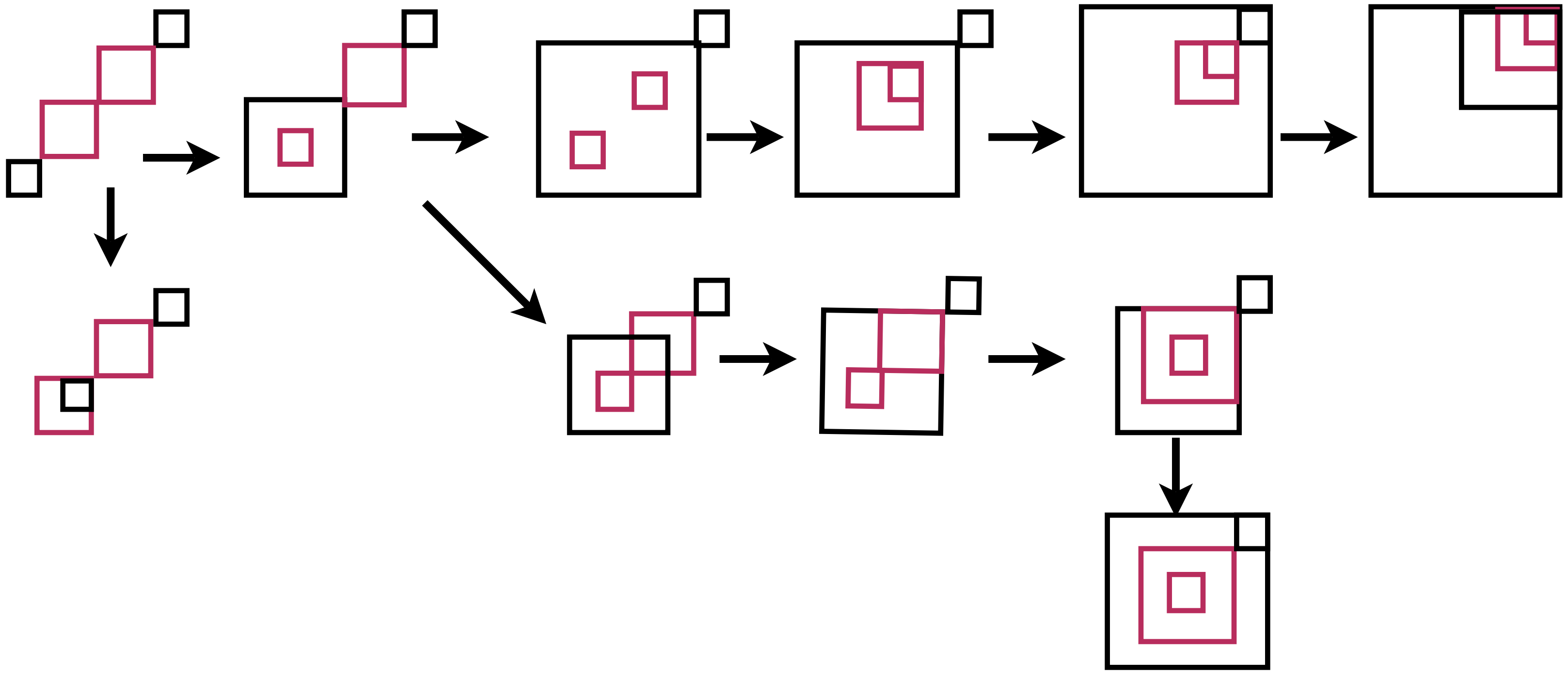


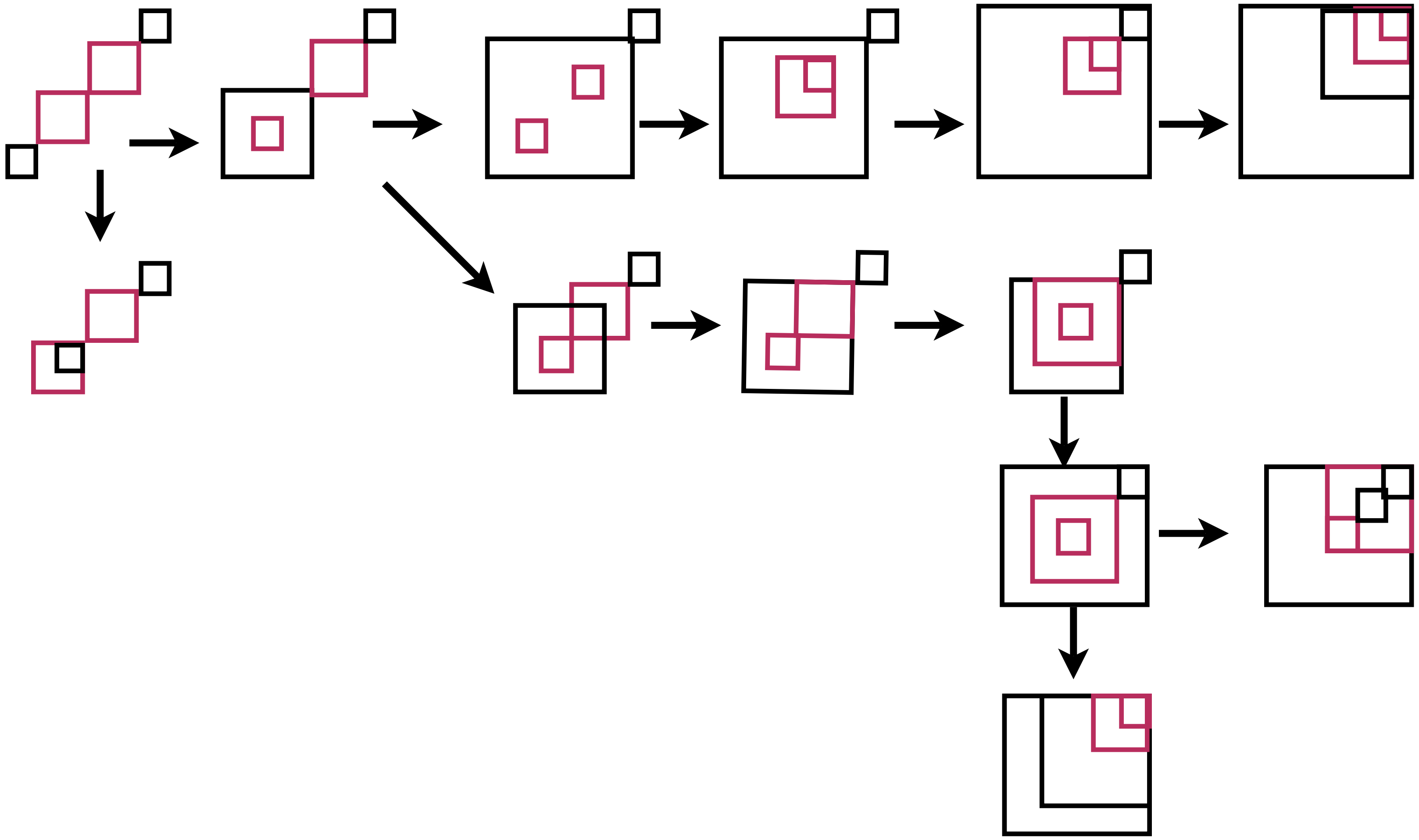


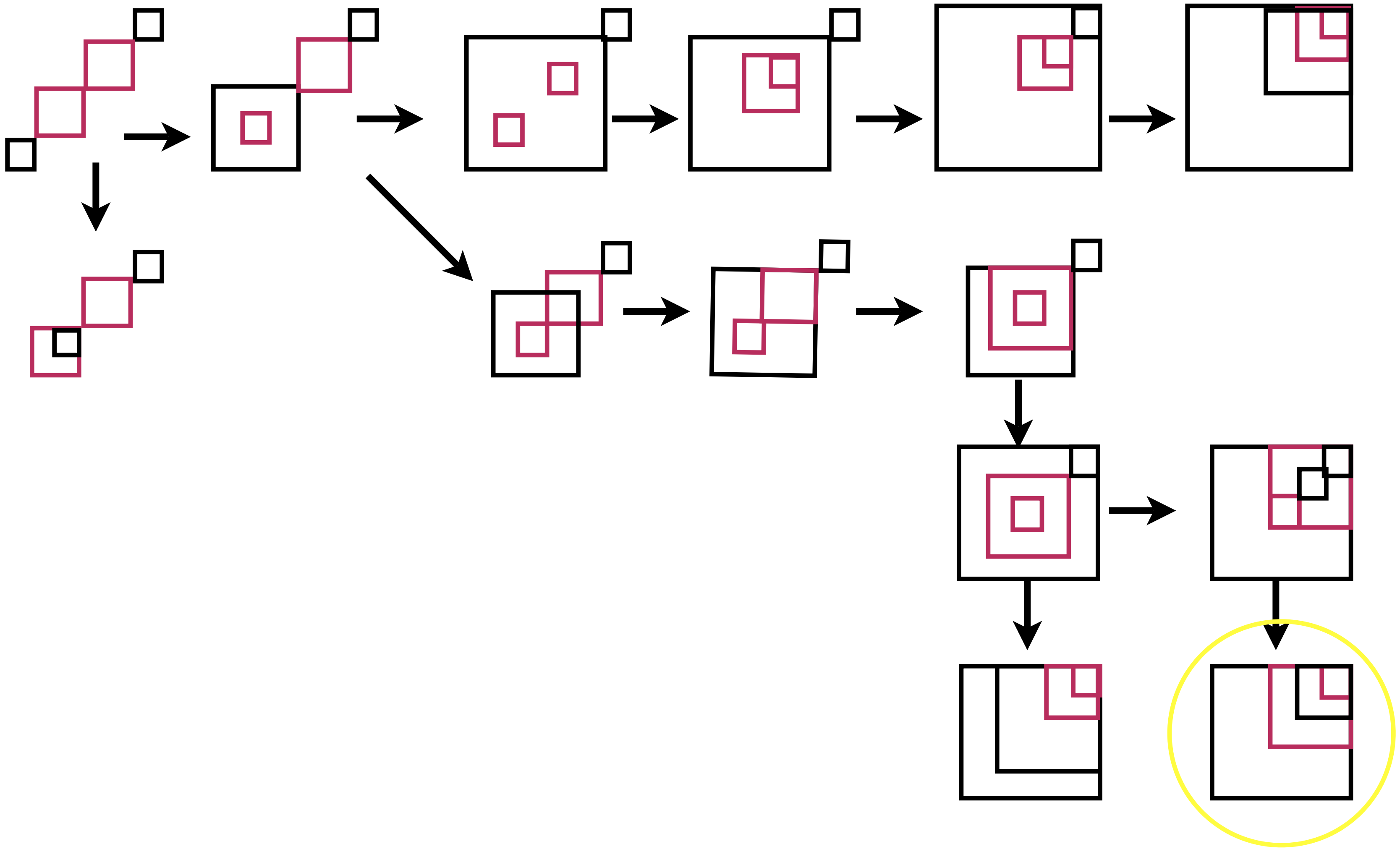


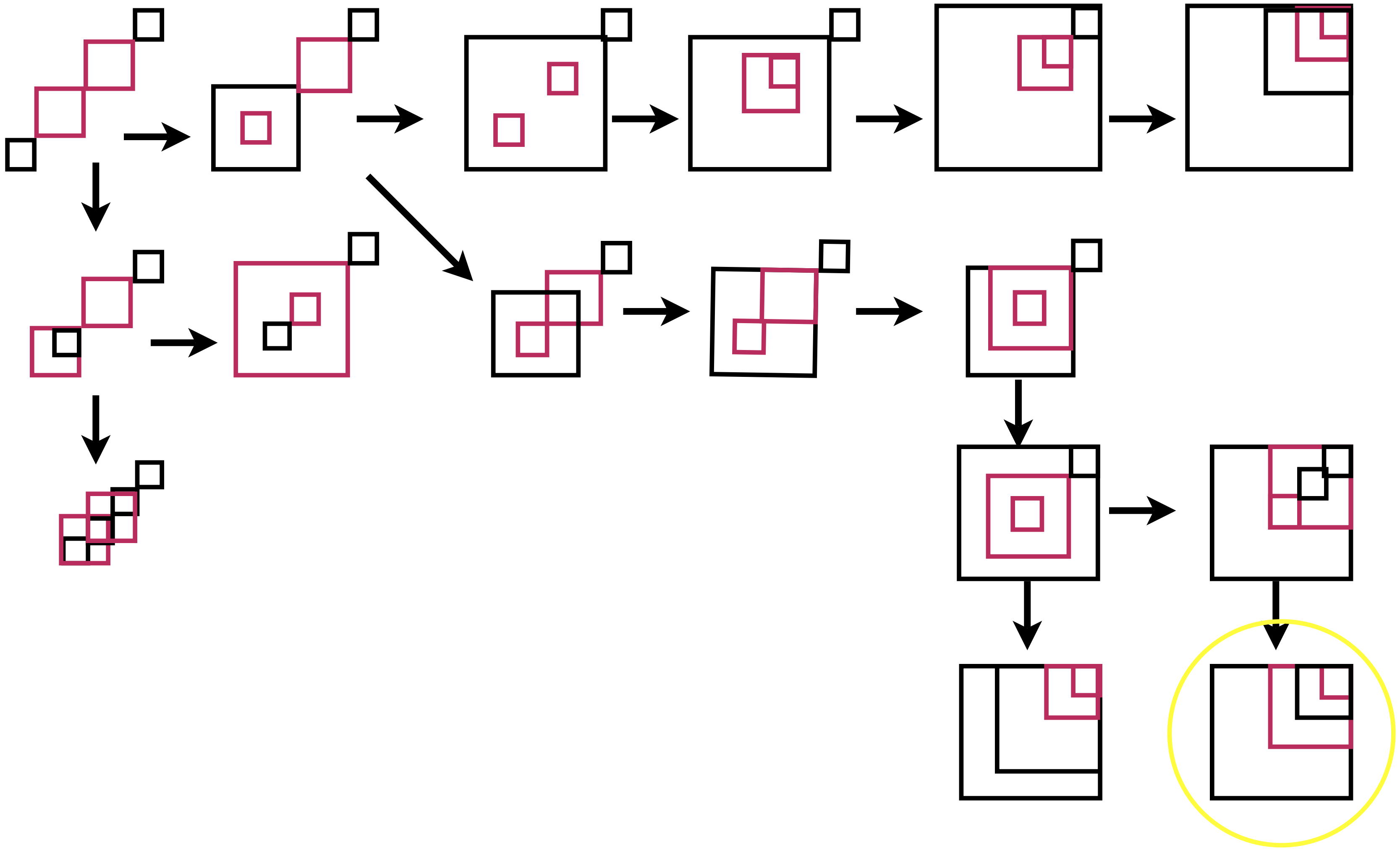


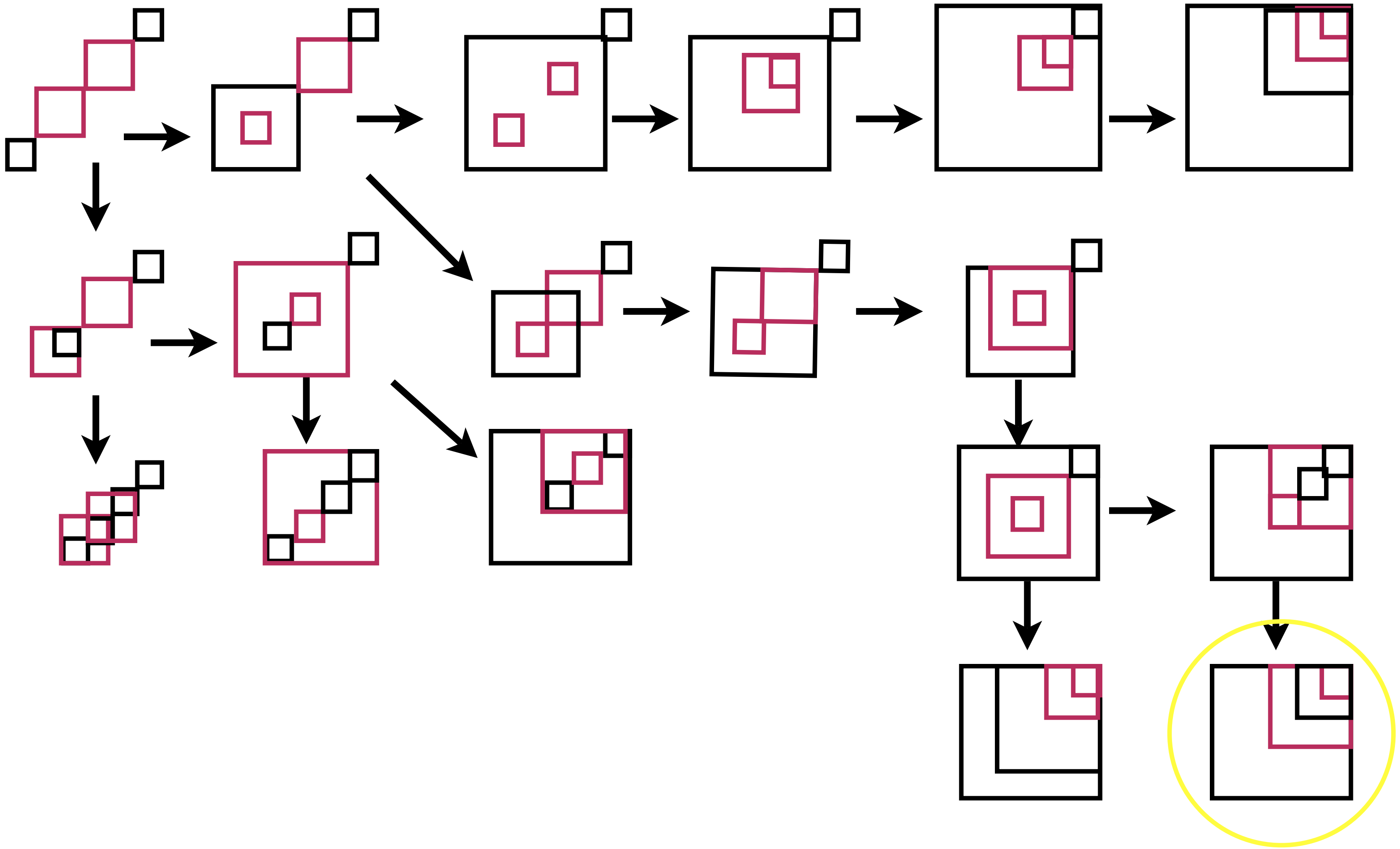


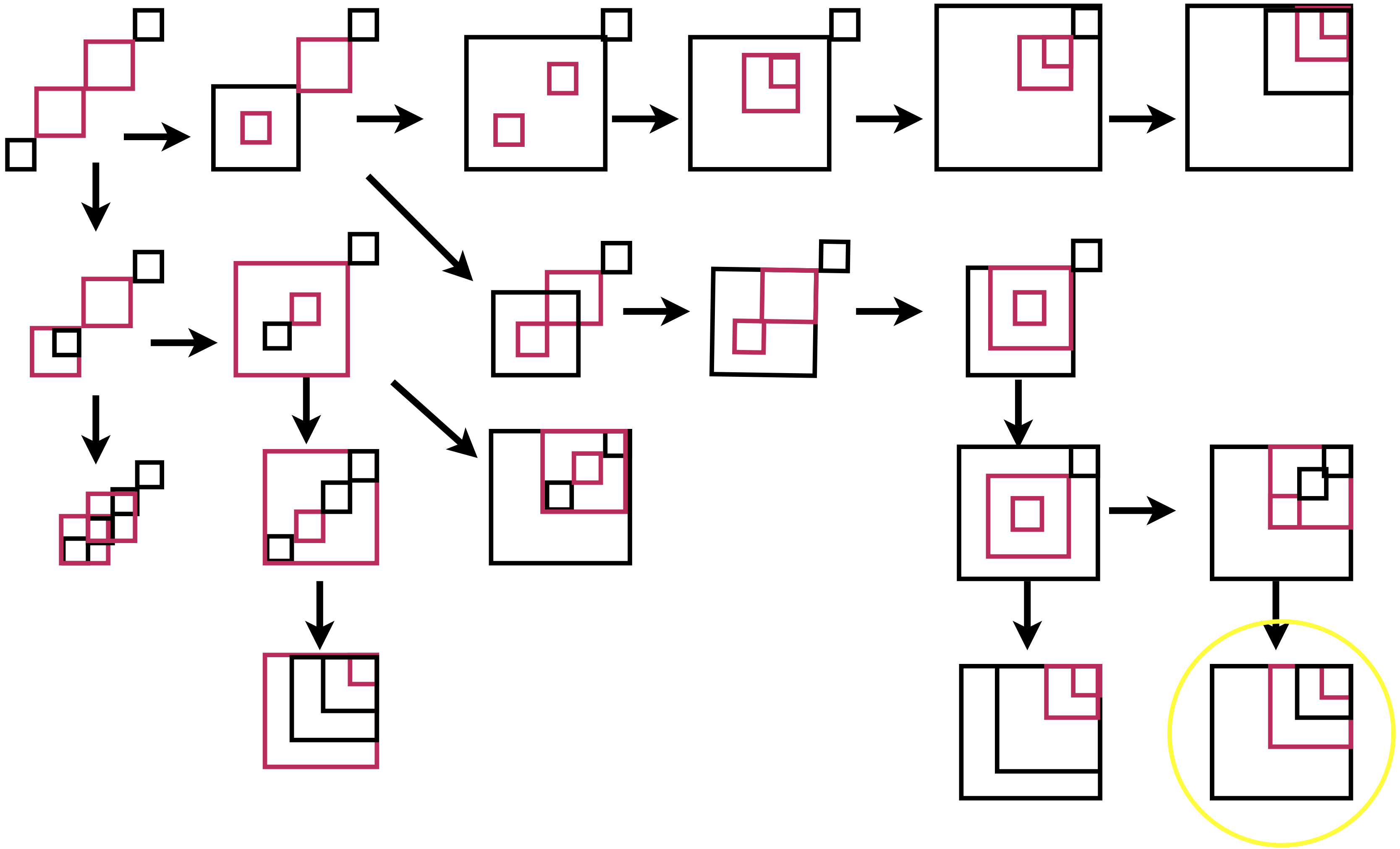


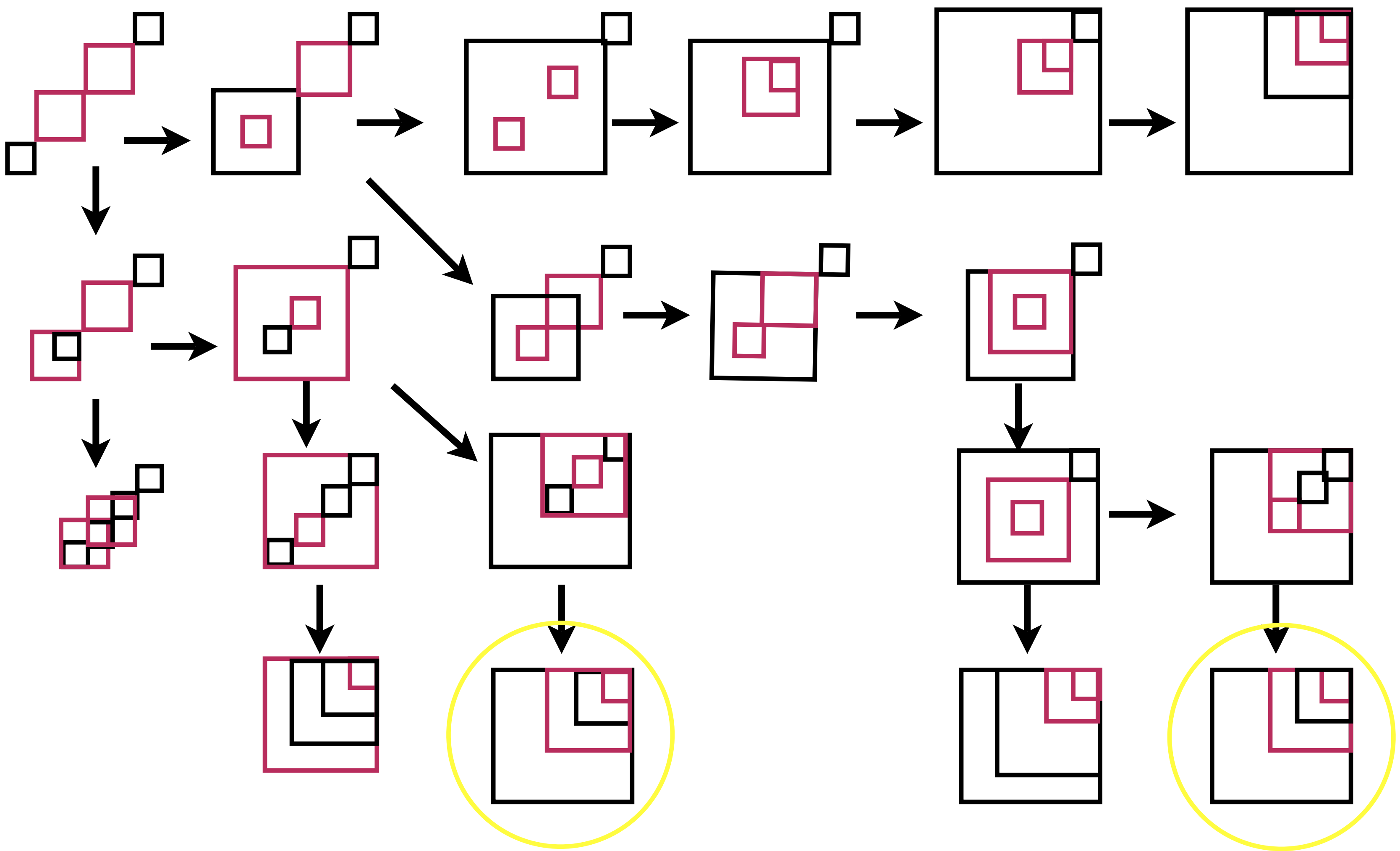


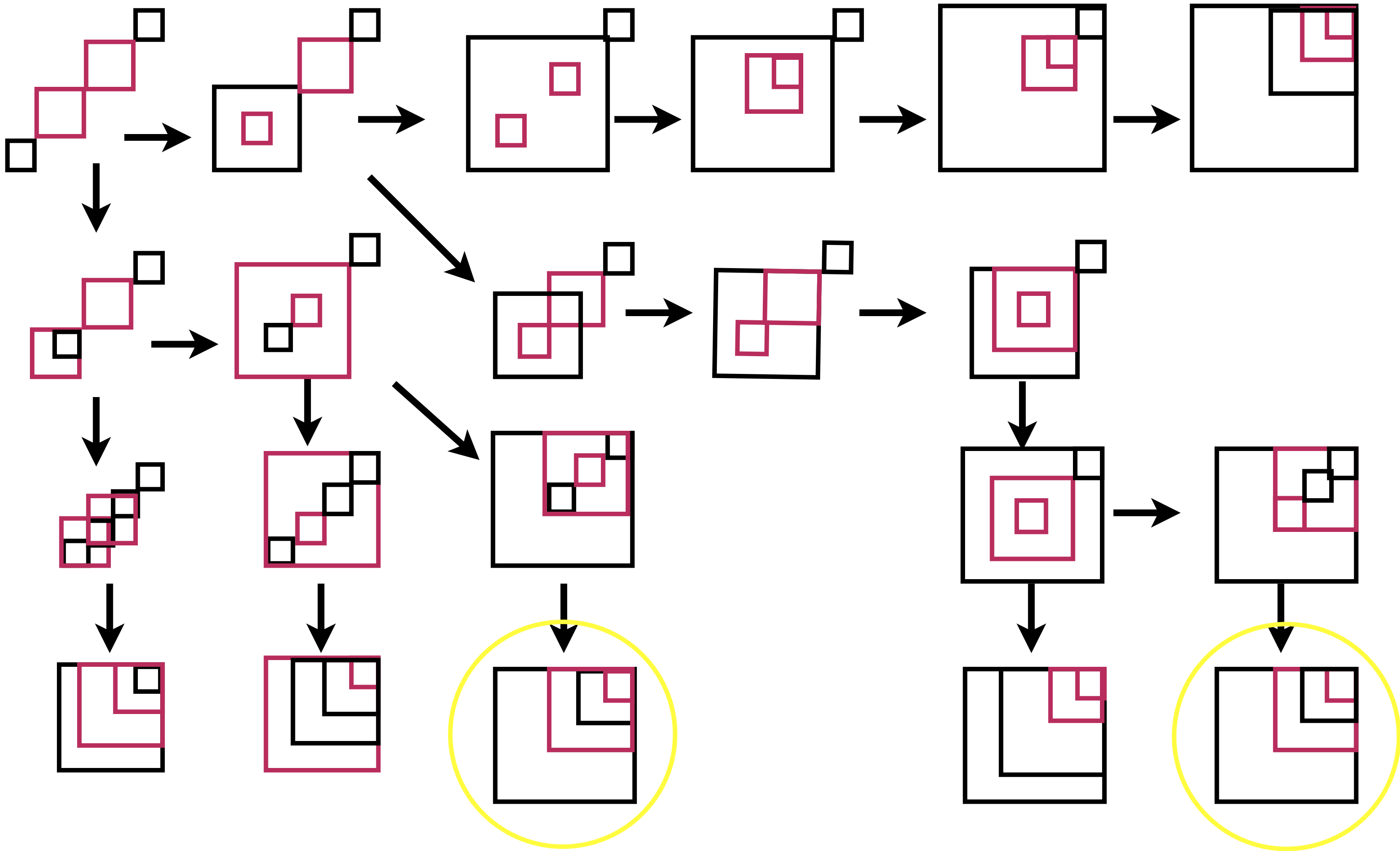


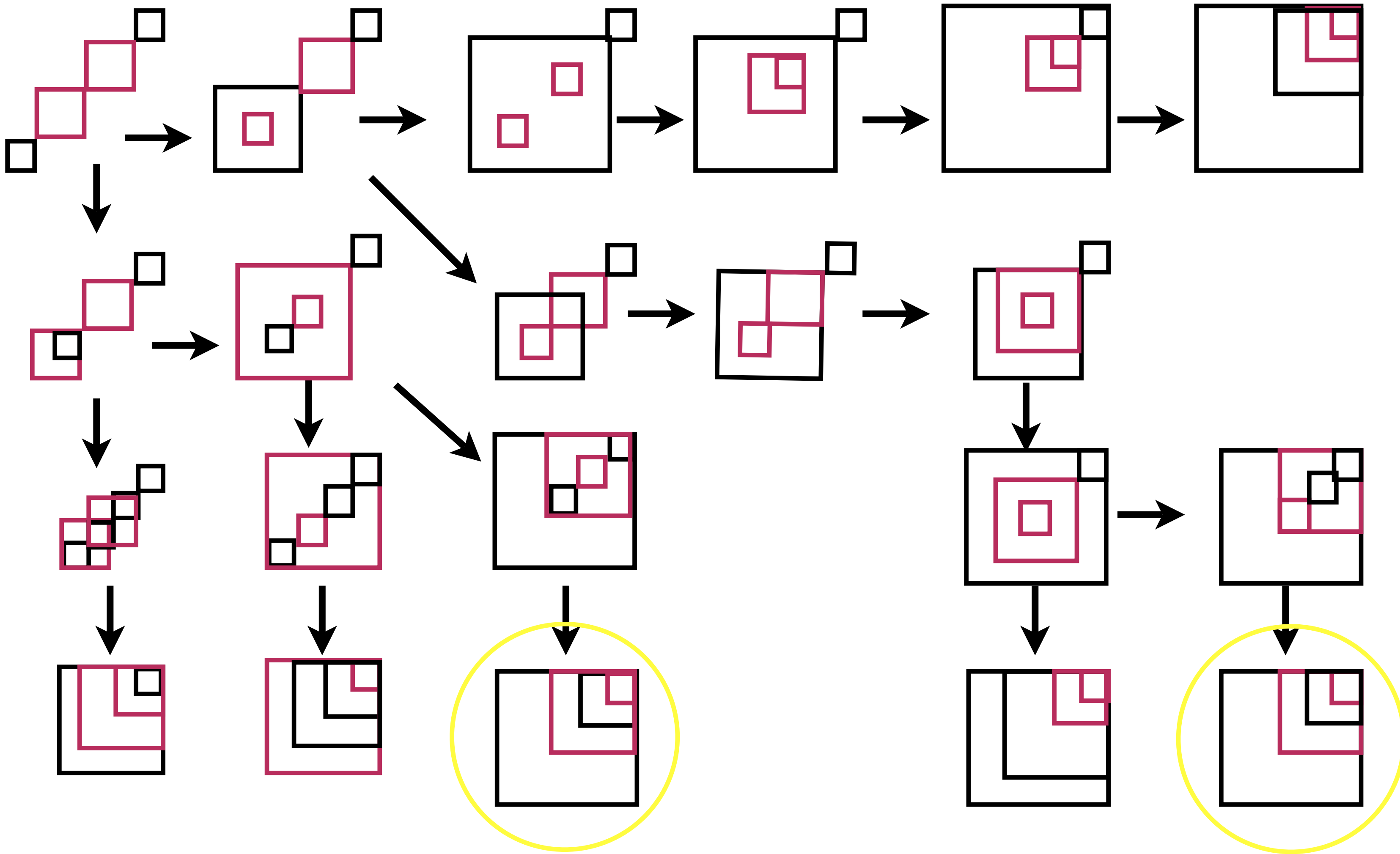






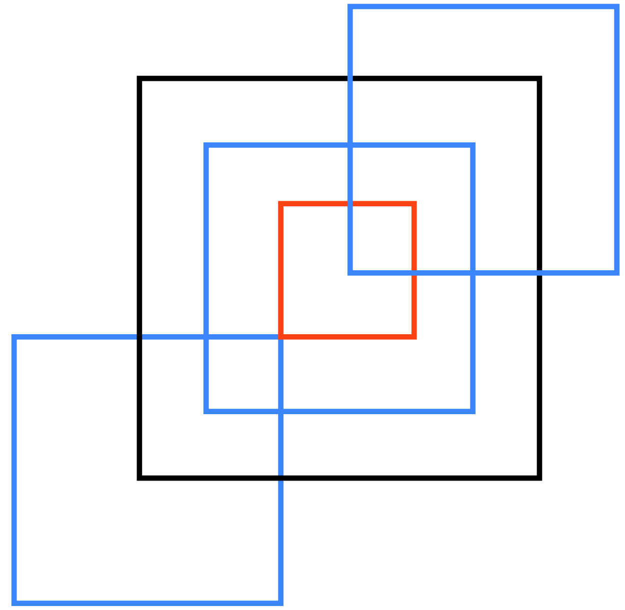


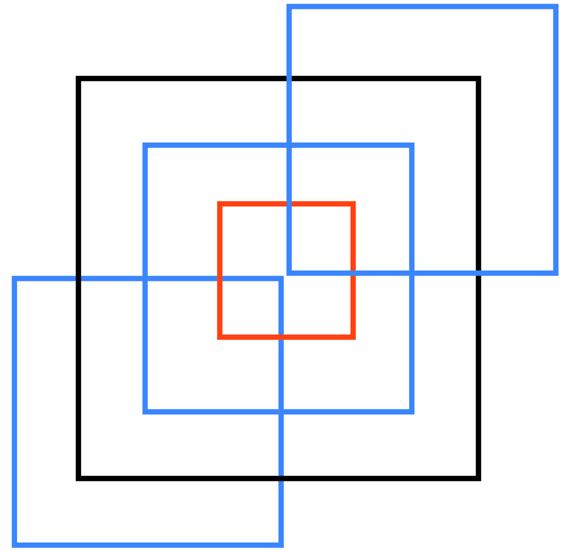


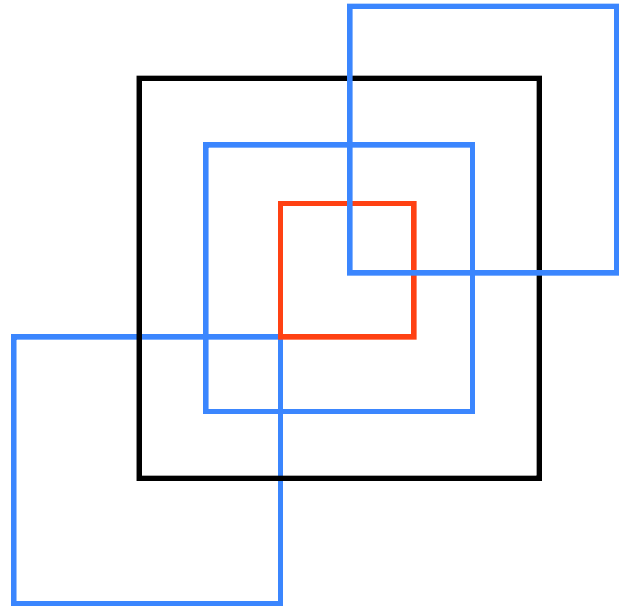


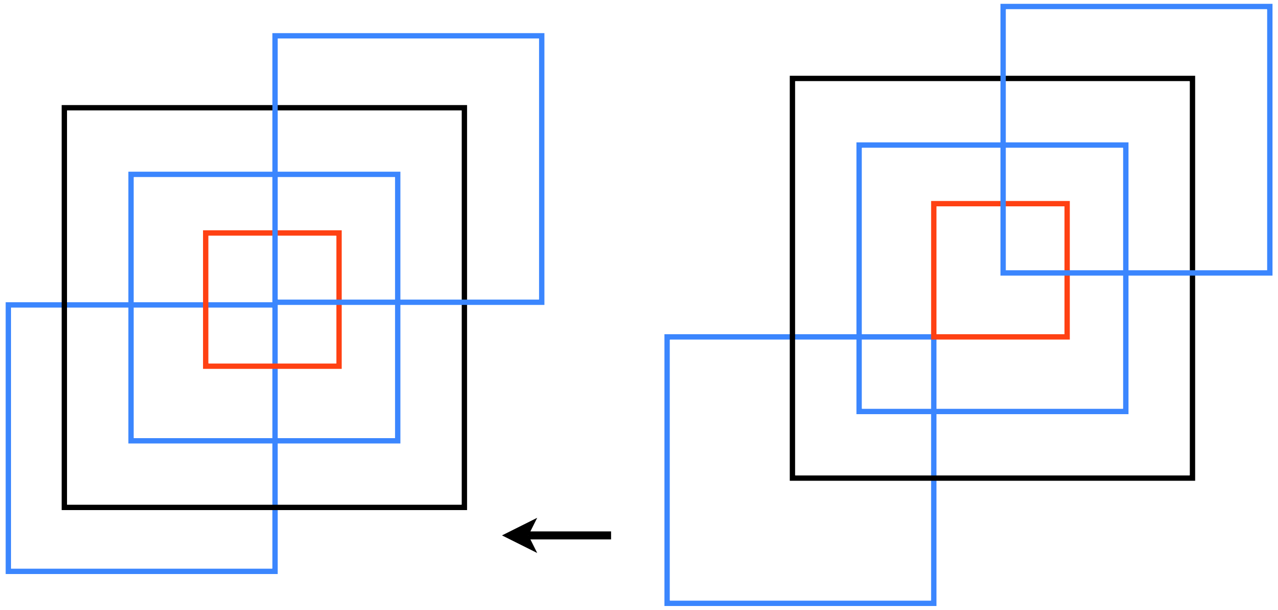
Theorem (Buch-Kresch-Tamvakis)

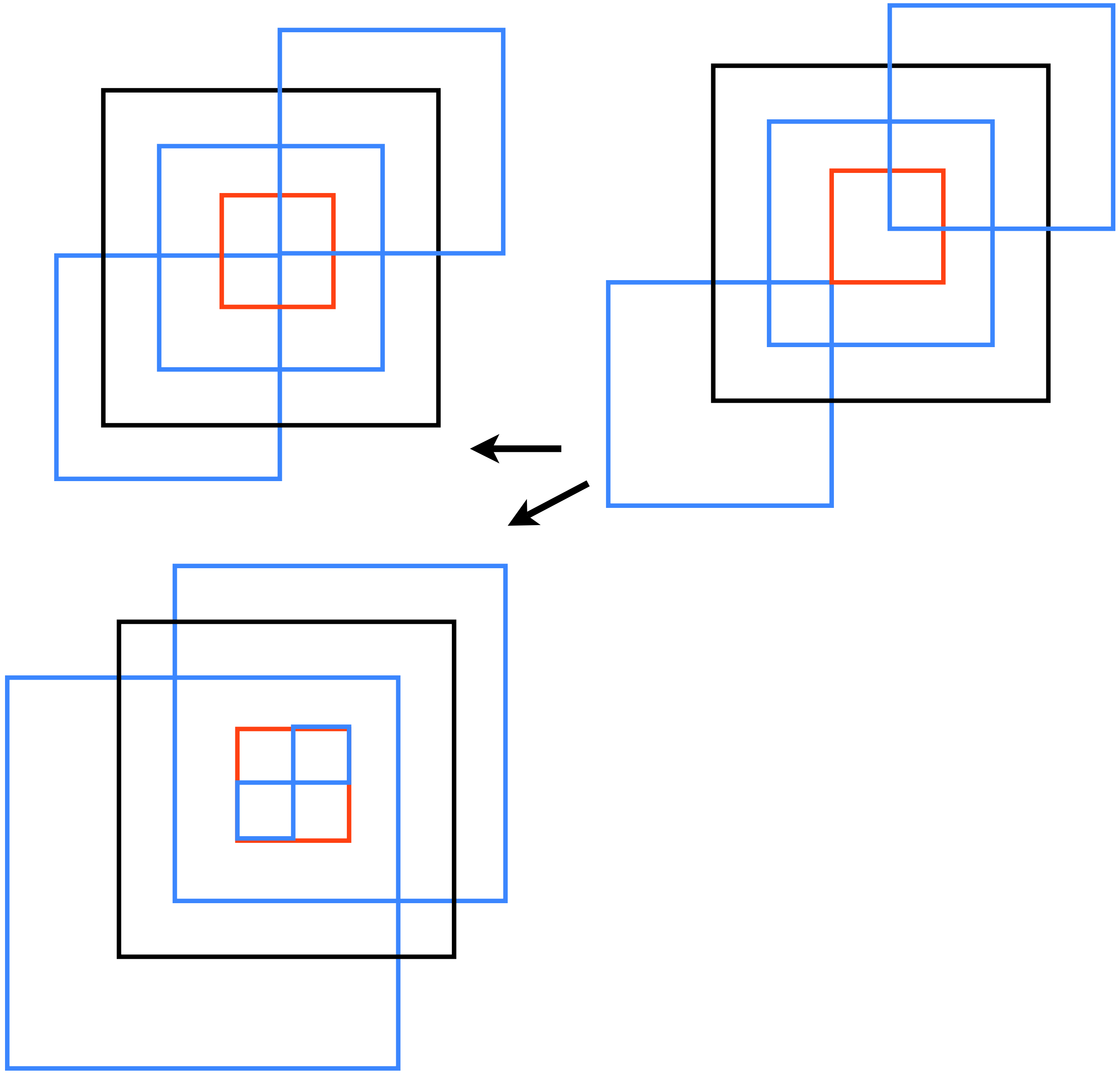
This rule also gives a positive rule for the quantum cohomology of Grassmannians.

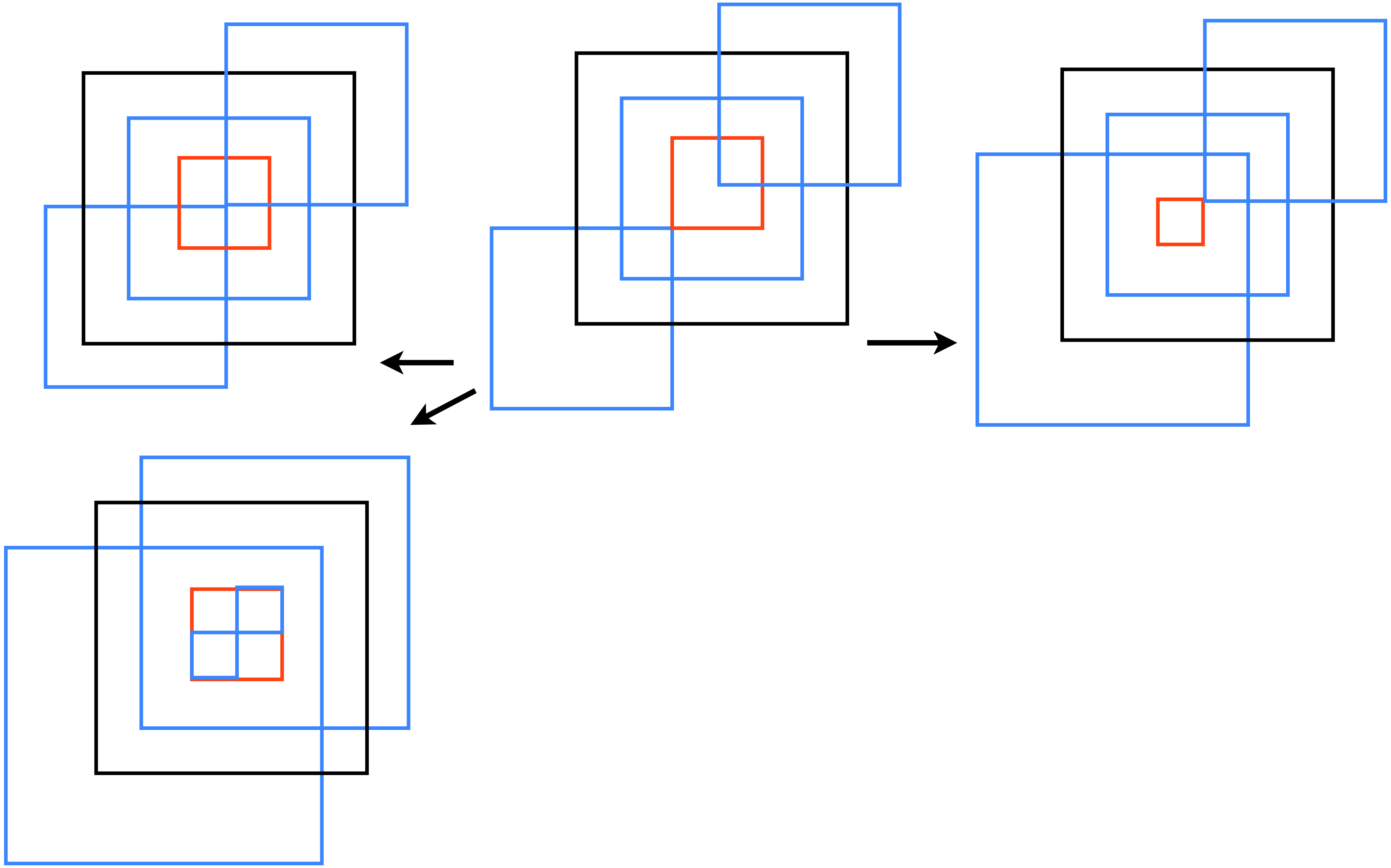


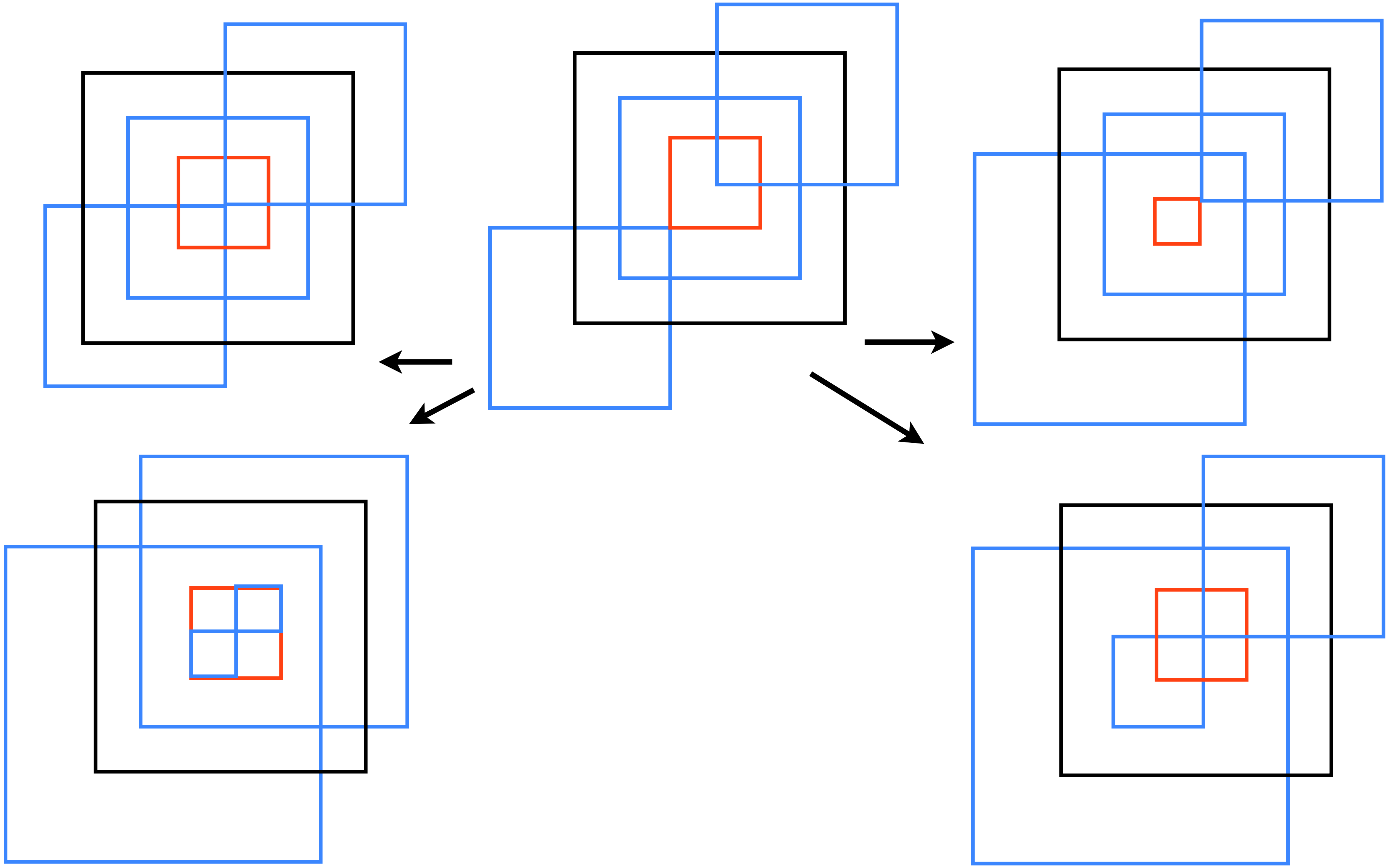


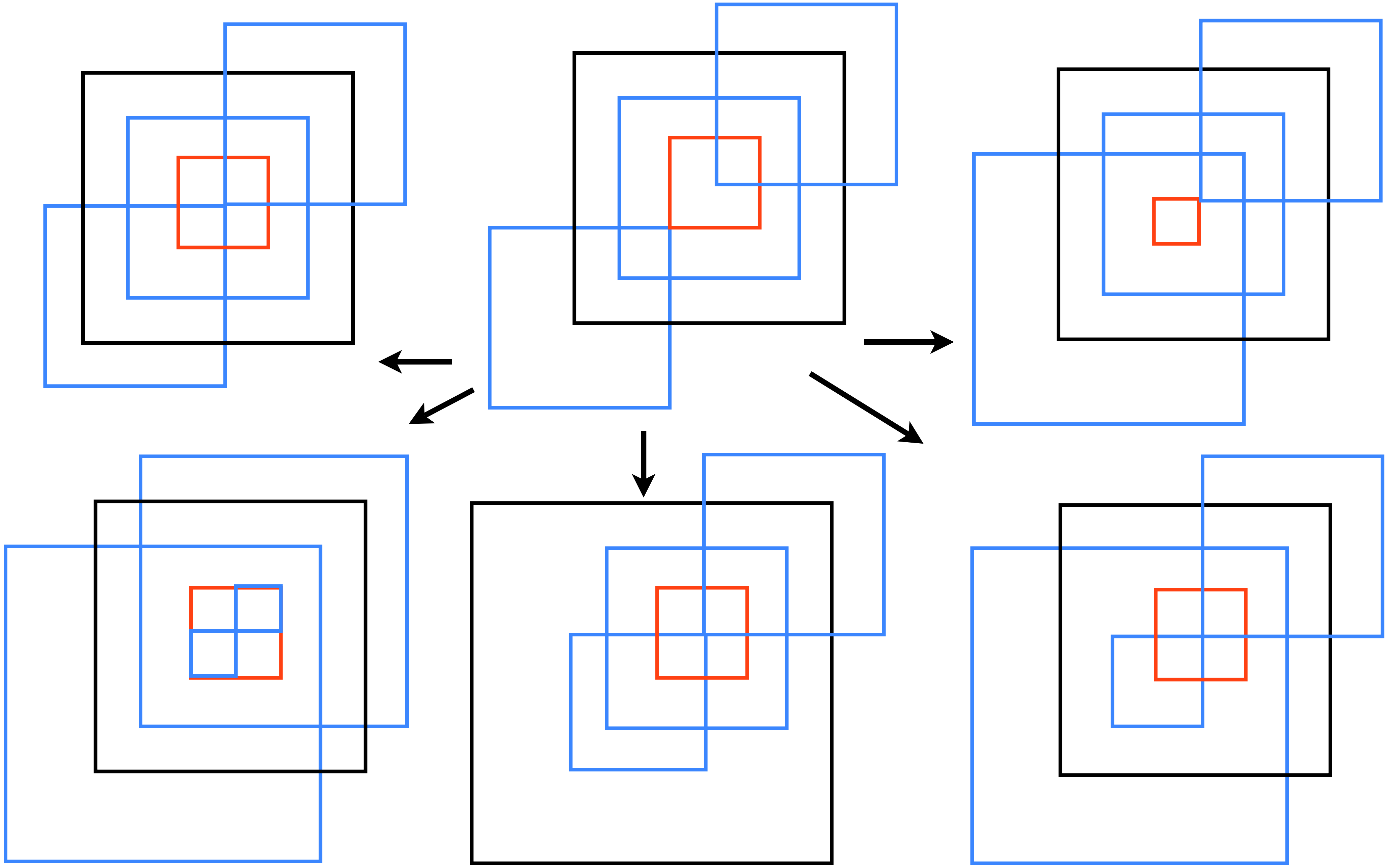


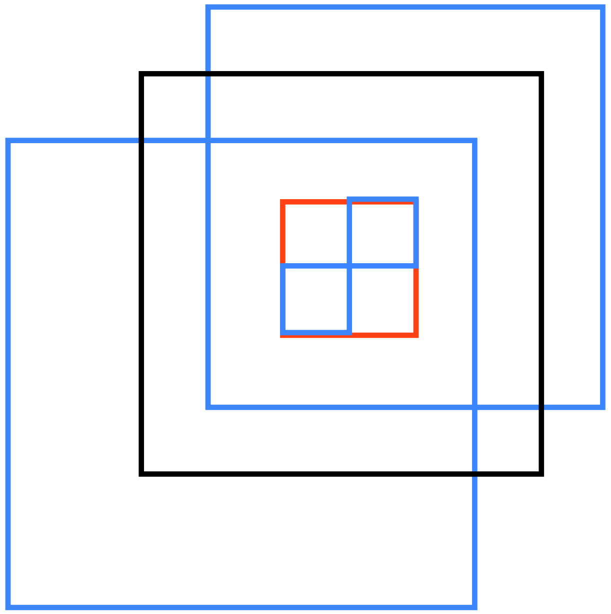


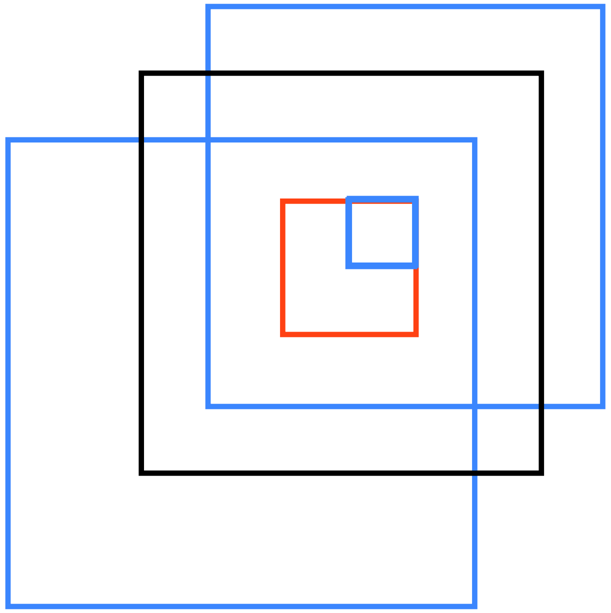


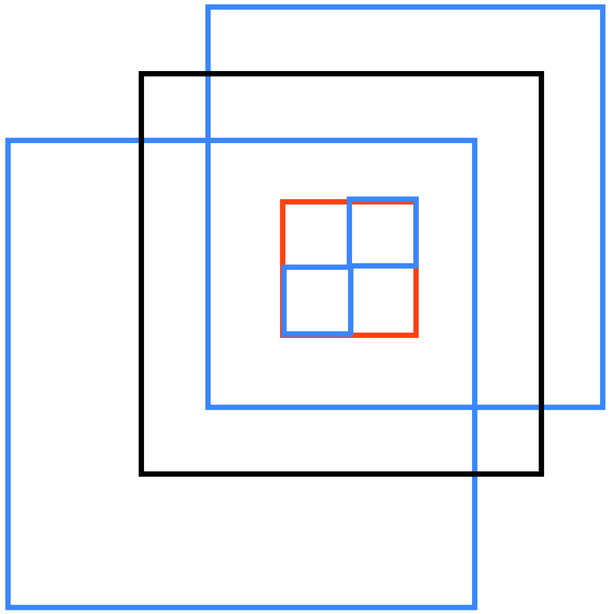


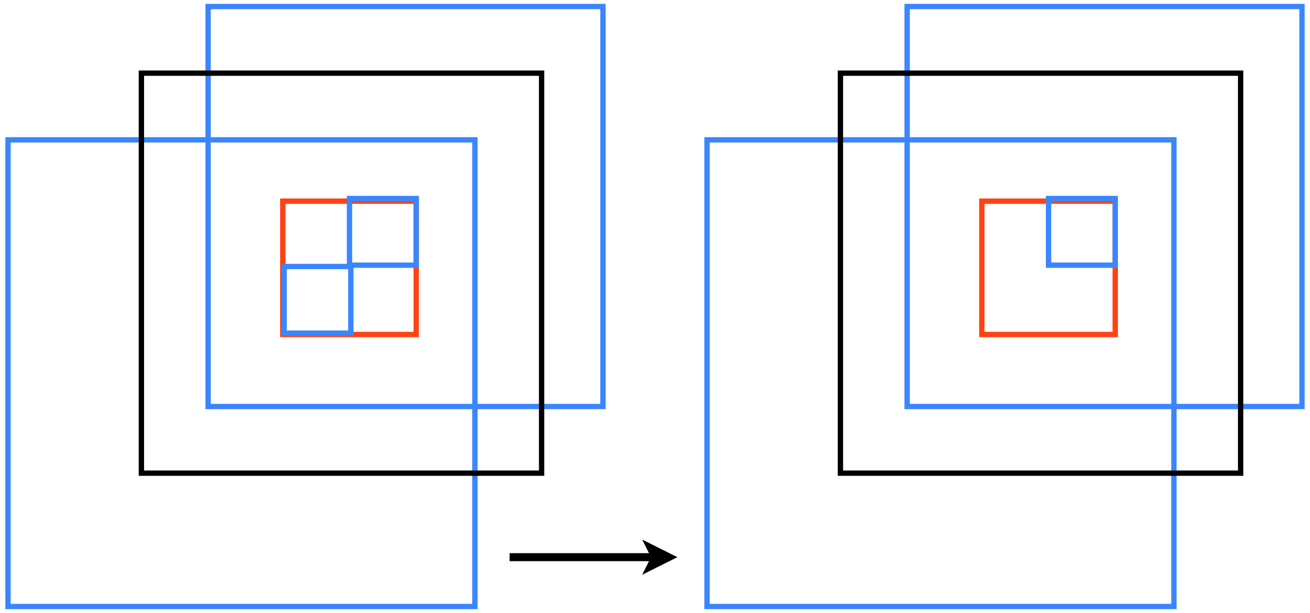


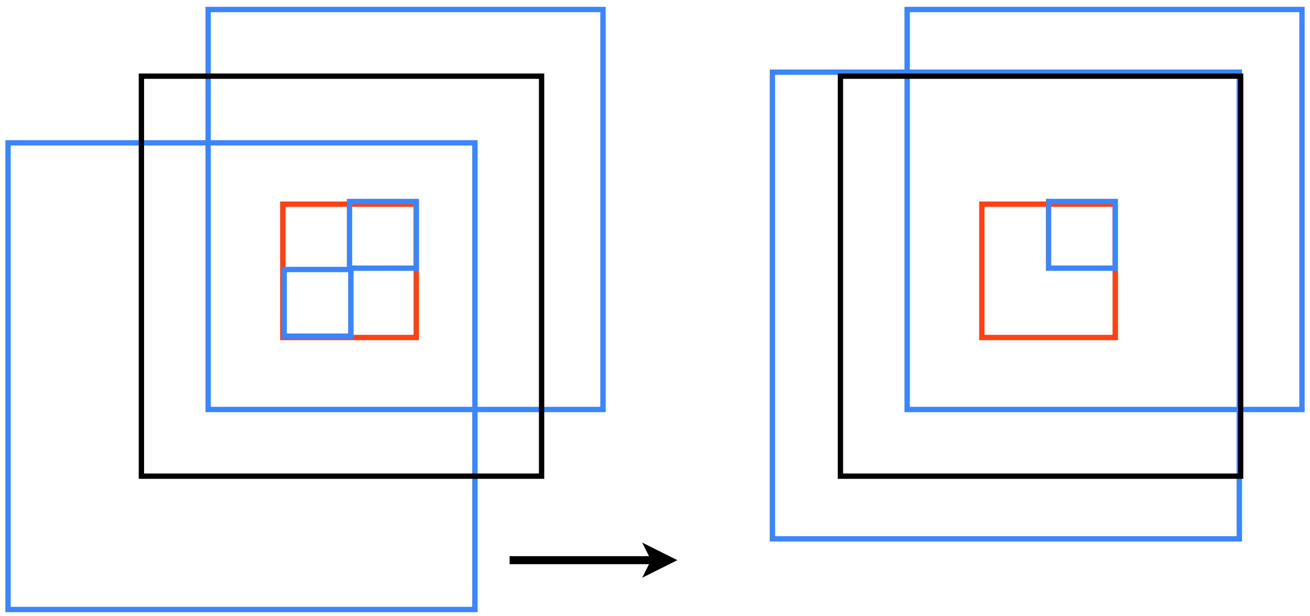


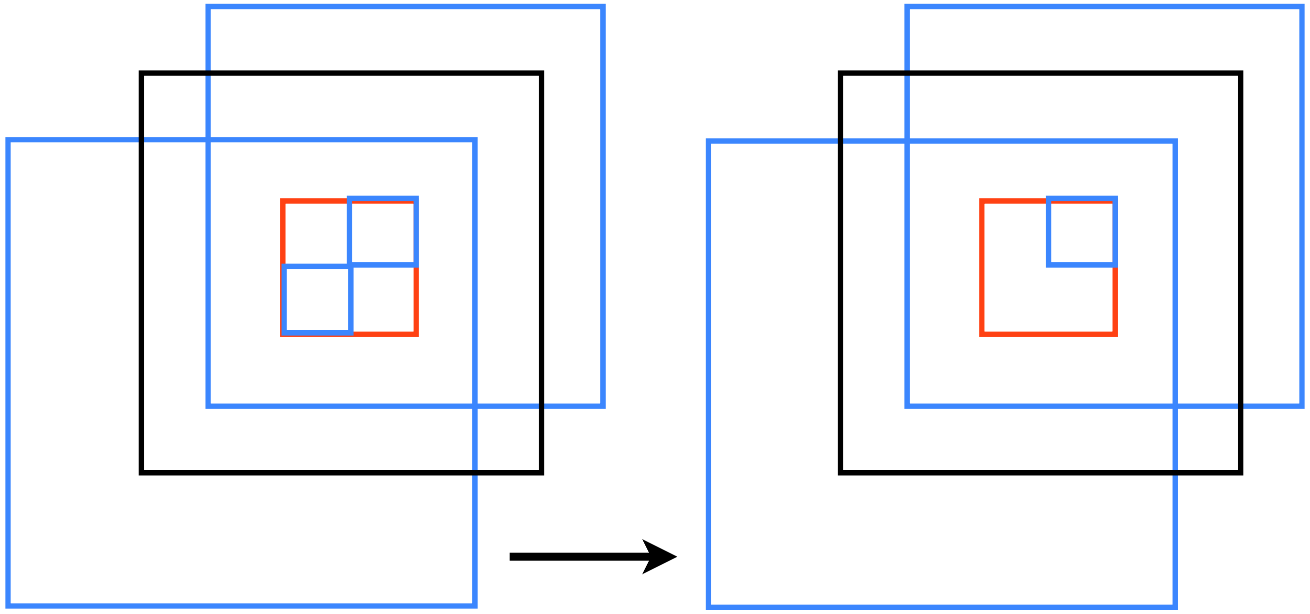


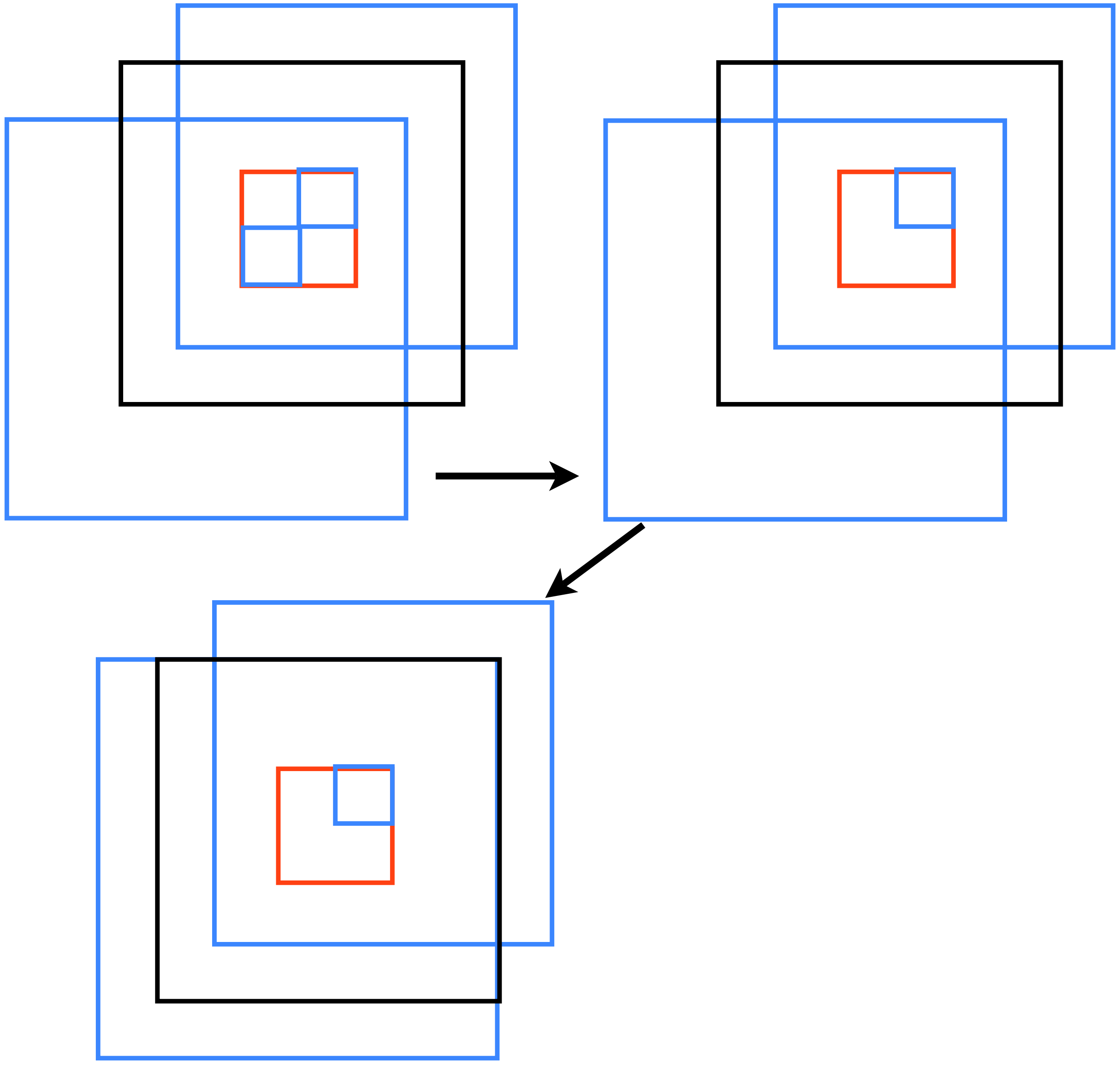


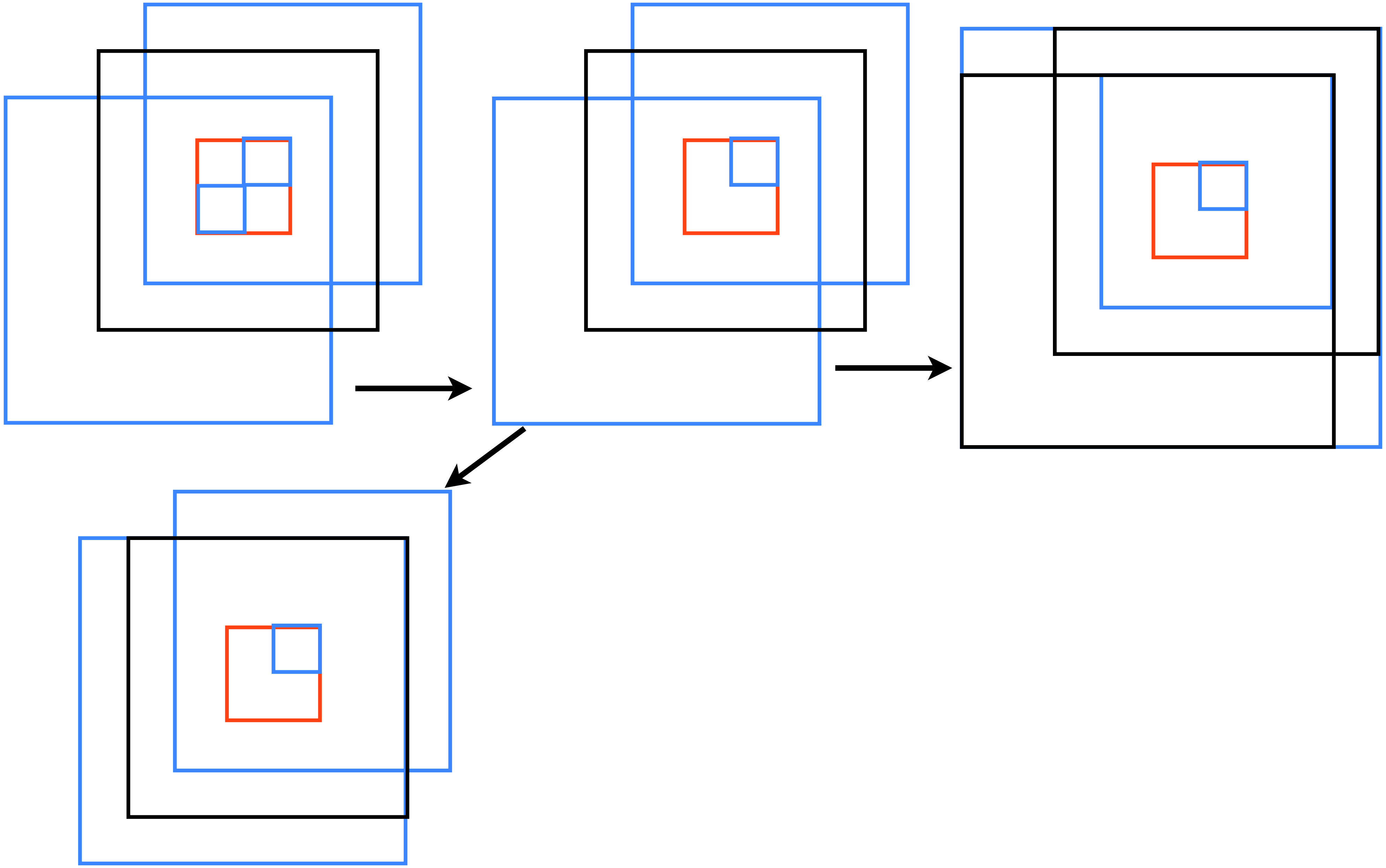


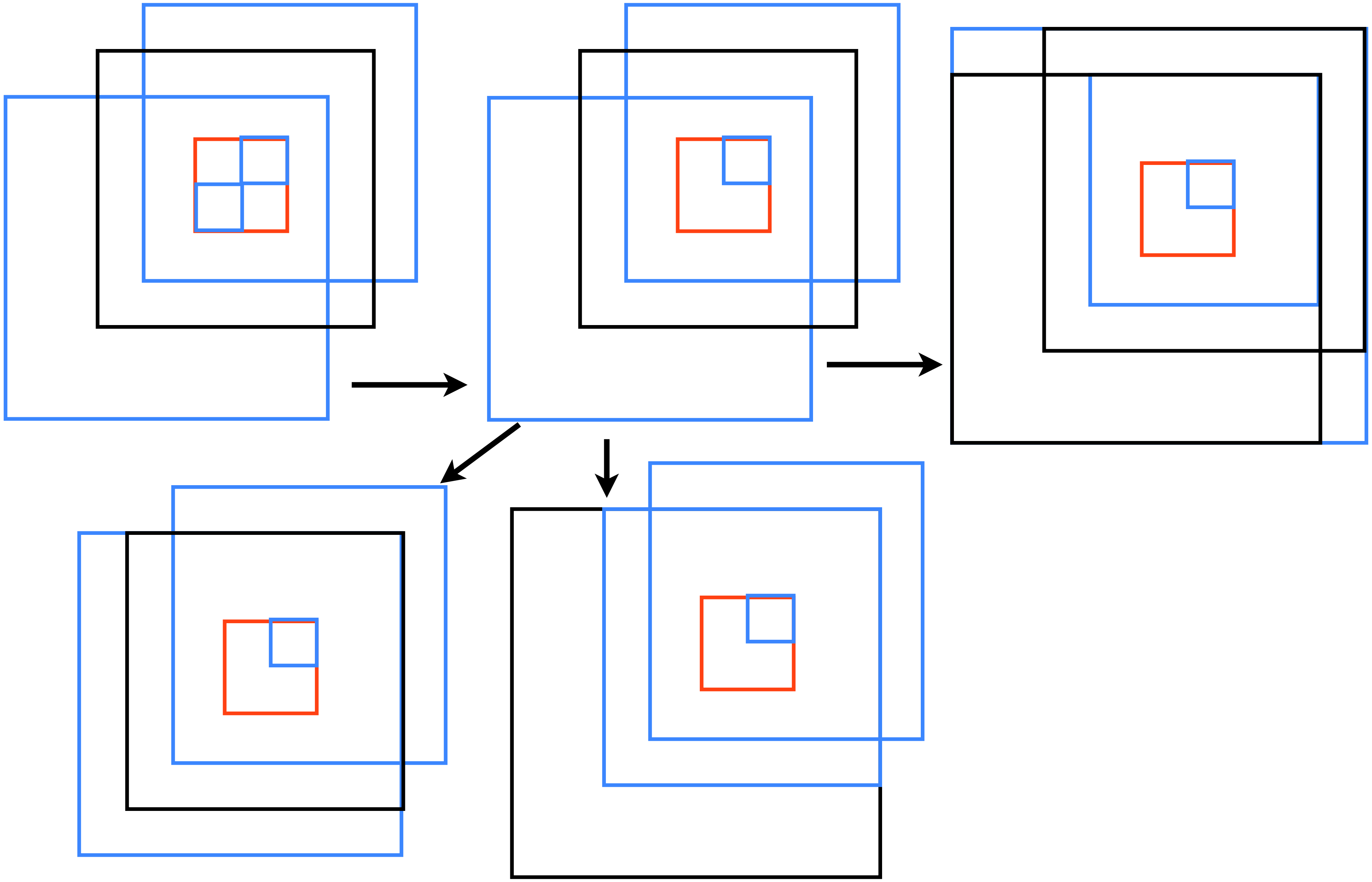


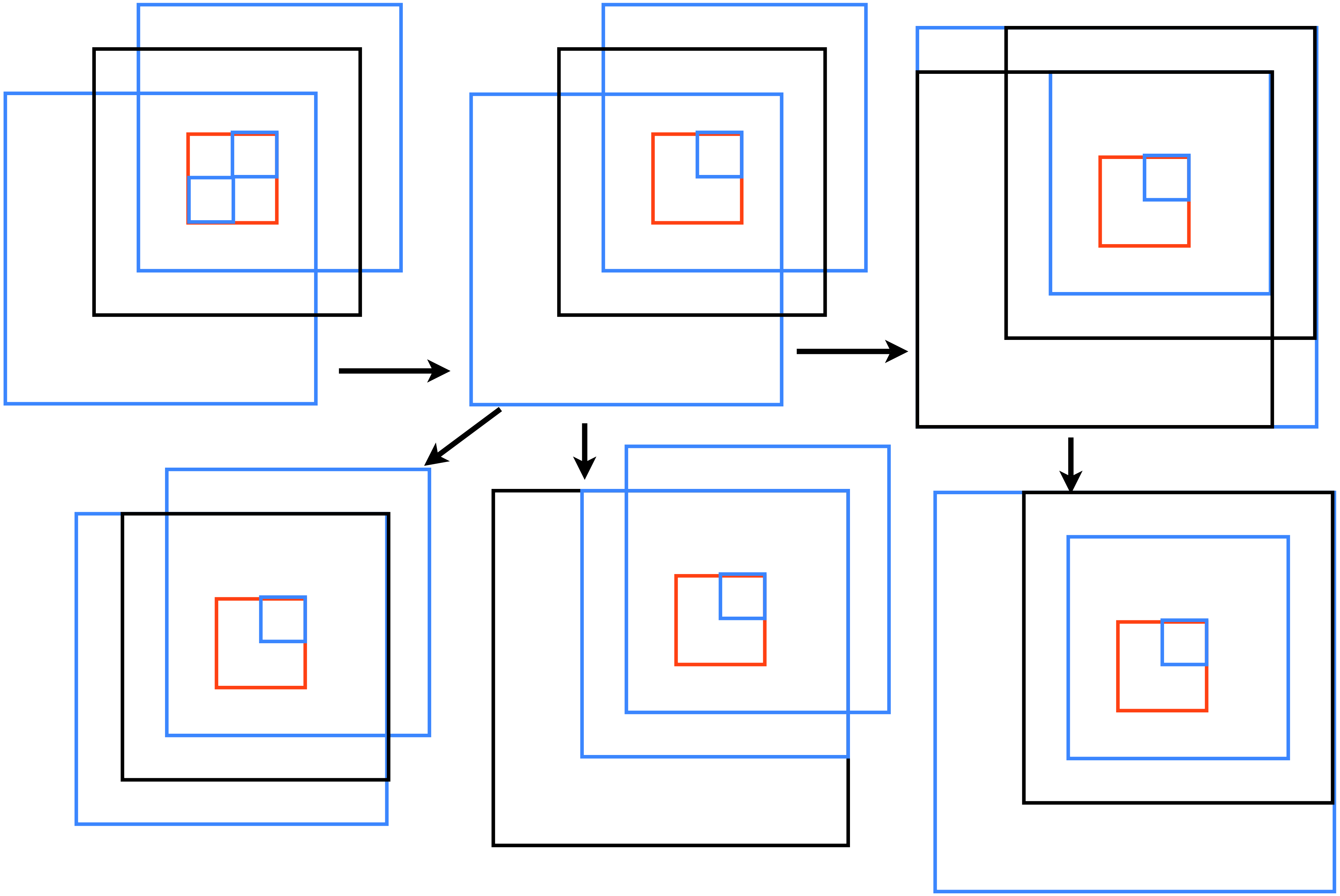


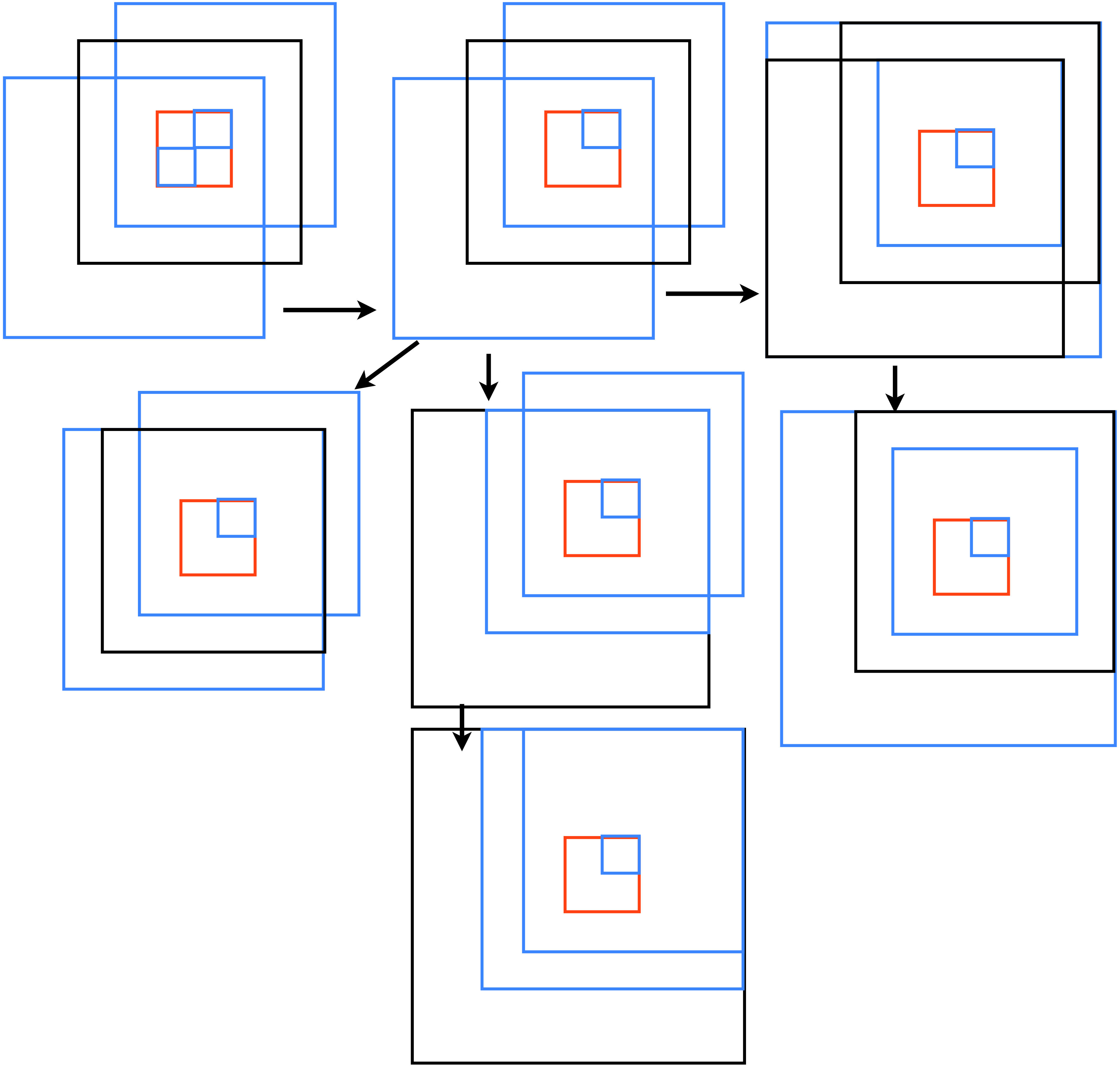


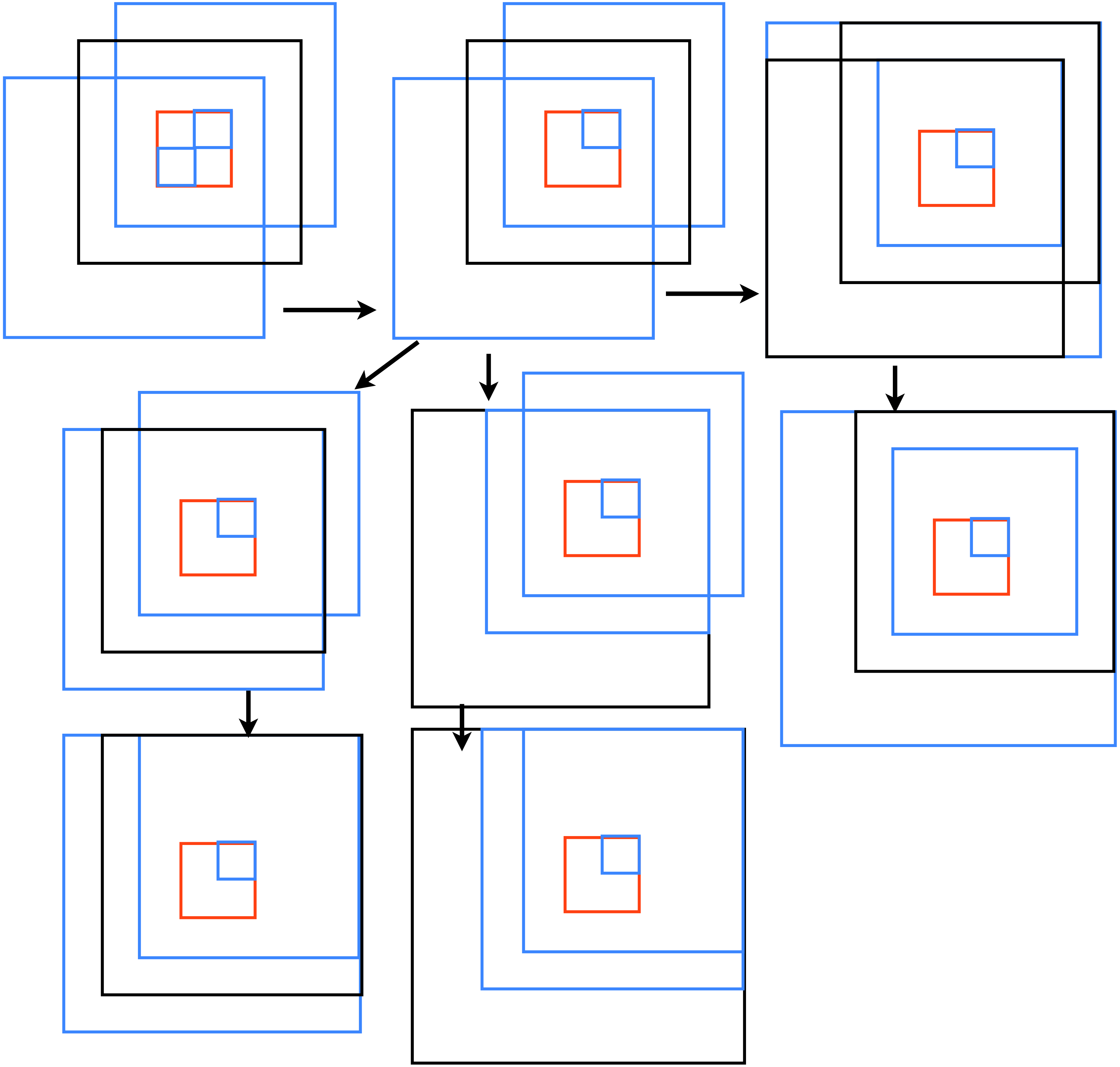


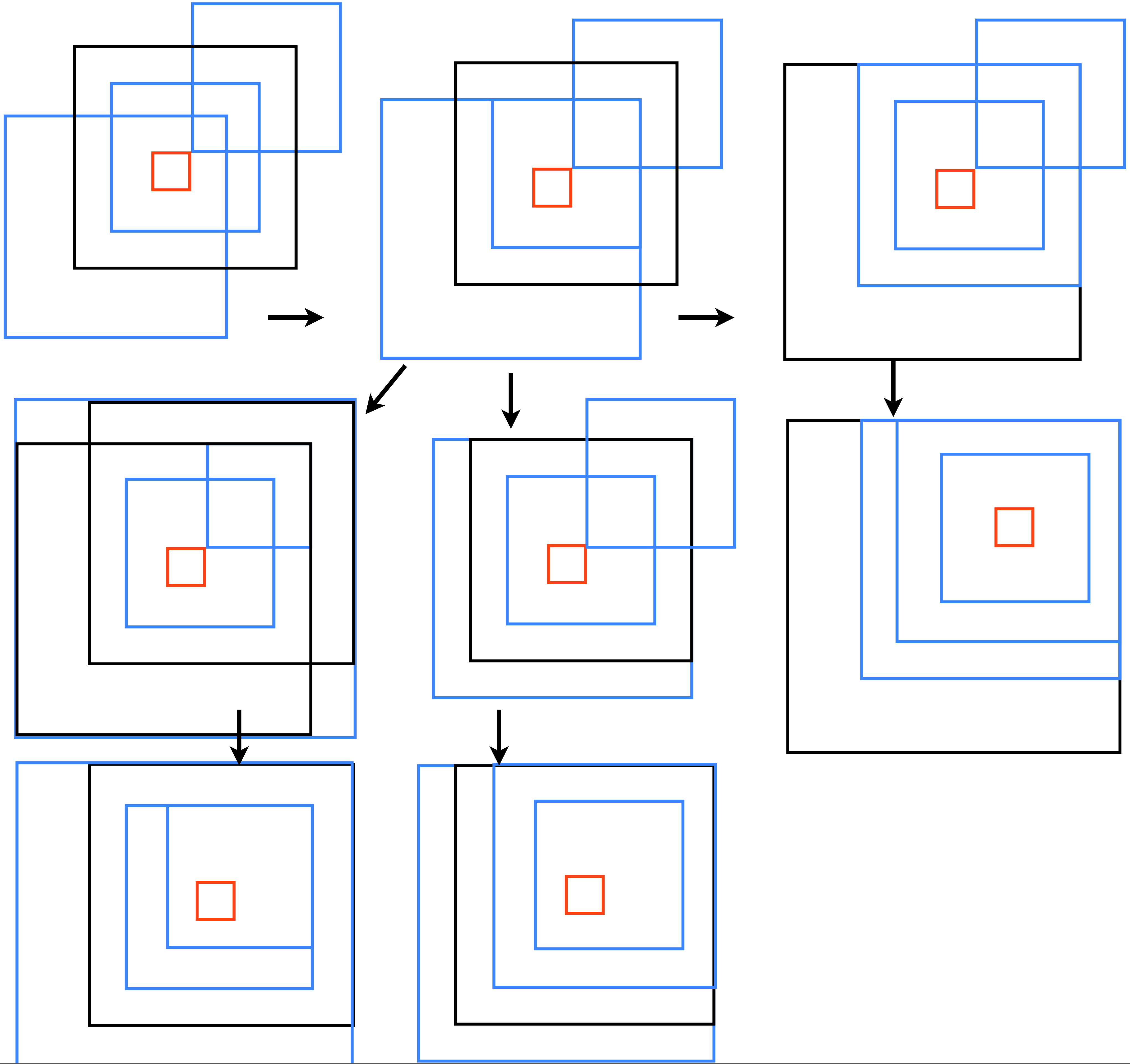


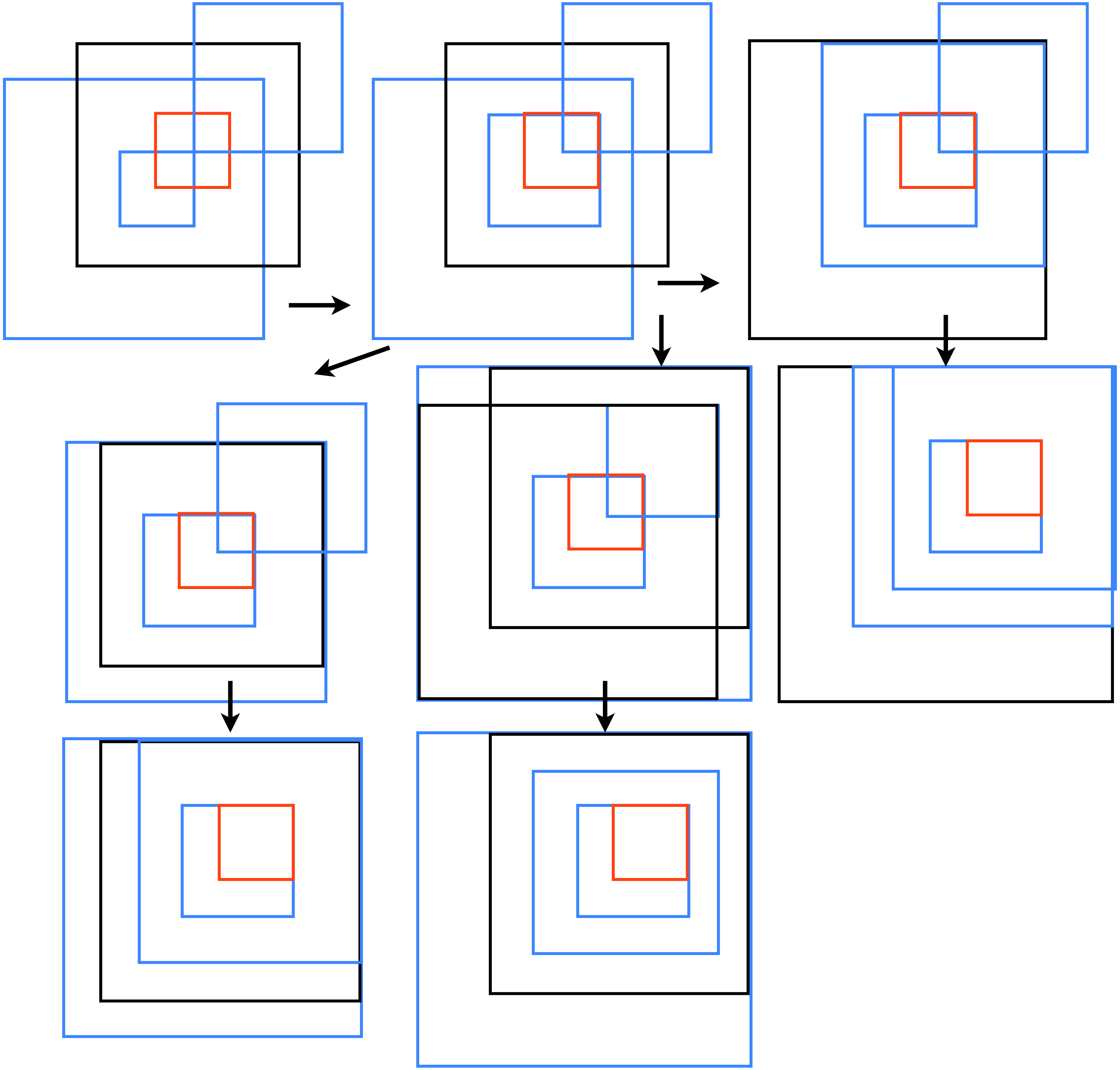


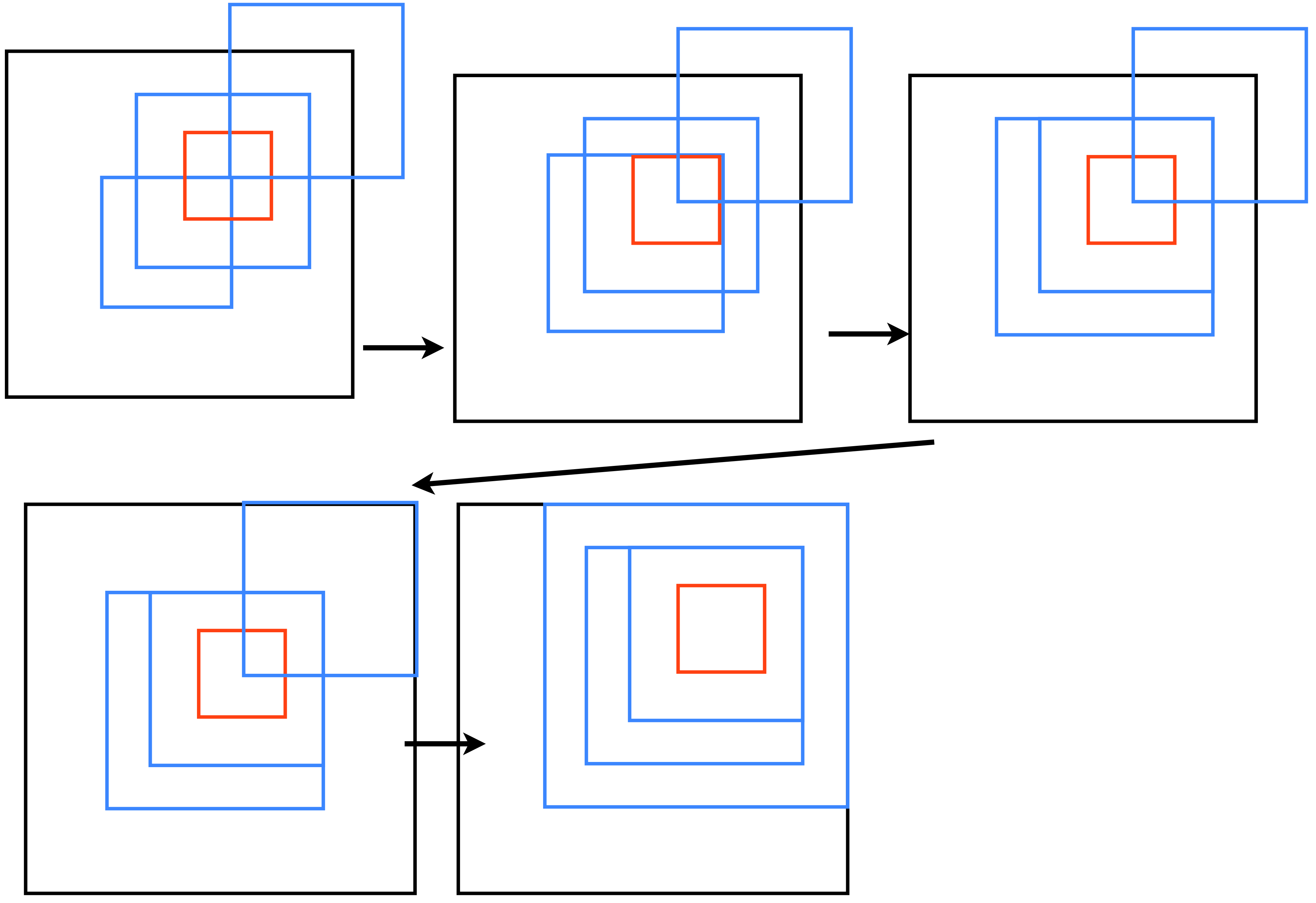


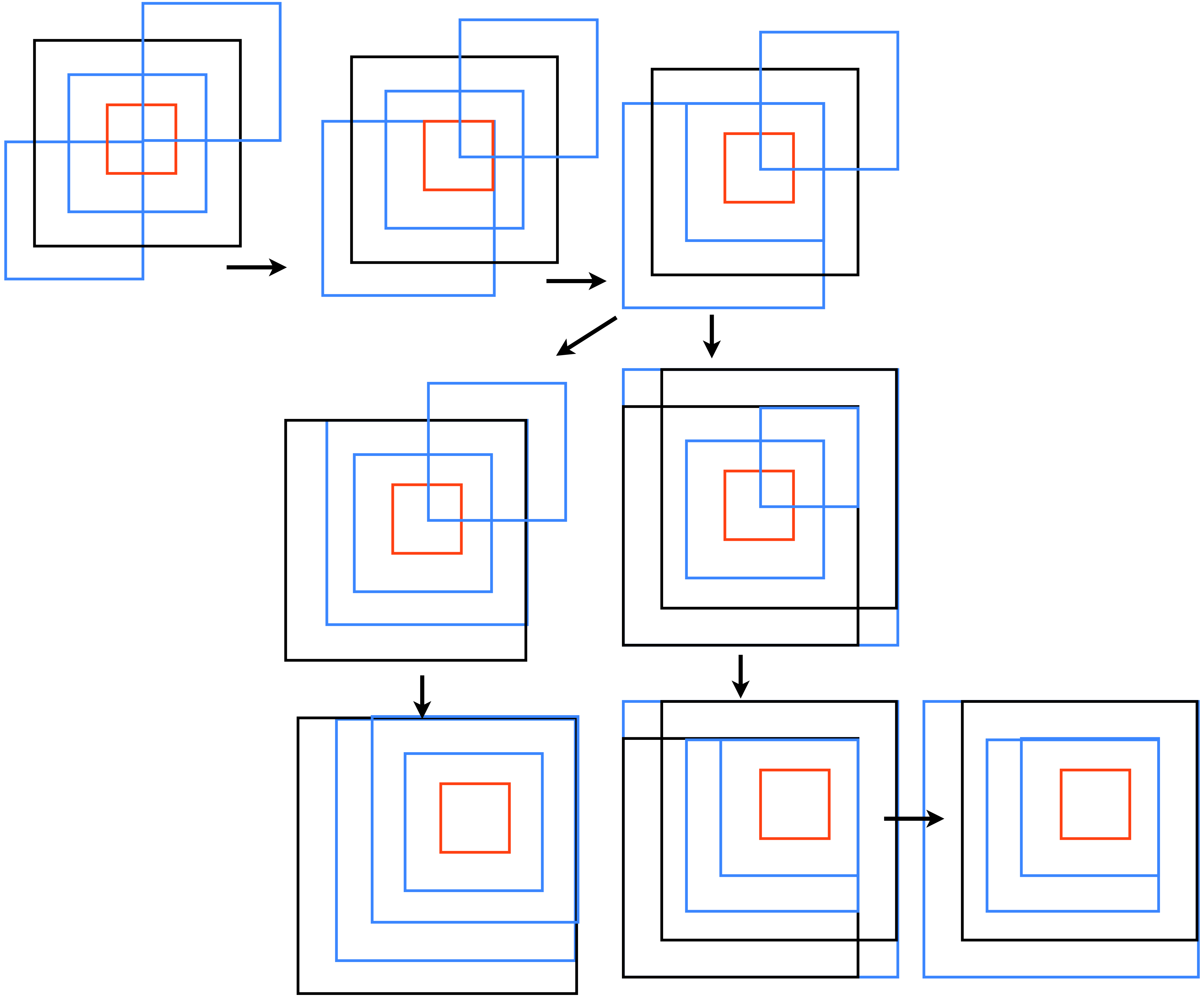












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The cohomology of orthogonal flag varieties is generated by Schubert varieties.

Q defines a smooth quadric hypersurface in $\mathbb{P}V$. $OG(k, n)$ is the Fano variety of $(k-1)$-dimensional projective linear spaces on Q.

It is useful to consider singular quadrics/degenerate forms. Let Q^r_d denote a corank r sub-quadric of Q whose span has (vector space) dimension d.

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1. The dimension of a linear space contained in Q_d^r is bounded by $r + \lfloor \frac{d-r}{2} \rfloor$. An isotropic linear space of dimension j intersects the singular locus of Q_d^r in a subspace of dimension greater than or equal to $j - \lfloor \frac{d-r}{2} \rfloor$.

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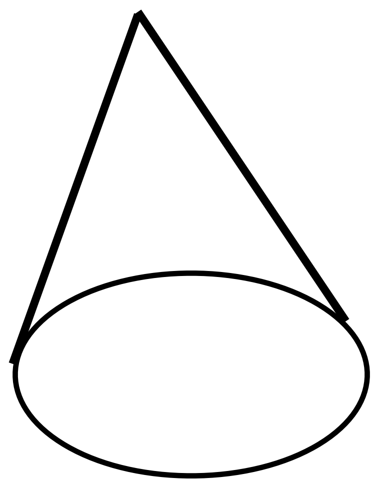

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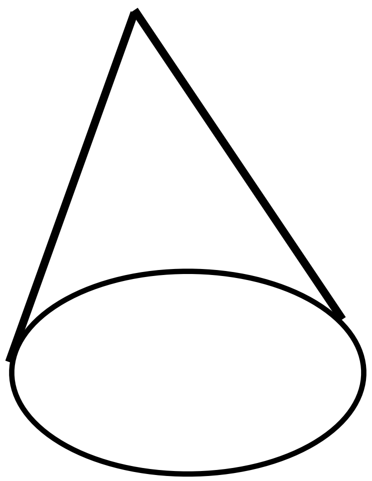

Restriction Varieties

A quadric sequence for $OG(k, n)$ is a totally ordered sequence of isotropic linear spaces and sub-quadrics of Q

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The restriction variety $V(L_\bullet, Q_\bullet)$ is the Zariski closure of the locus

$$\begin{aligned} V^0(L_\bullet, Q_\bullet) := \{W \in OG(k,n) \quad | \quad & \dim(W \cap L_{l_j}) = j, \\ & \dim(W \cap Q^{r_i}_{d_i}) = k - i + 1, \\ & \dim(W \cap Q^{r_i, sing}_{d_i}) = x_i\} \end{aligned}$$

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Notation:

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$$L_2 \subset L_4 \subset L_6 \subset Q^4_9 \subset Q^2_{11}$$

To define restriction varieties in orthogonal flag varieties $OF(k_1, \ldots, k_h; n)$ enrich the data by a choice of color:

$$L_{l_1}[c_1] \subset L_{l_2}[c_2] \subset \cdots \subset L_{l_s}[c_s] \subset Q^{r_{k-s}}_{d_{k-s}}[c_{s+1}] \subset \cdots \subset Q^{r_1}_{d_1}[c_{k_h}]$$

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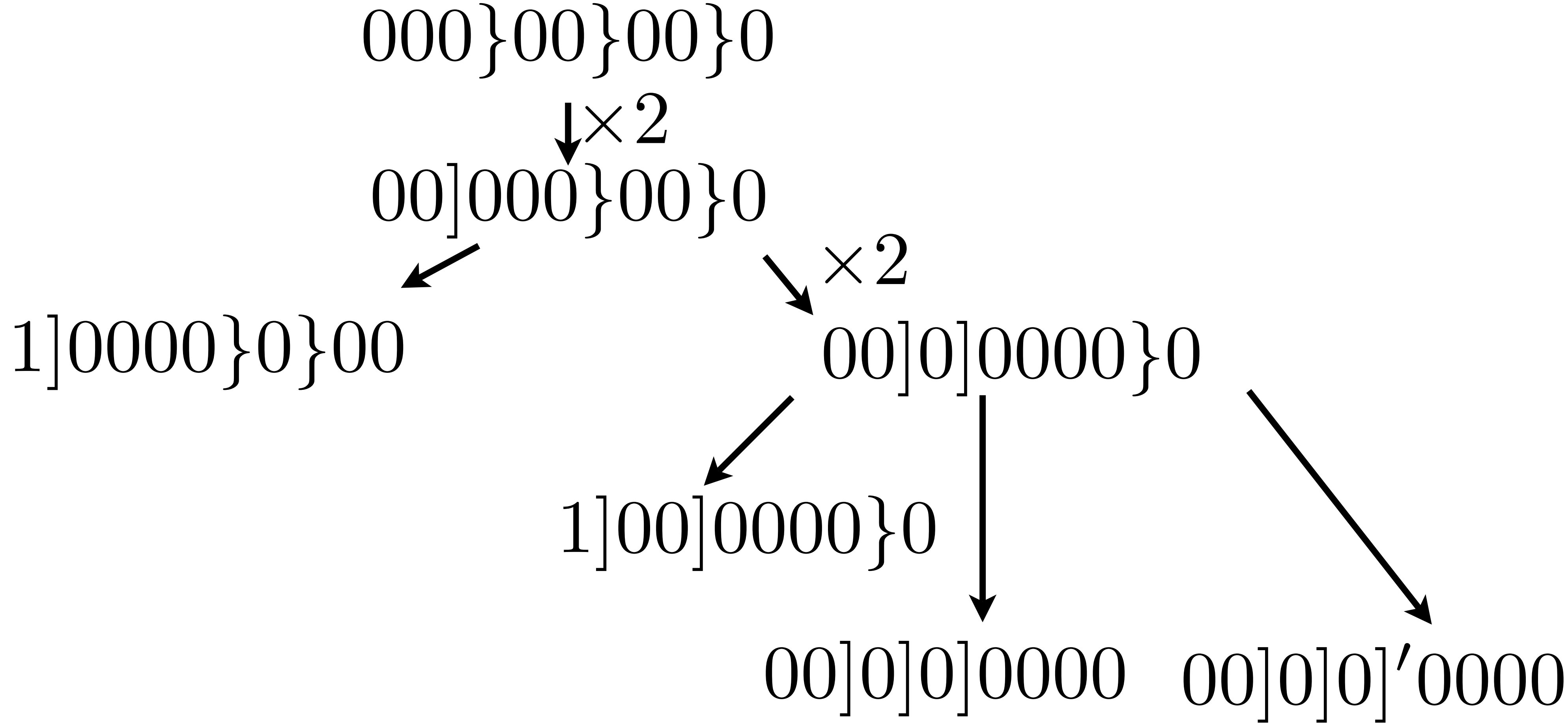
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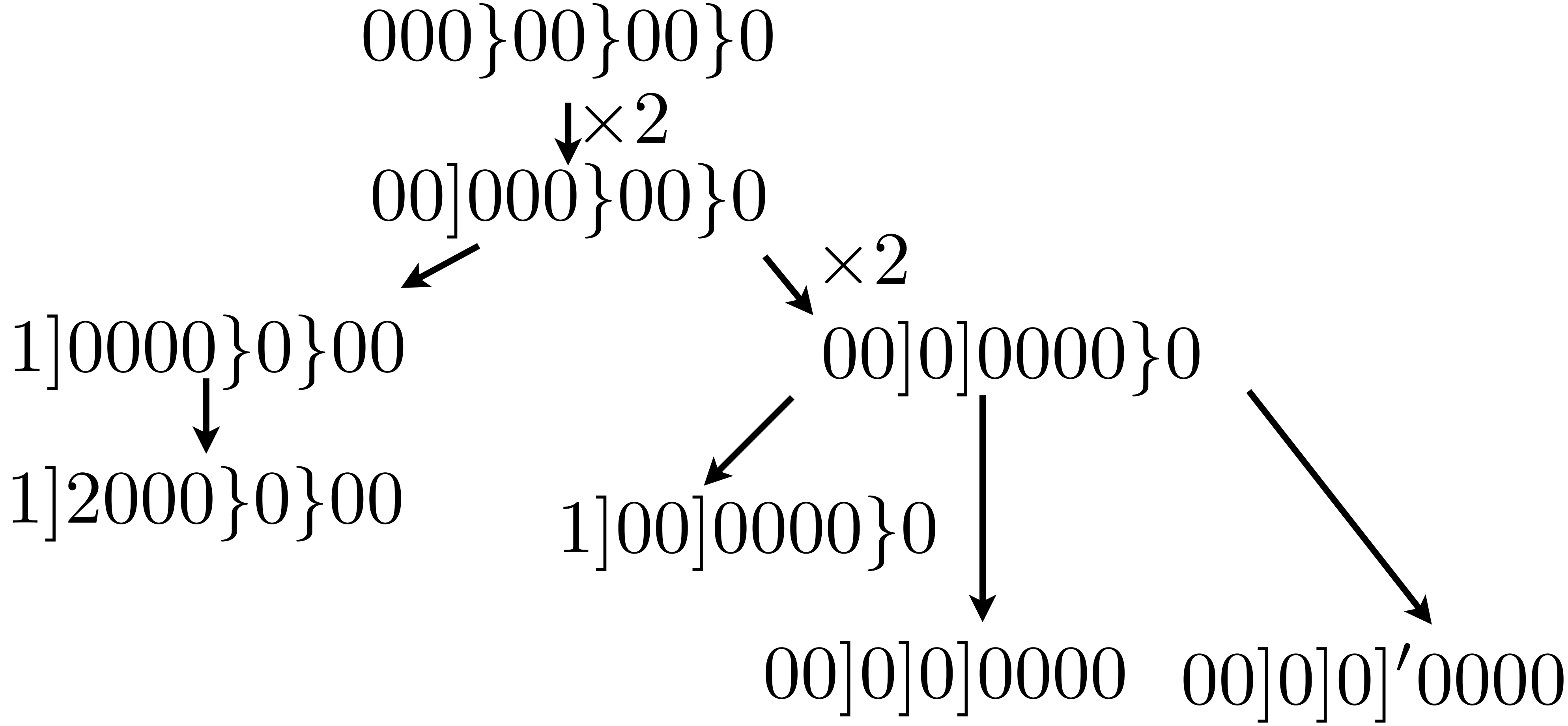
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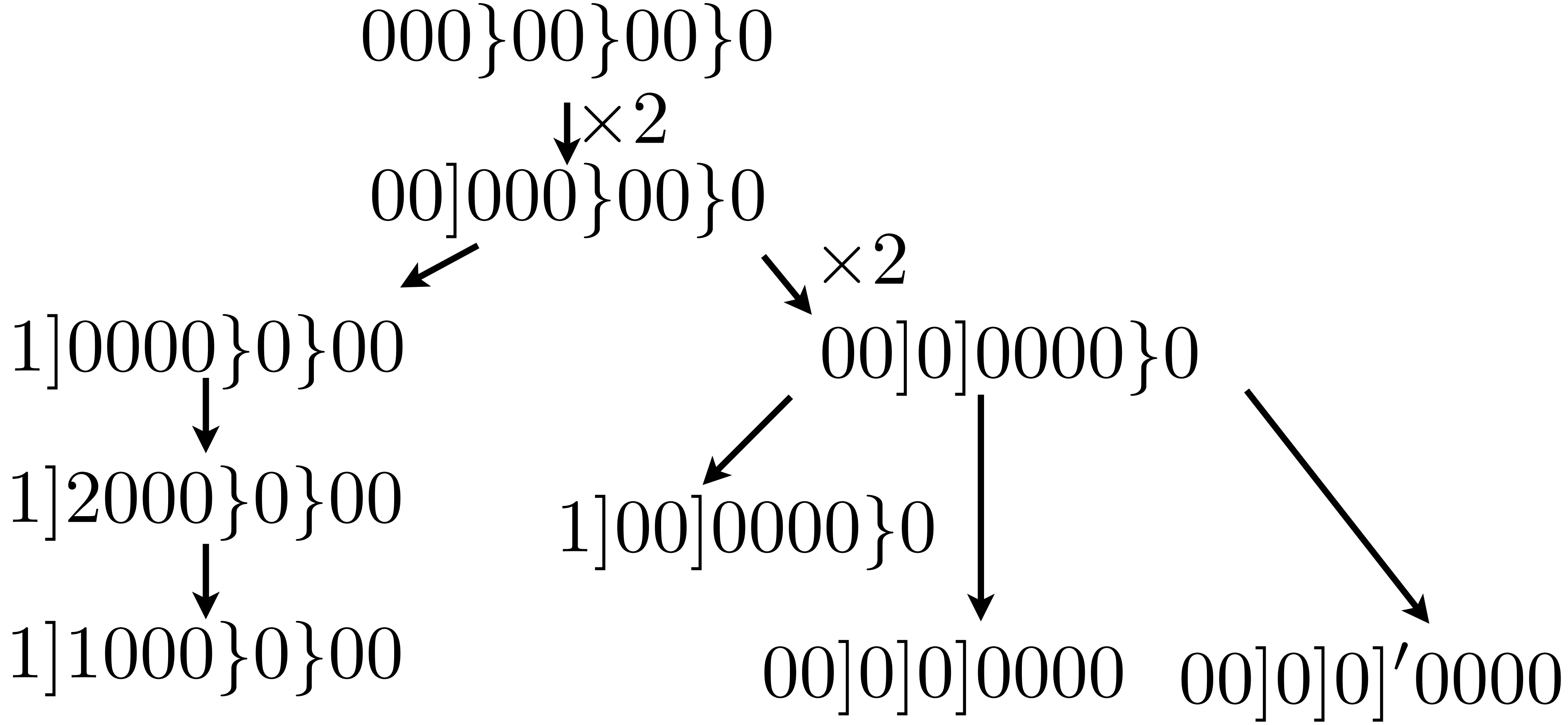
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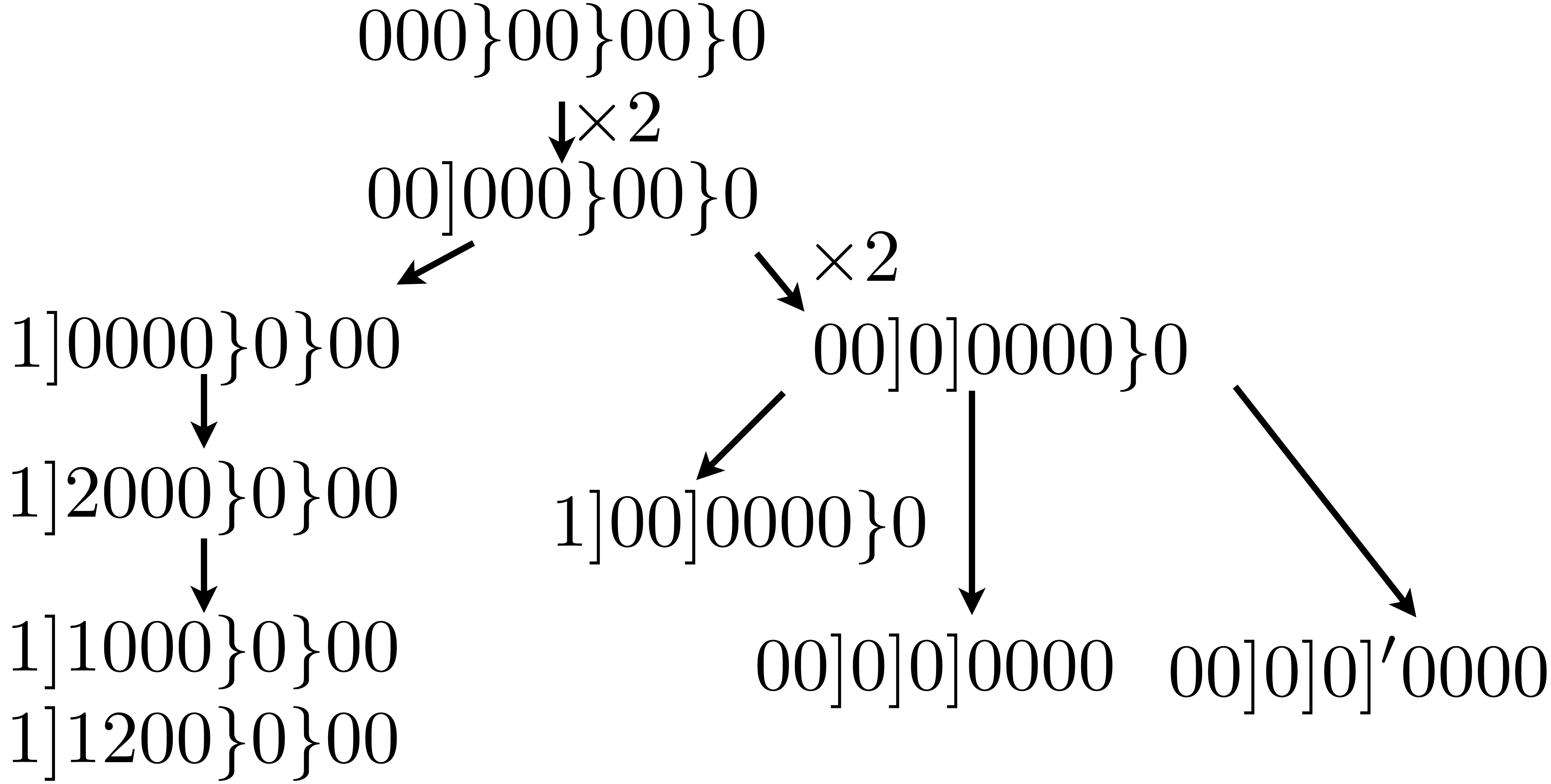
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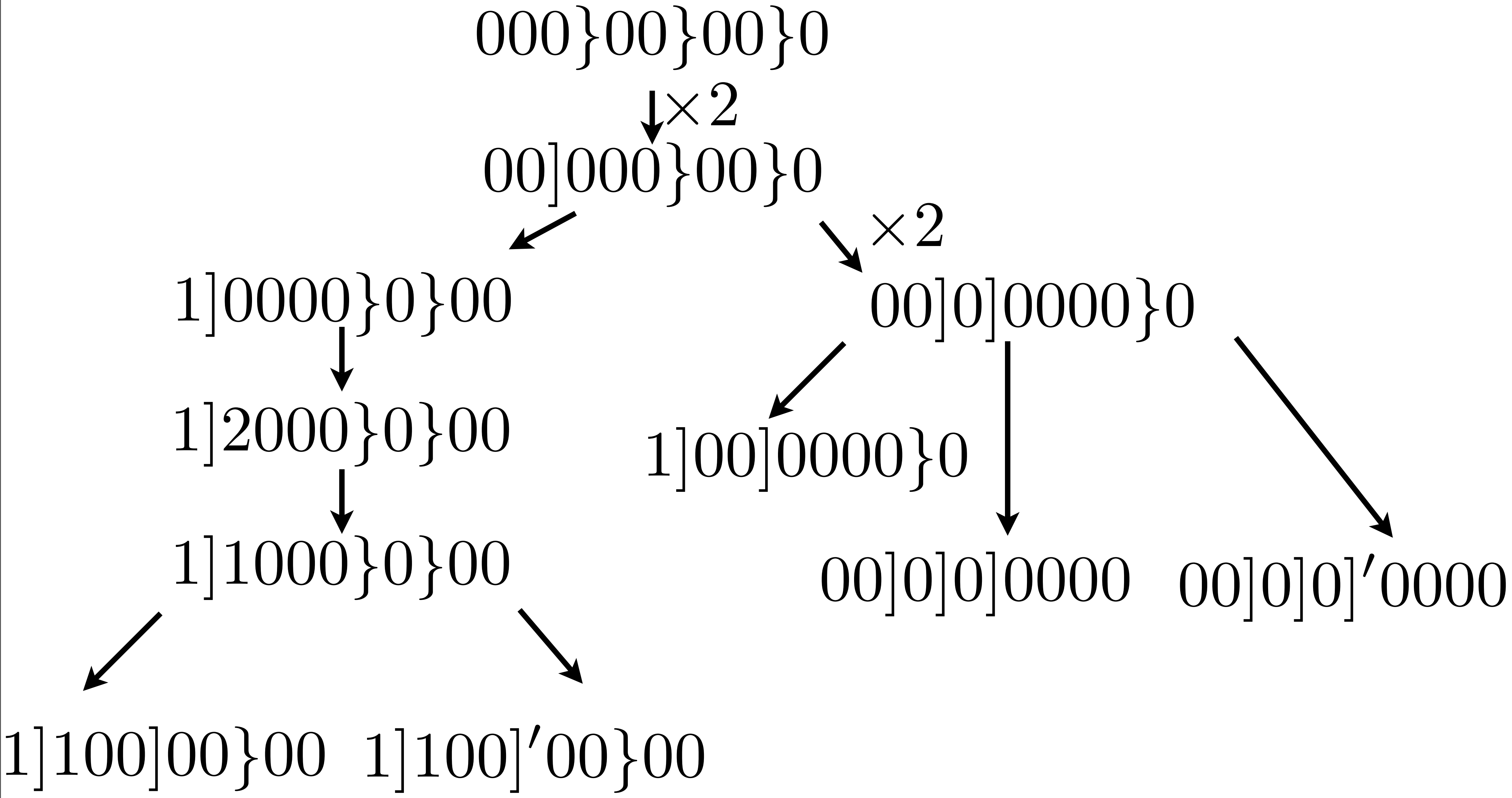
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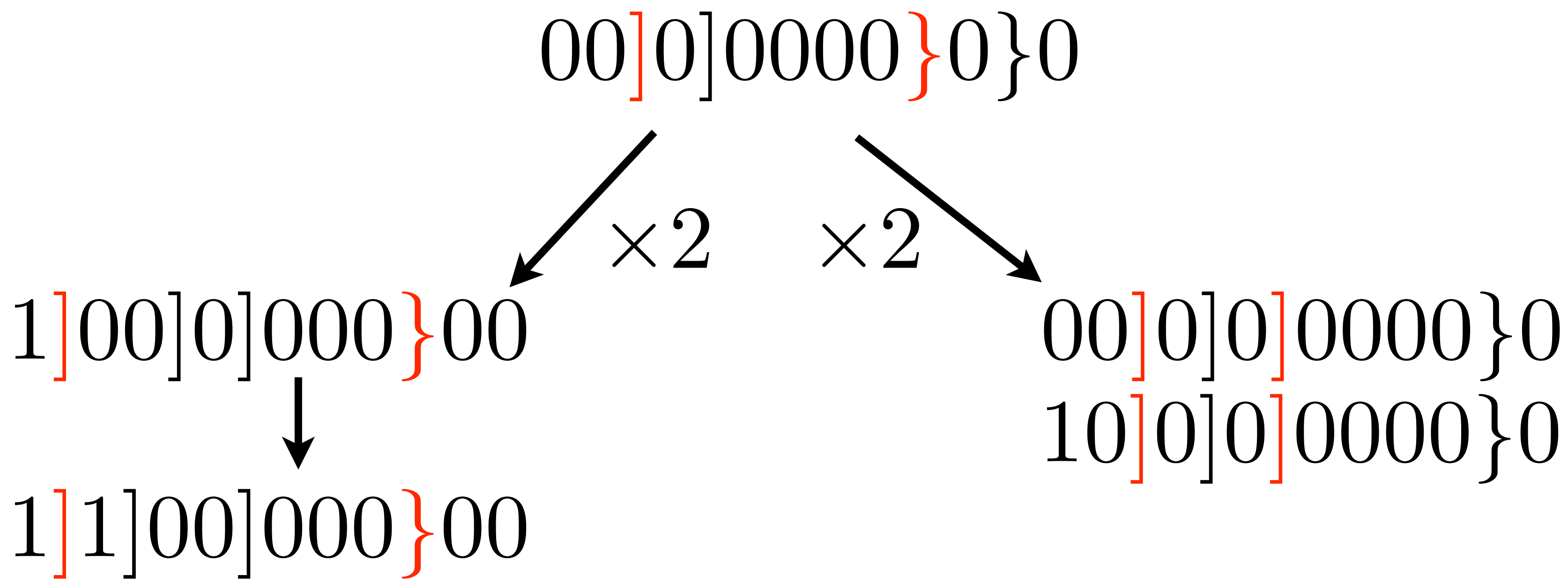
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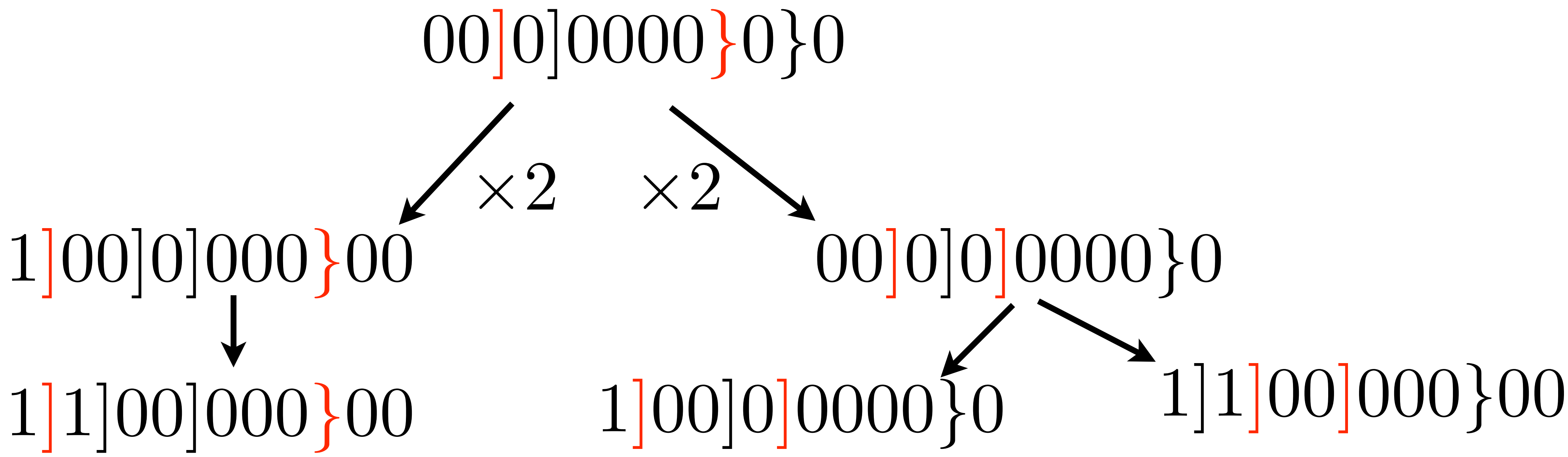
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- One can reverse the process to obtain a presentation of the cohomology ring of $OF(k_1, \dots, k_h; n)$ when n is odd and a presentation of the invariant part of the cohomology when n is even.

OG(4,11)

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$$\frac{1}{2}\;00]0000\}000\}0\}0 - \frac{1}{2}\;1]00000\}00\}0\}00 - \frac{1}{2}\;1]0000\}000\}00\}0$$

$$\mathrm{OG}(4{,}11)$$

$$11]22]000\}00\}00 \; = \; 11]00]0000\}0\}00 \; =$$

$$22]00]0000\}00\}0 \; - 1]200]0000\}00\}0 \; =$$

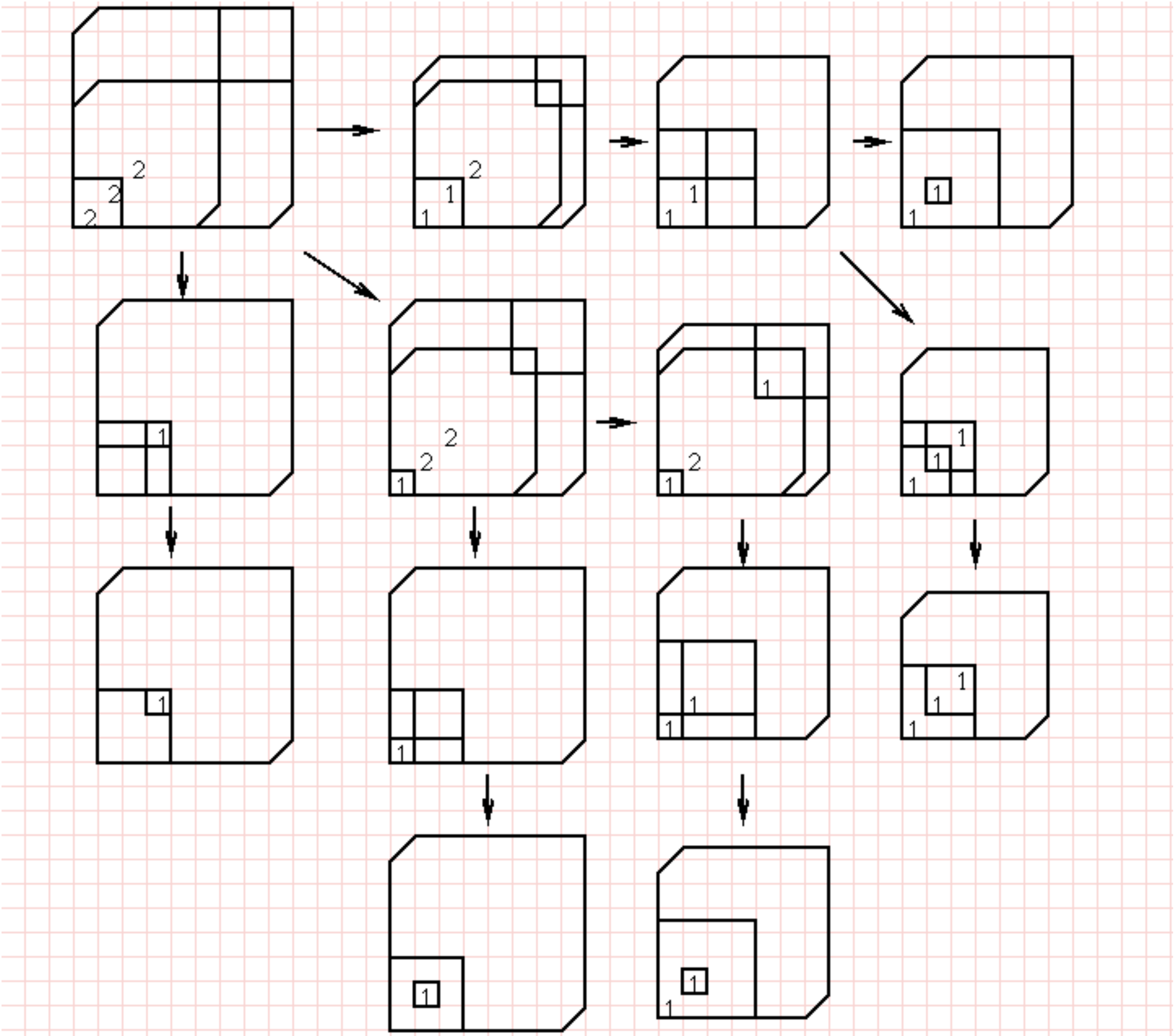
$$00]00]00000\}0\}0 \; - \; 1]000]0000\}00\}0 \; =$$

$$\frac{1}{2}\; 33]000\}0000\}0\}0 \; - \frac{1}{2}\; 1]3000\}000\}00\}0 \; =$$

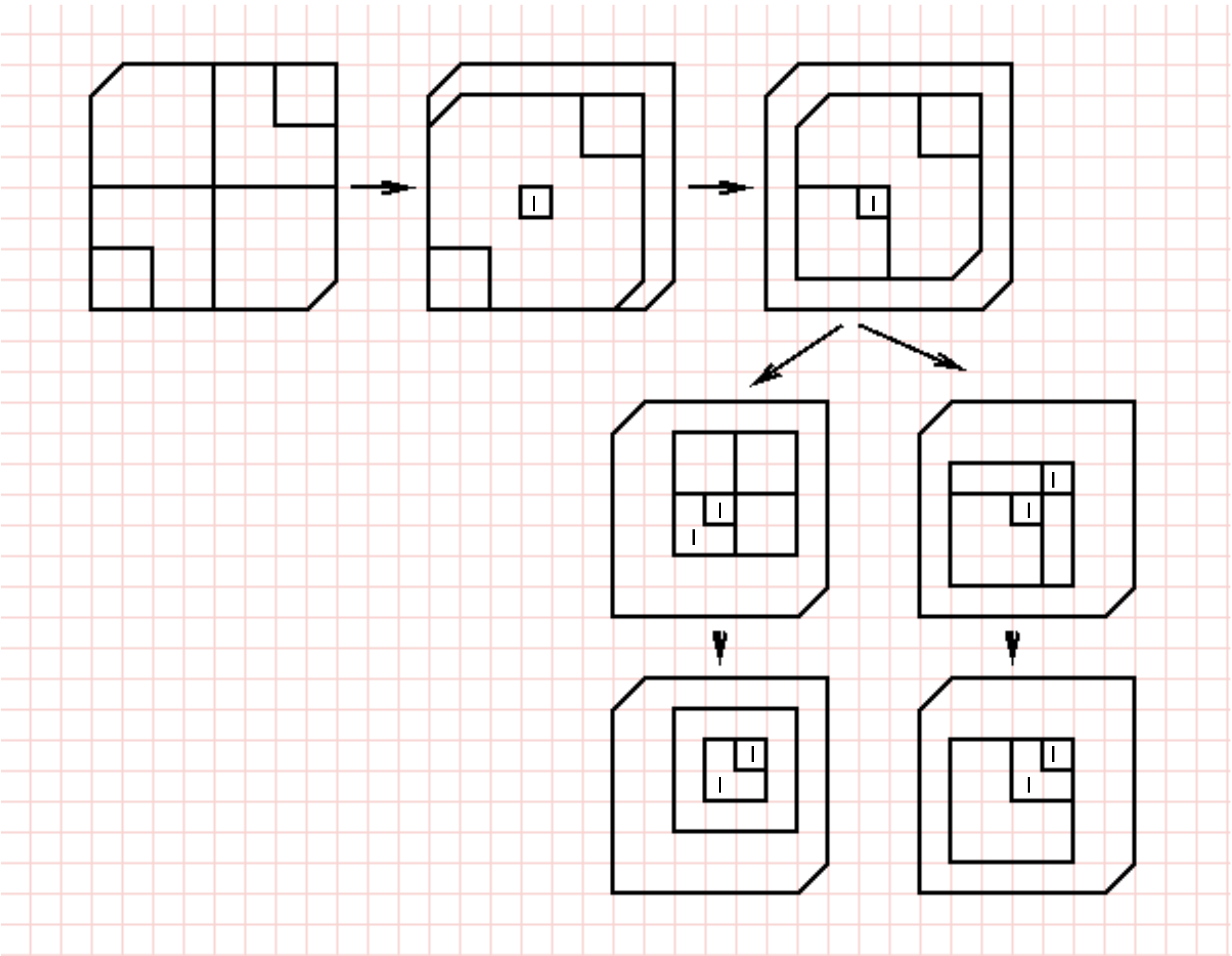
$$\frac{1}{2}\; 00]0000\}000\}0\}0 - \frac{1}{2}\; 1]00000\}00\}0\}00 - \frac{1}{2}\; 1]0000\}000\}00\}0 \; =$$

$$\frac{1}{4}\; 000\}000\}000\}0\}0 - \frac{1}{4}\; 00\}00000\}00\}0\}0 - \frac{1}{4}\; 00\}0000\}000\}00\}$$

OG(3,9)



OG(4,9)



$$\sigma_{3,1} \cdot \sigma_{3,1} = 2\sigma_{4,3,1}$$

Thank you